

A Redundant Klee-Minty Construction with All the Redundant Constraints Touching the Feasible Region

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Abstract

By introducing some redundant Klee-Minty constructions, we have previously shown that the central path may visit every vertex of the Klee-Minty cube having $2^n - 2$ “sharp” turns in dimension n . In all of the previous constructions, the maximum of the distances of the redundant constraints to the corresponding facets is an exponential number of the dimension n , and those distances are decaying geometrically. In this paper, we provide a new construction in which all of the distances are set to zero, i.e., all of the redundant constraints touch the feasible region.

Key words: Linear optimization, Klee-Minty cube, interior point methods, redundant, central path.

1 Introduction

The *simplex method* was introduced by Dantzig [1] in 1947 for solving linear optimization problems (LPs). In 1972, Klee and Minty [7] showed that the simplex method may take an exponential number of iterations to find an optimal solution. More precisely, they presented an LP over an n -dimensional squashed cube and proved that a variant of the simplex method visits all of its 2^n vertices before reaching the optimal solution. The pivot rule used in [7] was the most negative reduced cost pivot rule that is frequently referred to as “Dantzig rule”. Variants of the Klee-Minty cube have been used to prove exponential running time for most pivot rules; see [12] and the references therein for details. Stimulated mostly by the Klee-Minty worst-case example, the search for a polynomial algorithm for solving LPs has started.

In 1979, Khachiyan [6] proved that the *ellipsoid method* solves LPs in polynomial time. In spite of its polynomial iteration complexity, the ellipsoid method turned out to be inefficient in computational practice. In 1984, Karmarkar [5] proposed a polynomial time algorithm with a better complexity bound that sparked the research on polynomial time *interior point methods* (IPMs). Unlike the simplex method that goes along the edges of the polytope corresponding to the feasible region, IPMs pass through the interior of this polytope. Starting at a neighborhood of the *analytic center*¹, most IPMs follow the so-called *central path*¹ and converge to the analytic center of the optimal face; see for example [10, 13]. It is well known that the number of iterations needed to have the duality gap smaller than ϵ is bounded above by $O(\sqrt{N} \ln \frac{v_0}{\epsilon})$, where N and v_0 denote the number of inequalities and the duality gap at the starting point, respectively.

¹See page 3 for the definition of the analytic center and the central path.

The standard rounding procedure [10] can be used to compute an exact optimal solution with the choice of $\epsilon \leq 2^{-O(L)}$, where L is the bit-length of the input data. In this case, the iteration complexity becomes $O(\sqrt{N}L)$.

In [2], the authors show that the central path of a redundant representation of the Klee-Minty cube may trace the *simplex path*—the edge-path followed by the simplex method. More precisely, an exponential number of redundant constraints parallel to the facets passing through the optimal vertex are added to the Klee-Minty cube to force the central path to visit an arbitrarily small neighborhood of each vertex of that cube, thus having $2^n - 2$ “sharp” turns. The distances of the redundant constraints to the corresponding facets are chosen to be uniform (and at least $d \geq n2^{n+1}$), and the number of the inequalities is required to be at least $N = O(n^2 2^{6n})$, which is further improved to $N = O(n2^{3n})$ in [4] by a meticulous analysis. In [3], those distances are allowed to decay geometrically as $d = n(2^{n+4}, \dots, 2^{n-k+5}, \dots, 2^5)^T$, and the number of the redundant constraints is significantly reduced to $N = O(n^3 2^{2n})$. A simplified construction, where the number of the redundant constraints is further reduced to $N = O(n2^{2n})$, is presented in [9] by placing the redundant constraints parallel to the coordinate hyperplanes at geometrically decaying distances $d \approx (2^{n-1}, \dots, 2^{n-k}, \dots, 2, 0)^T$.

In this paper, we present a new redundant Klee-Minty construction with $d = 0$. In other words, all of the redundant constraints touch the feasible region. The number of the redundant constraints is required to be $N = O(2^{n^2})$, which is exponentially larger than those of the previous constructions.

The rest of the paper is organized as follows. In Section 2, we introduce our redundant Klee-Minty problems and present the main results as three propositions, whose proofs are provided in Section 3. We use the following notations. The largest integer smaller than a scalar α is denoted by $\lfloor \alpha \rfloor$. For any vector $x = (x_1, \dots, x_n)^T$, the vector $(\lfloor x_1 \rfloor, \dots, \lfloor x_n \rfloor)^T$ is denoted by $\lfloor x \rfloor$. The unique maximizer of a strictly concave function $f(x)$ over a convex set $S \subseteq \mathbb{R}^n$ is denoted by $\arg \max_{x \in S} f(x)$.

2 The Main Results

We consider the following Klee-Minty problem [7], with the convention $y_0 = 0$, where τ is a small positive factor by which the unit cube $[0, 1]^n$ is squashed.

$$\begin{aligned} \min \quad & y_n \\ \text{subject to} \quad & \tau y_{k-1} \leq y_k \leq 1 - \tau y_{k-1}, \quad \text{for } k = 1, \dots, n. \end{aligned} \tag{1}$$

The polytope represented by all of the $2n$ inequalities of (1) is denoted by \mathcal{C}_0^n . Variants of the simplex method may take $2^n - 1$ iterations to solve this problem; see the survey paper [12] and the references therein. Starting from the vertex $(0, \dots, 0, 1)^T$, they may visit all of the vertices of the polytope ordered by the decreasing values of the last coordinate y_n until reaching the optimal point, which is the origin.

We consider redundant constraints induced by the halfplanes $y_k \geq 0$ repeated h_k times, for $k = 1, \dots, n$. Therefore, the problem that we are interested in is

$$\begin{aligned} \min \quad & y_n \\ \text{subject to} \quad & \tau y_{k-1} \leq y_k \leq 1 - \tau y_{k-1}, \quad \text{for } k = 1, \dots, n, \\ & 0 \leq y_k, \quad \text{repeated } h_k \text{ times, for } k = 1, \dots, n. \end{aligned} \tag{2}$$

Let $h = (h_1, \dots, h_n)^T$ be the repetition vector. We denote by \mathcal{C}_h^n the polytope represented by all of the constraints of (2). Obviously, problem (1) is a special case of (2) with $h = 0$. Observe that all of the redundant constraints of (2) touch the feasible region.

Let $y = (y_1, \dots, y_n)^T$. The *analytic center* of the polytope \mathcal{C}_h^n is defined to be the unique solution of the strictly concave maximization problem

$$\chi^h = \arg \max_y \sum_{k=1}^n (\ln s_k + \ln \bar{s}_k + h_k \ln y_k),$$

where, for $k = 1, \dots, n$,

$$s_k = y_k - \tau y_{k-1}, \quad \bar{s}_k = 1 - y_k - \tau y_{k-1}.$$

It can be seen that $\chi^h \neq \chi^0$ for $h \neq 0$ due to the fact that the polytopes represented by \mathcal{C}_h^n and \mathcal{C}_0^n coincide, although their actual algebraic representations are different.

Remark 2.1. To avoid confusion, we denote the identical feasible regions of (1) and (2) by \mathcal{C}^n , which is also well known as the Klee-Minty cube. Note that the analytic center of \mathcal{C}^n cannot be defined without making its algebraic representation clear, while the analytic centers of \mathcal{C}_h^n and \mathcal{C}_0^n are well defined.

From the necessary and sufficient optimality conditions (the gradient is equal to zero), the analytic center χ^h is the unique solution of the system

$$\begin{cases} \frac{1}{s_k} - \frac{\tau}{s_{k+1}} - \frac{1}{\bar{s}_k} - \frac{\tau}{\bar{s}_{k+1}} + \frac{h_k}{y_k} = 0, & \text{for } k = 1, \dots, n-1, \\ \frac{1}{s_n} - \frac{1}{\bar{s}_n} + \frac{h_n}{y_n} = 0, \\ s_k > 0, \bar{s}_k > 0, y_k > 0, & \text{for } k = 1, \dots, n. \end{cases} \quad (3)$$

The *central path* of (2) is defined to be the set

$$\mathcal{P}_h = \left\{ y(\mu) \mid y(\mu) = \arg \max_y -y_n + \mu \sum_{k=1}^n (\ln s_k + \ln \bar{s}_k + h_k \ln y_k), \quad \mu > 0 \right\}.$$

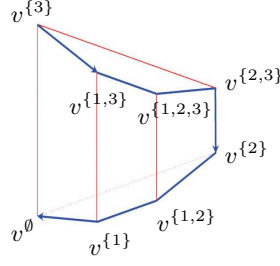
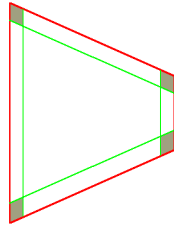
It is easy to see that any point $y(\mu)$ on the central path \mathcal{P}_h satisfies all of the equations of (3), except the last one.

By analogy with the unit cube $[0, 1]^n$, we denote the vertices of the Klee-Minty cube \mathcal{C}^n by using subsets of $\{1, \dots, n\}$. For $S \subseteq \{1, \dots, n\}$, a vertex $v^S = (v_1^S, \dots, v_n^S)^T$ of \mathcal{C}^n is defined by

$$v_k^S = \begin{cases} 1 - \tau v_{k-1}^S, & \text{if } k \in S \\ \tau v_{k-1}^S, & \text{otherwise,} \end{cases} \quad k = 1, \dots, n,$$

with the convention that $v_0^S = 0$. The definition is illustrated by Figure 1. The δ -neighborhood of a vertex v^S is defined, see Figure 2, by

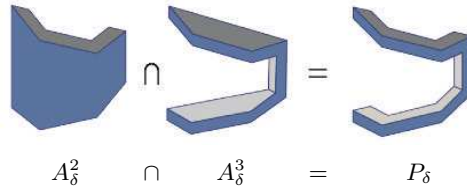
$$\mathcal{N}_\delta(v^S) = \left\{ y \in \mathcal{C}^n \mid \begin{cases} \bar{s}_k \leq \delta, & \text{if } k \in S \\ s_k \leq \delta, & \text{otherwise} \end{cases} \quad k = 1, \dots, n \right\}.$$

Figure 1: The vertices of \mathcal{C}^3 and the simplex path P_0 .Figure 2: The δ -neighborhoods of the 4 vertices of \mathcal{C}^2 .

To ease the analysis, we provide a mathematical definition for the simplex path and its δ -neighborhood in \mathcal{C}^n . For this purpose, we first define, for $k = 2, \dots, n$, the sets

$$T_\delta^k = \{y \in \mathcal{C}^n \mid \bar{s}_k \leq \delta\}, \quad C_\delta^k = \{y \in \mathcal{C}^n \mid \bar{s}_k > \delta, s_k > \delta\}, \quad B_\delta^k = \{y \in \mathcal{C}^n \mid s_k \leq \delta\},$$

and the set $\hat{C}_\delta^k = \{y \in \mathcal{C}^n \mid \bar{s}_k \leq \delta, s_{k-1} \leq \delta, \dots, s_1 \leq \delta\}$, for $k = 1, \dots, n$. Visually, the sets T_δ^k , C_δ^k , and B_δ^k can be considered as the top, central, and bottom parts of \mathcal{C}^n , and obviously $\mathcal{C}^n = T_\delta^k \cup C_\delta^k \cup B_\delta^k$, for $k = 1, \dots, n$. Then, a δ -neighborhood of the simplex path, see Figure 3, might be given as $P_\delta = \bigcap_{k=2}^n A_\delta^k$, where $A_\delta^k = T_\delta^k \cup \hat{C}_\delta^{k-1} \cup B_\delta^k$, for $k = 2, \dots, n$. The simplex path itself, see Figure 1, is precisely determined by $P_0 = \bigcap_{k=2}^n A_0^k$.

Figure 3: The set P_δ , the δ -neighborhood of the simplex path, for \mathcal{C}^3 .

In the rest of this section, we focus on (2) with the following choice of parameters

$$\bar{\tau} = \frac{n}{2(n+1)}, \quad \bar{\delta} = \frac{1}{4(n+1)}, \quad \bar{h} = \left[\left(\frac{2}{\bar{\delta}}, \frac{4}{\bar{\tau}\bar{\delta}^2}, \dots, \frac{2n}{\bar{\tau}n(n-1)/2\bar{\delta}n} \right)^T \right].$$

The resulting redundant minimization problem, which depends only on n , is referred to as problem (RP^n) . For the sake of simplicity, we denote \mathcal{C}_h^n of (RP^n) by $\bar{\mathcal{C}}^n$. Obviously, $\bar{\tau} + \bar{\delta} < 1/2$,

so the $\bar{\delta}$ -neighborhoods of the 2^n vertices of \bar{C}^n are non-overlapping. The analytic center of \bar{C}^n is denoted by χ^n , and the central path of (RP^n) is denoted by \mathcal{P}^n . The following proposition gives the number of inequalities of (RP^n) . Its proof is given in Section 3.3.

Proposition 2.1. *The number of the inequalities in (RP^n) is $N = O(2^{n^2})$.*

Proposition 2.2 is to ensure that the analytic center χ^n is in the $\bar{\delta}$ -neighborhood of the vertex $v^{\{n\}}$, which is precisely \hat{C}_δ^n . The proof of the proposition is presented in Section 3.4.

Proposition 2.2. *The analytic center χ^n of \bar{C}^n is in the $\bar{\delta}$ -neighborhood of $v^{\{n\}}$, i.e., $\chi^n \in \hat{C}_\delta^n$.*

Proposition 2.3 states that the central path \mathcal{P}^n of (RP^n) takes at least $2^n - 2$ turns before converging to the origin as it stays in the $\bar{\delta}$ -neighborhood of the simplex path. Thus, the central path \mathcal{P}^n visits the $\bar{\delta}$ -neighborhoods of all of the 2^n vertices of C^n . The proof of the proposition is presented in Section 3.5.

Proposition 2.3. *The central path \mathcal{P}^n of (RP^n) stays in the $\bar{\delta}$ -neighborhood of the simplex path of C^n , i.e., $\mathcal{P}^n \subset P_{\bar{\delta}}$.*

3 The Proofs of Propositions 2.1, 2.2, and 2.3

3.1 Preliminary Results

We show that the central path \mathcal{P}_h of (2) is bent along the simplex path of the Klee-Minty cube C^n so that it visits the δ -neighborhood of every vertex of that cube, providing that h satisfies

$$h_k \tau^{k-1} \delta \geq 1 + 2\tau^k + \sum_{i=1}^{k-1} h_i, \quad \text{for } k = 1, \dots, n, \quad (4)$$

where by convention $\sum_{i=1}^0 h_i = 0$. For $k = 1, \dots, n$, the k^{th} inequality of (4) ensures that the central path \mathcal{P}_h is pushed enough toward the set \hat{C}_δ^k . In the following lemma, we prove some implications of inequality system (4).

Lemma 3.1. *For any $k = 2, \dots, n$, the k^{th} inequality of (4) implies all of the following inequalities*

$$h_k \tau^{k-1} \delta \geq 2\tau^k + \tau^{m-1} \left(1 + \sum_{i=m}^{k-1} h_i \right), \quad \text{for } m = 1, \dots, k-1.$$

Proof. The proof immediately follows as $\tau^{m-1} < 1$ and $h_i > 0$ for $i = 1, \dots, m-1$. \square

We now present the main theorem of this section.

Theorem 3.2. *Let $\tau + \delta < 1/2$ and h satisfy (4). Then, for problem (2), we have*

$$\mathcal{P}_h \cap C_\delta^{k+1} \subseteq \hat{C}_\delta^k, \quad \text{for } k = 1, \dots, n-1.$$

Proof. We show that for all $k = 1, \dots, n-1$, any point on the central path \mathcal{P}_h that satisfies $s_{k+1} > \delta$ and $\bar{s}_{k+1} > \delta$, also satisfies

$$\bar{s}_k \leq \delta, \quad s_{k-1} \leq \delta, \quad \dots, \quad s_1 \leq \delta.$$

Recall that any point on the central path \mathcal{P}_h satisfies the first $n-1$ equations of (3). From the k^{th} equation of (3), we have

$$\frac{h_k}{y_k} = -\frac{1}{s_k} + \frac{1}{\bar{s}_k} + \frac{\tau}{s_{k+1}} + \frac{\tau}{\bar{s}_{k+1}},$$

which, since $s_{k+1} > \delta$ and $\bar{s}_{k+1} > \delta$, implies that

$$h_k \leq \frac{1}{\bar{s}_k} + \frac{2\tau}{\delta}.$$

Since by (4) the inequality $h_k \geq (1 + 2\tau)/\delta$ holds, we get $\bar{s}_k \leq \delta$.

Let $1 \leq m \leq k-1$. Adding the k^{th} equation of (3) multiplied by τ^{k-1} to the j^{th} equation of (3) multiplied by $-\tau^{j-1}$, for all $j = m, \dots, k-1$, we have

$$\frac{h_k \tau^{k-1}}{y_k} - \sum_{i=m}^{k-1} \frac{h_i \tau^{i-1}}{y_i} \leq \frac{\tau^{m-1}}{s_m} + \frac{2\tau^k}{\delta}.$$

Using the facts that $y_i \geq \tau^{i-m} y_m$, for any $i = m, \dots, n$, and $y_m \geq s_m$, we obtain

$$h_k \tau^{k-1} - \sum_{i=m}^{k-1} \frac{h_i \tau^{m-1}}{s_m} \leq \frac{\tau^{m-1}}{s_m} + \frac{2\tau^k}{\delta},$$

or equivalently

$$h_k \tau^{k-1} \leq \left(1 + \sum_{i=m}^{k-1} h_i\right) \frac{\tau^{m-1}}{s_m} + \frac{2\tau^k}{\delta},$$

which, using the inequalities of Lemma 3.1, implies that $s_m \leq \delta$. □

3.2 Existence of an Integer-valued Repetition Vector h

In this subsection, we show that there exists an integer-valued repetition vector h that satisfies (4). In the rest of this subsection, we assume that $\tau + \delta < 1/2$. Let us start with the following lemma which gives inequalities that are used in the subsequent lemmas.

Lemma 3.3. *For any $k = 1, 2, \dots$, we have*

$$1 + 2\tau + \sum_{i=1}^k \frac{i}{\tau^{i(i-1)/2} \delta^i} \leq \frac{k+1}{\tau^{k(k-1)/2} \delta^k}.$$

Proof. The proof is by induction on k . The inequality is obviously true for $k = 1$ as $\tau + \delta < 1/2$. Assuming that the inequality holds for all $k = 1, \dots, j$, we have

$$1 + 2\tau + \sum_{i=1}^{j+1} \frac{i}{\tau^{i(i-1)/2}\delta^i} \leq \frac{j+1}{\tau^{j(j-1)/2}\delta^j} + \frac{j+1}{\tau^{j(j+1)/2}\delta^{j+1}} = \frac{(j+1)(1+\tau^j\delta)}{\tau^{j(j+1)/2}\delta^{j+1}} \leq \frac{j+2}{\tau^{j(j+1)/2}\delta^{j+1}}.$$

Therefore, the inequality holds for $j = k + 1$, and the proof is completed. \square

The following lemma provides an explicit solution of (4).

Lemma 3.4. *The vector $h = (\frac{1}{\delta}, \frac{2}{\tau\delta^2}, \dots, \frac{n}{\tau^{n(n-1)/2}\delta^n})^T$ satisfies (4).*

Proof. Obviously $h_1 \geq 1/\delta$. Therefore, it suffices to show that, for $k = 2, \dots, n$,

$$\frac{k\tau^{k-1}\delta}{\tau^{k(k-1)/2}\delta^k} \geq 1 + 2\tau + \sum_{i=1}^{k-1} \frac{i}{\tau^{i(i-1)/2}\delta^i}.$$

This inequality is equivalent to

$$\frac{k}{\tau^{(k-1)(k-2)/2}\delta^{k-1}} \geq 1 + 2\tau + \sum_{i=1}^{k-1} \frac{i}{\tau^{i(i-1)/2}\delta^i},$$

which hold as shown in Lemma 3.3. \square

We look for an integer-valued solution \bar{h} of (4), since every component of h represents the repetition number of the corresponding coordinate-plane. The following lemma shows how to construct an integer-valued solution from an arbitrary solution h of (4).

Lemma 3.5. *If h is a solution of (4), then so is the integer-valued vector $\bar{h} = \lfloor 2h \rfloor$.*

Proof. For any $k = 1, \dots, n$, there exists $0 \leq \varepsilon_k < 1$ such that $\bar{h}_k = 2h_k - \varepsilon_k$. By assumption, h satisfies (4). Therefore, multiplying each side of (4) by 2 and substituting $2h_k$ by $\bar{h}_k + \varepsilon_k$, for all $k = 1, \dots, n$, we get

$$\bar{h}_k\tau^{k-1} + \varepsilon_k\tau^{k-1}\delta \geq 2 + 4\tau^k + \sum_{i=1}^{k-1} \bar{h}_i + \sum_{i=1}^{k-1} \varepsilon_i,$$

implying

$$\bar{h}_k\tau^{k-1}\delta \geq 2 + 4\tau^k - \varepsilon_k\tau^{k-1}\delta + \sum_{i=1}^{k-1} \bar{h}_i + \sum_{i=1}^{k-1} \varepsilon_i \geq 2 + 4\tau^k - \delta + \sum_{i=1}^{k-1} \bar{h}_i \geq 1 + 2\tau^k + \sum_{i=1}^{k-1} \bar{h}_i.$$

Therefore, the integer-valued vector \bar{h} satisfies (4). \square

3.3 Proof of Proposition 2.1

The number of the inequalities of $(\mathbb{R}P^n)$ is

$$\begin{aligned} N &= 2n + \sum_{k=1}^n \bar{h}_k, \\ &\leq 2n + \sum_{k=1}^n 2k \left(\frac{2n+2}{n} \right)^{k(k-1)/2} (4n+4)^k, \\ &\leq 2n + 2n(n+1)^n 2^{2n} \left(1 + \frac{1}{n} \right)^{n(n-1)/2} \sum_{k=1}^n 2^{k(k-1)/2}. \end{aligned}$$

Since $(1 + 1/n)^n \leq e$ and $\sum_{k=1}^n 2^{k(k-1)/2} \leq 2^{n^2/2}$, we get

$$N \leq 2n + (n+1)^{n+1} 2^{2n+1} e^{(n-1)/2} 2^{n^2/2},$$

implying that $N = O(2^{n^2})$. □

3.4 Proof of Proposition 2.2

The proof is similar to the proof of Theorem 3.2. The analytic center χ^n of $\bar{\mathcal{C}}^n$ is the unique solution of (3). We prove that any point that satisfies (3) also satisfies

$$\bar{s}_n \leq \bar{\delta}, \quad s_{n-1} \leq \bar{\delta}, \quad \dots, \quad s_1 \leq \bar{\delta}.$$

From Lemmas 3.4 and 3.5, the vector \bar{h} satisfies all of the inequalities of (4). From the n^{th} equation of (3), we have

$$\frac{\bar{h}_n}{y_n} \leq \frac{1}{\bar{s}_n}.$$

Since $\bar{h}_n \geq 1/\bar{\delta}$ holds from (4), we get $\bar{s}_n \leq \bar{\delta}$.

Let $1 \leq m \leq n-1$. Adding the n^{th} equation of (3) multiplied by $\bar{\tau}^{n-1}$ to the j^{th} equation of (3) multiplied by $-\bar{\tau}^{j-1}$, for all $j = m, \dots, n-1$, we have

$$\frac{\bar{h}_n \bar{\tau}^{n-1}}{y_n} - \sum_{i=m}^{n-1} \frac{\bar{h}_i \bar{\tau}^{i-1}}{y_i} \leq \frac{\bar{\tau}^{m-1}}{s_m} + \frac{2\bar{\tau}^n}{\bar{\delta}}.$$

Using the facts that $y_i \geq \bar{\tau}^{i-m} y_m$, for any $i = m, \dots, n$, and $y_m \geq s_m$, we obtain

$$\bar{h}_n \bar{\tau}^{n-1} - \sum_{i=m}^{n-1} \frac{\bar{h}_i \bar{\tau}^{m-1}}{s_m} \leq \frac{\bar{\tau}^{m-1}}{s_m} + \frac{2\bar{\tau}^n}{\bar{\delta}},$$

or equivalently

$$\bar{h}_n \bar{\tau}^{n-1} \leq \left(1 + \sum_{i=m}^{n-1} \bar{h}_i \right) \frac{\bar{\tau}^{m-1}}{s_m} + \frac{2\bar{\tau}^n}{\bar{\delta}},$$

which, using the inequalities of Lemma 3.1, implies that $s_m \leq \bar{\delta}$. □

3.5 Proof of Proposition 2.3

We show that the central path \mathcal{P}^n of $(\mathbb{R}P^n)$ is contained in the $\bar{\delta}$ -neighborhood of the simplex path $P_{\bar{\delta}} = \bigcap_{k=2}^n A_{\bar{\delta}}^k$. By Proposition 2.2, the starting point χ^n of \mathcal{P}^n , which is the analytic center of $\bar{\mathcal{C}}^n$, belongs to $\hat{\mathcal{C}}_{\bar{\delta}}^n = \mathcal{N}_{\bar{\delta}}(v^{\{n\}})$. Since $\mathcal{C}^n = \bigcap_{k=2}^n (T_{\bar{\delta}}^k \cup C_{\bar{\delta}}^k \cup B_{\bar{\delta}}^k)$, we have

$$\mathcal{P}^n = \bigcap_{k=2}^n (T_{\bar{\delta}}^k \cup C_{\bar{\delta}}^k \cup B_{\bar{\delta}}^k) \cap \mathcal{P}^n = \bigcap_{k=2}^n (T_{\bar{\delta}}^k \cup (C_{\bar{\delta}}^k \cap \mathcal{P}^n) \cup B_{\bar{\delta}}^k) \cap \mathcal{P}^n.$$

By Theorem 3.2, $\mathcal{P}^n \subset \bigcap_{k=2}^n (T_{\bar{\delta}}^k \cup \hat{\mathcal{C}}_{\bar{\delta}}^{k-1} \cup B_{\bar{\delta}}^k) = \bigcap_{k=2}^n A_{\bar{\delta}}^k = P_{\bar{\delta}}$. \square

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