

# Size-constrained graph partitioning polytope. Part I: Dimension and trivial facets

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## Abstract

We consider the problem of clustering a set of items into subsets whose sizes are bounded from above and below. We formulate the problem as a graph partitioning problem and propose an integer programming model for solving it. This formulation generalizes several well-known graph partitioning problems from the literature like the clique partitioning problem, the equi-partition problem and the  $k$ -way equi-partition problem. In this paper and its sequel, we analyze the structure of the corresponding polytope and prove several results concerning the facial structure. Our analysis yields important results for the closely related equi-partition and  $k$ -way equi-partition polytopes as well.

## 1 Introduction

The problem of partitioning a set of items into clusters to achieve an objective (e.g., to obtain homogeneous clusters and extract valuable information on each of the clusters, to divide the items into clusters that have minimal interaction in-between, etc.) is encountered in a wide range of disciplines such as marketing, economics, biology, psychology, politics, etc. Owing to this wide spectrum of applications, the clustering problem has been studied extensively in Operations Research.

Three classes of clustering problems can be differentiated in Operations Research applications. In the first class of problems, the items to be clustered are articulated by numerical attributes. Generally, the items are represented by vectors in a real space which correspond to a set of attribute measurements. This class of problems find applications in many areas like pattern recognition (see [4] for a general treatment) and marketing (for example, [18]), among others.

In the second class of problems, there are no attributes defined at all and only certain pairwise relations between the items are provided. For example in some telecommunications networks, the nodes are clustered on the basis of the communication level between

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pairs of nodes. Generally, the aim in such problems is to cluster the items (routers, users, customers, etc.) so as to minimize the communication between the clusters (for example, see [11], [17]), or minimize total communication cost at all within the whole network (for example, see [21]). Other application areas of this class of clustering problems are compiler design, parallel computing and electronic circuit design (for example, see [5], [6], [10]).

In the third class, the data relevant to clustering involve attributes that are defined categorically. Namely, attributes do not assume numerical values and are evaluated in categories (for example, see [15]). For example, if the objects under consideration are people, hair color is an example of an attribute which is best stated categorically (brown, black, blond, etc.); for consumers, the attitude against a product or brand can be more meaningfully articulated categorically (positive, negative, neutral); in some applications, it might be more meaningful to state the size categorically as “large”, “medium” or “small” rather than some numerical measures. Fortunately, this class of problems can also be analyzed in the second class of clustering problems above owing to some qualitative data analysis techniques which convert such type of data into pairwise relations between the items. An example of such a conversion is given by Grötschel and Wakabayashi ([7]).

Generally in Operations Research literature, the clustering problems in the second and third classes above are classified as *graph partitioning problems*. In this scheme, the items to be clustered are represented by the nodes of a graph, and pairwise relations among them are represented by edge weights. The aim is to find the ‘best’ clustering, which optimizes a certain objective function, of the nodes of the graph into subsets.

In practical clustering applications, the subsets to be obtained might be subject to various constraints such as restrictions on the number of subsets, restrictions on the sizes of the subsets, requirement that certain items be or not be in the same subset, etc. In this paper and its sequel, we confine ourselves to clustering problems with restrictions on the subset sizes. Namely, the clustering problems that are addressed in this paper and its companion involve parameters  $F_U$  and  $F_L$ , which constrain the sizes of the subsets to be obtained from above and below, respectively. For instance, in a specific clustering application, the subsets we obtain might represent voting districts for a certain elective procedure, or distribution regions belonging to a distribution network. It might not be possible to support voting districts or distribution regions larger than a pre-specified size  $F_U$  because some resources at disposal (like human resources for the elective procedure, or vehicles for the distribution network) might not be adequate in quantity or in quality. In addition, reasons like economies of scale or need for balanced partitions, where the sizes of the subsets obtained are ‘almost’ equal to each other, might give rise to minimum size requirements as well on the subsets. For instance, we might want to impose the restriction that the subsets have size at least  $F_L$  in order to make the voting districts large enough to allow meaningful statistical inferences, or, in order to make the distribution regions large enough to benefit from economies of scale, or, in order to obtain almost equally sized subsets that will allow applying similar administrative procedures.

Besides bounds  $F_L$  and  $F_U$  on the subset sizes, we assume as well that the data relevant to clustering is provided in the form of pairwise relations among the items. This

locates our problem in the second and third class of clustering problems described above. Following the prevalent practice in the literature, we formulate it as a graph partitioning problem. The formal definition and an integer programming formulation of the problem are given in Section 3. But before, in Section 2 we present the notation used throughout this paper and its sequel.

## 2 Notation

We represent the set of items to be clustered by the set  $V = \{1, 2, \dots, n\}$ . Consider a graph  $G = (V, E)$  on  $V$ . Each edge  $e \in E$  is also represented by means of its adjacent nodes  $u$  and  $v$ : the terms  $\{u, v\}$  and  $\{v, u\}$  are used interchangeably to refer to the edge  $e$ .

A *partition*,  $\pi$ , is a collection of non-empty subsets  $N_1, N_2, \dots, N_k$  of  $V$  such that  $N_i \cap N_j = \emptyset$  for all  $i \neq j$ , and  $\bigcup_{i=1}^k N_i = V$ . A *size constrained partition* (shortly, an *sc-partition*) is a partition with  $F_L \leq |N_i| \leq F_U$  for all  $i = 1, \dots, k$ . We associate a characteristic vector  $w^\pi \in \{0, 1\}^{|E|}$  with each partition  $\pi$ , where  $w_{u,v}^\pi = 1$  if  $u, v \in N_i$  for some  $i \in \{1, 2, \dots, k\}$ , and 0 otherwise. We use the representations  $\pi$  and  $(N_1, N_2, \dots, N_k)$ , and its characteristic vector  $w^\pi$  interchangeably to refer to a partition. We use the term *subclique* to refer to the subsets (i.e.,  $N_i$ 's) in the partitions.

Let  $Q \subseteq V$  and  $S \subseteq V$ . We denote the set of all edges whose both endnodes are in  $Q$  by  $E(Q)$ , i.e.,  $E(Q) = \{\{u, v\} \in E : u, v \in Q\}$ . The set of edges with one endnode in  $Q$  and the other endnode in  $S$  are denoted by  $E(Q, S)$ , i.e.,  $E(Q, S) = \{\{u, v\} \in E : u \in Q, v \in S\}$ . For ease of notation, a set with only one element is represented with the element itself in mathematical expressions: namely, for instance,  $Q - u$  stands for  $Q - \{u\}$ . Adjacent edges of a node  $u$  is denoted by  $\delta(u)$ , i.e.,  $\delta(u) = E(u, V - u)$ . For any  $x \in \mathbb{R}^{|E|}$ , let  $x(F) = \sum_{e \in F} x_e$  for  $F \subseteq E$ .

Throughout,  $\text{conv}(X)$  and  $\text{aff}(X)$  denotes the convex hull and the affine hull of a set  $X \subset \mathbb{R}^{|E|}$ , respectively. The equality set  $M(P)$  of a polytope  $P$  is a maximal set of linearly independent hyperplanes that contain  $P$ . The dimension of  $P$  is denoted by  $\text{dim}(P)$ .

## 3 Problem definition and formulation

The problem we address in this paper and its companion consists of finding the sc-partition  $\pi$  that optimizes

$$\sum_{e \in E} c_e w_e^\pi.$$

This is a quite common objective function for most of the graph partitioning problems in the literature. Some articles interpret the minimization of this objective function as clustering of the most similar items together when the edge weights  $c_e$ 's represent a measure of dissimilarity (like distance) between the items (see [8], [7], [13] and [12] among others). Some others interpret the maximization of it as division of a network or

a compiler into clusters which have minimal interaction in-between (see [5], [6], [10] and [11] among others).

This objective function, which is a sum of the weights of a set of edges, allows us to assume that  $G = (V, E)$  is a complete graph. Because, if it is not, we can always convert it to one by adding the missing edges with weights 0. Solving the problem on this complete graph would be completely equivalent to solving it on  $G$ . Hence, in the sequel, we assume that  $G = (V, E)$  is a complete graph defined on  $n$  nodes.

The integer programming formulation we propose for our problem is as follows:

$$\begin{aligned} & \text{minimize or maximize} && \sum_{e \in E} c_e w_e \\ & \text{subject to} \\ & w_{u,v} + w_{u,t} - w_{v,t} \leq 1 && \forall u, v, t \in V : u, v, t \text{ different} && (3.1a) \\ & w(\delta(u)) \geq F_L - 1 && u \in V && (3.1b) \\ & w(\delta(u)) \leq F_U - 1 && u \in V && (3.1c) \\ & w_{u,v} \in \{0, 1\} && \forall \{u, v\} \in E && (3.1d) \end{aligned}$$

where the variable  $w_{u,v}$  takes value 1 if nodes  $u$  and  $v$  are in the same subclique, and 0 otherwise. Constraints (3.1a), called triangle inequalities, ensure that if any two edges of a triangle linking three nodes in  $V$  are contained in one subclique (i.e., all three of the nodes are packed in the same subclique), then the third edge of the triangle is also contained in that subclique. Constraints (3.1b) and (3.1c) stand for the size restrictions on the subcliques. It is easy to see that the set of characteristic vectors of sc-partitions of  $G$  coincide exactly with set of feasible solutions of formulation (3.1).

This paper and its sequel are devoted to the analysis of the structure of the polytope

$$\mathcal{P}^{lu}(n, F_L, F_U) = \text{conv}\{w^\pi \in \mathbb{R}^{|E|} : w^\pi \text{ feasible for (3.1) (i.e., } \pi \text{ is an sc-partition)}\}.$$

Obviously, this polytope is uniquely determined by the three parameters  $n$ ,  $F_L$  and  $F_U$ .

Formulation (3.1) and  $\mathcal{P}^{lu}(n, F_L, F_U)$  generalize the following several graph partitioning problems from the literature and their corresponding polytopes.

- For  $F_L = 1$  and  $F_U = n$  (i.e., when (3.1b) and (3.1c) become redundant), this formulation turns into the so-called *clique partitioning problem* which was introduced by Grötschel and Wakabayashi ([8],[7]; see also [1], [16]). This problem addresses the most basic clustering problem where there is no restriction on the number of subsets or the sizes of the subsets (i.e., for a feasible solution  $(N_1, \dots, N_k)$  of the clique partition problem,  $1 \leq k \leq n$ , and  $1 \leq |N_i| \leq n$  for all  $i = 1, \dots, k$ ). For further use, we define

$$\mathcal{P}(n) = \text{conv}\{w^\pi \in \mathbb{R}^{|E|} : w^\pi \text{ satisfies (3.1a) and (3.1d) (i.e., } \pi \text{ is a partition)}\}.$$

- When  $F_U = n$  (i.e., when the upper bound on the subclique sizes is removed), we get the so-called *clique partitioning problem with minimum clique size requirement*

of Ji and Mitchell [9]. Let the corresponding polytope be denoted as  $\mathcal{P}^l(n, F_L)$ , i.e.,

$$\mathcal{P}^l(n, F_L) = \text{conv}\{w^\pi \in \mathbb{R}^{|E|} : w^\pi \text{ satisfies (3.1a), (3.1b) and (3.1d)}\}.$$

- When  $F_L = 1$  (i.e., the lower bound is removed), we end up with the so-called *simple graph partitioning problem* of Sørensen ([20] and [19]). We denote the corresponding polytope as

$$\mathcal{P}^u(n, F_U) = \text{conv}\{w^\pi \in \mathbb{R}^{|E|} : w^\pi \text{ satisfies (3.1a), (3.1c) and (3.1d)}\}.$$

- The equipartition problem consists of dividing a graph  $G$  into two subgraphs with sizes  $\lfloor \frac{n}{2} \rfloor$  and  $\lceil \frac{n}{2} \rceil$ . Let  $\mathcal{P}^{\text{equi}}(n)$  denote the equi-partition polytope, that is, the convex hull of characteristic vectors  $w^\pi$  such that  $\pi = (N_1, N_2)$ ,  $|N_1| = \lfloor \frac{n}{2} \rfloor$  and  $|N_2| = \lceil \frac{n}{2} \rceil$ . Conforti et al. ([2], [3]) provide a detailed analysis of  $\mathcal{P}^{\text{equi}}(n)$ .
- The  $k$ -way equipartition problem ( $k \geq 3$ ) consists of dividing a graph  $G$  with  $(n \bmod k) = 0$  into  $k$  equally sized subgraphs. And, similarly, let  $\mathcal{P}^{k\text{-way}}(n, k)$  denote the  $k$ -way equi-partition polytope, which is the convex hull of  $w^\pi$  such that  $\pi = (N_1, \dots, N_k)$  and  $|N_i| = F = \frac{n}{k}$  for  $i = 1, \dots, k$ . For a detailed treatment of  $\mathcal{P}^{k\text{-way}}(n, k)$ , see [13] and [12] by Mitchell.

In this paper and its sequel, we attribute special importance to the equi-partition polytope  $\mathcal{P}^{\text{equi}}(n)$  and the  $k$ -way equi-partition polytope  $\mathcal{P}^{k\text{-way}}(n, k)$ . Because, in the sequel paper, we present several results, which are new to the literature, on  $\mathcal{P}^{\text{equi}}(n)$  and  $\mathcal{P}^{k\text{-way}}(n, k)$ . In Section 4, we state conditions on  $n$ ,  $F_L$  and  $F_U$  that make  $\mathcal{P}^{lu}(n, F_L, F_U)$  equivalent to  $\mathcal{P}^{\text{equi}}(n)$  or  $\mathcal{P}^{k\text{-way}}(n, k)$ .

For ease of notation, throughout we shortly use  $\mathcal{P}^{lu}$  to refer to  $\mathcal{P}^{lu}(n, F_L, F_U)$  unless otherwise stated. In the next section, we start with the analysis of  $\mathcal{P}^{lu}$  by proving feasibility conditions and dimension.

## 4 Non-emptiness and dimension of $\mathcal{P}^{lu}$

Not all  $(n, F_L, F_U)$  triples yield a nonempty  $\mathcal{P}^{lu}$ . For example, for  $n = 31$ ,  $F_L = 8$  and  $F_U = 10$  it is not possible to find an sc-partition, i.e.,  $\mathcal{P}^{lu}(31, 8, 10) = \emptyset$ . Lemma 4.1 states the necessary and sufficient condition  $(n, F_L, F_U)$  should satisfy for  $\mathcal{P}^{lu}$  to be non-empty. We assume in this and the next section that  $F_L \geq 2$ .

**Lemma 4.1.**  $\mathcal{P}^{lu} \neq \emptyset$  if and only if  $\lfloor \frac{n}{F_L} \rfloor \geq \lceil \frac{n}{F_U} \rceil$ .

*Proof.* A partition  $\pi = (N_1, \dots, N_k)$  is an sc-partition if and only if  $kF_L \leq n \leq kF_U$ , which holds if and only if  $\lfloor \frac{n}{F_L} \rfloor \geq k \geq \lceil \frac{n}{F_U} \rceil$ .  $\square$

Note that, non-emptiness of  $\mathcal{P}^{lu}$  is dependent only on the values of  $n$ ,  $F_L$  and  $F_U$ , and not on the set  $V$  itself. This implicitly implies that non-emptiness of  $\mathcal{P}^{lu}$  is preserved through permutations of  $V$ . One can also infer from the proof of this lemma that an sc-partition has at least  $\left\lceil \frac{n}{F_U} \right\rceil$  and at most  $\left\lfloor \frac{n}{F_L} \right\rfloor$  subcliques.

We now pass on to determining the dimension of  $\mathcal{P}^{lu}$ . When  $F_U = F_L$ , the problem reduces to the  $k$ -way equipartition problem whose dimension is shown to be  $\binom{n}{2} - n$  by Mitchell [13]. Hence, assume in the rest of this section that  $F_U \geq F_L + 1$ .

We now give two technical lemmas that will be used in establishing the dimension of  $\mathcal{P}^{lu}$  and in some of the facetness proofs in the following sections.

**Lemma 4.2.** *Let  $P$  denote a face of  $\mathcal{P}^{lu} \neq \emptyset$ . Let  $\pi = (N_1, N_2, \dots, N_k)$  be an sc-partition such that  $w^\pi \in P$ . Let  $\alpha w = \alpha_0$  ( $\alpha \neq 0$ ) be a hyperplane in  $\mathbb{R}^{\binom{n}{2}}$  and suppose  $P \subset \{w \in \mathbb{R}^{\binom{n}{2}} : \alpha w = \alpha_0\}$ . Let  $\{i, j\} \subseteq N_1$  and  $h \in N_2$ . Then  $\alpha_{i,h} = \alpha_{j,h}$  provided that the following are satisfied for  $i, j$  and  $h$ :*

- $w^{\pi^1} \in P$  where  $\pi^1 = (N_1^*, N_2^*, \dots, N_k^*)$ ,  $N_1^* = N_1 - \{i\}$ ,  $N_2^* = N_2 \cup \{i\}$ ,  $N_l^* = N_l \forall l \in \{3, \dots, k\}$ ;
- $w^{\pi^2} \in P$  where  $\pi^2 = (N_1^*, N_2^*, \dots, N_k^*)$ ,  $N_1^* = N_1 - \{j\}$ ,  $N_2^* = N_2 \cup \{j\}$ ,  $N_l^* = N_l \forall l \in \{3, \dots, k\}$ ;
- $w^{\pi^3} \in P$  where  $\pi^3 = (N_1^*, N_2^*, \dots, N_k^*)$ ,  $N_1^* = N_1 - \{i\} \cup \{h\}$ ,  $N_2^* = N_2 - \{h\} \cup \{i\}$ ,  $N_l^* = N_l \forall l \in \{3, \dots, n\}$ ;
- $w^{\pi^4} \in P$  where  $\pi^4 = (N_1^*, N_2^*, \dots, N_k^*)$ ,  $N_1^* = N_1 - \{j\} \cup \{h\}$ ,  $N_2^* = N_2 - \{h\} \cup \{j\}$ ,  $N_l^* = N_l \forall l \in \{3, \dots, n\}$ .

*Proof.* Using the fact that  $\alpha w^\pi = \alpha_0$  and  $\alpha w^{\pi^1} = \alpha_0$ , we obtain

$$\alpha_{i,j} + \alpha(i, N_1 - \{i, j\}) = \alpha_{i,h} + \alpha(i, N_2 - \{h\}). \quad (4.1)$$

From  $\alpha w^\pi = \alpha_0$  and  $\alpha w^{\pi^2} = \alpha_0$  we have

$$\alpha_{i,j} + \alpha(j, N_1 - \{i, j\}) = \alpha_{j,h} + \alpha(j, N_2 - \{h\}). \quad (4.2)$$

Similarly, from  $\alpha w^\pi = \alpha_0$  and  $\alpha w^{\pi^3} = \alpha_0$  we have

$$\alpha_{i,j} + \alpha(i, N_1 - \{i, j\}) + \alpha(h, N_2 - \{h\}) = \alpha_{j,h} + \alpha(h, N_1 - \{i, j\}) + \alpha(i, N_2 - \{h\}), \quad (4.3)$$

and from  $\alpha w^\pi = \alpha_0$  and  $\alpha w^{\pi^4} = \alpha_0$  we have

$$\alpha_{i,j} + \alpha(j, N_1 - \{i, j\}) + \alpha(h, N_2 - \{h\}) = \alpha_{i,h} + \alpha(h, N_1 - \{i, j\}) + \alpha(j, N_2 - \{h\}). \quad (4.4)$$

From equations (4.1) and (4.3) we obtain

$$\alpha_{i,h} + \alpha(h, N_2 - \{h\}) = \alpha_{j,h} + \alpha(h, N_1 - \{i, j\}). \quad (4.5)$$

From equations (4.2) and (4.4) we obtain

$$\alpha_{j,h} + \alpha(h, N_2 - \{h\}) = \alpha_{i,h} + \alpha(h, N_1 - \{i, j\}). \quad (4.6)$$

Equations, (4.5) and (4.6) imply  $\alpha_{i,h} = \alpha_{j,h}$ .  $\square$

**Lemma 4.3.** *Let  $\alpha w = \alpha_0$  ( $\alpha \neq 0$ ) be a hyperplane in  $\mathbb{R}^{\binom{n}{2}}$ . Suppose that  $\mathcal{P}^{lu} \subset \{w \in \mathbb{R}^{\binom{n}{2}} : \alpha^T w = \alpha_0\}$ . If there exists an integer  $k \geq 2$  such that  $kF_L < n < kF_U$ , then  $\alpha_{e'} = \alpha_{e''}$  for all  $e', e'' \in E$ .*

*Proof.* The inequalities  $kF_L < n < kF_U$  imply that there exists an sc-partition  $\pi = (N_1, \dots, N_k)$  such that  $|N_1| > F_L$  and  $|N_2| < F_U$ . Since such an sc-partition would be feasible for all different permutations of  $V$ , setting  $P = \mathcal{P}^{lu}$  in Lemma 4.2 yields the result.  $\square$

The following two theorems determine the dimension of  $\mathcal{P}^{lu}$ . The first one proves the dimension when  $\lfloor \frac{n}{F_L} \rfloor = \lceil \frac{n}{F_U} \rceil$ , i.e., when all sc-partitions in  $\mathcal{P}^{lu}$  have the same number of subcliques. The second one proves the dimension when  $\lfloor \frac{n}{F_L} \rfloor > \lceil \frac{n}{F_U} \rceil$ , namely, when the number of subcliques in the sc-partitions can take on several values.

**Theorem 4.1.** *Suppose  $\lfloor \frac{n}{F_L} \rfloor = \lceil \frac{n}{F_U} \rceil \geq 2$ .*

- (i) *If  $(n \bmod F_L) = 0$  or  $(n \bmod F_U) = 0$ , then  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$ .*
- (ii) *Suppose  $F_U - F_L = 1$ . If  $(n \bmod F_L) \neq 0$  and  $(n \bmod F_U) \neq 0$ , then  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$ .*
- (iii) *Suppose  $F_U - F_L \geq 2$ .*
  - (a) *If  $(n \bmod F_L) = 1$  or  $(n \bmod F_U) = F_U - 1$ , then  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$ .*
  - (b) *If  $(n \bmod F_L) > 1$  and  $0 < (n \bmod F_U) < F_U - 1$ , then  $\dim(\mathcal{P}^{lu}) = \binom{n}{2}$  (i.e., it is full-dimensional).*

*Proof.* All sc-partitions have  $k = \lfloor \frac{n}{F_L} \rfloor = \lceil \frac{n}{F_U} \rceil$  subcliques. Note that when  $\lfloor \frac{n}{F_L} \rfloor = \lceil \frac{n}{F_U} \rceil = 1$ ,  $\mathcal{P}^{lu}$  reduces to a single point, in which case the dimension is equal to 0.

- (i) If  $(n \bmod F_L) = 0$  or  $(n \bmod F_U) = 0$ , then in any sc-partition all subcliques have size  $F_L$  or  $F_U$ , respectively. This means,  $\mathcal{P}^{lu}$  reduces to the  $k$ -way equipartition polytope whose dimension is equal to  $\binom{n}{2} - n$  by [13].
- (ii) We have  $kF_L < n < kF_U$ . By Lemma 4.3, the only hyperplane that contains  $\mathcal{P}^{lu}$  is

$$w(E) = (n \bmod F_L) \binom{F_U}{2} + [k - (n \bmod F_L)] \binom{F_L}{2}.$$

- (iii) (a) When  $(n \bmod F_L) = 1$ , again we have  $kF_L < n < kF_U$  and hence by Lemma 4.3, the only hyperplane that contains  $\mathcal{P}^{lu}$  is

$$w(E) = \binom{F_L + 1}{2} + (k - 1) \binom{F_L}{2}.$$

A symmetric reasoning can be done when  $(n \bmod F_U) = F_U - 1$ .

- (b) Arguing similarly, we infer that the only hyperplane that contains  $\mathcal{P}^{lu}$  is in the form  $w(E) = C$  where  $C$  is a constant. But, when  $(n \bmod F_L) > 1$  and  $0 < (n \bmod F_U) < F_U - 1$ , there always exist sc-partitions  $\pi_1$  and  $\pi_2$  with  $w^{\pi_1}(E) \neq w^{\pi_2}(E)$ . For example, suppose  $(n \bmod F_L) = 2$ . Consider an sc-partition  $\pi_1 = (N_1, \dots, N_k)$  where  $|N_1| = |N_2| = F_L + 1$  and  $|N_i| = F_L$  for  $i = 3, \dots, k$ . We now construct another sc-partition  $\pi_2$  by moving one node from  $N_2$  to  $N_1$ , i.e.,  $\pi_2 = (\tilde{N}_1, \dots, \tilde{N}_k)$  where  $|\tilde{N}_1| = F_L + 2$ ,  $|\tilde{N}_i| = F_L$  for  $i = 2, \dots, k$ . Clearly,  $w^{\pi_2}(E) = w^{\pi_1}(E) + 1$ .

□

**Theorem 4.2.** Suppose that  $\lfloor \frac{n}{F_L} \rfloor \geq \lceil \frac{n}{F_U} \rceil + 1$ .

(i) Suppose that  $\lceil \frac{n}{F_U} \rceil = 1$  and  $\lfloor \frac{n}{F_L} \rfloor = 2$ .

(a) If  $(n \bmod F_L) = 0$ , then  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n - 1)$ .

(b) If  $(n \bmod F_L) \neq 0$ , then  $\dim(\mathcal{P}^{lu}) = \binom{n}{2}$  (i.e., it is full-dimensional).

(ii) Suppose that  $\lfloor \frac{n}{F_L} \rfloor = \lceil \frac{n}{F_U} \rceil + 1$  and  $\lceil \frac{n}{F_U} \rceil \geq 2$ .

(a) If  $(n \bmod F_L) = 0$  and  $(n \bmod F_U) = 0$ , then  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n - 1)$ .

(b) If  $(n \bmod F_L) \neq 0$  or  $(n \bmod F_U) \neq 0$ , then  $\dim(\mathcal{P}^{lu}) = \binom{n}{2}$  (i.e., it is full-dimensional).

(iii) If  $\lfloor \frac{n}{F_L} \rfloor \geq \lceil \frac{n}{F_U} \rceil + 2$ , then  $\dim(\mathcal{P}^{lu}) = \binom{n}{2}$  (i.e., it is full-dimensional).

*Proof.* We distinguish two cases:

*Case 1:*  $n \bmod F_L \neq 0$ ,

*Case 2:*  $n \bmod F_L = 0$ .

Case 1: Since  $n \bmod F_L \neq 0$ , we have  $F_L \lfloor \frac{n}{F_L} \rfloor < n$ . And,  $n \leq F_U \lceil \frac{n}{F_U} \rceil$  implies  $n < F_U \lfloor \frac{n}{F_L} \rfloor$  since  $\lfloor \frac{n}{F_L} \rfloor \geq \lceil \frac{n}{F_U} \rceil + 1$ . Then, by Lemma 4.3, the only hyperplane that contains  $\mathcal{P}^{lu}$  is  $w(E) = C$  for some constant  $C$ .

Now, consider an sc-partition  $\pi_1$  with  $\lfloor \frac{n}{F_L} \rfloor$  subcliques. Since  $\lfloor \frac{n}{F_L} \rfloor \geq \lceil \frac{n}{F_U} \rceil + 1$ , it is possible to construct another sc-partition  $\pi_2$  with  $\lfloor \frac{n}{F_L} \rfloor - 1$  subcliques by distributing the nodes of the smallest subclique of  $\pi_1$  over the others. It is easy to see that  $w^{\pi_1}(E) < w^{\pi_2}(E)$ , because while moving nodes from the smallest subclique of  $\pi_1$  to the others, we are actually increasing the number of edges contained within the subcliques. Then,  $\mathcal{P}^{lu}$  is full-dimensional. This proves parts (i - b), (ii - b) and (iii) of the theorem for *Case 1* (*Case 1* does not apply to parts (i - a) and (ii - a)).

Case 2: On the one hand, since  $(n \bmod F_L) = 0$ , we have  $\lfloor \frac{n}{F_L} \rfloor = \frac{n}{F_L}$ , which implies  $F_L \left( \frac{n}{F_L} - 1 \right) < n$ . On the other hand, we have

$$n \leq F_U \lceil \frac{n}{F_U} \rceil \leq F_U \left( \frac{n}{F_L} - 1 \right).$$



Then, we can apply Lemma 4.3 if  $n < F_U \left\lceil \frac{n}{F_U} \right\rceil$  (i.e., if  $(n \bmod F_U) \neq 0$ ) or  $\left\lceil \frac{n}{F_U} \right\rceil < \frac{n}{F_L} - 1$ . But, we can always create sc-partitions  $\pi_1$  and  $\pi_2$  like we do in the proof of *Case 1*, and existence of two such sc-partitions contradicts Lemma 4.3. This proves parts (ii - b) and (iii) of the theorem for this case. Now remains parts (i - a) and (ii - a). (*Case 2* does not apply to part (i - b)).

Part (i - a): We have,  $\mathcal{P}^{lu} = \text{conv}(A_1 \cup \{w^\pi\})$  where

$$A_1 = \{w \in \{0, 1\}^{|E|} : w \text{ satisfies (3.1a) and } w(\delta(u)) = F_L - 1 \forall u \in V\}$$

and  $\pi$  is the sc-partition with only one subclique, i.e.,  $w_e^\pi = 1$  for all  $e \in E$ . The convex hull of  $A_1$ ,  $\text{conv}(A_1)$ , is the polytope of a  $k$ -way equipartition problem whose dimension is shown to be  $\binom{n}{2} - n$  by Mitchell [13]. It is easy to verify that

$$\text{aff}(A_1) = \{w \in \mathbb{R}^{|E|} : w(\delta(u)) = F_L - 1 \forall u \in V\}.$$

Clearly,  $w^\pi \notin \text{aff}(A_1)$  and so, dimension of  $\mathcal{P}^{lu}$  is equal to  $\binom{n}{2} - (n - 1)$ .

Part (ii - a): We have  $\mathcal{P}^{lu} = \text{conv}(A_1 \cup A_2)$  where

$$A_1 = \{w \in \{0, 1\}^{|E|} : w \text{ satisfies (3.1a) and } w(\delta(u)) = F_L - 1 \forall u \in V\},$$

and

$$A_2 = \{w \in \{0, 1\}^{|E|} : w \text{ satisfies (3.1a) and } w(\delta(u)) = F_U - 1 \forall u \in V\}.$$

Both of  $A_1$  and  $A_2$  are the feasible sets of  $k$ -way equipartition problems. It is easy to verify that

$$\text{aff}(A_1) = \{w \in \mathbb{R}^{|E|} : w(\delta(u)) = F_L - 1 \forall u \in V\}$$

and

$$\text{aff}(A_2) = \{w \in \mathbb{R}^{|E|} : w(\delta(u)) = F_U - 1 \forall u \in V\}.$$

We know that both of  $\text{aff}(A_1)$  and  $\text{aff}(A_2)$  have dimension  $d = \binom{n}{2} - n$ . The fact that they are parallel affine sets directly implies that  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n - 1)$ .  $\square$

Now, we can state the conditions that make  $\mathcal{P}^{lu}$  equivalent to  $\mathcal{P}^{equi}(n)$  or  $\mathcal{P}^{k-way}(n, k)$ .

$$\mathcal{P}^{lu} = \begin{cases} \mathcal{P}^{equi}(n) & \text{when } n \text{ is odd, } \dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1 \text{ and } \left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil = 2; \\ \mathcal{P}^{equi}(n) & \text{when } n \text{ is even, } \dim(\mathcal{P}^{lu}) = \binom{n}{2} - n \text{ and } \left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil = 2; \\ \mathcal{P}^{k-way}(n, k), & \text{when } \dim(\mathcal{P}^{lu}) = \binom{n}{2} - n. \end{cases}$$

We summarize the results of Theorems 4.1 and 4.2 in Tables 1 and 2, respectively. While Table 1 includes all the results related with Theorem 4.1, Table 2 only displays results corresponding to parts (i - a), (i - b), (ii - a) and (ii - b) of Theorem 4.2 (i.e., the parts corresponding to the case  $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil + 1$ ).

	$(n \bmod F_L) = 0$	$(n \bmod F_L) = 1$	$1 < (n \bmod F_L)$
$(n \bmod F_U) = 0$	$\binom{n}{2} - n$	$\binom{n}{2} - n$	$\binom{n}{2} - n$
$(n \bmod F_U) = F_U - 1$		$\binom{n}{2} - 1$	$\binom{n}{2} - 1$
$0 < (n \bmod F_U) < F_U - 1$		$\binom{n}{2} - 1$ or $\binom{n}{2}$	$\binom{n}{2} - 1$ or $\binom{n}{2}$

Table 1: Dimension of  $\mathcal{P}^{lu}$  when  $\lfloor \frac{n}{F_L} \rfloor = \lfloor \frac{n}{F_U} \rfloor$  (Theorem 4.1). Whenever  $0 < (n \bmod F_U) < F_U - 1$  and  $1 < (n \bmod F_L)$ , dimension depends on the value of  $F_U - F_L$ : if  $F_U - F_L \geq 2$ ,  $\mathcal{P}^{lu}$  is full-dimensional; and if  $F_U - F_L = 1$ , it has dimension  $\binom{n}{2} - 1$ .

	$(n \bmod F_L) = 0$	$(n \bmod F_L) \neq 0$	
	$\binom{n}{2} - (n - 1)$	$\binom{n}{2}$	When $\lfloor \frac{n}{F_U} \rfloor = 1$
$(n \bmod F_U) = 0$	$\binom{n}{2} - (n - 1)$	$\binom{n}{2}$	When $\lfloor \frac{n}{F_U} \rfloor \geq 2$
$(n \bmod F_U) \neq 0$	$\binom{n}{2}$		

Table 2: Dimension of  $\mathcal{P}^{lu}$  when  $\lfloor \frac{n}{F_L} \rfloor = \lfloor \frac{n}{F_U} \rfloor + 1$  (parts  $(i - a)$ ,  $(i - b)$ ,  $(ii - a)$  and  $(ii - b)$  of Theorem 4.2). Note that, whenever  $\lfloor \frac{n}{F_L} \rfloor \geq \lfloor \frac{n}{F_U} \rfloor + 2$ ,  $\mathcal{P}^{lu}$  is full-dimensional no matter what the values of  $(n \bmod F_L)$ ,  $(n \bmod F_U)$  and  $\lfloor \frac{n}{F_U} \rfloor$  are (part  $(iii)$  of Theorem 4.2).

## 5 More on $\mathcal{P}^{lu}$

In this section, we deepen the analysis of the polytope  $\mathcal{P}^{lu}$ . First, we give a characterization of the full-dimensional  $\mathcal{P}^{lu}$ 's. Secondly, we give expressions for the equality sets of  $\mathcal{P}^{lu}$ ,  $M(\mathcal{P}^{lu})$ .

Note that, when  $\lfloor \frac{n}{F_U} \rfloor = 1$ ,  $w^\pi = \mathbf{1} \in \mathcal{P}^{lu}$  (i.e., not partitioning the set  $V$  and packing all items in one subclique is feasible). From this point on, we avoid this trivial solution and assume that  $n > F_U$ .

### 5.1 Characterization of Full-dimensional $\mathcal{P}^{lu}$

In this section, we establish four categories that full-dimensional  $\mathcal{P}^{lu}$ 's fall into. This characterization is going to be very helpful in proving the conditions that make several classes of valid inequalities facet-defining in Section 6. We need the following definitions to be able to prove this result.

**Definition 5.1.** *Whenever  $F_U - F_L = 1$ , an sc-partition  $\pi = (N_1, N_2, \dots, N_k)$  ( $k \geq 4$ ) is called 1-loose if  $|\{i : |N_i| = F_L\}| \geq 2$  and  $|\{i : |N_i| = F_U\}| \geq 2$ .*

**Definition 5.2.** *Whenever  $F_U - F_L \geq 2$ , an sc-partition  $\pi = (N_1, N_2, \dots, N_k)$  is called 2-loose if it has at least two subcliques, say  $N_i$  and  $N_j$ , such that  $F_L < |N_i| < F_U$  and*

$$F_L < |N_j| < F_U.$$

**Definition 5.3.** *Whenever an sc-partition is 1-loose or 2-loose, it is called a loose sc-partition.*

We now present a proposition that characterizes  $\mathcal{P}^{lu}$ 's with loose sc-partitions.

**Proposition 5.1.** *The polytope  $\mathcal{P}^{lu}$  contains loose partitions if and only if there exists an integer  $k$  for which  $kF_L + 1 < n < kF_U - 1$ .*

We omit the proof because it is straightforward. Propositions 5.2 and 5.3 show that existence of 1-loose or 2-loose sc-partitions in  $\mathcal{P}^{lu}$  implies that  $\mathcal{P}^{lu}$  is full-dimensional.

**Proposition 5.2.** *Suppose that  $F_U - F_L = 1$  and  $\mathcal{P}^{lu}$  has a feasible 1-loose sc-partition. Then,  $\mathcal{P}^{lu}$  is full dimensional only if  $\lfloor \frac{n}{F_L} \rfloor > \lfloor \frac{n}{F_U} \rfloor$ .*

*Proof.* From Theorem 4.1, we know that  $\mathcal{P}^{lu}$  is not full-dimensional if  $\lfloor \frac{n}{F_L} \rfloor = \lfloor \frac{n}{F_U} \rfloor$ . So, suppose  $\lfloor \frac{n}{F_L} \rfloor > \lfloor \frac{n}{F_U} \rfloor$ . Existence of a feasible 1-loose sc-partition in  $\mathcal{P}^{lu}$  implies that we can employ Lemma 4.3. Consider an arbitrary sc-partition  $\pi_1$  with  $\lfloor \frac{n}{F_L} \rfloor$  subcliques. We can always find another sc-partition  $\pi_2$  with  $\lfloor \frac{n}{F_L} \rfloor - 1$  subcliques since

$$F_L \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) < n \leq F_U \left\lfloor \frac{n}{F_U} \right\rfloor \leq F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right).$$

But, then  $w^{\pi_1}(E) < w^{\pi_2}(E)$ . □

**Proposition 5.3.** *Suppose that  $F_U - F_L \geq 2$ . If  $\mathcal{P}^{lu}$  has a feasible 2-loose sc-partition, then it is full dimensional.*

*Proof.* Let  $\pi_1 = (N_1, N_2, \dots, N_k)$  be a feasible 2-loose sc-partition in  $\mathcal{P}^{lu}$ . Then  $\mathcal{P}^{lu}$  satisfies the conditions of Lemma 4.3. Without loss of generality, assume that  $F_L < |N_1| \leq |N_2| < F_U$ . Now, we can create another sc-partition  $\pi_2$  by shifting a node from  $N_1$  to  $N_2$ . But, we contradict Lemma 4.3 since  $w^{\pi_1}(E) < w^{\pi_2}(E)$ . □

The converse of these propositions does not hold; for example, when  $|V| = 49$ ,  $F_L = 8$  and  $F_U = 10$ , the sc-partitions have either 5 or 6 subcliques. The ones with 5 subcliques have 4 subcliques with size 10 and one subclique with size 9; the ones with 6 subcliques have 5 subcliques with size 8 and one subclique with size 9 (i.e., no feasible 2-loose sc-partitions). But, from part (i - b) of Theorem 4.2, the polytope corresponding to this instance is full-dimensional.

The following two theorems, one written for the case  $F_U - F_L \geq 2$  and the other for the case  $F_U - F_L = 1$ , describe the categories that full-dimensional  $\mathcal{P}^{lu}$ 's fall into.

**Theorem 5.1.** *Suppose that  $F_U - F_L \geq 2$  and  $\mathcal{P}^{lu}$  is full-dimensional. Then, it complies with exactly one of the following:*

**(FD-1)** It has feasible 2-loose sc-partitions,

**(FD-2)**  $(n \bmod F_L) = 1$ ,  $(n \bmod F_U) = 0$  and  $\lfloor \frac{n}{F_L} \rfloor = \lfloor \frac{n}{F_U} \rfloor + 1$ ;

**(FD-3)**  $(n \bmod F_L) = 1$ ,  $(n \bmod F_U) = F_U - 1$  and  $\lfloor \frac{n}{F_L} \rfloor = \lfloor \frac{n}{F_U} \rfloor + 1$ ;

**(FD-4)**  $(n \bmod F_L) = 0$ ,  $(n \bmod F_U) = F_U - 1$  and  $\lfloor \frac{n}{F_L} \rfloor = \lfloor \frac{n}{F_U} \rfloor + 1$ .

*Proof.* First, let  $k = \lfloor \frac{n}{F_L} \rfloor = \lfloor \frac{n}{F_U} \rfloor$ . According to part (iii - b) of Theorem 4.1, we must have  $F_U - F_L \geq 2$  and  $kF_U - 2 \geq n \geq kF_L + 2$ . Then, by Proposition 5.1,  $\mathcal{P}^{lu}$  contains 2-loose sc-partitions (i.e., **(FD-1)**).

Now, suppose  $\lfloor \frac{n}{F_L} \rfloor > \lfloor \frac{n}{F_U} \rfloor$ . We distinguish three cases:  $(n \bmod F_L) > 1$ ,  $(n \bmod F_L) = 1$  and  $(n \bmod F_L) = 0$ .

Case 1  $(n \bmod F_L) > 1$ : In this case, obviously we have  $F_L \lfloor \frac{n}{F_L} \rfloor + 2 \leq n$ . On the other hand, we have

$$n \leq F_U \lfloor \frac{n}{F_U} \rfloor \leq F_U \left( \lfloor \frac{n}{F_L} \rfloor - 1 \right) = F_U \lfloor \frac{n}{F_L} \rfloor - F_U \leq F_U \lfloor \frac{n}{F_L} \rfloor - 2,$$

which by Proposition 5.1 implies that  $\mathcal{P}^{lu}$  contains 2-loose sc-partitions (i.e., **(FD-1)**).

Case 2  $(n \bmod F_L) = 1$ : We have

$$n = F_L \lfloor \frac{n}{F_L} \rfloor + 1.$$

Adding  $0 > 1 - F_L$  to this equation yields

$$n > F_L \left( \lfloor \frac{n}{F_L} \rfloor - 1 \right) + 2.$$

We also have,  $n \leq F_U \lfloor \frac{n}{F_U} \rfloor$ .

- First, consider  $\lfloor \frac{n}{F_L} \rfloor > \lfloor \frac{n}{F_U} \rfloor + 1$ . We have

$$F_U \lfloor \frac{n}{F_U} \rfloor \leq \left( \lfloor \frac{n}{F_L} \rfloor - 1 \right) F_U - F_U \leq \left( \lfloor \frac{n}{F_L} \rfloor - 1 \right) F_U - 2,$$

which proves then that  $\mathcal{P}^{lu}$  contains 2-loose sc-partitions by Proposition 5.1.

- Secondly, consider  $\lfloor \frac{n}{F_L} \rfloor = \lfloor \frac{n}{F_U} \rfloor + 1$ . If

$$n \leq F_U \lfloor \frac{n}{F_U} \rfloor - 2 = F_U \left( \lfloor \frac{n}{F_L} \rfloor - 1 \right) - 2,$$

then  $\mathcal{P}^{lu}$  contains 2-loose sc-partitions. On the other hand, if  $n = F_U \lfloor \frac{n}{F_U} \rfloor - 1$ ,

$\mathcal{P}^{lu}$  complies with **(FD-3)**; and, if  $n = F_U \lfloor \frac{n}{F_U} \rfloor$ ,  $\mathcal{P}^{lu}$  complies with **(FD-2)**.

Case 3 ( $n \bmod F_L = 0$ ): First, note that we must have  $\left\lceil \frac{n}{F_U} \right\rceil \geq 2$ , because otherwise,  $\mathcal{P}^{lu}$  is not full-dimensional by part (i - a) of Theorem 4.2. From  $(n \bmod F_L) = 0$ , we infer  $F_L(\frac{n}{F_L} - 1) + 2 \leq n$ .

- Consider first  $\left\lfloor \frac{n}{F_L} \right\rfloor > \left\lfloor \frac{n}{F_U} \right\rfloor + 1$ . From  $n \leq F_U \left\lceil \frac{n}{F_U} \right\rceil$ , we have

$$n \leq F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) - F_U < F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) - F_U,$$

which proves existence of 2-loose sc-partitions in  $\mathcal{P}^{lu}$ .

- Now, consider  $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lfloor \frac{n}{F_U} \right\rfloor + 1$ . If  $n \leq F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) - 2$ ,  $\mathcal{P}^{lu}$  has loose sc-partitions; if,  $n = F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right) - 1$ ,  $\mathcal{P}^{lu}$  complies with **(FD-4)**. Note that,  $\mathcal{P}^{lu}$  is not full-dimensional when  $n = F_U \left( \left\lfloor \frac{n}{F_L} \right\rfloor - 1 \right)$  (i.e.,  $(n \bmod F_U) = 0$ ) by part (ii - a) of Theorem 4.2.

□

**Theorem 5.2.** *Suppose  $F_U - F_L = 1$  and  $\mathcal{P}^{lu}$  is full-dimensional. Then it complies with exactly one of **(FD-2)**, **(FD-3)**, **(FD-4)**, and,*

**(FD-5):** *it has feasible 1-loose sc-partitions and  $\left\lfloor \frac{n}{F_L} \right\rfloor > \left\lfloor \frac{n}{F_U} \right\rfloor$ .*

We skip the proof because it is very similar to that of Theorem 5.1.

In the rest of this paper and in its sequel, when we say that  $\mathcal{P}^{lu}$  complies with **(FD-1)** we implicitly assume that  $F_U - F_L \geq 2$ . Similarly, when we say that  $\mathcal{P}^{lu}$  complies with **(FD-5)**, we implicitly assume that  $F_U - F_L = 1$ .

*Remark:* In Theorems 5.1 and 5.2, we can still show full-dimensionality for **(FD-1)**, **(FD-2)**, **(FD-3)** and **(FD-5)** when  $\left\lfloor \frac{n}{F_U} \right\rfloor = 1$ . On the other hand, **(FD-4)** implies full-dimensionality only if  $\left\lfloor \frac{n}{F_U} \right\rfloor \geq 2$ .

## 5.2 Equality Sets of $\mathcal{P}^{lu}$

In this section we express the equality sets of  $\mathcal{P}^{lu}$  when it is not full-dimensional. These characterizations are going to be helpful in proving the conditions under which several valid inequalities are facet defining in Section 6.

When  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$ , the equality set of  $\mathcal{P}^{lu}$  consists of a single hyperplane  $w(E) = C$  where  $C$  is an appropriate constant. When  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$  the equality set consists of the hyperplanes

$$w(\delta(u)) = F \quad \forall u \in V,$$

where  $F \in \{F_L, F_U\}$ . These equality sets can trivially be inferred from the proofs of Theorems 4.1 and 4.2. However, when  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n - 1)$ , the equality set is not trivial, and is shown in the following proposition.

**Proposition 5.4.** *Suppose  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n - 1)$ . Then, the equality set of  $\mathcal{P}^{lu}$  consists of  $n - 1$  linearly independent hyperplanes in the form  $\sum_{(i,j) \in E} (\lambda_i + \lambda_j) w_{i,j} = 0$*

where  $\lambda_i \in \mathbb{R}$  for  $i \in V$  and  $\sum_{i \in V} \lambda_i = 0$ .

*Proof.* If  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n - 1)$ , then feasible solutions in  $\mathcal{P}^{lu}$  lie in either of the following parallel affine spaces (by the proof of Theorem 4.2):

$$\begin{aligned} A_1 &= \{w \in \mathbb{R}^{\binom{n}{2}} \mid w(\delta(u)) = F_L - 1 \quad \forall u \in V\}; \\ A_2 &= \{w \in \mathbb{R}^{\binom{n}{2}} \mid w(\delta(u)) = F_U - 1 \quad \forall u \in V\}. \end{aligned}$$

We now assign a coefficient  $\lambda_u$  ( $u \in V$ ) to each of the hyperplanes defining  $A_1$  and take linear combinations of these hyperplanes to obtain

$$\sum_{u \in V} \lambda_u w(\delta(u)) = \sum_{u \in V} \lambda_u (F_L - 1). \quad (5.1)$$

Now, we search for  $\vec{\lambda}$ 's for which  $A_2$  is contained in the affine space defined by (5.1). All points in  $A_2$  satisfy  $w(\delta(u)) = F_U - 1$  for all  $u \in V$ . Inserting this into (5.1) yields

$$\sum_{u \in V} \lambda_u (F_U - 1) = \sum_{u \in V} \lambda_u (F_L - 1).$$

This equality implies  $\sum_{u \in V} \lambda_u = 0$ . □

## 6 Trivial Facets for $\mathcal{P}^{lu}$

Having set the grounds, we now pass on to the facetness proofs. In the sequel, we assume that  $F_L \geq 3$ . Recall that, we also assume  $n > F_U$  (i.e. sc-partitions in  $\mathcal{P}^{lu}$  have at least two subcliques). We start with several technical lemmas.

**Lemma 6.1.** *Suppose that  $F_U - F_L \geq 2$ . Suppose also that  $g^T w \leq h$  is a valid inequality for  $\mathcal{P}^{lu}$ . Let  $P = \{w \in \mathcal{P}^{lu} \mid g^T w = h\}$ . Suppose further that there exists  $k \in \mathbb{Z}_+$  such that  $kF_L + 1 < n < kF_U - 1$  (i.e., there exist 2-loose sc-partitions in  $\mathcal{P}^{lu}$ ). Pick such a 2-loose sc-partition  $\pi = (N_1, N_2, \dots, N_k)$  such that  $w^\pi \in P$ ,  $i \in N_1$ ,  $j \in N_2$ ,  $F_L < |N_1| < F_U$  and  $F_L < |N_2| < F_U$  for  $i, j \in V$ . Then  $g_{i,j} = 0$  if the following hold:*

- $w^{\pi^*} \in P$  where  $\pi^* = (N_1^*, \dots, N_k^*)$ ,  $N_1^* = (N_1 - i) \cup j$ ,  $N_2^* = (N_2 - j) \cup i$  and  $N_l^* = N_l$  for all  $l \in \{3, \dots, k\}$ .
- $w^{\hat{\pi}} \in P$  where  $\hat{\pi} = (\hat{N}_1, \dots, \hat{N}_k)$ ,  $\hat{N}_1 = N_1 - i$ ,  $\hat{N}_2 = N_2 \cup i$  and  $\hat{N}_l = N_l$  for all  $l \in \{3, \dots, k\}$ .
- $w^{\tilde{\pi}} \in P$  where  $\tilde{\pi} = (\tilde{N}_1, \dots, \tilde{N}_k)$ ,  $\tilde{N}_1 = N_1 \cup j$ ,  $\tilde{N}_2 = N_2 - j$ ,  $\tilde{N}_l = N_l$  for all  $l \in \{3, \dots, k\}$ .

*Proof.* From the equalities  $g^T w^\pi = h$  and  $g^T w^{\pi^*} = h$ , we obtain

$$g(i, N_1 - i) + g(j, N_2 - j) = g(i, N_2 - j) + g(j, N_1 - i). \quad (6.1)$$

From the equalities  $g^T w^\pi = h$  and  $g^T w^{\hat{\pi}} = h$ , we obtain

$$g(i, N_1 - i) = g(i, N_2 - j) + g_{i,j}. \quad (6.2)$$

From the equalities  $g^T w^\pi = h$  and  $g^T w^{\tilde{\pi}} = h$ , we get

$$g(j, N_2 - j) = g(j, N_1 - i) + g_{i,j}. \quad (6.3)$$

Now summing the equalities (6.2) and (6.3) and comparing the sum with (6.1), we get  $g_{i,j} = 0$ .  $\square$

**Lemma 6.2.** *Suppose that  $g^T w \leq h$  is a valid inequality for  $\mathcal{P}^{lu}$ . Let  $P = \{w \in \mathcal{P}^{lu} | g^T w = h\}$ . Suppose  $P \subseteq \{w \in \mathcal{P}^{lu} | g^T w = h\}$ . Let  $u, v$  and  $x$  be three nodes in  $V$ .*

- *Let  $\pi = (N_1, N_2, \dots, N_k)$  be an sc-partition such that  $\{u, v\} \subset N_1$ ,  $x \in N_2$  and  $|N_1| = |N_2| + 1$ ;*
- *let  $\tilde{\pi} = (\tilde{N}_1, \dots, \tilde{N}_k)$  where  $\tilde{N}_1 = (N_1 - v) \cup x$ ,  $\tilde{N}_2 = (N_2 - x) \cup v$ ,  $\tilde{N}_l = N_l$  for  $l \in \{3, \dots, k\}$ ;*
- *let  $\pi^* = (N_1^*, \dots, N_k^*)$  where  $N_1^* = (N_2 - x) \cup \{u, v\}$ ,  $N_2^* = (N_1 - \{u, v\}) \cup x$ ,  $N_l^* = N_l$  for  $l \in \{3, \dots, k\}$ ;*
- *let  $\hat{\pi} = (\hat{N}_1, \dots, \hat{N}_k)$  where  $\hat{N}_1 = (N_1^* - v) \cup x$ ,  $\hat{N}_2 = (N_2^* - x) \cup v$ ,  $N_l^* = N_l$  for  $l = \{3, \dots, k\}$ .*

*If  $\{w^\pi, w^{\tilde{\pi}}, w^{\pi^*}, w^{\hat{\pi}}\} \subset P$  then  $g_{u,v} = g_{u,x}$ .*

*Proof.* Comparing the equalities,  $g^T w^\pi = h$  and  $g^T w^{\tilde{\pi}} = h$ , we obtain

$$g_{u,v} + g(v, N_1 - \{u, v\}) + g(x, N_2 - x) = g_{u,x} + g(x, N_1 - \{u, v\}) + g(v, N_2 - x),$$

which is equivalent to  $g_{u,v} - g_{u,x} = A$  where  $A = [g(x, N_1 - \{u, v\}) - g(x, N_2 - x)] - [g(v, N_1 - \{u, v\}) - g(v, N_2 - x)]$ . Comparing the equalities  $g^T w^{\pi^*} = h$  and  $g^T w^{\hat{\pi}} = h$ , we obtain

$$g_{u,v} + g(v, N_1^* - \{u, v\}) + g(x, N_2^* - x) = g_{u,x} + g(x, N_1^* - \{u, v\}) + g(v, N_2^* - x),$$

which is equivalent to  $g_{u,v} - g_{u,x} = B$  where  $B = [g(x, N_1^* - \{u, v\}) - g(x, N_2^* - x)] - [g(v, N_1^* - \{u, v\}) - g(v, N_2^* - x)]$ . Since  $N_1^* - \{u, v\} = N_2 - x$  and  $N_2^* - x = N_1 - \{u, v\}$ , we have  $A = -B$ . Hence,  $g_{u,x} = g_{u,v}$ .  $\square$

**Lemma 6.3.** *Suppose that  $g^T w \leq h$  is a valid inequality for  $\mathcal{P}^{lu}$ . Let  $P = \{w \in \mathcal{P}^{lu} | g^T w = h\}$ . Let  $u, v, x$  and  $y$  be four nodes in  $V$ .*

- *Let  $\pi = (N_1, N_2, \dots, N_k)$  be an sc-partition such that  $\{u, x\} \in N_1$ ,  $\{v, y\} \in N_2$ ;*

- let  $\tilde{\pi} = (\tilde{N}_1, \dots, \tilde{N}_k)$  where  $\tilde{N}_1 = (N_1 - u) \cup v$ ,  $\tilde{N}_2 = (N_2 - v) \cup u$ ,  $\tilde{N}_l = N_l$  for  $l \in \{3, \dots, k\}$ ;
- let  $\pi^* = (N_1^*, \dots, N_k^*)$  where  $N_1^* = (N_1 - x) \cup y$ ,  $N_2^* = (N_2 - y) \cup x$ ,  $N_l^* = N_l$  for  $l \in \{3, \dots, k\}$ ;
- let  $\hat{\pi} = (\hat{N}_1, \dots, \hat{N}_k)$  where  $\hat{N}_1 = (N_1^* - u) \cup v$ ,  $\hat{N}_2 = (N_2^* - v) \cup u$ ,  $N_l^* = N_l$  for  $l \in \{3, \dots, k\}$ .

If  $\{w^\pi, w^{\tilde{\pi}}, w^{\pi^*}, w^{\hat{\pi}}\} \subset P$  then  $g_{u,x} + g_{v,y} = g_{u,y} + g_{v,x}$ .

*Proof.* Comparing the equalities,  $g^T w^\pi = h$  and  $g^T w^{\tilde{\pi}} = h$  we obtain

$$g_{u,x} + g(u, N_1 - \{u, x\}) + g_{v,y} + g(v, N_2 - \{v, y\}) = g_{u,y} + g(u, N_2 - \{v, y\}) + g_{v,x} + g(v, N_1 - \{u, x\}),$$

which is equivalent to  $(g_{u,x} + g_{v,y}) - (g_{u,y} + g_{v,x}) = A$  where

$$A = [g(u, N_2 - \{v, y\}) - g(u, N_1 - \{u, x\})] - [g(v, N_2 - \{v, y\}) - g(v, N_1 - \{u, x\})].$$

Comparing the equalities  $g^T w^{\pi^*} = h$  and  $g^T w^{\hat{\pi}} = h$ , we obtain

$$g_{u,y} + g(u, N_1^* - \{u, y\}) + g_{v,x} + g(v, N_2^* - \{v, x\}) = g_{u,x} + g(u, N_2^* - \{v, x\}) + g_{v,y} + g(v, N_1^* - \{u, y\}),$$

which is equivalent to  $(g_{u,x} + g_{v,y}) - (g_{u,y} + g_{v,x}) = B$  where

$$B = [g(u, N_1^* - \{u, y\}) - g(u, N_2^* - \{v, x\})] - [g(v, N_1^* - \{u, y\}) - g(v, N_2^* - \{v, x\})].$$

Since  $N_1^* - \{u, y\} = N_1 - \{u, x\}$  and  $N_2^* - \{v, x\} = N_1 - \{v, y\}$ , we have  $A = -B$ . Hence,  $g_{u,x} + g_{v,y} = g_{u,y} + g_{v,x}$ .  $\square$

**Lemma 6.4.** *Suppose that  $g^T w \leq h$  is a valid inequality for  $\mathcal{P}^{lu}$ . Let  $p \in V$  and  $Q \subseteq V$ . Suppose that  $g(p, \tilde{Q}_1) + \alpha = g(p, \tilde{Q}_2)$  for any disjoint  $\tilde{Q}_1, \tilde{Q}_2 \subset Q$  with  $|\tilde{Q}_1| = \sigma_1$ ,  $|\tilde{Q}_2| = \sigma_2$  where  $\alpha \in \mathbb{R}$ ,  $\sigma_1, \sigma_2 \in \mathbb{Z}$ ,  $\sigma_1 > 0$  and  $\sigma_2 \geq 0$ . Then,  $g_{p,q_1} = g_{p,q_2}$  for any  $q_1, q_2 \in Q$ .*

*Proof.* Pick arbitrarily  $\tilde{Q}_1$  and  $\tilde{Q}_2$  with sizes  $\sigma_1$  and  $\sigma_2$  respectively. Pick arbitrarily  $q_1 \in \tilde{Q}_1$  and  $q_2 \in R$  where

$$R = \begin{cases} \tilde{Q}_2, & \text{if } \tilde{Q}_2 \neq \emptyset, \\ Q - \tilde{Q}_1, & \text{if } \tilde{Q}_2 = \emptyset. \end{cases}$$

Consider the sets  $\tilde{Q}_1^* = (\tilde{Q}_1 - q_1) \cup q_2$ ,  $R^* = (R - q_2) \cup q_1$ . Comparing  $g(p, \tilde{Q}_1) + \alpha = g(p, \tilde{Q}_2)$  and  $g(p, \tilde{Q}_1^*) + \alpha = g(p, \tilde{Q}_2^*)$  yields the result.  $\square$

Note that, when  $\tilde{Q}_2 = \emptyset$ , it is crucial to have in this lemma that  $\tilde{Q}_1 \subsetneq Q$ .

In the sequel, we prove the conditions under which several classes of valid inequalities become facet defining for  $\mathcal{P}^{lu}$ . Each class is investigated in a separate section. Commonly, for each of the valid inequalities, we assume that the face defined by the valid inequality under consideration, say  $a^T w \leq b$ , is contained in the face defined by some valid inequality  $g^T w \leq h$ . In our proofs, with reference to Theorem 3.6 in Section I.4.3 of [14], we basically show that  $g^T w = h$  is equal to a linear combination of  $a^T w = b$  and the hyperplanes in  $M(\mathcal{P}^{lu})$ .



## 6.1 The Non-Negativity Constraints

In this section, we provide the conditions that make the non-negativity constraints facet defining. We denote the face corresponding to a non-negativity constraint  $w_e \geq 0$  by  $P_e$ , i.e.,  $P_e = \{\mathbf{w} \in \mathcal{P}^{lu} : w_e = 0\}$ . We assume that there exists a valid inequality  $g^T \mathbf{w} \leq h$  such that

$$P_e \subseteq \{\mathbf{w} \in \mathcal{P}^{lu} | g^T \mathbf{w} = h\}.$$

In Theorems 6.1, 6.2, 6.4 and 6.3, we show that  $g^T \mathbf{w} = h$  can be written as a linear combination of  $w_e = 0$  and the equalities in  $M(\mathcal{P}^{lu})$ , if any.

Before moving on to these theorems, we present the following useful proposition.

**Proposition 6.1.** *If  $\lfloor \frac{n}{F_L} \rfloor = 2$ , the non-negativity constraints  $w_e \geq 0$  are not facet defining for  $\mathcal{P}^{lu}$ .*

*Proof.* Recall that we assume  $n > F_U$  (i.e.,  $\lfloor \frac{n}{F_U} \rfloor \geq 2$ ). So, when  $\lfloor \frac{n}{F_L} \rfloor = 2$ , we can conclude that all sc-partitions have 2 subcliques. Then, for each edge  $e = \{u, v\}$  we have

$$P_e \subseteq \{\mathbf{w} \in \mathcal{P}^{lu} : w_{x,u} + w_{x,v} - w_{u,v} = 1 \quad \forall x \in V - \{u, v\}\}.$$

□

**Theorem 6.1.** *Suppose  $\mathcal{P}^{lu}$  is full-dimensional. Then  $w_e \geq 0$  is facet-defining for  $\mathcal{P}^{lu}$  if and only if one of the following hold:*

- $n \geq 3F_L + 1$ , or,
- $n = 3F_L \leq 2F_U - 2$ .

*Proof.* Let  $e = \{u, v\}$  for  $u, v \in V$ . We present the proof separately for **(FD-1)**-**(FD-5)**. Note that, by Proposition 6.1 we have  $n \geq 3F_L$ .

**(FD-1)** : Note that, in this case we assume  $F_U - F_L \geq 2$ . We analyze two cases separately:

- (a)  $\mathcal{P}^{lu}$  has 2-loose sc-partitions with at least 3 sub-cliques: Note that, in this case,  $n$  is at least  $3F_L + 2$ . Pick an arbitrary edge  $\{i, j\} \in E - \{u, v\}$ . Pick a 2-loose sc-partition  $\pi = (N_1, N_2, \dots, N_k)$  with at least three subcliques and where  $u \in N_1, i \in N_1, j \in N_2, v \in N_3$  ( $i$  is not necessarily different from  $u$ ),  $F_L < |N_1| < F_U, F_L < |N_2| < F_U$ . By Lemma 6.1 and symmetry, we conclude that  $g_{i,j} = 0$  for any  $\{i, j\} \in E - \{u, v\}$ . Thus,  $w_{u,v} \geq 0$  defines a facet of  $\mathcal{P}^{lu}$ .
- (b) All 2-loose sc-partitions in  $\mathcal{P}^{lu}$  have at most 2 sub-cliques: Pick two arbitrary nodes  $x, y \in V - \{u, v\}$ . Pick a 2-loose sc-partition, say  $\pi^1 = (N_1^1, N_2^1)$  such that  $\{u, x\} \subset N_1^1$  and  $\{v, y\} \subset N_2^1$ . By Lemma 6.1,  $g_{x,y} = 0$ . By arbitrariness of  $x$  and  $y$ ,  $g_e = 0$  for all  $e \in E(V - \{u, v\})$ .

Pick another sc-partition, say  $\pi^2 = (N_1^2, N_2^2, \dots, N_k^2)$  ( $k \geq 3$ ), such that  $u \in N_1^2, \{v, x\} \subset N_2^2$  and  $y \in N_3^2$ . Switching  $x$  and  $y$  in  $\pi^1$ , we get  $g_{v,x} = g_{v,y}$  for all  $x, y \in V - \{u, v\}$ , which can be rewritten as  $g_{v,z} = \alpha \in \mathbb{R}$  for all

$z \in V - \{u, v\}$ . Similarly, we can also infer  $g_{u,z} = \beta \in \mathbb{R}$  for all  $z \in V - \{u, v\}$ . Now, in  $\pi^1$ , shifting  $x$  from  $N_1^1$  to  $N_2^1$  shows that  $\alpha = \beta$ . Since  $\{w^{\pi^1}, w^{\pi^2}\} \subset P_e$ , comparing  $g^T w^{\pi^1} = h$  and  $g^T w^{\pi^2} = h$  yields  $\alpha = \beta = 0$ , due to the fact that  $w^{\pi^2}(\{u, v\}, V - \{u, v\}) < w^{\pi^1}(\{u, v\}, V - \{u, v\}) = n - 2$ . Note that, this proof for **(FD-1)** is applicable even if  $3F_L \leq n \leq 3F_L + 1$  and  $n \leq 2F_U - 2$ .

**(FD-2)** : Proposition 6.1 and **(FD-2)** imply that  $n \geq 3F_L + 1$ .

Pick two arbitrary nodes  $x, y \in V - \{u, v\}$ . Pick an sc-partition  $\pi = (N_1, N_2, \dots, N_k)$  with  $k = \left\lfloor \frac{n}{F_L} \right\rfloor \geq 3$ ,  $\{v, x\} \subset N_1$ ,  $y \in N_2$ ,  $u \in N_3$ ,  $|N_1| = F_L + 1$  and  $|N_i| = F_L$  for  $i = 2, \dots, k$ . Using Lemma 6.2, we infer  $g_{v,x} = g_{x,y}$ . By symmetry, we infer this result for  $u$  as well and generalize this result as  $g_e = \alpha$  for all  $e \in E - \{u, v\}$ .

Now, consider sc-partitions  $\pi^1 = (N_1^1, \dots, N_{k_1}^1)$  with  $k_1 = \left\lfloor \frac{n}{F_L} \right\rfloor$ ,  $u \in N_1^1$ ,  $v \in N_2^1$ ; and,  $\pi^2 = (N_1^2, \dots, N_{k_2}^2)$  with  $k_2 = \left\lfloor \frac{n}{F_U} \right\rfloor = k_1 - 1$ ,  $u \in N_1^2$ ,  $v \in N_2^2$ . Clearly,  $\{w^{\pi^1}, w^{\pi^2}\} \subset P_e$ . Comparing  $g^T w^{\pi^1} = h$  and  $g^T w^{\pi^2} = h$ , we infer that  $\alpha = 0$ .

**(FD-3)** : The proof for this case is the same as the proof of case **(FD - 2)**.

**(FD-4)** : Pick three nodes  $x, y, z \in V - \{u, v\}$  arbitrarily. Pick an sc-partition  $\pi = (N_1, N_2, \dots, N_k)$  with  $k \geq 2$  such that  $\{v, z\} \subset N_1$ ,  $\{u, x, y\} \subset N_2$ ,  $|N_1| = F_U - 1$ ,  $|N_i| = F_U$  for all  $i = 2, \dots, k$ . Applying Lemma 6.2, we obtain  $g_{x,y} = g_{y,z}$ , which by arbitrariness of  $x, y$  and  $z$  implies  $g_e = \beta$  for all  $e \in E(V - \{u, v\})$  where  $\beta \in \mathbb{R}$ . Now, shifting  $x$  from  $N_2$  to  $N_1$  in  $\pi$  we get  $g_{u,x} = g_{v,x}$ . Since  $x$  is an arbitrary node, we infer  $g_{u,x} = g_{v,x} = \alpha_x \in \mathbb{R}$  for all  $x \in V - \{u, v\}$ . Now, pick another sc-partition  $\pi^* = (N_1^*, \dots, N_{k+1}^*)$  with  $|N_i^*| = F_L$  for  $i = 1, \dots, k + 1$ ,  $u \in N_1^*$ ,  $\{x, v\} \subset N_2^*$  and  $y \in N_3^*$ . Now, switching  $x$  and  $y$  in  $\pi^*$ , we get  $\alpha_x = \alpha_y$  which implies  $\alpha_x = \alpha_y = \alpha$  for all  $x, y \in V - \{u, v\}$ .

- Now, assume that  $\left\lfloor \frac{n}{F_L} \right\rfloor \geq 4$  (i.e.,  $n \geq 4F_L$ ). Pick an sc-partition  $\tilde{\pi} = (\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_k)$  where  $k \geq 3$ ,  $|\tilde{N}_1| = F_U - 1$  and  $|\tilde{N}_i| = F_U$  for  $i = 2, \dots, k$ . Suppose  $u \in \tilde{N}_1$ ,  $\{x, y\} \subset \tilde{N}_2$  and  $v \in \tilde{N}_3$ . Shifting  $y$  from  $\tilde{N}_2$  to  $\tilde{N}_1$  we get  $\beta(F_U - 1) = \beta(F_U - 2) + \alpha$  which implies  $\alpha = \beta$ . Hence, we infer that, if  $\alpha = \beta \neq 0$ ,  $P_e$  is contained in an hyperplane  $g_{u,v}w_{u,v} + \alpha w(E - \{u, v\}) = h$  for some constants  $h \in \mathbb{R}$  and  $g_{u,v} \in \mathbb{R}$ . But, this contradicts the fact that  $\{\pi^*, \tilde{\pi}\} \subset P_e$  do not both lie on this hyperplane since  $w^{\pi^*}(E - \{u, v\}) < w^{\tilde{\pi}}(E - \{u, v\})$ . Then,  $\alpha = \beta = 0$ .
- Finally, assume that  $\left\lfloor \frac{n}{F_L} \right\rfloor = 3$  (i.e.,  $n = 3F_L = 2F_U - 1$ ). In this case,  $P_e$  is contained in the hyperplane

$$g_{u,v}w_{u,v} + \alpha w(E(\{u, v\}, V - \{u, v\})) + \beta w(E(V - \{u, v\})) = h \quad (6.4)$$

for some constants  $g_{u,v} \in \mathbb{R}$  and  $h \in \mathbb{R}$ . Now, consider two sc-partitions  $\pi^1 = (N_1^1, N_2^1, N_3^1)$  and  $\pi^2 = (N_1^2, N_2^2)$  implied by **(FD-4)**, where  $|N_i^1| = F_L$

for  $i = 1, 2, 3$ ,  $|N_1^2| = F_U - 1$ ,  $|N_2^2| = F_U$ ,  $u \in N_1^1 \cap N_1^2$ ,  $v \in N_2^1 \cap N_2^2$ . For  $\pi^1$ , the hyperplane equation (6.4) turns into

$$2(F_L - 1)\alpha + \left[ 2\binom{F_L - 1}{2} + \binom{F_L}{2} \right] \beta = h;$$

and, for  $\pi^2$  it becomes

$$(2F_U - 3)\alpha + \left[ \binom{F_U - 1}{2} + \binom{F_U - 2}{2} \right] \beta = h.$$

$P_e$  is contained in the hyperplane represented by the equation (6.4) obtained by setting

$$\begin{aligned} \alpha &= 2\binom{F_L - 1}{2} + \binom{F_L}{2} - \binom{F_U - 1}{2} - \binom{F_U - 2}{2} \\ &= (F_L - 1) \left( \frac{3}{2}F_L - 2 \right) - (F_U - 2)^2, \end{aligned}$$

and  $\beta = 2(F_U - F_L) - 1$ . Hence, when  $\mathcal{P}^{lu}$  complies with **(FD-4)** and  $n = 3F_L = 2F_U - 1$ ,  $w_{u,v} \geq 0$  is not facet defining. Indeed, when  $n = 3F_L = 2F_U - 1$  it is possible to show that

$$\begin{aligned} \alpha w_{u,v} + \alpha w(E(\{u, v\}, V - \{u, v\})) + \beta w(E(V - \{u, v\})) &\geq \\ &(F_L - 1) \left( \frac{3}{2}F_L - 2 \right) (2F_U - 3) - 2(F_L - 1)(F_U - 2)^2 \end{aligned}$$

is facet defining if  $\beta - \alpha \geq 0$ ; and,

$$\begin{aligned} \alpha w_{u,v} + \alpha w(E(\{u, v\}, V - \{u, v\})) + \beta w(E(V - \{u, v\})) &\leq \\ &(F_L - 1) \left( \frac{3}{2}F_L - 2 \right) (2F_U - 3) - 2(F_L - 1)(F_U - 2)^2 \end{aligned}$$

is facet defining if  $\beta - \alpha < 0$ .

**(FD-5)** : Note that, in this case we assume  $F_U - F_L = 1$ . The proof for this case is very similar to the proof of case **(FD-2)**. □

**Theorem 6.2.** *Suppose that  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$ . Then, the nonnegativity constraints  $w_e \geq 0$  are facet defining for  $\mathcal{P}^{lu}$  if and only if  $n \geq 3F_L + 1$ .*

*Proof.* By Proposition 6.1, we consider  $n \geq 3F_L$ . But, when  $n = 3F_L$ ,  $\dim(\mathcal{P}^{lu})$  is never equal to  $\binom{n}{2} - 1$ . Hence, we have  $n \geq 3F_L + 1$ . We give the proof for only the case  $(n \bmod F_L) = 1$ . The proof for  $(n \bmod F_U) = F_U - 1$  is very similar.

Let  $e = \{u, v\}$  for  $u, v \in V$ . Pick two arbitrary nodes  $x, y \in V - \{u, v\}$ . Using  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$ , we can set  $g_{x,y} = 0$ . Now, pick an sc-partition  $\pi = (N_1, N_2, \dots, N_k)$  such that  $|N_1| = F_L + 1$ ,  $|N_i| = F_L$  for  $i = 2, \dots, k$ ,  $\{x, v\} \subset N_1$ ,  $y \in N_2$  and  $u \in N_3$ . Using Lemma 6.2 we obtain  $g_{v,x} = g_{x,y}$ . By symmetry, we can also infer  $g_{u,x} = g_{x,y}$ , which generalizes into  $g_e = 0$  for all  $e \in E - \{u, v\}$  for we have already set  $g_{x,y} = 0$ . □

**Theorem 6.3.** *Suppose  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$ . Then the non-negativity constraints  $w_e \geq 0$  are facet-defining for  $\mathcal{P}^{lu}$  if and only if  $n \geq 3F_L$ .*

*Proof.* When  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$ , we have  $n \geq 3F_L$  by Theorem 4.2 and the assumption that  $n > F_U$ .

Let  $e = \{u, v\}$  and without loss of generality, assume  $u = n, v = n-1$ . Let  $k = \frac{n}{F_U} = \frac{n}{F_L} - 1$ . We know from Proposition 5.4 that  $M(\mathcal{P}^{lu})$  consists of linearly independent hyperplanes in the form  $\sum_{(u,v) \in E} (\lambda_u + \lambda_v) w_{u,v} = 0$  where  $\lambda \in \mathbb{R}^n$  and  $\sum_{u \in V} \lambda_u = 0$ . We choose  $n-1$  such hyperplanes  $H_1, H_2, \dots, H_{n-1}$  which are determined by the  $\lambda$ 's displayed in the rows of the following matrix:

$$\begin{array}{l} H_1 \longrightarrow \\ H_2 \longrightarrow \\ H_3 \longrightarrow \\ \vdots \\ H_{n-1} \longrightarrow \end{array} \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_{n-1} & \lambda_n \\ 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{bmatrix}.$$

One can verify that  $H_1, H_2, \dots, H_{n-1}$  are linearly independent. Now, consider the set of edges  $S = \{\{1, n\}, \{2, n\}, \dots, \{n-2, n\}, \{1, n-1\}\}$ . The following matrix displays the coefficients of the edges belonging to  $S$  in the equations representing  $H_1$  to  $H_{n-1}$ :

$$\begin{array}{l} H_1 \longrightarrow \\ H_2 \longrightarrow \\ H_3 \longrightarrow \\ H_4 \longrightarrow \\ \vdots \\ H_{n-3} \longrightarrow \\ H_{n-2} \longrightarrow \\ H_{n-1} \longrightarrow \end{array} \begin{bmatrix} \{1, n\} & \{2, n\} & \{3, n\} & \{4, n\} & \dots & \{n-2, n\} & \{1, n-1\} \\ 1 & -1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & -1 \\ -1 & -1 & -1 & -1 & \dots & -1 & 1 \end{bmatrix}.$$

This matrix is non-singular. Hence, we can set  $g_e = 0$  for all  $e \in S$  (again, by Theorem 3.6 in Section I.4.3 of [14]). Now, pick four arbitrary nodes  $x, y, z, t \in V - \{n-1, n\}$  and an sc-partition  $\pi = (N_1, N_2, \dots, N_{k+1})$  where  $|N_i| = F_L$  for all  $i = 1, 2, \dots, k+1$ ,  $\{n, x\} \subset N_1$ ,  $\{n-1, y\} \subset N_2$  and  $\{z, t\} \subset N_3$ . Using Lemma 6.3 we infer  $g_{x,n} + g_{y,n-1} = g_{y,n} + g_{x,n-1}$ . Since  $g_{x,n} = g_{y,n} = 0$ , we have  $g_{x,n-1} = g_{y,n-1}$ . But we have already set  $g_{1,n-1} = 0$ , hence, we obtain  $g_{q,n-1} = 0$  for all  $q \in V - \{n-1, n\}$ . Applying Lemma 6.3 once more, we can obtain  $g_{n-1,y} + g_{z,t} = g_{n-1,z} + g_{y,t}$ , which implies  $g_e = \alpha \in \mathbb{R}$  for all  $e \in E(V - \{n-1, n\})$ .

Finally, construct another sc-partition by distributing the nodes of  $N_{k+1}$  over the others and call this new sc-partition  $\pi^*$ . Clearly,  $w^{\pi^*} \in P_e$ . Comparing  $g^T w^\pi = h$  and  $g^T w^{\pi^*} = h$  yields  $\alpha = 0$ .  $\square$

**Theorem 6.4.** *Suppose  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$ . Then the non-negativity constraints  $w_e \geq 0$  are facet-defining for  $\mathcal{P}^{lu}$  if and only if  $n \geq 3F_L$ .*

*Proof.* When  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$ ,  $M(\mathcal{P}^{lu})$  consists of the hyperplanes  $H_1, H_2, \dots, H_n$ , where  $H_u$  is defined as

$$w(\delta(u)) = F - 1$$

for  $F \in \{F_L, F_U\}$ . Let  $e = \{u, v\}$ , and assume without loss of generality that  $v = n - 1$  and  $u = n$ . Let  $S = \{\{1, n\}, \{2, n\}, \dots, \{n - 1, n\}, \{1, n - 1\}\}$ . In the following matrix, the coefficients of the edges belonging to  $S$  in the equations representing  $H_1$  to  $H_n$  are displayed:

$$\begin{array}{r}
H_1 \rightarrow \\
H_2 \rightarrow \\
H_3 \rightarrow \\
H_4 \rightarrow \\
\vdots \\
H_{n-3} \rightarrow \\
H_{n-2} \rightarrow \\
H_{n-1} \rightarrow \\
H_n \rightarrow
\end{array}
\begin{array}{cccccccc}
\{1, n\} & \{2, n\} & \{3, n\} & \dots & \{n-2, n\} & \{1, n-1\} & \{1, n-2\} \\
\left[ \begin{array}{cccccccc}
1 & 0 & 0 & \dots & 0 & 1 & 1 \\
0 & 1 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 1 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & \dots & 1 & 0 & 1 \\
0 & 0 & 0 & \dots & 0 & 1 & 0 \\
1 & 1 & 1 & \dots & 1 & 0 & 0
\end{array} \right]
\end{array}
.$$

This matrix is non-singular. With reference to Theorem 3.6 in Section I.4.3 of [14], we use this fact to set  $g_e = 0$  for all  $e \in S$ .

Now, pick four arbitrary nodes  $x, y, z, t \in V - \{n, n - 1\}$ ; and, consider an sc-partition  $\pi = (N_1, N_2, \dots, N_k)$  where  $k = \frac{n}{F}$ ,  $|N_i| = F$  for all  $i = 1, 2, \dots, k$ ,  $\{n, x\} \subset N_1$ ,  $\{n - 1, y\} \subset N_2$  and  $\{z, t\} \subset N_3$ . Applying Lemma 6.3 to this sc-partition yields  $g_{n,x} + g_{n-1,y} = g_{n,y} + g_{n-1,x}$ . But, since we have already set  $g_{n,x} = g_{n,y} = 0$ , we obtain  $g_{n-1,x} = g_{n-1,y}$ . But then, since  $\{1, n - 1\} \in S$  and we have already set  $g_{1,n-1} = 0$ , this implies that  $g_{q,n-1} = 0$  for all  $q \in V - \{n - 1, n\}$ . Now, applying Lemma 6.3 once more gives  $g_{n-1,y} + g_{z,t} = g_{n-1,t} + g_{y,z}$ , which, by  $g_{1,n-2} = 0$ , directly implies  $g_e = 0$  for all  $e \in E(V - \{n - 1, n\})$ .  $\square$

In Table 3, we summarize the results of this section.

$\dim(\mathcal{P}^{lu})$	'if' or 'iff'	Condition(s) of facetness
$\binom{n}{2}$ (i.e., full-dim.)	iff	$n \geq 3F_L + 1$ , or, $n = 3F_L \leq 2F_U - 2$
$\binom{n}{2} - 1$	iff	$n \geq 3F_L + 1$
$\binom{n}{2} - (n - 1)$	iff	$n \geq 3F_L$
$\binom{n}{2} - n$	iff	$n \geq 3F_L$

Table 3: Conditions that make the non-negativity constraints facets for different values of  $\dim(\mathcal{P}^{lu})$ .

## 6.2 Upper Bound Constraints

In this section, we denote by  $P$  the face defined by a constraint in (3.1c), i.e.,

$$P = \{\mathbf{w} \in \mathcal{P}^{lu} | w(\delta(u)) = F_U - 1\} \quad \text{for } u \in V. \quad (6.5)$$

For some  $(n, F_L, F_U)$  triples, it is possible that  $P = \emptyset$ . For such instances, the right hand side of the constraints (3.1c) has to be strengthened to ensure that the  $P$ 's are non-empty. For example, consider the triple  $(33, 8, 10)$ , which imply  $\left\lfloor \frac{n}{F_U} \right\rfloor = \left\lfloor \frac{n}{F_L} \right\rfloor = 4$ . We

have  $\max_i \{|N_i|\} = 9$  for any sc-partition  $\pi = (N_1, N_2, N_3, N_4)$ . Clearly, in this example we need to use 9 rather than 10 in the right hand side of (3.1c) to ensure that  $P \neq \emptyset$ . In other words, we need to be able to find the correct right hand side value for which (3.1c) remain valid and  $P$ 's become non-empty.

Now, we introduce a series of numbers,  $\phi_i^U$ , where  $\phi_1^U$  gives us this correct right hand side value we look for. The other elements of this series (i.e.,  $\phi_i^U$  for  $i > 1$ ) will be utilized in Sections 9 and 12. Consider the definition

$$\phi_i^U = \max_{\substack{\pi=(N_1, \dots, N_k): \\ w^\pi \in \mathcal{P}^{lu}(R_{i-1}, F_L, F_U)}} \left\{ \max_{j=1, \dots, k} \{|N_j|\} \right\}.$$

where  $\phi_0 = F_U$ ,  $R_0 = n$  and  $R_i = R_{i-1} - \phi_i^U$ . According to this definition,  $\phi_1^U$  gives the maximal size the subsets can attain in sc-partitions (i.e., in feasible solutions of  $\mathcal{P}^{lu}(n, F_L, F_U)$ ),  $\phi_2^U$  gives the second maximal size the subsets can attain in sc-partitions, and so on. Equivalently we can also say that  $\phi_2^U$  is the maximal size the subsets can attain in feasible solutions of  $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$ ,  $\phi_3^U$  is the maximal size the subsets can attain in feasible solutions of  $\mathcal{P}^{lu}(n - \phi_1^U - \phi_2^U, F_L, F_U)$ , and so on.

The following formula can be used for computing the values of  $\phi_i^U$ :

$$\phi_i^U = \max \left\{ \phi \in \mathbb{Z}_+ \mid F_L \leq \phi \leq F_U, \left\lfloor \frac{R_{i-1} - \phi}{F_L} \right\rfloor \geq \left\lceil \frac{R_{i-1} - \phi}{\phi} \right\rceil \right\}. \quad (6.6)$$

Note that,  $R_k = 0$  for  $k = \left\lfloor \frac{n}{F_U} \right\rfloor$ , and hence  $\phi_i^U$  for  $i > \left\lfloor \frac{n}{F_U} \right\rfloor$  is not defined.

Now, we define  $P_u^U$ 's as follows:

$$P_u^U = \{ \mathbf{w} \in \mathcal{P}^{lu} \mid w(\delta(u)) = \phi_1^U - 1 \} \quad \forall u \in V.$$

Due to the definition of  $\phi_1^U$ ,  $P_u^U$  is never empty for all  $u \in V$ .

Assume that there exists a valid inequality  $g^T w \leq h$  such that

$$P_u^U \subseteq \{ \mathbf{w} \in \mathcal{P}^{lu} : g^T w = h \}$$

for a node  $u$ . In Theorems 6.5, 6.6 and 6.7 we show that  $g^T w = h$  can be written as a linear combination of  $w(\delta(u)) = \phi_1^U$  and the equalities in  $M(\mathcal{P}^{lu})$ , if any. Beforehand, we present a technical lemma.

**Lemma 6.5.** *If  $\dim(n - \phi_1^U, F_L, F_U) \neq \binom{n - \phi_1^U}{2} - (n - \phi_1^U)$ ,  $\phi_1^U = F_U$ .*

*Proof.* Suppose that  $\phi_1^U < F_U$ . If  $\dim(n - \phi_1^U, F_L, F_U) \neq \binom{n - \phi_1^U}{2} - (n - \phi_1^U)$ , there exists  $\pi = (N_1, \dots, N_k)$  in  $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$  such that  $|N_l| > F_L$  for some  $l \in \{1, \dots, k\}$ . Then, we can construct an sc-partition  $\pi^* = (N_1^*, \dots, N_{k+1}^*)$  in  $\mathcal{P}^{lu}(n, F_L, F_U)$  such that  $|N_i^*| = |N_i|$  for  $i = 1, \dots, k$  and  $|N_{k+1}^*| = \phi_1^U$ . But, then we would be able to find another sc-partition  $\tilde{\pi} = (\tilde{N}_1, \dots, \tilde{N}_k)$  by shifting a node from  $N_l^*$  to  $N_{k+1}^*$ . Then, we would have  $|\tilde{N}_{k+1}| = \phi_1^U + 1$ , which contradicts the definition of  $\phi_1^U$ .  $\square$

**Theorem 6.5.** *Suppose  $\mathcal{P}^{lu}$  is full-dimensional and  $F_U - F_L \geq 2$ . Then, (3.1c) are facet defining for  $\mathcal{P}^{lu}$  if and only if*

$$(i) \dim(\mathcal{P}^{lu}(n - F_U, F_L, F_U)) = \binom{n - F_U}{2} - (n - F_U - 1), \text{ or,}$$

$$(ii) \dim(\mathcal{P}^{lu}(n - F_U, F_L, F_U)) = \binom{n - F_U}{2}.$$

*Proof.* Due to Lemma 6.5, the conditions of this theorem dictate us to consider (3.1c) rather than their strengthened version  $w(\delta(u)) \leq \phi_1^U$ . We pick a node  $u$  and prove that the conditions (i) and (ii) make its corresponding inequality in (3.1c) facet defining.

- (i) By Theorem 4.2, we have  $\frac{n - F_U}{F_L} = \frac{n - F_U}{F_U} + 1$ . Let  $k = \frac{n - F_U}{F_U}$ . Pick four nodes  $p, q, x, y \in V - u$  arbitrarily. Pick an sc-partition  $\pi = (N_1, \dots, N_{k+2})$  in  $\mathcal{P}^{lu}(n, F_L, F_U)$  such that  $|N_1| = F_U$ ,  $|N_i| = F_L$  for  $i = 2, \dots, k + 2$ ,  $u \in N_1$ ,  $\{p, x\} \subset N_2$  and  $\{q, y\} \subset N_3$ . Applying Lemma 6.3 gives  $g_{p,x} + g_{q,y} = g_{p,y} + g_{q,x}$ , which implies  $g_{p,z} - g_{q,z} = \alpha \in \mathbb{R}$  for all  $z \in V - \{u, p, q\}$ . Now, pick another sc-partition  $\pi^* = (N_1^*, N_2^*, \dots, N_{k+2}^*)$  in  $\mathcal{P}^{lu}(n, F_L, F_U)$  such that  $|N_1^*| = F_U$ ,  $|N_i^*| = F_L$  for  $i = 2, \dots, k + 2$ ,  $\{u, q\} \subset N_1^*$  and  $p \in N_2^*$ . Switching  $p$  and  $q$  in  $\pi^*$  gives

$$g_{q,u} + g(q, N_1^* - \{u, q\}) + g(p, N_2^* - p) = g_{p,u} + g(p, N_1^* - \{u, q\}) + g(q, N_2^* - p). \quad (6.7)$$

Now, construct yet another sc-partition  $\tilde{\pi} = (\tilde{N}_1, \dots, \tilde{N}_{k+1})$  in  $\mathcal{P}^{lu}(n, F_L, F_U)$  by distributing the nodes of  $N_{k+2}^*$  in  $\pi^*$  over  $N_2^*, \dots, N_{k+1}^*$ . By Theorem 4.2 we have  $|\tilde{N}_i| = F_U$  for  $i = 1, \dots, k + 1$ . Clearly, we also have  $\{u, q\} \subset \tilde{N}_1$  and  $p \in \tilde{N}_2$ . Switching  $p$  and  $q$  in  $\tilde{\pi}$  gives

$$g_{q,u} + g(q, N_1^* - \{u, q\}) + g(p, (N_2^* - p) \cup \bar{N}_2^*) = g_{p,u} + g(p, N_1^* - \{u, q\}) + g(q, (N_2^* - p) \cup \bar{N}_2^*),$$

where  $\bar{N}_2^* = \tilde{N}_2 - N_2^*$  with  $|\bar{N}_2^*| = F_U - F_L$ . Comparing this latter equation with (6.7) yields

$$g(p, \bar{N}_2^*) = g(q, \bar{N}_2^*).$$

Plugging  $g_{p,z} - g_{q,z} = \alpha$  into this equality implies that  $\alpha = 0$ , from which we infer  $g_{p,z} = g_{q,z}$  for all  $z \in V - \{u, p, q\}$ . But, since  $p$  and  $q$  are arbitrary, this immediately gives  $g_e = \beta \in \mathbb{R}$  for all  $e \in E(V - u)$ . Further, comparing  $g^T w^{\pi^*} = h$  and  $g^T w^{\tilde{\pi}} = h$  we obtain  $\beta = 0$ . Then, (6.7) directly implies  $g_{u,p} = \gamma \in \mathbb{R}$  for all  $p \in V - u$ .

- (ii)
- If  $\mathcal{P}^{lu}(n - F_U, F_L, F_U)$  has a 2-loose sc-partition (i.e., it complies with **(FD-1)**), applying Lemma 6.1 to such a 2-loose sc-partition and generalizing, one can infer  $g_e = 0$  for all  $e \in E(V - u)$ . Now, consider two arbitrary nodes  $x, y \in V - u$  and an sc-partition  $\pi = (N_1, \dots, N_k)$  in  $\mathcal{P}^{lu}(n, F_L, F_U)$  such that  $\{u, x\} \in N_1$ ,  $y \in N_2$  and  $|N_1| = F_U$ . Switching  $x$  and  $y$  in  $\pi$  yields  $g_{u,x} = g_{u,y}$ . Since  $x$  and  $y$  are arbitrary, we infer  $g_e = \alpha \in \mathbb{R}$  for  $e \in \delta(u)$ .
  - If  $(n - F_U, F_L, F_U)$  satisfies **(FD-2)**, pick an sc-partition in  $\mathcal{P}^{lu}(n, F_L, F_U)$ ,  $\pi^1 = (N_1^1, N_2^1, \dots, N_{k+2}^1)$ , such that  $k = \left\lceil \frac{n - F_U}{F_U} \right\rceil \geq 1$ ,  $u \in N_1^1$ ,  $|N_1^1| = F_U$ ,

$|N_2^1| = F_L + 1$  and  $|N_i^1| = F_L$  for  $i = 3, \dots, k+2$ . Applying Lemma 6.2 to this sc-partition and generalizing, we infer  $g_e = \alpha \in \mathbb{R}$  for all  $e \in E(V - u)$ . Now, pick another sc-partition  $\pi^2 = (N_1^2, N_2^2, \dots, N_{k+1}^2)$  in  $\mathcal{P}^{lu}(n, F_L, F_U)$ , where  $N_1^2 = N_1^1$ ,  $|N_i^2| = F_U$  for  $i = 1, 2, \dots, k+1$ . Comparing  $g^T w^{\pi^1} = h$  and  $g^T w^{\pi^2} = h$  we infer that  $\alpha = 0$  for all  $e \in E(V - u)$ . Now, pick two arbitrary nodes  $x, y \in V - u$  such that, without loss of generality,  $x \in N_1^1$  and  $y \in N_2^1$ . Switching  $x$  and  $y$  in  $\pi^1$ , we obtain  $g_{u,x} = g_{u,y}$ . This implies  $g_{u,v} = \beta \in \mathbb{R}$  for all  $v \in V - u$  and hence that (6.8) is facet defining. We can use this proof when  $(n - F_U, F_L, F_U)$  satisfies **(FD-3)** or **(FD-5)** as well, by choosing  $\pi^1$  and  $\pi^2$  accordingly.

- If  $(n - F_U, F_L, F_U)$  satisfies **(FD-4)**, pick an sc-partition  $\pi^1 = (N_1^1, N_2^1, \dots, N_{k+1}^1)$  in  $\mathcal{P}^{lu}(n, F_L, F_U)$  where  $k = \left\lceil \frac{n - F_U}{F_U} \right\rceil \geq 2$  (recall from the Remark of Theorem 5.1 that **(FD-4)** implies full-dimensionality of  $\mathcal{P}^{lu}(n - F_U, F_L, F_U)$  if and only if  $\left\lceil \frac{n - F_U}{F_U} \right\rceil \geq 2$ ),  $u \in N_1^1$ ,  $|N_1^1| = F_U$ ,  $|N_2^1| = F_U - 1$ ,  $|N_i^1| = F_U$  for  $i = 3, \dots, k+1$ . Applying Lemma 6.2 to this sc-partition and generalizing, we infer that  $g_e = \alpha$  for all  $e \in E(V - u)$ . Now, pick another sc-partition  $\pi^2 = (N_1^2, N_2^2, \dots, N_{k+2}^2)$  in  $\mathcal{P}^{lu}(n, F_L, F_U)$  where  $N_1^2 = N_1^1$ ,  $|N_i^2| = F_L$  for  $i = 2, \dots, k+2$ . Comparing  $g^T w^{\pi^1} = h$  and  $g^T w^{\pi^2} = h$  we infer  $\alpha = 0$ . Now, in a similar manner to the previous paragraph (where facetness for **(FD-2)** is proved), we can show that  $g_{u,v} = \beta \in \mathbb{R}$  for all  $v \in V - u$ .

Now, suppose that  $\dim(\mathcal{P}^{lu}(n - F_U, F_L, F_U))$  is either  $\binom{n - F_U}{2} - 1$  or  $\binom{n - F_U}{2} - (n - F_U)$ . Observe from Section 5.2 that,  $\mathcal{P}^{lu}(n - F_U, F_L, F_U)$  is contained in the hyperplane  $w(\bar{E}) = C$  where  $C$  is a constant and  $\bar{E}$  represents the edge set of a complete graph with  $n - F_U$  nodes. Then,

$$P_u^U \subseteq \{\mathbf{w} \in \mathbb{R}^{\binom{n}{2}} \mid w(E) = C + \binom{F_U}{2}\}.$$

Note that,  $w(E) \leq C + \binom{F_U}{2}$  is valid for  $\mathcal{P}^{lu}$ . □

**Theorem 6.6.** *Suppose  $\dim(\mathcal{P}^{lu}) = \frac{n(n-1)}{2} - 1$ . Then*

$$w(\delta(u)) \leq \phi_1^U \quad \forall u \in V \tag{6.8}$$

are facet defining for  $\mathcal{P}^{lu}$  if and only if  $\left\lceil \frac{n}{F_L} \right\rceil = \left\lceil \frac{n}{F_U} \right\rceil \geq 3$  and  $(n \bmod F_U) = F_U - 1$ .

*Proof.* First, note that if  $\dim(\mathcal{P}^{lu}) = \frac{n(n-1)}{2} - 1$ ,  $\left\lceil \frac{n}{F_L} \right\rceil = \left\lceil \frac{n}{F_U} \right\rceil$  by Theorem 4.1. Moreover, we have either  $(n \bmod F_L) = 1$  or  $(n \bmod F_U) = F_U - 1$ . Now, we show that (6.8) are facet defining if  $\left\lceil \frac{n}{F_L} \right\rceil = \left\lceil \frac{n}{F_U} \right\rceil \geq 3$  and  $(n \bmod F_U) = F_U - 1$ .

Since  $\dim(\mathcal{P}^{lu}) = \frac{n(n-1)}{2} - 1$  we can fix  $g_{e^*} = 0$  for some  $e^* \in E(V - u)$ . Now, pick four arbitrary nodes  $v, t, x, y \in V - u$  and an sc-partition  $\pi = (N_1, N_2, \dots, N_k)$  ( $k \geq 3$ ) such that  $\{u, y\} \subset N_1$ ,  $\{v, t\} \subset N_2$ ,  $x \in N_3$ ,  $|N_1| = |N_2| = F_U$ ,  $|N_3| = F_U - 1$  and  $|N_i| = F_U$  for  $i = 4, \dots, k$ . Applying Lemma 6.2 to  $\pi$ , we can infer that  $g_{v,t} = g_{v,x}$  for all



$v, t, x \in V - u$ . Since  $v, t$  and  $x$  are arbitrary and  $g_{e^*} = 0$  for some  $e^* \in E(V - u)$ , we infer  $g_e = 0$  for  $e \in E(V - u)$ . Switching  $v$  and  $y$  in  $\pi$ , we get  $g_{u,v} = g_{u,y}$ , which implies  $g_{u,v} = \alpha \in \mathbb{R}$  for all  $v \in V - u$ .

When  $(n \bmod F_L) = 1$  or  $\lfloor \frac{n}{F_L} \rfloor = \lceil \frac{n}{F_U} \rceil = 2$ ,  $P_u^U$  is contained in the faces defined by the upper 2-star inequalities introduced in Section 12.  $\square$

**Theorem 6.7.** *When  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n - 1)$ , (6.8) are facet defining for  $\mathcal{P}^{lu}$ .*

*Proof.* When  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n - 1)$ ,  $\mathcal{P}^{lu}(n, F_L, F_U) = \mathcal{P}^{lu}(n, F_L, F_L) \cup \mathcal{P}^{lu}(n, F_U, F_U)$ . Since  $P_u^U = \mathcal{P}^{lu}(n, F_U, F_U)$ , we have  $\dim(P_u^U) = \binom{n}{2} - n$ .  $\square$

Table 4 summarizes this section.

$\dim(\mathcal{P}^{lu})$	'if' or 'iff'	Condition(s) of facetness
$\binom{n}{2}$ (i.e., full dim.)	iff	<ul style="list-style-type: none"> <li>• <math>\dim(\mathcal{P}^{lu}(n - F_U, F_L, F_U)) = \binom{n}{2} - n</math>, or,</li> <li>• <math>\dim(\mathcal{P}^{lu}(n - F_U, F_L, F_U)) = \binom{n}{2}</math>.</li> </ul>
$\binom{n}{2} - 1$	iff	$\lfloor \frac{n}{F_U} \rfloor = \lceil \frac{n}{F_U} \rceil \geq 3$ , and $(n \bmod F_U) = F_U - 1$ .
$\binom{n}{2} - (n - 1)$	-	no condition (always facet).

Table 4: Conditions that make the upper-bound constraints facet-defining for different values of  $\dim(\mathcal{P}^{lu})$ .

### 6.3 Lower Bound Constraints

In this section, we express the conditions that make the strengthened version of the lower bound constraints (3.1b) facet defining. The proofs of the theorems (except a part of the proof of Theorem 6.9) are not presented because they are very similar to their counterparts in the previous section.

In this section, we introduce the series  $\phi_i^L$ , which is in a sense the symmetric of  $\phi_i^U$  of the previous section. We define  $\phi_i^L$  in the following manner:

$$\phi_i^L = \min_{\substack{\pi=(N_1, \dots, N_k): \\ w^\pi \in \mathcal{P}^{lu}(R_{i-1}, F_L, F_U)}} \left\{ \min_{j=1, \dots, k} \{|N_j|\} \right\}.$$

where  $\phi_0 = F_L$ ,  $R_0 = n$  and  $R_i = \max\{0, R_{i-1} - \phi_i^L\}$ . Similarly,  $\phi_1^L$  gives the *minimal* size the subsets can attain in sc-partitions (i.e., in feasible solutions of  $\mathcal{P}^{lu}(n, F_L, F_U)$ ), and,  $\phi_2^L$  gives the second *minimal* size the subsets can attain in sc-partitions, and so on. We can equivalently say that  $\phi_2^L$  is the minimal size the subsets can attain in feasible solutions of  $\mathcal{P}^{lu}(n - \phi_1^L, F_L, F_U)$ ,  $\phi_3^L$  is the maximal size the subsets can attain in feasible solutions of  $\mathcal{P}^{lu}(n - \phi_1^L - \phi_2^L, F_L, F_U)$ , and so on. Likewise,  $\phi_i^U$  for  $i > 1$  will be utilized in Section 11.

The following formula, which is symmetric of (6.6), can be employed for computing the values of  $\phi_i^L$ 's:

$$\phi_i^L = \min \left\{ \phi \in \mathbb{Z}_+ \mid F_L \leq \phi \leq F_U, \left\lfloor \frac{R_{i-1} - \phi}{\phi} \right\rfloor \geq \left\lceil \frac{R_{i-1} - \phi}{F_U} \right\rceil \right\}. \quad (6.9)$$

Note that, for  $k = \left\lfloor \frac{n}{F_L} \right\rfloor$ ,  $R_k = 0$  and hence,  $\phi_i^L$  for  $i > k$  is not defined.

Now follows the lemma and the theorems that summarize the main results of this section regarding the lower bound constraints.

**Lemma 6.6.** *If  $\dim(n - \phi_1^L, F_L, F_U) \neq \binom{n - \phi_1^L}{2} - (n - \phi_1^L)$ ,  $\phi_1^L = F_L$ .*

**Theorem 6.8.** *Suppose  $\mathcal{P}^{lu}$  is full-dimensional. Then, (3.1b) are facet defining for  $\mathcal{P}^{lu}$  if and only if*

- (i)  $\dim(\mathcal{P}^{lu}(n - F_L, F_L, F_U)) = \binom{n}{2} - n$ , or,
- (ii)  $\dim(\mathcal{P}^{lu}(n - F_L, F_L, F_U)) = \binom{n}{2}$ .

**Theorem 6.9.** *Suppose  $\dim(\mathcal{P}^{lu}) = \frac{n(n-1)}{2} - 1$ . Let  $\phi^* = \min\{\phi \in \mathbb{Z}_+ \mid F_L \leq \phi < F_U, \left\lfloor \frac{n-\phi}{F_U} \right\rfloor \leq \left\lfloor \frac{n-\phi}{\phi} \right\rfloor\}$ . Then,*

$$w(\delta(u)) \geq \phi_1^L - 1 \quad \forall u \in V \quad (6.10)$$

are facet defining for  $\mathcal{P}^{lu}$  if and only if  $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lfloor \frac{n}{F_U} \right\rfloor \geq 3$  and  $(n \bmod F_L) = 1$ .

*Proof.* When  $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lfloor \frac{n}{F_U} \right\rfloor = 2$  or  $(n \bmod F_U) = F_U - 1$ ,  $P_u^L = \{\mathbf{w} \in \mathcal{P}^{lu} \mid w(\delta(u)) \geq \phi_1^L - 1\}$  is contained in the faces defined by the lower 2-star inequalities in Section 11.  $\square$

**Theorem 6.10.** *Suppose  $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n - 1)$ . Then (6.10) are facet defining for  $\mathcal{P}^{lu}$ .*

Table 5 summarizes this section.

$\dim(\mathcal{P}^{lu})$	'if' or 'iff'	Condition(s) of facetness
$\binom{n}{2}$ (i.e., full dim.)	iff	<ul style="list-style-type: none"> <li>• <math>\dim(\mathcal{P}^{lu}(n - F_L, F_L, F_U)) = \binom{n}{2} - n</math>, or,</li> <li>• <math>\dim(\mathcal{P}^{lu}(n - F_L, F_L, F_U)) = \binom{n}{2}</math>.</li> </ul>
$\binom{n}{2} - 1$	iff	$\left\lfloor \frac{n}{F_U} \right\rfloor = \left\lfloor \frac{n}{F_L} \right\rfloor \geq 3$ , and $(n \bmod F_L) = 1$ .
$\binom{n}{2} - (n - 1)$	-	no condition (always facet).

Table 5: Conditions that make the lower-bound constraints facet-defining for different values of  $\dim(\mathcal{P}^{lu})$ .

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