Size-constrained graph partitioning polytope. Part II: Non-trivial facets

F. Aykut Özsoy, Martine Labbé*

Université Libre de Bruxelles, Département d'Informatique

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Abstract

We consider the problem of clustering a set of items into subsets whose sizes are bounded from above and below. We formulate the problem as a graph partitioning problem and propose an integer programming model for solving it. This formulation generalizes several well-known graph partitioning problems from the literature like the clique partitioning problem, the equi-partition problem and the k-way equi-partition problem. In this paper, we analyze the structure of the corresponding polytope and prove several results concerning the facial structure. Our analysis yields important results for the closely related equi-partition and k-way equi-partition polytopes as well.

This is the second part of the two papers addressing the study of the facial structure of the *size-constrained graph partitioning polytope*. All the definitions and notation of the first paper (i.e., "part I") apply for this paper as well. Hence, we start numbering the sections of this paper from where we have left in the first paper.

In this paper, we prove facetness for $\mathcal{P}^{lu}$ of several classes of valid inequalities: 2-partition inequalities, lower and upper general clique inequalities, cycle inequalities and the lower and upper 2-star inequalities. As already stated in the first paper, the equi-partition polytope $\mathcal{P}^{equi}(n)$ and the k-way equi-partition polytope $\mathcal{P}^{k-way}(n,k)$ are special cases of $\mathcal{P}^{lu}$. Certain theorems of this paper prove for the first time in the literature that some of the aforementioned valid inequalities are facet defining for these two polytopes as well. We highlight such results we obtain as corollaries to corresponding theorems.

*Corresponding author: Universit Libre de Bruxelles, Boulevard du Triomphe, CP 210/01, 1050 Bruxelles, Belgique. E-mail:mlabbe@ulb.ac.be

7 2-Partition inequalities

This section is devoted to the 2-partition inequalities, which are defined in the following manner over two disjoint subsets S and T of V:

$$w(E(S,T)) - w(E(S)) - w(E(T)) \leq \min(|S|,|T|) \qquad \text{for } S,T \subset V : S \cap T = \emptyset. \quad (7.1)$$

They are introduced by Grötschel and Wakabayashi [3] for the clique partitioning polytope $\mathcal{P}(n)$ and shown to be facet defining for this polytope if and only if $|S| \neq |T|$. Oosten et al. [7] introduce some generalizations of these inequalities (the weighted 2-partition inequalities), which are also facet defining for $\mathcal{P}(n)$. Ji and Mitchell [4] and Sørensen [8] show that these inequalities are facet defining for $\mathcal{P}^l(n, F_L)$ and $\mathcal{P}^u(n, F_U)$ as well, respectively.

One special feature of the 2-partition inequalities is their having the triangle inequalities as special cases: for disjoint $S,T \subset V$ with $|S| = 1$ and $|T| = 2$, (7.1) turns into a triangle inequality.

In this section, we denote by $P_{S,T}$ the face defined by a 2-partition inequality over the sets S and T, i.e.,

$$P_{S,T} = \left\{ \mathbf{w} \in \mathcal{P}^{lu} | w(E(S,T)) - w(E(S)) - w(E(T)) = \min(|S|,|T|) \right\}.$$

Now, in Theorems 7.1-7.6 we prove several sufficient conditions that make a 2-partition inequality facet defining. As before, we assume that there exists a valid inequality $g^T w \leq h$ such that

$$P_{S,T} \subseteq \{\mathbf{w} \in \mathcal{P}^{lu} | g^T w = h\}.$$

We will show that $g^T w = h$ is a linear combination of the 2-partition inequality under consideration and the equalities in $M(\mathcal{P}^{lu})$, if any.

But, before, in Lemma 7.1 we give a characterization of the sc-partitions that lie on $P_{S,T}$.

Lemma 7.1. *Consider two disjoint subsets $S,T \subset V$ and the corresponding face $P_{S,T}$. Suppose without loss of generality that $|S| < |T|$. Then, $w^\pi \in P_{S,T}$ for an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ if and only if $0 \leq |N_i \cap T| - |N_i \cap S| \leq 1$ for all $i = 1, 2, \ldots, k$.*

Proof. Let $s_i^\pi = |N_i \cap S|$ and $t_i^\pi = |N_i \cap T|$ for all $i = 1, 2, \ldots, k$. Then,

$$w^\pi(E(S,T)) - w^\pi(E(S)) - w^\pi(E(T)) = \sum_{i=1}^{k} \left(s_i^\pi t_i^\pi - \binom{s_i^\pi}{2} - \binom{t_i^\pi}{2} \right).$$

Rearranging we get

$$w^\pi(E(S,T)) - w^\pi(E(S)) - w^\pi(E(T)) = \sum_{i=1}^{k} \left(\frac{(s_i^\pi + t_i^\pi) - (t_i^\pi - s_i^\pi)^2}{2} \right). \quad (7.2)$$

Let $\mu_i^\pi = \min\{s_i^\pi, t_i^\pi\}$ and $\delta_i^\pi = |t_i^\pi - s_i^\pi|$. Then, we can rewrite (7.2) as

$$w^\pi(E(S,T)) - w^\pi(E(S)) - w^\pi(E(T)) = \sum_{i=1}^{k} \left(\mu_i^\pi - \binom{\delta_i^\pi}{2} \right),$$

which attains its maximum value $|S|$ if and only if $\mu_i^\pi = s_i^\pi$ and $0 \le \delta_i^\pi \le 1$ for all $i = 1, 2, \ldots, k$. □

Theorem 7.1. *Suppose that $F_U - F_L \ge 2$. Suppose further that $\mathcal{P}^{lu}$ is full-dimensional and complies with* **(FD-1)***. Let S and T be two proper and disjoint subsets of V such that $|S| < |T|$ and $|V - (S \cup T)| \ge 2$. Let $\Delta = |T| - |S|$. Pick an integer k such that $kF_L + 1 < n < kF_U - 1$ (note that such a k exists since $\mathcal{P}^{lu}$ complies with* **(FD-1)***). The 2-partition inequality (7.1) defined over S and T is facet defining for $\mathcal{P}^{lu}$ if*

- *$\Delta < k$, and,*
- *there exist an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $F_L < |N_i| < F_U$ for $i = 1, k$, $F_L \le |N_i| \le F_U$ for $i = 2, \ldots, k-1$, and*

$$|S| \le \left(\left\lfloor \frac{|N_1| - 1}{2} \right\rfloor + \sum_{i=2}^{k-\Delta} \left\lfloor \frac{|N_i|}{2} \right\rfloor + \sum_{i=k-\Delta+1}^{k-1} \left\lfloor \frac{|N_i| - 1}{2} \right\rfloor + \left\lfloor \frac{|N_k| - 2}{2} \right\rfloor \right). \quad (7.3)$$

Proof. Let $L = \min\{2, |S|\}$. There exist integer μ_i ($i = 1, \ldots, k$) values such that

1. $\mu_i > 0$ for $i = 1, \ldots, L$ and $\mu_i \ge 0$ for $i = L+1, \ldots, k$,
2. $\sum_{i=1}^{k} \mu_i = |S|$,
3. $2\mu_1 + 1 \le |N_1|$,
4. $2\mu_i \le |N_i|$ for $i = 2, \ldots, k - \Delta$,
5. $2\mu_i + 1 \le |N_i|$ for $i = k - \Delta + 1, \ldots, k - 1$,
6. $2\mu_k + 2 \le |N_k|$.

Pick π in such a way that

- $|N_i \cap S| = \mu_i$ for $i = 1, 2, \ldots, k$,
- $|N_i \cap T| = \begin{cases} \mu_i, & \text{for } i = 1, 2, \ldots, k - \Delta, \\ \mu_i + 1, & \text{for } i = k - \Delta + 1, \ldots, k. \end{cases}$

By Lemma 7.1, $w^\pi \in P_{S,T}$. The conditions on μ_1 and μ_k imply $N_1 - (S \cup T) \ne \emptyset$ and $N_k - (S \cup T) \ne \emptyset$. Pick arbitrarily $u \in N_1 - (S \cup T)$ and $v \in N_k - (S \cup T)$. From Lemma 6.1, we infer $g_{u,v} = 0$, which leads to $g_e = 0$ for all $e \in E(V - (S \cup T))$ due to arbitrariness of u and v. We can apply Lemma 6.1 also to an arbitrary $s \in N_1 \cap S$ and v to infer $g_{s,v} = 0$, which generalizes into $g_e = 0$ for all $e \in E(S, V - (S \cap T))$.

Moving u from N_1 to N_k yields

$$g(u, N_1 \cap T) = g(u, N_k \cap T). \tag{7.4}$$

Due to Lemma 6.4, (7.4) implies that $g_{u,t'} = g_{u,t''}$ for all $t', t'' \in T$, which generalizes into $g_e = \alpha \in \mathbb{R}$ for all $e \in E(T, V - (S \cup T))$. Then, (7.4) is equivalent to

$$\mu_1 \alpha = (\mu_k + 1)\alpha. \tag{7.5}$$

Now, obtain another sc-partition $\pi^* = (N_1^*, \dots, N_k^*)$ from π by shifting an arbitrary $t^* \in N_k \cap T$ from N_k to N_1. Shifting u from N_1^* to N_k^*, we get

$$g(u, N_1^* \cap T) = g(u, N_k^* \cap T),$$

which is equivalent to $(\mu_1 + 1)\alpha = \mu_k \alpha$. Solving this equation and (7.5) simultaneously yields $\alpha = 0$.

Now, pick an arbitrary $t' \in N_k \cap T$ and shift it from N_k to N_1 in π to get

$$g(t', N_k \cap (T - t')) + g(t', N_k \cap S) = g(t', N_1 \cap T) + g(t', N_1 \cap S). \tag{7.6}$$

Claim 1: Equation (7.6) implies that $g_{e'} = -g_e = \beta \in \mathbb{R}$ for all $e' \in E(T)$ and $e \in E(S, T)$.
Proof: When $k = 2$ and $\mu_2 = 0$ (i.e., $N_k \cap (T - t') = \emptyset$), obviously we have $|S| = 1$ and $|T| = 2$, and the result is trivial. When $k \geq 3$ or $\mu_k > 0$ (i.e., $N_k \cap (T - t') \neq \emptyset$), condition (i) on μ_i values allows us to apply Lemma 6.4 twice and conclude that $g_{e'} = \beta \in \mathbb{R}$ for all $e' \in E(T)$ and $g_e = \gamma$ for all $e \in E(S, T)$. Then (7.6) can be rewritten as

$$(\mu_k - \mu_1)(\beta + \gamma) = 0. \tag{7.7}$$

If $\mu_k = 0$ or $\mu_k \neq \mu_1$, (7.7) implies $\beta = -\gamma$ since $\mu_1 > 0$. Suppose $\mu_k = \mu_1 > 0$. We obtain a new sc-partition $\tilde{\pi} = (\tilde{N}_1, \dots, \tilde{N}_k)$ from π in the following way:

- If $|N_1 - (S \cup T)| \geq 2$, pick $\{u, x\} \subset N_1 - (S \cup T)$ arbitrarily. Obtain $\tilde{\pi}$ by switching $\{u, x\}$ in N_1 and $\{\tilde{s}, \tilde{t}\}$ in N_k, where $\tilde{s} \in N_k \cap S$ and $\tilde{t} \in N_k \cap T$.
- If $|N_1 - (S \cup T)| = 1$ (i.e., $|N_1| = 2\mu_1 + 1$), we have $|N_1| + 1 \leq |N_k|$. This implies that $|N_1| < F_U - 1$ and $|N_k| > F_L + 1$. Obtain $\tilde{\pi}$ by switching u in N_1 and $\{\tilde{s}, \tilde{t}\}$ in N_k.

In either case, shifting t' from $\tilde{N}_k$ to $\tilde{N}_1$ yields $(\mu_k - \mu_1 - 2)(\beta + \gamma) = 0$. Solving this and (7.7) simultaneously, we get $\beta = -\gamma$. This completes the proof of the claim.

Claim 2: $g_e = \beta$ for all $e \in E(S)$.
Proof: Shift s from N_1 to N_k in π to get

$$g(s, N_1 \cap S) - \mu_1 \beta = g(s, N_k \cap S) - \mu_k \beta. \tag{7.8}$$

Due to Lemma 6.4, this equality implies that $g_{s,s'} = \theta \in \mathbb{R}$ for all $s' \in S - s$. Then, (7.8) turns into

$$(\mu_1 - \mu_k)(\theta - \beta) = 0. \tag{7.9}$$

If $\mu_k \neq \mu_1$, then we infer $\theta = \beta$. Suppose $\mu_k = \mu_1 > 0$. Consider the sc-partition $\tilde{\pi}$ obtained above in the proof of Claim 1. Shifting s from $\tilde{N}_1$ to $\tilde{N}_k$ in $\tilde{\pi}$ gives $(\mu_1 - \mu_k + 2)(\theta - \beta) = 0$. Solving this simultaneously with (7.9) yields $\theta = \beta$. This completes the proof of the claim and the theorem. □

When

$$|S| \leq \left(\left\lfloor \frac{F_L - 2}{2} \right\rfloor + \Delta \left\lfloor \frac{F_L - 1}{2} \right\rfloor + (k - \Delta - 1) \left\lfloor \frac{F_L}{2} \right\rfloor \right), \tag{7.10}$$

this theorem proves that $P_{S,T}$ is a facet.

On the other hand, when $|S|$ violates (7.10), it is not a trivial task to check if there exists an sc-partition that satisfies (7.3). For this case, we introduce a very simple polynomial algorithm to check existence of such an sc-partition.

Let $\pi = (N_1, \ldots, N_k)$ be an sc-partition such that

$$F_L < |N_i| < F_U \qquad \text{for } i = 1, k, \tag{7.11}$$

$$F_L \leq |N_i| \leq F_U \qquad \text{for } i = 2, \ldots, k-1. \tag{7.12}$$

Let

$$\tau_i(\pi) = \begin{cases} \left\lfloor \frac{|N_i|-1}{2} \right\rfloor, & \text{for } i = 1 \text{ and } i = k - \Delta + 1, \ldots, k-1, \\ \\ \left\lfloor \frac{|N_i|}{2} \right\rfloor, & \text{for } i = 2, \ldots, k - \Delta, \\ \\ \left\lfloor \frac{|N_i|-2}{2} \right\rfloor, & \text{for } i = k. \end{cases} \tag{7.13}$$

If $\sum_{i=1}^{k} \tau_i \geq |S|$ for π, this means $P_{S,T}$ is a facet.

The aim of our algorithm is to construct an sc-partition π that is in accordance with (7.11) and (7.12) and that maximizes $\sum_{i=1}^{k} \tau_i(\pi)$. Clearly, $\tau_i(\pi)$'s depend only on the subclique sizes $|N_1|, \ldots, |N_k|$ of π. Hence, the aim of our algorithm is essentially to construct valid subclique sizes (i.e., $|N_i|$'s for $i = 1, \ldots, k$) in accordance with (7.11) and (7.12) such that $\sum_{i=1}^{k} \tau_i(\pi)$ is maximized.

Initially in the algorithm, we put F_L nodes in each of the k subcliques of π (i.e., we assign $|N_i| = F_L$ for $i = 1, \ldots, k$) and distribute the remaining $r = n - kF_L$ nodes over these subcliques. During the course of this construction, we make use of a concept which we call *the growth potential*. We define the growth potential corresponding to a subclique i as the increase in the value of $\tau_i(\pi)$ induced by unit increase in the value of $|N_i|$. For example, when $|N_1| = 8$, $\tau_1(\pi) = 3$. If we add 1 to $|N_1|$, $\tau_1(\pi)$ increases to 4. Hence, when $|N_1| = 8$ we say that the growth potential of subclique N_1 is 1. By definition, the growth potential can only take on 0-1 values. We denote the growth potential of a subclique N_i by $\psi(|N_i|)$, where

$$\psi(|N_i|) = \begin{cases} ((1 + |N_i|) \bmod 2), & \text{for } i = 1 \text{ and } i = k - \Delta + 1, \ldots, k-1. \\ (|N_i| \bmod 2), & \text{for } i = 2, \ldots, k - \Delta \text{ and } i = k. \end{cases}$$

Besides, we use the following constants for denoting the maximum allowed subclique sizes as implied by (7.11) and (7.12):

$$\chi_i = \begin{cases} F_U - 1, & \text{for } i = 1, k; \\ F_U, & \text{for } i = 2, \ldots, k-1. \end{cases}$$

The algorithm is based on the following four index sets $\mathcal{A}$, $\mathcal{Y}$, $\mathcal{S}$ and $\mathcal{U}$:

$$\begin{aligned} \mathcal{A} &= \{i : \psi(|N_i|) = 1, |N_i| < \chi_i\}, \\ \mathcal{Y} &= \{i : \psi(|N_i|) = 0, |N_i| < \chi_i - 1\}, \\ \mathcal{S} &= \{i : \psi(|N_i|) = 0, |N_i| = \chi_i - 1\}, \\ \mathcal{U} &= \{i : |N_i| = \chi_i\}. \end{aligned}$$

Note that, no matter what the values of $|N_i|$'s are, we always have

$$\mathcal{A} \cup \mathcal{Y} \cup \mathcal{S} \cup \mathcal{U} = \{1, \ldots, k\}.$$

For a subclique $j \in \mathcal{A}$, increasing $|N_j|$ by one makes j leave $\mathcal{A}$ and move into one of $\mathcal{Y}$, $\mathcal{S}$ or $\mathcal{U}$. If $j \in \mathcal{Y}$ and if we increase $|N_j|$ by one, j leaves $\mathcal{Y}$ and moves into $\mathcal{A}$. Clearly, if $j \in \mathcal{S}$ and we increase $|N_j|$ by one, j shifts into $\mathcal{U}$. And, we can not increase $|N_j|$ by one for a subclique N_j with $j \in \mathcal{U}$.

Among these four sets, $\mathcal{A}$ retains a special feature in that if $j \in \mathcal{A}$ and we increase $|N_j|$ by one, $\tau_j(\pi)$ increases by 1, too. This is not the case for $j \in \mathcal{Y} \cup \mathcal{S} \cup \mathcal{U}$. For this reason, in order to obtain an increase in $\sum_{i=1}^{k} \tau_i$, we need to add nodes to subcliques with indices in $\mathcal{A}$. This is the basic logic we use in the algorithm, which we present below:

Algorithm:

Input: $n, k, |S|, \Delta$;
Output: Valid subclique sizes $|N_1|, \ldots, |N_k|$ in accordance with condition (i);
Variable: integer r;
Constants: χ_i for $i = 1, \ldots, k$ as defined above;
Functions: $\psi(|N_i|)$ for $i = 1, \ldots, k$ as defined above;
Initialization: $|N_i| = F_L$ for $i = 1, \ldots, k$; $r = n - kF_L$;
Step 1: For all $j \in \mathcal{A}$ do $|N_j| = |N_j| + \min(1, r)$ and $r = r - \min(1, r)$;
Step 2: Pick a $j \in \mathcal{Y}$ do $|N_j| = |N_j| + \min(1, r)$ and $r = r - \min(1, r)$;
Step 3: If $(\mathcal{A} \cup \mathcal{Y} \neq \emptyset)$ then go to Step 1, else go to Step 4.
Step 4: For all $j \in \mathcal{S}$ do $|N_j| = |N_j| + \min(1, r)$ and $r = r - \min(1, r)$;
Return: $|N_i|$ values for $i = 1, \ldots, k$.

In Step 1 we add one node to each of the subcliques whose indices belong to $\mathcal{A}$. For each node added, $\sum_{i=1}^{k} \tau_i(\pi)$ increases by 1. Once Step 1 is completed, growth potentials of all the subcliques are 0 (i.e., $\mathcal{A}$ is empty and we can not induce an increase in $\sum_{i=1}^{k} \tau_i(\pi)$ by adding a node to a subclique). At this point, the best action to be taken would be to move a subclique into $\mathcal{A}$. This is exactly what Step 2 carries out by adding one node to a subclique in $\mathcal{Y}$. In Step 3, we check if there remains any subcliques in $\mathcal{A} \cup \mathcal{Y}$ (i.e., if there remains any subcliques with growth potential equal to 1, or, whose growth potential would be equal to 1 when size is increased by 1). If there are any, we return to Step 1 and continue in the same manner. If there are not any, we can infer that $\{1, \ldots, k\} = \mathcal{S} \cup \mathcal{U}$ and we can put the remaining r nodes we have into the subcliques in $\mathcal{S}$ only. Step 4 carries out this terminating task.

We plug the $|N_i|$'s returned by the algorithm into (7.13) to obtain $\tau_i(\pi)$'s. As we have mentioned above, if $\sum_{i=1}^{k} \tau_i(\pi) \geq |S|$, then we conclude that $P_{S,T}$ is a facet.

Now, we continue with the other theorems of this section.

Theorem 7.2. *Suppose that $\mathcal{P}^{lu}$ is full-dimensional and complies with* **(FD-2)**. *Let S and T be two proper and disjoint subsets of V such that $|S| < |T|$. Let $\Delta = |T| - |S|$ and $k = \left\lfloor \frac{n}{F_L} \right\rfloor$. The 2-partition inequality (7.1) is facet defining for $\mathcal{P}^{lu}$ if $\Delta < k - 1$ and*

$$|S| \leq (\Delta + 1) \left\lfloor \frac{F_L - 1}{2} \right\rfloor + (k - \Delta - 2) \left\lfloor \frac{F_L}{2} \right\rfloor. \tag{7.14}$$

Proof. Let $L = \min\{2, |S|\}$. By (7.14), there exist integer μ_i ($i = 1, \ldots, k-1$) values such that

1. $\mu_i > 0$ for $i = 1, \ldots, L$, $\mu_i \geq 0$ for $i = L+1, \ldots, k-1$,
2. $\sum_{i=1}^{k} \mu_i = |S|$,
3. $2\mu_1 + 2 \leq F_L + 1$,
4. $2\mu_i \leq F_L$ for $i = 2, \ldots, k - \Delta - 1$,
5. $2\mu_i + 1 \leq F_L$ for $i = k - \Delta, \ldots, k-1$.

Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that

- $|N_1| = F_L + 1$ and $|N_i| = F_L$ for $i = 2, \ldots, k$,
- $|N_i \cap S| = \mu_i$ for $i = 1, 2, \ldots, k-1$,
- $|N_i \cap T| = \begin{cases} \mu_i, & \text{for } i = 1, 2, \ldots, k - \Delta - 1, \\ \mu_i + 1, & \text{for } i = k - \Delta, \ldots, k-1. \end{cases}$

By Lemma 7.1, $w^\pi \in P_{S,T}$. Conditions on μ_1 imply that $|N_1 - (S \cup T)| \geq 2$. Pick arbitrarily $u, v \in N_1 - (S \cup T)$ and $x \in N_k - (S \cup T)$. We can apply Lemma 6.2 to infer $g_{u,v} = g_{u,x}$, which generalizes into $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - (S \cup T)$.

Moving u from N_1 to N_{k-1} in π gives

$$\begin{aligned} g(u, N_1 \cap S) \quad + \quad & g(u, N_1 \cap T) + g(u, N_1 - (S \cup T \cup u)) = \\ & g(u, N_{k-1} \cap S) + g(u, N_{k-1} \cap T) + g(u, N_{k-1} - (S \cup T)). \end{aligned} \tag{7.15}$$

Applying Lemma 6.4 to this equation, one can infer that $g_{u,t} = \beta$ for all $t \in T$ and $g_{u,s} = \gamma$ for all $s \in S$. This turns (7.15) into

$$(\mu_{k-1} - \mu_1)(\gamma + \beta - 2\alpha) = \alpha - \beta. \tag{7.16}$$

Now, obtain another sc-partition $\tilde{\pi} = (\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k)$ from π by switching $v \in N_1 - (S \cup T \cup u)$ and $t \in N_{k-1} \cap T$. Moving u from $\tilde{N}_1$ to $\tilde{N}_{k-1}$ yields

$$(\mu_{k-1} - \mu_1)(\gamma + \beta - 2\alpha) = \beta - \alpha, \tag{7.17}$$

which, together with the former equation, implies

- $\alpha = \beta = \gamma$ if $\mu_1 \neq \mu_{k-1}$, and,
- $\alpha = \beta$ if $\mu_1 = \mu_{k-1}$.

If $\mu_1 = \mu_{k-1}$, move u from N_1 to N_k in π to obtain $\mu_1(\beta + \gamma) = 2\mu_1\alpha$, which obviously leads to $\gamma = \alpha$.

Next, we obtain a new sc-partition $\pi^* = (N_1^*, N_2^*, \ldots, N_{k-1}^*)$ from π by removing subclique N_k by distributing its nodes over the other subcliques. Comparing $g^T w^\pi = h$ with $g^T w^{\pi^*} = h$, we obtain $\alpha = \beta = \gamma = 0$.

Now, shifting $t \in \tilde{N}_1 \cap T$ from $\tilde{N}_1$ to $\tilde{N}_{k-1}$ in $\tilde{\pi}$ yields

$$g(t, \tilde{N}_1 \cap S) + g(t, \tilde{N}_1 \cap T) = g(t, \tilde{N}_{k-1} \cap S) + g(t, \tilde{N}_{k-1} \cap T). \tag{7.18}$$

Applying Lemma 6.4 to this equation and generalizing, we get $g_e = \sigma \in \mathbb{R}$ for all $e \in E(S,T)$ and $g_e = \omega \in \mathbb{R}$ for all $e \in E(T)$. But, then (7.18) turns into $(\mu_1 - \mu_{k-1})(\sigma + \omega) = 0$, which immediately implies that $\sigma = -\omega$ if $\mu_1 \neq \mu_{k-1}$. If $\mu_1 = \mu_{k-1}$, one can shift t from $\tilde{N}_1$ to $\tilde{N}_k$ to get $\mu_1(\sigma + \omega) = 0$, which immediately implies that $\sigma = -\omega$ since $\mu_1 > 0$.

Finally, we show $g_e = -\sigma$ for all $e \in E(S)$. It is always possible to obtain a new sc-partition $\pi' = (N_1', \ldots, N_k')$ by rearranging π such that

- $|N_1' \cap S| = 2$ if $|S| = 2$;
- $|N_1' \cap S| \geq 2$ and $|N_2' \cap S| > 0$ if $|S| \geq 3$.

Let $\mu_1' = |N_1' \cap S|$ and $\mu_{k-1}' = |N_{k-1}' \cap S|$. Shifting $s \in N_1' \cap S$ from N_1' to N_{k-1}' in π' yields

$$g(s, N_1' \cap (S - s)) + (\mu_1')\sigma = g(s, N_{k-1}' \cap S) + (\mu_{k-1}' + 1)\sigma.$$

If $|S| = 2$ this equation already implies $g_e = -\sigma$ for $e \in E(S)$. If $|S| \geq 3$, one can apply Lemma 6.4 to infer that $g_e = -\sigma$ for all $e \in E(S)$. □

Theorem 7.3. *Suppose that $F_U - F_L \geq 2$. Suppose further that $\mathcal{P}^{lu}$ is full-dimensional and complies with* **(FD-3)** *or* **(FD-4)**. *Let S and T be two proper and disjoint subsets of V such that $|S| < |T|$. Let $\Delta = |T| - |S|$ and $k = \left\lceil \frac{n}{F_U} \right\rceil$. The 2-partition inequality (7.1) is facet defining for $\mathcal{P}^{lu}$ if $\Delta < k$ and*

$$|S| \leq (k - \Delta) \left\lfloor \frac{F_L}{2} \right\rfloor + \Delta \left\lfloor \frac{F_L - 1}{2} \right\rfloor - 1.$$

Proof. The proof is very similar to the proof of Theorem 7.2. □

Remark:

- When $\mathcal{P}^{lu}$ complies with **(FD-1)**, the triangle inequalities (i.e., $|S| = 1$ and $|T| = 2$) are facet defining if $\left\lfloor \frac{n}{F_L} \right\rfloor \geq 2$ (i.e., $n \geq 2F_L + 2$).
- When $\mathcal{P}^{lu}$ complies with **(FD-2)**, the triangle inequalities are facet defining if $\left\lfloor \frac{n}{F_L} \right\rfloor \geq 3$ (i.e., $n \geq 3F_L + 1$).
- When $\mathcal{P}^{lu}$ complies with **(FD-3)** or **(FD-4)**, the triangle inequalities are facet defining if $\left\lfloor \frac{n}{F_L} \right\rfloor \geq 3$ (i.e., $n \geq 3F_L + 1$ for **(FD-3)** and $n \geq 3F_L$ for **(FD-4)**).

Theorem 7.4. *Suppose that dim($\mathcal{P}^{lu}$) $= \binom{n}{2} - 1$. Let S and T be two proper and disjoint subsets of V such that $|S| < |T|$. Let $\Delta = |T| - |S|$ and*

$$k = \begin{cases} \left\lfloor \frac{n}{F_L} \right\rfloor, & \text{if } (n \text{ mod } F_L) = 1, \\ \left\lceil \frac{n}{F_U} \right\rceil, & \text{if } (n \text{ mod } F_U) = F_U - 1, \end{cases}$$

The 2-partition inequality (7.1) is facet defining for $\mathcal{P}^{lu}$ if $\Delta < k$ and

$$|S| \leq \begin{cases} \Delta \left\lfloor \frac{F_L - 1}{2} \right\rfloor + (k - \Delta - 1) \left\lfloor \frac{F_L}{2} \right\rfloor + \left\lfloor \frac{F_L - 2}{2} \right\rfloor & \text{if } (n \text{ mod } F_L) = 1, \\ \left\lfloor \frac{F_U - 2}{2} \right\rfloor + (k - \Delta - 1) \left\lfloor \frac{F_U}{2} \right\rfloor + (\Delta - 1) \left\lfloor \frac{F_U - 1}{2} \right\rfloor + \left\lfloor \frac{F_U - 4}{2} \right\rfloor & \text{if } (n \text{ mod } F_U) = F_U - 1. \end{cases} \tag{7.19}$$

Proof. Let $L = \min\{2, |S|\}$. By (7.19), there exist integer μ_i ($i = 1, \ldots, k$) values such that

1. $\mu_i > 0$ for $i = 1, \ldots, L$, $\mu_i \geq 0$ for $i = L + 1, \ldots, k$,
2. $\sum_{i=1}^{k} \mu_i = |S|$,
3. $2\mu_1 + 2 \leq \begin{cases} F_L + 1, & \text{if } (n \bmod F_L) = 1, \\ F_U, & \text{if } (n \bmod F_U) = F_U - 1; \end{cases}$,
4. $2\mu_i \leq \begin{cases} F_L, & \text{if } (n \bmod F_L) = 1, \\ F_U, & \text{if } (n \bmod F_U) = F_U - 1; \end{cases}$ for $i = 2, \ldots, k - \Delta$,
5. $2\mu_i + 1 \leq \begin{cases} F_L, & \text{if } (n \bmod F_L) = 1, \\ F_U, & \text{if } (n \bmod F_U) = F_U - 1; \end{cases}$ for $i = k - \Delta + 1, \ldots, k - 1$,
6. $2\mu_k + 2 \leq \begin{cases} F_L, & \text{if } (n \bmod F_L) = 1, \\ F_U - 2, & \text{if } (n \bmod F_U) = F_U - 1. \end{cases}$

Using the fact that dim($\mathcal{P}^{lu}$) $= \binom{n}{2} - 1$, we assign $g_{e^*} = 0$ for some $e^* \in E(V - (S \cup T))$. Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ with

- $|N_1| = F_L + 1$ and $|N_i| = F_L$ for $i = 2, \ldots, k$ if $(n \bmod F_L) = 1$,
- $|N_i| = F_U$ for $i = 1, \ldots, k-1$ and $|N_k| = F_U - 1$ if $(n \bmod F_U) = F_U - 1$,
- $|N_i \cap S| = \mu_i$ for $i = 1, 2, \ldots, k$,
- $|N_i \cap T| = \begin{cases} \mu_i, & \text{for } i = 1, 2, \ldots, k - \Delta, \\ \mu_i + 1, & \text{for } i = k - \Delta + 1, \ldots, k. \end{cases}$

By Lemma 7.1, $w^\pi \in P_{S,T}$. Values of μ_1 and μ_k allow us to apply Lemma 6.2 to infer $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - (S \cup T))$. But, then $g_{e^*} = 0$ implies $\alpha = 0$.

Moving $u \in N_1 - (S \cup T)$ from N_1 to N_k in π, and proceeding in the same way as for (7.15) in the proof of Theorem 7.2, one can show that $g_e = \beta \in \mathbb{R}$ for all $e \in E(T, V - (S \cup T))$ and $g_e = \gamma \in \mathbb{R}$ for all $e \in E(S, V - (S \cup T))$. Moreover, continuing in the same manner, one can obtain

$$(\mu_k - \mu_1)(\gamma + \beta - 2\alpha) = \alpha - \beta, \tag{7.20}$$
$$(\mu_k - \mu_1)(\gamma + \beta - 2\alpha) = \beta - \alpha, \tag{7.21}$$

which are essentially very similar to (7.16) and (7.17). These two equalities imply $\beta = \alpha = 0$. Now, we obtain a new sc-partition $\tilde{\pi} = (\tilde{N}_1, \ldots, \tilde{N}_k)$ from π by switching $v \in N_k - (S \cup T)$ and $s \in N_1 \cap S$. Moving $u \in \tilde{N}_1 - (S \cup T)$ to $\tilde{N}_k$ in $\tilde{\pi}$ yields $(\mu_k - \mu_1 + 2)\gamma = 0$, which, together with (7.21) implies $\gamma = 0$.

Switching $u \in N_1 - (S \cup T)$ and $t \in N_k \cap T$ in π yields

$$g(t, N_1 \cap S) + g(t, N_1 \cap T) = g(t, N_k \cap S) + g(t, N_k \cap (T - t)). \tag{7.22}$$

From Lemma 6.4, one can infer that $g_e = \sigma$ for all $e \in E(S, T)$ and $g_e = \omega$ for all $e \in E(T)$. This turns (7.22) into $(\mu_1 - \mu_k)(\sigma + \omega) = 0$. If $\mu_1 \neq \mu_k$, then we have $\sigma = -\omega$. Suppose $\mu_1 = \mu_k$. Due to restrictions on the values of μ_1 and μ_k, when $\mu_1 = \mu_k$, we have $|N_1 - (S \cup T)| \geq 3$. Pick $\{v, x\} \subset N_1 - (S \cup T \cup u)$ and obtain a new sc-partition $\widehat{\pi} = (\widehat{N}_1, \ldots, \widehat{N}_k)$ from π by switching $\{v, x\}$ and $\{\widehat{s}, \widehat{t}\} \subset N_k$ where $\widehat{s} \in N_k \cap S$ and $\widehat{t} \in N_k \cap (T - t)$. Switching u and t in $\widehat{\pi}$ yields $(\mu_1 - \mu_k + 2)(\sigma + \omega) = 0$, which leads to $\sigma = -\omega$.

We can show that $g_e = -\sigma$ for all $e \in E(S)$ just like we do in the proof of Theorem 7.2. □

According to this theorem, the triangle inequalities are always facet defining for $\mathcal{P}^{lu}$ when its dimension is $\binom{n}{2} - 1$.

Another very important result of Theorem 7.4 is that the 2-partition inequalities are facet defining for the equi-partition polytope $\mathcal{P}^{equi}(n)$ when n is odd and certain other conditions are satisfied. We highlight this result, which is new to the literature, by means of the following corollary.

Corollary 7.1. *Consider the equi-partition polytope $\mathcal{P}^{equi}(n)$ where $n = 2p + 1$ (i.e., n is odd). Let S and T be two disjoint subsets of V such that $|T| = |S| + 1$ and*

$$|S| \leq \left\lfloor \frac{p-1}{2} \right\rfloor + \left\lfloor \frac{p-2}{2} \right\rfloor.$$

Then, the 2-partition inequality (7.1) is facet defining for $\mathcal{P}^{equi}(n)$.

This corollary shows in particular that the triangle inequalities are facet defining for $\mathcal{P}^{equi}(n)$ when n is odd.

Theorem 7.5. *Suppose that dim($\mathcal{P}^{lu}$) $= \binom{n}{2} - (n-1)$. Let S and T be two proper and disjoint subsets of V such that $|S| < |T|$. Let $\Delta = |T| - |S|$ and $k = \frac{n}{F_L}$. The 2-partition inequality (7.1) is facet defining for $\mathcal{P}^{lu}$ if $\Delta < k-1$ and*

$$|S| \leq 2\left\lfloor \frac{F_L - 2}{2} \right\rfloor + (k - \Delta - 2)\left\lfloor \frac{F_L}{2} \right\rfloor + (\Delta - 1)\left\lfloor \frac{F_L - 1}{2} \right\rfloor \tag{7.23}$$

Proof. Let $L = \min\{2, |S|\}$. There exist integer μ_i $(i = 1, \ldots, k-1)$ values such that

1. $\mu_i > 0$ for $i = 1, \ldots, L$, $\mu_i \geq 0$ for $i = L+1, \ldots, k-1$,
2. $\sum_{i=1}^{k} \mu_i = |S|$,
3. $2\mu_1 + 2 \leq F_L$,
4. $2\mu_i \leq F_L$ for $i = 2, \ldots, k - \Delta - 1$,
5. $2\mu_i + 1 \leq F_L$ for $i = k - \Delta, \ldots, k-2$,
6. $2\mu_{k-1} + 2 \leq F_L$.

By part (*iii*) of Theorem 4.2, we know $(k-1)F_U = kF_L$ where $k = \frac{n}{F_L} = \frac{n}{F_U} + 1$. Let $\pi = (N_1, N_2, \ldots, N_k)$ be an sc-partition with $|N_i| = F_L$ for $i = 1, 2, \ldots, k$. Suppose the following conditions hold for π:

- $|N_i \cap S| = \mu_i$ for $i = 1, 2, \ldots, k$,
- $|N_i \cap T| = \begin{cases} \mu_i, & \text{for } i = 1, 2, \ldots, k - \Delta - 1, \\ \mu_i + 1, & \text{for } i = k - \Delta, \ldots, k-1. \end{cases}$

By Lemma 7.1, $w^\pi \in P_{S,T}$.

Without loss of generality, suppose $n \notin S \cup T$. Consider the set of edges $Q = \{\{1,n\}, \{2,n\}, \ldots, \{n-1,n\}\}$. Recall the hyperplanes $H_1, H_2, \ldots, H_{n-1}$ we considered belonging to $M(\mathcal{P}^{lu})$ in Theorem 6.3. The coefficients in the equations representing $H_1, \ldots, H_{n-1}$ of the edges in Q are as follows:

$$\begin{array}{cccccc} \{1,n\} & \{2,n\} & \{3,n\} & \ldots & \{n-2,n\} & \{n-1,n\} \end{array}$$
$$\begin{bmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \\ -1 & -1 & -1 & \ldots & -1 & 0 \end{bmatrix}.$$

This matrix is non-singular. Hence, set $g_e = 0$ for all $e \in Q$.

Suppose that $n \in N_1$ in π. Pick $u \in N_1 - (S \cup T)$ and $\{x, y\} \subset N_k$ arbitrarily (note that $N_k \cap (S \cup T) = \emptyset$). By Lemma 6.3, we have $g_{n,u} + g_{x,y} = g_{n,x} + g_{u,y}$. Since we have already set $g_{n,u} = g_{n,x} = 0$, this implies $g_{x,y} = g_{u,y}$, which generalizes into $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - (S \cup T \cup n)$.

Now, pick an sc-partition $\pi^1 = (N_1^1, N_2^1, \dots, N_k^1)$ such that $n \in N_k^1$ and $u \in N_1^1 - (S \cup T)$. Switching n and u in π^1 gives

$$g(u, N_1^1 \cap S) + g(u, N_1^1 \cap T) + \alpha(F_L - 2\mu_1 - 1) = \alpha(F_L - 1). \tag{7.24}$$

Applying Lemma 6.4, we infer that $g_e = \beta \in \mathbb{R}$ for all $e \in E(V - (S \cup T), T)$ and $g_e = \gamma \in \mathbb{R}$ for all $e \in E(V - (S \cup T), S)$. Then (7.24) turns into

$$\beta + \gamma = 2\alpha. \tag{7.25}$$

Now, consider the sc-partition $\pi^2 = (N_1^2, \dots, N_k^2)$ obtained by switching $t \in N_{k-1} \cap T$ and $x \in N_k - (S \cup T)$. Switching u and n in π^2 yields

$$(F_L - 2\mu_1 - 1)\alpha + \mu_1\beta + \mu_1\gamma = (F_L - 2)\alpha + \beta.$$

Comparing this equation with (7.25) yields $\beta = \gamma = \alpha$.

Now, consider the sc-partition π^* obtained from π by distributing N_k over the other subsets. Comparing $g^T w^\pi = h$ with $g^T w^{\pi^*} = h$ gives $\alpha = \beta = \gamma = 0$.

Now, switching $t \in N_{k-1} \cap T$ and $u \in N_1 - (S \cup T)$ gives

$$g(t, N_1 \cap S) + g(t, N_1 \cap T) = g(t, N_{k-1} \cap S) + g(t, N_{k-1} \cap T). \tag{7.26}$$

This equality and Lemma 6.4 leads us to $g_e = \sigma \in \mathbb{R}$ for all $e \in E(S, T)$ and $g_e = \omega \in \mathbb{R}$ for all $e \in E(T)$. Then, (7.26) turns into

$$(\mu_1 - \mu_{k-1})(\sigma + \omega) = 0. \tag{7.27}$$

If $\mu_1 \neq \mu_{k-1}$ then we have $\sigma = -\omega$. Suppose $\mu_1 = \mu_{k-1}$. Then, obtain a new sc-partition $\pi' = (N_1', \dots, N_k')$ by switching $\{x, y\} \subset N_k$ and $\{s', t'\} \subset N_{k-1}$ in π where $s \in N_{k-1} \cap S$ and $t \in N_{k-1} \cap T$. Switching $t \in N_{k-1}' \cap T$ and $u \in N_1' - (S \cup T)$ yields $(\mu_1 - \mu_{k-1} + 1)(\sigma + \omega) = 0$, which, together with (7.27) implies $\sigma = -\omega$.

Finally, we show that $g_e = -\sigma$ for all $e \in E(S)$. Pick an sc-partition $\tilde{\pi} = (\tilde{N}_1, \dots, \tilde{N}_k)$ where $n \in N_{k-1}$ and

- $|N_1 \cap S| = 2$ if $|S| = 2$,
- $|N_1 \cap S| \geq 2$ and $|N_2 \cap S| > 0$ if $|S| \geq 3$.

Note that, we can always find such an sc-partition $\tilde{\pi}$. Switching $s \in \tilde{N}_1 \cap S$ and n in $\tilde{\pi}$ yields

$$g(s, \tilde{N}_1 \cap S) + g(s, \tilde{N}_1 \cap T) = g(s, \tilde{N}_{k-1} \cap S) + g(s, \tilde{N}_{k-1} \cap T).$$

Applying Lemma 6.4 to this equality yields $g_e = -\sigma$ for all $e \in E(S)$. □

Remark: When $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$, the triangle inequalities are facet defining for $\mathcal{P}^{lu}$ if $n \geq 3F_L$.

Theorem 7.6. *Suppose that $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$. Let S and T be two proper and disjoint subsets of V such that $|S| < |T|$. Let $\Delta = |T| - |S|$ and $k = \frac{n}{F}$ where*

$$F = \begin{cases} F_L, & \text{if } (n \bmod F_L) = 0, \\ F_U, & \text{if } (n \bmod F_U) = 0. \end{cases}$$

The 2-partition inequality (7.1) is facet defining for $\mathcal{P}^{lu}$ if $\Delta < k$ and

$$|S| \leq (k - \Delta - 1)\left\lfloor \frac{F}{2} \right\rfloor + (\Delta - 1)\left\lfloor \frac{F-1}{2} \right\rfloor + \left\lfloor \frac{F-2}{2} \right\rfloor + \left\lfloor \frac{F-3}{2} \right\rfloor. \tag{7.28}$$

Proof. Let $L = \min\{2, |S|\}$. There exist integer μ_i $(i = 1, \ldots, k)$ values such that

1. $\mu_i > 0$ for $i = 1, \ldots, L$, $\mu_i \geq 0$ for $i = L+1, \ldots, k$,
2. $\sum_{i=1}^{k} \mu_i = |S|$,
3. $2\mu_1 + 2 \leq F$,
4. $2\mu_i \leq F$ for $i = 2, \ldots, k - \Delta$,
5. $2\mu_i + 1 \leq F$ for $i = k - \Delta + 1, \ldots, k-1$,
6. $2\mu_k + 3 \leq F$.

Without loss of generality, assume that $\{n, n-1, n-2\} \subset V - (S \cup T)$. Let $Q = \{\{1,n\}, \{2,n\}, \ldots, \{n-1,n\}, \{n-2,n-1\}\}$. When $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$, $M(\mathcal{P}^{lu})$ consists of hyperplanes $H_1, \ldots, H_n$ where H_u is defined as

$$w(\delta(u)) = F - 1.$$

The columns of the edges belonging to Q in the equalities corresponding to $H_1, \ldots, H_n$ are as follows:

$$\begin{array}{ccccccc} \{1,n\} & \{2,n\} & \{3,n\} & \ldots & \{n-2,n\} & \{n-1,n\} & \{n-2,n-1\} \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 1 \\ 1 & 1 & 1 & \ldots & 1 & 1 & 0 \end{bmatrix}.$$

This matrix is non-singular. Hence, we set $g_e = 0$ for all $e \in Q$.

Pick an sc-partition $\pi = (N_1, \ldots, N_k)$ such that

- $|N_i \cap S| = \mu_i$ for $i = 1, 2, \ldots, k$,

- $|N_i \cap T| = \begin{cases} \mu_i, & \text{for } i = 1, 2, \ldots, k - \Delta, \\ \mu_i + 1, & \text{for } i = k - \Delta + 1, \ldots, k; \end{cases}$
- $n \in N_1$.

By Lemma 6.4, $w^\pi \in P_{S,T}$.

Pick arbitrarily $u \in N_1 - (S \cup T)$ and $\{x, y\} \subset N_k - (S \cup T)$. Lemma 6.3 implies that $g_{u,n} + g_{x,y} = g_{u,y} + g_{n,x}$. Since $\{u, n\}$ and $\{n, x\}$ are in Q, we infer $g_{x,y} = g_{u,y}$, which generalizes into $g_e = 0$ for all $e \in E(V - (S \cup T \cup n))$ since we have already set $g_{n-2,n-1} = 0$.

Now, switch x and n in π to obtain

$$g(x, N_1 \cap S) + g(x, N_1 \cap T) = g(x, N_k \cap S) + g(x, N_k \cap T). \tag{7.29}$$

Lemma 6.4 implies that $g_e = \beta \in \mathbb{R}$ for all $e \in E(T, V - (S \cup T))$ and $g_e = \gamma \in \mathbb{R}$ for all $e \in E(S, V - (S \cup T))$. Then, (7.29) turns into

$$(\mu_k - \mu_1 + 1)\beta + (\mu_k - \mu_1)\gamma = 0. \tag{7.30}$$

Now, we obtain a new sc-partition $\pi^1 = (N_1^1, \ldots, N_k^1)$ from π by switching $t \in N_k \cap T$ and $u \in N_1 - (S \cup T)$. Switching x and n in π^1 yields

$$(\mu_k - \mu_1 - 1)\beta + (\mu_k - \mu_1)\gamma = 0.$$

Solving this equation simultaneously with (7.30) gives $\beta = 0$. Now, obtain yet another sc-partition $\pi^2 = (N_1^2, \ldots, N_k^2)$ from π by switching $s \in N_1 \cap S$ and $y \in N_k - (S \cup T)$. Switching x and n in π^2 yields

$$(\mu_k - \mu_1 + 2)\gamma = 0,$$

which, when compared with (7.30), implies $\gamma = 0$.

Switching $t \in N_k \cap T$ and $u \in N_1 - (S \cup T)$ in π gives

$$g(t, N_1 \cap S) + g(t, N_1 \cap T) = g(t, N_k \cap S) + g(t, N_k \cap (T - t)). \tag{7.31}$$

Again, applying Lemma 6.4, we infer $g_e = \sigma$ for all $e \in E(S, T)$ and $g_e = \omega$ for all $e \in E(T)$. Then, (7.31) can be rewritten as

$$(\mu_k - \mu_1)(\sigma + \omega) = 0. \tag{7.32}$$

If $\mu_k \neq \mu_1$, we have $\sigma = -\omega$. Suppose $\mu_k = \mu_1$. Obtain a new sc-partition $\pi^* = (N_1^*, \ldots, N_k^*)$ by switching $\{s^*, t^*\} \subset N_1$ and $\{x, y\} \subset N_k - (S \cap T)$, where $s^* \in N_1 \cap S$ and $t^* \in N_1 \cap T$. Switching $t \in N_k \cap T$ and $u \in N_1 - (S \cup T)$ yields

$$(\mu_k - \mu_1 + 2)(\sigma + \omega) = 0,$$

which implies $\omega = -\sigma$. In a very similar manner, one can also show that $g_e = -\sigma$ for all $e \in E(S)$. □

This theorem suggests that, under certain conditions, the 2-partition inequalities and in particular the triangle inequalities are facet defining for the equi-partition polytope $\mathcal{P}^{equi}(n)$ for even n, and the k-way equi-partition polytope ($k \geq 3$). We highlight these two important results in the following corollaries. To our knowledge, these results as well are new to the literature.

Corollary 7.2. *Consider the equi-partition polytope $\mathcal{P}^{equi}(n)$ where $n = 2p$ (i.e., n is even). Let S and T be two disjoint and proper subsets of V such that $|T| = |S| + 1$ and*

$$|S| \leq \left\lfloor \frac{p-2}{2} \right\rfloor + \left\lfloor \frac{p-3}{2} \right\rfloor.$$

Then, the 2-partition inequality (7.1) is facet defining for $\mathcal{P}^{equi}(n)$.

Corollary 7.3. *Consider the k-way equi-partition polytope $\mathcal{P}^{k-way}(n,k)$ where $k \geq 3$ and $n = kF$. Let S and T be two disjoint and proper subsets of V such that $\Delta = |T| - |S| < k$ and (7.28) is satisfied. Then, the 2-partition inequality (7.1) is facet defining for $\mathcal{P}^{k-way}(n,k)$.*

8 The lower general clique inequalities

Let Q be, throughout this section, a subset of V such that $|Q| > \left\lfloor \frac{n}{F_L} \right\rfloor$ and $\left(|Q| \bmod \left\lfloor \frac{n}{F_L} \right\rfloor\right) \neq 0$. Likewise, let k, r, p and q be defined, throughout this section, in the following manner:

- $k = \left\lfloor \frac{n}{F_L} \right\rfloor$;
- $r = n \bmod F_L$, i.e., $n = kF_L + r$,
- $p = \left\lfloor \frac{|Q|}{k} \right\rfloor$ ($p \geq 1$ always due to choice of $|Q|$), and,
- $q = (|Q| \bmod k)$ ($q > 0$ always due to choice of $|Q|$).

Consider the following inequality:

$$w(E(Q)) \geq (k-q)\binom{p}{2} + q\binom{p+1}{2}. \tag{8.1}$$

These inequalities are first introduced by Chopra and Rao [1] for a graph partitioning problem. Chopra and Rao call these inequalities 'the general clique inequalities'. But in this context, we prefer using the term 'the lower general clique inequalities' because in Section 9 we introduce another set of valid inequalities, 'the upper general clique inequalities', which have the same left hand side as and the opposite sense of (8.1).

Chopra and Rao [1] introduce these inequalities for the polytope corresponding to the problem of partitioning G into at most $\bar{l}$ subcliques without any size restrictions, i.e., they deal with the following polytope:

$$conv\{\mathbf{w}^\pi : \pi = (N_1, \ldots, N_l), l \leq \bar{l}, 1 \leq |N_i| \leq n \text{ for } i = 1, \ldots, l\}.$$

Obviously, when $\bar{l} = k$, $\mathcal{P}^{lu}$ is a subset of this polytope. And, this means that (8.1) is valid for $\mathcal{P}^{lu}$ as well.

Ji and Mitchell [4] prove the conditions for these inequalities to be facet defining for $\mathcal{P}^{l}(n, F_L)$. Theorems (8.1)-(8.3) of this section are devoted to proving the conditions that make (8.1) facet defining for $\mathcal{P}^{lu}$. We denote by P_Q the face defined by a lower general clique inequality written for a subset Q, i.e.,

$$P_Q = \left\{ \mathbf{w} \in \mathcal{P}^{lu} : w(E(Q)) = (k-q)\binom{p}{2} + q\binom{p+1}{2} \right\}.$$

As usual, we assume that there exists a valid inequality $g^T w = h$ such that

$$P_Q \subseteq \{\mathbf{w} \in \mathcal{P}^{lu} : g^T w = h\}.$$

We aim to show that $g^T w = h$ can be written as a linear combination of (8.1) and the hyperplane equations in $M(\mathcal{P}^{lu})$, if any. Before moving on to the theorems, we give a simple characterization of P_Q in the following lemma.

Lemma 8.1. *For any sc-partition $\pi = (N_1, \ldots, N_l)$, $w^\pi \in P_Q$ if and only if*

1. $l = k$,
2. $|\{i : |N_i \cap Q| = p\}| = k - q$, *and,*
3. $|\{i : |N_i \cap Q| = p + 1\}| = q$.

Proof. Sufficiency is obvious. We prove necessity. Note that any sc-partition π^* that minimizes $w^\pi(E(Q))$ necessarily has k subcliques, because, minimizing $w^\pi(E(Q))$ requires that the nodes of Q be dispersed to the maximal number of subcliques.

Secondly, for $\pi^1 = (N_1^1, \ldots, N_k^1)$ with $w^{\pi^1} \in P_Q$, we necessarily have

$$\big||N_i^1 \cap Q| - |N_j^1 \cap Q|\big| \leq 1 \qquad \forall i, j = 1, \ldots, k. \tag{8.2}$$

To see this, suppose that $|N_{i^*}^1 \cap Q| - |N_{j^*}^1 \cap Q| \geq 2$ for some i^* and j^*. Now, obtain another sc-partition $\pi^2 = (N_1^2, \ldots, N_k^2)$ from π^1 by switching $x \in N_{i^*}^1 \cap Q$ and $y \in N_{j^*}^1 - Q$. Clearly, $w^{\pi^1}(E(Q)) > w^{\pi^2}(E(Q))$. This contradicts the assumption that w^{π^1} minimizes $w^\pi(E(Q))$.

The conditions (ii) and (iii) follow from (8.2) and the fact that any π with $w^\pi \in P_Q$ has k subcliques. □

Theorem 8.1. *Suppose that the polytope $\mathcal{P}^{lu}$ is full-dimensional and suppose $F_U - F_L \geq 2$.*

1. *When $r = 0$, (8.1) is valid for $\mathcal{P}^{lu}$, but not facet-defining.*
2. *When $r = 1$, (8.1) is facet-defining for $\mathcal{P}^{lu}$ if and only if $Q = V$.*
3. *When $r > 1$, (8.1) is facet-defining for $\mathcal{P}^{lu}$.*

Proof. (i) When $r = 0$,

$$P_Q \subseteq \{\mathbf{w} \in \mathcal{P}^{lu} : w(\delta(u)) = F_L - 1\}$$

for any $u \in V$.

(ii) When $r = 1$, note that, all sc-partitions with k subcliques in $\mathcal{P}^{lu}$ are composed of $k-1$ subcliques with size F_L and one subclique with size $F_L + 1$.

We first show that $g_e = \alpha \in \mathbb{R}$ for all $e \in E(Q)$. Pick arbitrarily three nodes $u, v, x \in Q$. Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $w^\pi \in P_Q$, $\{u, v\} \subset N_1$, $x \subset N_2$, $|N_1| = F_L + 1$, $|N_i| = F_L$ for $i = 2, \ldots, k$, $|N_1 \cap Q| = p + 1$ and $|N_2 \cap Q| = p$. Applying Lemma 6.2 to π yields $g_{u,v} = g_{u,x}$. Since u, v and x are arbitrary, we have $g_e = \alpha \in \mathbb{R}$ for all $e \in E(Q)$.

Secondly, we show that $g_e = \beta \in \mathbb{R}$ for all $e \in E - E(Q)$. Pick arbitrarily $u \in Q$, $v, x \notin Q$. Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $w^\pi \in P_Q$, $\{u, v\} \subset N_1$, $x \in N_2$, $|N_1| = F_L + 1$, $|N_i| = F_L$ for $i = 2, \ldots, k$, $|N_1 \cap Q| = p + 1$ and $|N_2 \cap Q| = p$. Using Lemma 6.2, one can show $g_{v,x} = g_{v,u} = g_{u,x} = \beta \in \mathbb{R}$. Arbitrariness of u, v and x implies $g_e = \beta$ for all $e \in E - E(Q)$.

We infer that

$$P_Q \subseteq \{\mathbf{w} \in \mathcal{P}^{lu} : \alpha w(E(Q)) + \beta w(E - E(Q)) = C\}$$

for some constants α, β and C. Then, (8.1) is facet defining for $\mathcal{P}^{lu}$ if and only if $Q = V$.

(iii) When $r > 1$, we know from Theorem 5.1 that, all the sc-partitions with k subcliques are 2-loose sc-partitions (note that $F_U - F_L \geq 2$). We first show that $g_e = 0$ for all $e \in E(Q, V - Q)$. Pick two arbitrary nodes $u \in Q$ and $v \notin Q$. Pick a 2-loose sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $w^\pi \in P_Q$, $F_L < |N_1| < F_U$, $F_L < |N_2| < F_U$, $u \in N_1$, $v \in N_2$, $|N_1 \cap Q| = p+1$ and $|N_2 \cap Q| = p$. Using Lemma 6.1, we get $g(u, v) = 0$. Since u and v are picked arbitrarily, we can infer $g_e = 0$ for all $e \in E(Q, V - Q)$.

Now, picking the same sc-partition π and two arbitrary nodes $x \in N_1 - Q$ and $y \in N_2 - Q$, and applying Lemma 6.1, one can infer that $g(x, y) = 0$, i.e., $g_e = 0$ for all $e \in E(V - Q)$ since x and y are arbitrary.

Finally, pick three nodes $u, v, x \in Q$. Consider the sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ such that $w^\pi \in P_Q$, $u \in N_1$, $v, x \in N_2$, $F_L < |N_1| < F_U$, $F_L < |N_2| < F_U$, $|N_1 \cap Q| = p$ and $|N_2 \cap Q| = p+1$. Applying Lemma 4.2 to π, u, v and x by setting P to P_Q, one can infer that $g_e = \alpha \in \mathbb{R}$ for all $e \in E(Q)$.

□

Theorem 8.2. *If* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, *the lower general clique inequality (8.1) is facet defining for* $\mathcal{P}^{lu}$.

Proof. Using $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, we set $g'_e = 0$ for one $e' \in E - E(Q)$. Proceeding like in the proof of part (ii) in Theorem 8.1, we show $g_e = \beta$ for all $e \in E - E(Q)$ and $g_e = \alpha$ for all $e \in E(Q)$. But, $g'_e = 0$ implies $\beta = 0$. □

Note that, Theorem 8.2 proves as well that (8.1) is facet defining for $\mathcal{P}^{equi}(n)$. This result is already proved by Chopra and Rao [1]. Theorem 8.2 proposes just an alternative proof for this result.

Proposition 8.1. *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$. *The general clique inequality (8.1) is not facet defining for* $\mathcal{P}^{lu}$.

Proof. In this case,

$$P_Q \subseteq \{\mathbf{w} \in \mathcal{P}^{lu} : w(\delta(u)) = F_L - 1\}$$

for any $u \in V$. □

Theorem 8.3. *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$. *Let*

$$F = \begin{cases} F_L, & \text{if } (n \bmod F_L) = 0, \\ F_U, & \text{if } (n \bmod F_U) = 0. \end{cases}$$

The lower general clique inequality (8.1) is facet defining for $\mathcal{P}^{lu}$ *if and only if* $p < F - 1$.

Proof. First note that, when $p = F - 1$

$$P_Q \subseteq \{\mathbf{w} \in \mathcal{P}^{lu} : w_{u,v} = 0\}$$

for all $u, v \in V - Q$. Suppose $p < F - 1$. Without loss of generality, let $Q = \{1, 2, \dots, |Q|\}$ and $V - Q = \{|Q|+1, \dots, n\}$. Consider the set $S = \{\{1,n\}, \{2,n\}, \dots, \{n-1,n\}, \{1,n-1\}\}$. We know that $M(\mathcal{P}^{lu})$ consists of the hyperplanes $H_1, \dots, H_n$ where H_u is represented by the equation

$$w(\delta(u)) = F - 1.$$

In the following matrix, the coefficients of the edges belonging to S in the equations representing H_1 to H_n are displayed:

$$\begin{array}{r} \\ H_1 \rightarrow \\ H_2 \rightarrow \\ H_3 \rightarrow \\ \vdots \\ H_{n-2} \rightarrow \\ H_{n-1} \rightarrow \\ H_n \rightarrow \end{array} \begin{array}{c} \begin{array}{ccccccc} \{1,n\} & \{2,n\} & \{3,n\} & \dots & \{n-2,n\} & \{n-1,n\} & \{1,n-1\} \end{array} \\ \left[\begin{array}{ccccccc} 1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & 0 \end{array}\right] \end{array}.$$

This matrix is non-singular. With reference to Theorem 3.6 in Section I.4.3 of [6], we use this fact to set $g_e = 0$ for all $e \in S$.

Now, pick an arbitrary node $u \in Q$ and an sc-partition $\pi = (N_1, \dots, N_k)$ such that $w^\pi \in P_Q$. Suppose that $u \in N_1$, $n \in N_2$, $|N_1 \cap Q| = p + 1$ and $|N_2 \cap Q| = p$. Switching u and n, we get

$$g(u, (N_1 - u) \cap Q) + g(u, (N_1 - u) - Q) = g(u, (N_2 - n) \cap Q) + g(u, (N_2 - n) - Q).$$

Applying Lemma 6.4 to this equation yields $g_e = \alpha \in \mathbb{R}$ for $e \in E(Q)$ and $g_e = \beta$ for $e \in E(Q, (V-n)-Q)$. But, then since we have already fixed $g_{1,n-1} = 0$, we immediately have $\beta = 0$.

Now, pick an arbitrary node $x \in (V-n)-Q$. Pick an sc-partition $\pi^* = (N_1^*, \ldots, N_k^*)$ such that $w^\pi \in P_Q$, $x \in N_1^*$, $n \in N_2^*$, $|N_1 \cap Q| = p+1$ and $|N_2 \cap Q| = p$. Switching x and n yields

$$g(x, (N_1^* - x) - Q) = g(x, (N_2^* - n) - Q). \tag{8.3}$$

Applying Lemma 6.4 to this equation gives $g_e = \gamma$ for all $e \in E((V-n)-Q)$. Plugging this into (8.3), we infer $\gamma = 0$. □

Theorem 8.3 suggests that (8.1) is facet defining for $\mathcal{P}^{k-way}(n, k)$ as well. We highlight this result, which is new to the literature, in the following corollary.

Corollary 8.1. *The lower general clique inequality (8.1) is facet defining for $\mathcal{P}^{k-way}(n, k)$ if and only if*

1. *$(|Q| \text{ mod } k) \neq 0$, and,*
2. *$|Q| < n - k$.*

9 The Upper General Clique Inequalities

In this section, we introduce a new family of valid inequalities, the upper general clique inequalities, and prove conditions for which these inequalities become facet defining for $\mathcal{P}^{lu}$.

Let $Q \subseteq V$. Consider the ϕ_i^U values computed by means of the formula (6.6) in Section 6.2. Let k_Q and n_Q be defined throughout this section as follows:

- $k_Q = \max\{i | \sum_{l=1}^{i} \phi_l^U \leq |Q|\}$, and,
- $n_Q = |Q| - \sum_{l=1}^{k_Q} \phi_l^U$.

Contrary to the lower general clique inequality (8.1), the following upper general clique inequality aims to bound $w(E(Q))$ from above:

$$w(E(Q)) \leq \sum_{l=1}^{k_Q} \binom{\phi_l^U}{2} + \binom{n_Q}{2} \qquad \forall Q \subseteq V. \tag{9.1}$$

Note that in the statement of this inequality, we adopt the convention that $\binom{0}{2} = \binom{1}{2} = 0$.

Lemma 9.1. *The upper general clique inequality (9.1) is valid for $\mathcal{P}^{lu}$.*

Proof. The set Q should be split into the largest subcliques possible in order to maximize $w(E(Q))$. This can be achieved by putting ϕ_1^U nodes of Q in one subclique, ϕ_2^U nodes in another subclique, and so on. □

Throughout this section, $\mathcal{P}_Q$ stands for the face defined by (9.1). Next, we give a lemma that characterizes the sc-partitions whose characteristic vectors lie on P_Q. We skip the proof because it is trivial.

Lemma 9.2. *For an sc-partition $\pi = (N_1, \dots, N_k)$, $w^\pi \in P_Q$ if and only if $|N_i| = |N_i \cap Q| = \phi_i^U$ for $i = 1, \dots, k_Q$ and $|N_i| \geq |N_i \cap Q| = n_Q$ for $i = k_Q + 1$.*

Now, we continue with two lemmas that state some conditions on Q determining if (9.1) is or is not facet defining.

Lemma 9.3. *Suppose $F_U - F_L \geq 2$ and $Q = V$. The upper general clique inequality (9.1) is facet defining for $\mathcal{P}^{lu}$ if and only if it is full-dimensional and complies with* **(FD-1)** *and* **(FD-3)**.

Proof. When

- $\mathcal{P}^{lu}$ is full-dimensional and it complies with **(FD-2)**, or,
- $\mathcal{P}^{lu}$ is full-dimensional and it complies with **(FD-4)**, or,
- $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$,

we have

$$P_Q \subseteq \{\mathbf{w} \in \mathcal{P}^{lu} : w(\delta(u)) = F_U - 1\} \qquad \forall u \in V.$$

When $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$,

$$P_Q \subseteq \{\mathbf{w} \in \mathcal{P}^{lu} : w(\delta(u)) = F - 1\} \qquad \forall u \in V$$

where

$$F = \begin{cases} F_L, & \text{if } (n \bmod F_L) = 0, \\ F_U, & \text{if } (n \bmod F_U) = 0. \end{cases}$$

And, finally when $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, $P_Q = \mathcal{P}^{lu}$.

When $\mathcal{P}^{lu}$ is full-dimensional and it complies with one of **(FD-1)** and **(FD-3)**, there always exist $\phi_{i^*}^U$ and $\phi_{j^*}^U$ such that $\phi_{i^*}^U \neq \phi_{j^*}^U$. When $\phi_{i^*}^U = \phi_{j^*}^U + 1$, using Lemma 6.2, one can easily show that $g_e = \alpha \in \mathbb{R}$ for all $e \in E$.

Suppose that $\phi_{i^*}^U \geq \phi_{j^*}^U + 2$. Pick four arbitrary nodes x, y, z, t and an sc-partition $\pi = (N_1, N_2, \dots, N_k)$ where $k = \left\lceil \frac{n}{F_U} \right\rceil$, $\{x, z\} \subset N_{i^*}$, $\{y, t\} \subset N_{j^*}$, $|N_i| = \phi_i^U$ for $i = 1, 2, \dots, k$. Applying Lemma 6.3 to π yields

$$g_{x,z} - g_{x,t} = g_{y,z} - g_{y,t} = \alpha \in \mathbb{R} \qquad \forall z, t \in V - \{x, y\}. \tag{9.2}$$

But, switching x and y in π, we obtain

$$g(x, N_{i^*} - x) + g(y, N_{j^*} - y) = g(x, N_{j^*} - y) + g(y, N_{i^*} - x). \tag{9.3}$$

Plugging (9.2) into (9.3), we get $\alpha = 0$. Since x, y, z and t are arbitrary, we have $g_e = \gamma \in \mathbb{R}$ for all $e \in E$. $\square$

Lemma 9.4. *Suppose $F_U - F_L \geq 2$. Let Q be a proper subset of V. The upper general clique inequality (9.1) is not facet defining for $\mathcal{P}^{lu}$ if*

1. *$n_Q = 0$, or,*
2. *$|Q| \leq \phi_1^U + 1$.*

   When $\mathcal{P}^{lu}$ is full-dimensional, (9.1) is not facet defining for $\mathcal{P}^{lu}$ if

3. *$\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ is not full-dimensional.*

Proof.

1. When $n_Q = 0$, P_Q lies in the intersection of the hyperplanes $w_{u,v} = 0$ for all $u \in Q$ and $v \in V - Q$.

2. When $|Q| \leq \phi_1^U$, P_Q lies in the intersection of the hyperplanes $w_{u,v} = 1$ for all $u, v \in Q$. When $|Q| = \phi_1^U + 1$, P_Q is contained in the face defined by the cycle inequalities of Section 10. More precisely, let $Q = \{q_1, q_2, \ldots, q_{\phi_1^U+1}\}$; the face defined by the valid inequality $g^T w \leq \phi_1^U - 1$ with

$$g_{u,v} = \begin{cases} 1, & \text{if } u = q_i \text{ and } v = q_{i+1}, \text{ for all } i = 1, 2, \ldots, \phi_1^U + 1, \\ 0, & \text{otherwise}, \end{cases}$$

   where all indices are taken modulo $\phi_1^U + 1$, contains P_Q.

3. Let $\tilde{n} = n - \sum_{i=1}^{k_Q} \phi_i^U$ and $E_{\tilde{n}}$ be the edge set of the complete graph defined on $\tilde{n}$ nodes. When $\dim(\mathcal{P}^{lu}(\tilde{n}, F_L, F_U)) = \binom{\tilde{n}}{2} - \tilde{n}$, P_Q consists of w^π such that $\pi = (N_1, \ldots, N_k)$, $|N_1| = |N_1 \cap Q| = \phi_i^U$ for $i = 1, \ldots, k_Q$, $|N_{k_Q+1} \cap Q| = n_Q$, $|N_i| = F$ for $i = k_Q + 1, \ldots, k$ where F is equal to either F_L or F_U. This implies that P_Q is contained in the intersection of the hyperplanes $w(\delta(u)) = F - 1$ for all $u \in V - Q$.

   Now, suppose $\dim(\mathcal{P}^{lu}(\tilde{n}, F_L, F_U)) = \binom{\tilde{n}}{2} - (\tilde{n} - 1)$. All points w^π in P_Q where $\pi = (N_1, N_2, \ldots, N_k)$ have $|N_i| = |N_i \cap Q| = \phi_i^U$ for $i = 1, \ldots, k_Q$ and $|N_{k_Q+1} \cap Q| = n_Q$. Let P_1 consist of points $w^\pi \in P_Q$ such that $|N_i| = F_L$ for $i = k_Q + 1, \ldots, k$; and, let P_2 consist of points $w^\pi \in P_Q$ such that $|N_i| = F_U$ for $i = k_Q + 1, \ldots, k$. Since $P_Q = P_1 \cup P_2$, obviously, P_Q is contained in hyperplanes $w(\delta(u)) - w(\delta(v)) = 0$ for all $u, v \in V - Q$, which is in accordance with Proposition 5.4.

   Finally, suppose $\dim(\mathcal{P}^{lu}(\tilde{n}, F_L, F_U)) = \binom{\tilde{n}}{2} - 1$. Suppose that $\mathcal{P}^{lu}(\tilde{n}, F_L, F_U)$ is contained in an hyperplane $w(E_{\tilde{n}}) = C$ for some constant $C \in \mathbb{R}$. Then P_Q is contained in $w(E) = C + \sum_{i=1}^{k_Q} \binom{\phi_i^U}{2}$, where E, as before, denotes the edge set of the complete graph defined on n nodes.

□

The following Theorem 9.1 states the conditions that make (9.1) facet defining for a full-dimensional $\mathcal{P}^{lu}$. With reference to Lemma 9.4, we assume that $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ is full-dimensional, and hence by Theorem 5.1, it falls into one of four categories **(FD-1)**,**(FD-2)**-**(FD-4)**.

Theorem 9.1. *Suppose $\mathcal{P}^{lu}$ is full-dimensional and $F_U - F_L \geq 2$. Let Q be a subset of V which does not comply with any of the conditions (i)-(iii) of Lemma 9.4 (i.e., $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ is full-dimensional, $n_Q > 0$ and $|Q| > \phi_1^U + 1$).*

1. *When $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with* **(FD-1)**, *(9.1) is facet defining for $\mathcal{P}^{lu}$ if $1 \leq n_Q \leq F_L$.*

2. *When $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with* **(FD-2)**, *(9.1) is facet defining for $\mathcal{P}^{lu}$ if and only if*
   - $1 \leq n_Q < F_L$ *and* $k_Q \leq \left\lceil \frac{n}{F_U} \right\rceil - 1$, *or,*
   - $n_Q = F_L$ *and* $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 1$.

3. *When $F_U - F_L = 2$ and $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with* **(FD-3)**, *(9.1) is facet defining for $\mathcal{P}^{lu}$ if and only if*
   - $1 \leq n_Q < F_L$ *and* $k_Q \leq \left\lceil \frac{n}{F_U} \right\rceil - 1$, *or,*
   - $n_Q = F_L$ *and* $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 1$, *or,*
   - $n_Q = F_L + 1$ *and* $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 2$.

4. *When $F_U - F_L > 2$ and $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with* **(FD-3)**, *(9.1) is facet defining for $\mathcal{P}^{lu}$ if and only if*
   - $1 \leq n_Q < F_L$ *and* $k_Q \leq \left\lceil \frac{n}{F_U} \right\rceil - 1$, *or,*
   - $F_L \leq n_Q \leq F_L + 1$ *and* $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 1$.

5. *When $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with* **(FD-4)**, *(9.1) is facet defining for $\mathcal{P}^{lu}$ if and only if $1 \leq n_Q \leq F_L$ and $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 1$.*

Proof. We prove the cases (i) and (ii) only. The proofs of parts (iii), (iv) and (v) are similar.

1. Pick two arbitrary nodes $u, v \in V - Q$ and an sc-partition $\pi = (N_1, \ldots, N_h)$ where $\left\lceil \frac{n}{F_U} \right\rceil \leq h \leq \left\lfloor \frac{n}{F_L} \right\rfloor$, $|N_i| = |N_i \cap Q| = \phi_i^U$ for $i = 1, \ldots, k_Q$, $|N_{k_Q+1} \cap Q| = n_Q$, $F_L < |N_{k_Q+1}| < F_U$, $F_L < |N_{k_Q+2}| < F_U$. Note that such an sc-partition with $h \geq k_Q + 2$ exists since $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with **(FD-1)**. Suppose that $u \in N_{k_Q+1}$ and $v \in N_{k_Q+2}$ (there exists a $u \in N_{k_Q+1} - Q$ since we assume $n_Q \leq F_L$). By Lemma 6.1, $g_{u,v} = 0$. This means, we have $g_e = 0$ for all $e \in E(V - Q)$.

   Now, in π, shift v from N_{k_Q+2} to N_{k_Q+1} to obtain
   $$g(v, N_{k_Q+1} \cap Q) = 0. \tag{9.4}$$

When applied to this equation, Lemma 6.4 implies that $g_e = \alpha \in \mathbb{R}$ for $e \in E(Q, V-Q)$. But, then inserting this into (9.5) we immediately infer $g_e = 0$ for $e \in E(Q, V-Q)$.

Now, we will show $g_e = \beta \in \mathbb{R}$ for all $e \in E(Q)$. We prove separately for two cases: a) $n_Q \geq 2$, b) $n_Q = 1$ and $k_Q \geq 2$ (we do not consider $k_Q = 1$, because then (9.1) would not be facet defining by part (*ii*) of Lemma 9.4).

Case a): Pick four nodes $x, y, z, t \in Q$ and suppose that $\{x, z\} \subset N_{k_Q}$ and $\{y, t\} \subset N_{k_Q+1}$ in π. By Lemma 6.3, we have $g_{x,z} - g_{y,z} = g_{x,t} - g_{y,t}$, which implies

$$g_{x,q} - g_{y,q} = \gamma \in \mathbb{R} \qquad \forall q \in Q - \{x, y\}. \tag{9.5}$$

by arbitrariness of z and t. Switching x and y in π, we get

$$g(x, (N_{k_Q} \cap Q) - x) + g(y, (N_{k_Q+1} \cap Q) - y) = g(x, (N_{k_Q+1} \cap Q) - y) + g(y, (N_{k_Q} \cap Q) - x). \tag{9.6}$$

Plugging (9.5) into (9.6) yields $\gamma = 0$ since $\phi^U_{k_Q} = |N_{k_Q} \cap Q| \geq \phi^U_{k_Q+1} > |N_{k_Q+1} \cap Q|$ due to definition of k_Q. Then, we infer $g_e = \beta \in \mathbb{R}$ for all $e \in E(Q)$.

Case b): Pick four nodes $x, y, z, t \in Q$ and suppose that $t \in N_1$, $\{x, z\} \subset N_{k_Q}$ and $y \in N_{k_Q+1}$ in π. Obtain a new sc-partition $\pi^* = (N^*_1, \ldots, N^*_k)$ from π by switching z and t. Clearly, $w^{\pi^*} \in P_Q$. Switching x and y in π^* gives

$$g(x, (N^*_{k_Q} \cap Q) - x) + g(y, (N^*_{k_Q+1} \cap Q) - y) = g(x, (N^*_{k_Q+1} \cap Q) - y) + g(y, (N^*_{k_Q} \cap Q) - x).$$

Comparing this equation with (9.6) leads us to (9.5) once more. Now, arguing in a similar manner to the proof of case (*a*) we can obtain the result.

2. *Sufficiency:* Pick an sc-partition $\pi = (N_1, \ldots, N_{k+1})$ where $k = \left\lceil \frac{n}{F_U} \right\rceil$, $|N_i| = |N_i \cap Q| = \phi^U_i$ for $i = 1, \ldots, k_Q$, $|N_{k_Q+1}| = F_L + 1$, $|N_i| = F_L$ for $i = k_Q+2, \ldots, k+1$ and $|N_{k_Q+1} \cap Q| = n_Q$. Clearly, $w^\pi \in P_Q$. Note that such an sc-partition exists since $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi^U_i, F_L, F_U)$ complies with **(FD-2)**. We organize the proof in four steps.

    (1) We first show that $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V-Q)$. Assuming $n_Q < F_L$, pick three arbitrary nodes $u, v, t \in V - Q$ and suppose $\{u, v\} \subset N_{k_Q+1}$ and $t \in N_{k_Q+2}$. By Lemma 6.2, we infer $g_{u,v} = g_{u,t}$. Arbitrariness of these three nodes implies $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V-Q)$.

    Now, assume $n_Q = F_L$ and $k_Q < k - 1$. This latter implies that the number of subcliques in π is strictly larger than $k_Q + 2$ (i.e., $k_Q + 2 < k + 1$). Pick an arbitrary node $u \in V - Q$ and suppose $u \in N_{k_Q+1}$. Shift u from N_{k_Q+1} to N_{k_Q+2} to obtain

    $$g(u, N_{k_Q+1} - u) = g(u, N_{k_Q+2}). \tag{9.7}$$

    Since $k_Q + 2 < k + 1$, we can apply Lemma 6.4 and get $g_{u,x} = g_{u,y}$ for all $x, y \in (V-Q) - u$. Obviously, this result generalizes into $g_e = \alpha \in \mathbb{R}$ for $e \in E(V-Q)$.

(2) Now, we show $g_e = \alpha$ for $e \in E(Q, V-Q)$. Note that, for an arbitrary $u \in V-Q$, the equation (9.7) applies for any value of n_Q, no matter if it is smaller than F_L or not. The previous result $g_e = \alpha$ for $e \in E(V-Q)$ turns (9.7) into

$$g(u, N_{k_Q+1} - u) = F_L \alpha. \tag{9.8}$$

Applying Lemma 6.4 to this equation gives $g_{u,q_1} = g_{u,q_2}$ for all $q_1, q_2 \in Q$, which implies $g_e = \beta \in \mathbb{R}$ for $e \in E(Q, V-Q)$. But inserting this result into (9.8) gives $\beta = \alpha$.

(3) Next, we show $\alpha = 0$. Distribute the nodes in N_{k+1} over $N_{k_Q+1}, \ldots, N_k$ in π (note that, this is possible since $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with **(FD-2)**) and name this new sc-partition as $\tilde{\pi}$. Clearly, $w^{\tilde{\pi}} \in P_Q$. Comparing $g^T w^\pi = h$ and $g^T w^{\pi^*} = h$ yields $\alpha = 0$, because $w^\pi(E(V-Q,V)) < w^{\tilde{\pi}}(E(V-Q,V))$.

(4) Finally we can show that $g_e = \gamma \in \mathbb{R}$ for $e \in E(Q)$ in the same manner as in the proof of part (i).

Necessity: We consider three cases and prove separately:

a) $n_Q > F_L + 1$;

b) $n_Q = F_L + 1$;

c) $n_Q = F_L$ and $k_Q = \left\lceil \frac{n}{F_U} \right\rceil - 1$.

a) When $n_Q > F_L + 1$, P_Q consists of sc-partitions $\pi = (N_1, N_2, \ldots, N_k)$ with $k = \left\lceil \frac{n}{F_U} \right\rceil$, $|N_i| = F_U$ for $i = 1, 2, \ldots, k$, $|N_i \cap Q| = \phi_i^U = F_U$ for $i = 1, 2, \ldots, k_Q$ and $|N_{k_Q+1} \cap Q| = n_Q$. This means, P_Q lies in the intersection of the hyperplanes $w(\delta(u)) = F_U - 1$ for all $u \in V$.

b) Let P_1 be the set of sc-partitions $\pi = (N_1, \ldots, N_{k+1})$ with $k = \left\lceil \frac{n}{F_U} \right\rceil$, $|N_i| = |N_i \cap Q| = \phi_i^U = F_U$ for $i = 1, \ldots, k_Q$, $|N_{k_Q+1}| = |N_{k_Q+1} \cap Q| = F_L + 1$ and $|N_i| = F_L$ for $i = k_Q + 2, \ldots, k$. Let P_2 be the set of sc-partitions $\pi = (N_1, \ldots, N_k)$ with $|N_i| = F_U$ for $i = 1, \ldots, k$, $|N_i \cap Q| = \phi_i^U = F_U$ for $i = 1, \ldots, k_Q$, $|N_{k_Q+1} \cap Q| = n_Q = F_L + 1$. When $n_Q = F_L + 1$, we have $P_Q = P_1 \cup P_2$. In all the solutions in P_1 and P_2, all the nodes in $V - Q$ are always in subcliques with the same size. That is, we have $w(\delta(u)) = w(\delta(v))$ for all $u, v \in V - Q$, which shows that (9.1) is not facet defining in this case.

c) All $w^\pi \in P_Q$ with $\pi = (N_1, N_2, \ldots, N_k)$ has $|N_i| = |N_i \cap Q| = \phi_i^U = F_U$ for $i = 1, \ldots, k_Q$ and $|N_{k_Q+1} \cap Q| = n_Q = F_L$. Let P_1 consist of sc-partitions in P_Q with $k = k_Q + 2$, one of $|N_{k_Q+1}|$ and $|N_{k_Q+2}|$ is F_L and the other is $F_L + 1$; and, let P_2 consist of sc-partitions in P_Q with $k = k_Q + 1$ and $|N_{k_Q+1}| = F_U$. Since $\mathcal{P}^{lu}(n - \sum_{i=1}^{k_Q} \phi_i^U, F_L, F_U)$ complies with **(FD-2)**, we have $P_Q = P_1 \cup P_2$. In other words, $|V - Q| = F_L + 1$ and it is ensured that in any sc-partition of P_Q, at most one node of $V - Q$ is packed in a different subclique than the others (indeed, in the sc-partitions of P_2 it is ensured that all are packed in the same subclique).

Now, let $Q' = V - Q = \{q'_1, q'_2, \dots, q'_{F_L+1}\}$. Both P_1 and P_2 (and hence P_Q) are contained in the face defined by the 2-chorded cycle inequalities, which were introduced in [3] for $\mathcal{P}(n)$. More precisely, P_1 and P_2 are contained in an hyperplane $g^T w = 0$ such that

$$g_{u,v} = \begin{cases} 1, & \text{if } u = q'_i \text{ and } v = q'_{i+1}, \text{ for all } i = 1, 2, \dots, \phi_1^U + 1; \\ -1, & \text{if } u = q'_i \text{ and } v = q'_{i+2}, \text{ for all } i = 1, 2, \dots, \phi_1^U + 1; \\ 0, & \text{otherwise,} \end{cases}$$

where all indices are taken modulo $F_L + 1$.

□

Now, we prove necessary and sufficient conditions for (9.1) to be facet defining for $\mathcal{P}^{lu}$ with dimension $\binom{n}{2} - 1$. Note that in this case, $\mathcal{P}_Q = \mathcal{P}^{lu}$ (i.e., (9.1) constitutes the equality set of $\mathcal{P}^{lu}$) when $Q = V$. So, we assume in the following theorem that $Q \subset V$.

Theorem 9.2. *Suppose dim*$(\mathcal{P}^{lu}) = \frac{n(n-1)}{2} - 1$. *Let Q be a subset of V for which the conditions (i) and (ii) of Lemma 9.4 are not satisfied (i.e. $n_Q > 0$ and $|Q| > \phi_1^U + 1$). The upper general clique inequality (9.1) is not facet defining for $\mathcal{P}^{lu}$ when $(n \bmod F_L) = 1$; when $(n \bmod F_U) = F_U - 1$, it is facet defining for $\mathcal{P}^{lu}$ if and only if*

- $0 < n_Q < F_U - 1$ *and* $k_Q = \left\lceil \frac{n}{F_U} \right\rceil - 2$, *or,*
- $0 < n_Q \leq F_U - 1$ *and* $k_Q < \left\lceil \frac{n}{F_U} \right\rceil - 2$.

Proof. When $(n \bmod F_L) = 1$, P_Q lies in the hyperplanes $w(\delta(u)) = F_L - 1$ for all $u \in V - Q$. When $k_Q = \left\lceil \frac{n}{F_U} \right\rceil - 1$, P_Q lies in the hyperplanes $w(\delta(u)) = F_U - 2$ for all $u \in V - Q$. And, in a similar manner to part (ii) of Theorem 9.1, one can show that (9.1) is not facet defining for $\mathcal{P}^{lu}$ if $n_Q = F_U - 1$ and $k_Q = \left\lceil \frac{n}{F_U} \right\rceil - 2$.

When $(n \bmod F_U) = F_U - 1$, we set $g_{e'} = 0$ for one $e' \in E(V - Q)$. Proceeding in a similar manner to part (ii) of Theorem 9.1, one can show that $g_e = \alpha \in \mathbb{R}$ for all $e \in E - E(Q)$. But then, $g_{e'} = 0$ implies $g_e = 0$ for all $e \in E - E(Q)$. Proceeding like in part (i) of Theorem 9.1, one can show $g_e = \beta \in \mathbb{R}$ for all $e \in E(Q)$. □

Proposition 9.1 states that (9.1) is not facet defining at all when $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$. The following Theorem 9.3 states the necessary and sufficient conditions that make (9.1) facet defining for P^{lu} when its dimension is $\binom{n}{2} - n$.

Proposition 9.1. *The upper general clique inequality (9.1) is not facet defining for $\mathcal{P}^{lu}$ when dim*$(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$.

Proof. In this case, P_Q is contained in the hyperplanes $w(\delta(u)) = F_U - 1$ for all $u \in V$. □

Theorem 9.3. *Suppose $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$. Let*

$$F = \begin{cases} F_L, & \text{if } (n \text{ mod } F_L) = 0, \\ F_U, & \text{if } (n \text{ mod } F_U) = 0; \end{cases}$$

and let $k = \frac{n}{F}$. Let Q be a subset of V which does not comply with conditions (i) and (ii) in Lemma 9.4 (i.e., $n_Q > 0$ and $|Q| > F + 1$). The upper general clique inequality (9.1) is facet defining $\mathcal{P}^{lu}$ if and only if Q complies with one of the following:

1. *$n_Q = 1$ and $2 \leq k_Q < k - 1$, or*
2. *$2 \leq n_Q < F - 1$ and $1 \leq k_Q < k - 1$, or*
3. *$n_Q = F - 1$ and $1 \leq k_Q < k - 2$.*

Proof. If $k_Q = k - 1$,

$$P_Q = \{\mathbf{w} \in \mathcal{P}^{lu} : w_{u,v} = 1\} \qquad \forall u, v \in V - Q.$$

And, when $n_Q = F - 1$ and $k_Q = k - 2$, one can show in a similar manner to the proof of part (ii) of Lemma 9.4 that P_Q is contained in the face defined by the cycle inequalities.

Without loss of generality, let $Q = \{1, 2, \ldots, |Q|\}$ and $V - Q = \{|Q| + 1, \ldots, n\}$. Consider the set $S = \{\{1, n\}, \{2, n\}, \ldots, \{n - 1, n\}, \{1, n - 1\}\}$. Like in the proof of Theorem 8.3, one can fix the values of the g_e variables to 0 for all $e \in S$.

Now, we prove the separately for the parts (i), (ii) and (iii).

1. Pick $x, y, z \in V - Q$ and $p, q \in Q$ arbitrarily. Pick an sc-partition $\pi = (N_1, \ldots, N_k)$ such that $w^\pi \in P_Q$, $n_Q = 1$, $2 \leq k_Q < k - 1$, $p \in N_{k_Q}$, $\{n, z, q\} \subset N_{k_Q+1}$ and $\{x, y\} \subset N_{k_Q+2}$. Applying Lemma 6.3, one can infer that $g_{n,z} + g_{x,y} = g_{n,y} + g_{x,z}$, which leads to $g_{x,y} = g_{x,z}$ since $\{n, z\}, \{n, y\} \in S$. Since, x, y and z are arbitrary nodes, we infer that
$$g_e = \alpha \in \mathbb{R} \qquad \forall e \in E(V - Q). \tag{9.9}$$
Switching n and x in π yields
$$g(x, N_{k_Q+2} - x) = g(x, q) + g(x, N_{k_Q+1} - \{q, n\}). \tag{9.10}$$
If we plug (9.9) into (9.10) we get $g_{x,q} = \alpha$. Since x and q are arbitrary, this implies $g_e = \alpha$ for all $e \in E(Q, V - Q)$. But then since we have set $g_{1,n-1} = 0$, we get $\alpha = 0$.

   Now, switch p and q in π to obtain
$$g(p, N_{k_Q} - p) = g(q, N_{k_Q} - p). \tag{9.11}$$
Pick $q_1 \in N_1$ and $q_2 \in N_{k_Q}$ (note that, $k_Q \geq 2$). Obtain a new sc-partition $\tilde{\pi}$ by switching q_1 and q_2. Switching p and q in $\tilde{\pi}$ yields
$$g(p, N_{k_Q} - \{p, q_2\}) + g_{p,q_1} = g(q, N_{k_Q} - \{p, q_2\}) + g_{q,q_1}.$$
Comparing this equality with (9.11) gives $g_{p,q_1} - g_{q,q_1} = g_{p,q_2} - g_{q,q_2}$, which generalizes into $g_{p,\hat{q}} - g_{q,\hat{q}} = \beta \in \mathbb{R}$ for $\hat{q} \in Q - \{p, q\}$. However, due to (9.11), this implies that $\beta = 0$, which implies further that $g_e = \gamma \in \mathbb{R}$ for $e \in E(Q)$.

2. Proceeding in a similar manner to part (i), one can show that $g_e = 0$ for all $e \in E - E(Q)$. Now, we show $g_e = \beta \in \mathbb{R}$ for all $e \in E(Q)$. Pick three nodes $u, v, x \in Q$ arbitrarily. Pick an sc-partition $\pi = (N_1, \ldots, N_k)$ such that $w^\pi \in P_Q$, $2 \leq n_Q < F - 1$, $1 \leq k_Q < k - 1$, $\{u, x\} \subset N_{k_Q}$ and $\{v, n\} \subset N_{k_Q+1}$ arbitrarily. Now, applying Lemma 6.3 yields $g_{u,v} = g_{u,x}$ since $\{n, v\}, \{n, x\} \in S$, which implies $g_e = \beta \in \mathbb{R}$ for all $e \in E(Q)$.

3. The proof of this part is similar to those of parts (i) and (ii).

□

This theorem suggests that the upper general inequality (9.1 is facet defining for the k-way equi-partition polytope $\mathcal{P}^{k-way}(n, k)$. In fact, Mitchell [5] shows that (9.1) is valid for this polytope. Here in Theorem 9.3, we advance this result one step further and show for the first time that (9.1) is indeed facet defining for $\mathcal{P}^{k-way}(n, k)$. We highlight this contribution by means of the following corollary:

Corollary 9.1. *The upper general clique inequality (9.1) is facet defining for $\mathcal{P}^{k-way}(n, k)$ if and only if*

- $F_L + 2 \leq |Q| \leq n - F - 2$, *and,*
- $(|Q| \; mod \; F) \neq 0$.

10 The Cycle Inequalities

Let $G_C = (V_C, E_C)$ be a cycle in G. The cycle inequality

$$w(E_C) \leq |V_C| - 2 \tag{10.1}$$

is first introduced by Conforti, Rao and Sassano [2] for the equipartition polytope $\mathcal{P}^{equi}(n)$. They show that it is facet defining for $\mathcal{P}^{equi}(n)$ when n is odd and $|V_C| = \lceil \frac{n}{2} \rceil + 1$. In this section, we generalize this result for $\mathcal{P}^{lu}$. In the rest of this section, let ϕ_1^U and ϕ_2^U be defined as in (6.6), and all the indices be taken modulo $|V_C|$. And, let P_C denote the face of $\mathcal{P}^{lu}$ defined by (10.1) corresponding to the cycle G_C.

We first present a lemma that states the validity condition of (10.1) for $\mathcal{P}^{lu}$. We skip the proof because it is trivial.

Lemma 10.1. *The cycle inequality (10.1) is valid for $\mathcal{P}^{lu}$ if and only if $|V_C| \geq \phi_1^U + 1$.*

Now follows the theorems that state the conditions of facetness for $\mathcal{P}^{lu}$.

Theorem 10.1. *Suppose $\mathcal{P}^{lu}$ is full-dimensional and $F_U - F_L \geq 2$. Pick a cycle $G_C = (V_C, E_C)$ with $|V_C| = \phi_1^U + 1$. The cycle inequality (10.1) is facet defining for $\mathcal{P}^{lu}$ if $\phi_2^U \geq \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$ and $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$ is full-dimensional.*

Proof. Let $V_C = \{u_1, u_2, \ldots, u_{|V_C|}\}$ and

$$E_C = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{|V_C|-1}, u_{|V_C|}\}, \{\{u_{|V_C|}, u_1\}\}.$$

Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ with $|N_1| = \phi_1^U$, $N_1 \cap V_C = \{u_2, u_3, \ldots, u_{\phi_1^U+1}\}$ and $N_2 \cap V_C = \{u_1\}$. Clearly, $w^\pi(E_C) = |V_C| - 2$. If $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$ complies with **(FD-1)**, suppose that we have picked π such that $F_L < |N_2| < F_U$ and $F_L < |N_3| < F_U$. Using Lemma 6.1, we can show that $g_{v,v'} = 0$ for two arbitrary nodes v, v' in $V - V_C$. If $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$ complies with **(FD-2)**, **(FD-3)** or **(FD-4)**, suppose that we have picked π such that $|N_2| = |N_3| + 1$. Then, we can apply Lemma 6.2 so as to show that $g_{v,v'} = \alpha \in \mathbb{R}$ for any two arbitrary nodes v, v' in $V - V_C$. But, with **(FD-2)**, **(FD-3)** and **(FD-4)** it is always possible to construct another sc-partition with one less or one more number of subcliques than π (like in step (3) in the proof of part (ii) in Theorem 9.1). This implies $\alpha = 0$, namely, $g_e = 0$ for $e \in E(V - V_C)$.

Now, pick an arbitrary $v \in V - V_C$ and $u_i \in V_C$. Furthermore, pick an sc-partition $\pi' = (N_1', \ldots, N_k')$ such that $|N_1'| = \phi_1^U$, $N_1' \cap V_C = \{u_1, u_2, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{\phi_1^U+1}\}$, $N_2' \cap V_C = \{u_i\}$, $|N_2'| < F_U$, $|N_3'| > F_L$ and $v \in N_3'$. Note that full-dimensionality of $\mathcal{P}^{lu}(n - \phi_1^U, F_L, F_U)$ ensures existence of such an sc-partition. Shifting v from N_3' to N_2', we get $g_{v,u_i} = 0$, which generalizes into $g_e = 0$ for all $e \in E(V_C, V - V_C)$.

Now, we show $g_{u_i,u_{i+l}} = 0$ for $i = 1, 2, \ldots, |V_C|$ and $l = 2, 3, \ldots, \left\lfloor \frac{|V_C|}{2} \right\rfloor$. Pick an sc-partition $\tilde{\pi} = (\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_k)$ with $|\tilde{N}_1| = \phi_1^U$, $\tilde{N}_1 \cap V_C = \{u_{l'+1}, u_{l'+2}, \ldots, u_{|V_C|}\}$, $|\tilde{N}_2| = \phi_2^U$ and $\tilde{N}_2 \cap V_C = \{u_1, u_2, \ldots, u_{l'}\}$ where $3 \le l' \le \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$ (this is possible even for $l' = \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$ since $\phi_2^U \ge \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$). Clearly, $w^{\tilde{\pi}}(E_C) = |V_C| - 2$. Since $|V_C| = \phi_1^U + 1$ and $|\tilde{N}_1| = \phi_1^U$, there exists $\{v, v'\} \subset \tilde{N}_1 - V_C$. Switching v and u_1 yields

$$g(u_1, \{u_2, \ldots, u_{l'}\}) = g(u_1, \{u_{l'+1}, \ldots, u_{|V_C|}\}). \tag{10.2}$$

Now, consider another sc-partition $\pi^* = (N_1^*, \ldots, N_k^*)$ obtained from $\tilde{\pi}$ by switching $u_{l'}$ and v. Obviously, $w^{\pi^*}(E_C) = |V_C| - 2$. Now, switching u_1 and v' in π^* yields

$$g(u_1, \{u_2, \ldots, u_{l'}\}) - g_{u_1,u_{l'}} = g(u_1, \{u_{l'+1}, \ldots, |V_C|\}) + g_{u_1,u_{l'}}.$$

Comparing this with (10.2), we infer $g_{u_1,u_{l'}} = 0$. By symmetry, the same result can be obtained for any $u_i \in V_C$, that is, $g_{u_i,u_{i+l'-1}} = 0$ for any $i = 1, 2, \ldots, |V_C|$. This implies that $g_e = 0$ for all $e \in E(V_C) - E_C$. But, then (10.2) directly gives $g_{u_1,u_2} = g_{u_1,u_{|V_C|}}$, which generalizes into $g_e = \alpha \in \mathbb{R}$ for $e \in C$ by symmetry.

□

Theorem 10.2. *Suppose $dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. Pick a cycle $G_C = (V_C, E_C)$ with $|V_C| = \phi_1^U + 1$.*

1. *When $(n \bmod F_U) = F_U - 1$ and $F_U - F_L \ge 2$, the cycle inequality (10.1) is facet defining for $\mathcal{P}^{lu}$ if $\phi_2^U \ge \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$.*

2. *When* $(n \bmod F_U) = F_U - 1$ *and* $F_U - F_L = 1$*, the cycle inequality (10.1) is facet defining for* $\mathcal{P}^{lu}$ *if* $F_L \geq 4$ *and* $\phi_2^U \geq \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$.

3. *When* $(n \bmod F_L) = 1$*, the cycle inequality (10.1) is facet defining for* $\mathcal{P}^{lu}$ *if* $F_L \geq 4$ *and* $F_L \geq \left\lfloor \frac{|V_C|}{2} \right\rfloor + 1$.

Proof. We present the proof for case (i) only. The proofs for the other cases are similar. Note that in case (i),

- when $\left\lceil \frac{n}{F_U} \right\rceil = 2$, $\phi_1^U = F_U$; $\phi_2^U = F_U - 1$; and,
- when $\left\lceil \frac{n}{F_U} \right\rceil \geq 3$, $\phi_1^U = \phi_2^U = F_U$.

Let $V_C = \{u_1, u_2, \ldots, u_{|V_C|}\}$ and

$$E_C = \{\{u_1, u_2\}, \{u_2, u_3\}, \ldots, \{u_{|V_C|-1}, u_{|V_C|}\}, \{u_{|V_C|}, u_1\}\}.$$

Since $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, we can set $g_{e'} = 0$ for one $e' \in E(V - V_C)$. Pick three arbitrary nodes x, y and z in $V - V_C$. Pick an sc-partition $\pi = (N_1, \ldots, N_k)$ with, without loss of generality, $|N_1| = F_U$, $|N_2| = F_U - 1$, $N_1 \cap V_C = \{u_1, u_2, \ldots, u_{F_U-2}\}$, $N_2 \cap V_C = \{u_{F_U-1}, u_{F_U}, u_{F_U+1}\}$, $\{x, y\} \subset N_1$ and $t \in N_2$ (note that, $N_2 - V_C$ is non-empty because $|N_2| = F_U - 1 \geq (F_L + 2) - 1 \geq (3 + 2) - 1$, since we assume $F_L \geq 3$). Using Lemma 6.2, we get $g_e = \alpha \in \mathbb{R}$ for $e \in E(V - V_C)$ since x, y and z are arbitrary.

Pick another sc-partition $\pi^* = (N_1^*, \ldots, N_k^*)$ with $|N_1^*| = F_U$, $N_1^* \cap V_C = \{u_1, \ldots, u_{F_U-1}\}$, $N_2^* \cap C = \{u_{F_U}, u_{F_U+1}\}$ and $x \in N_1^*$. Shifting x from N_1^* to N_2^* yields

$$g(x, \{u_1, u_2, \ldots, u_{F_U-1}\}) = (F_U - 3)\alpha + g(x, \{u_{F_U}, u_{F_U+1}\}). \tag{10.3}$$

Now, we obtain a new sc-partition $\tilde{\pi}$ by only switching u_{F_U-1} and u_{F_U+1} in π^*. Shifting x from $\tilde{N}_1$ to $\tilde{N}_2$ in π we get

$$g(x, \{u_1, u_2, \ldots, u_{F_U-2}, u_{F_U+1}\}) = (F_U - 3)\alpha + g(x, \{u_{F_U-1}, u_{F_U}\}).$$

Comparing this with (10.3) we infer $g_{x,u_{F_U-1}} = g_{x,u_{F_U+1}}$, which generalizes into $g_{x,u_i} = g_{x,u_{i-2}}$ for $i = 1, 2, \ldots, |V_C|$ since u_{F_U+1} is arbitrary in the sense that we could have placed any two consecutive nodes of V_C into N_2^*.

When $|V_C|$ is odd, $g_{x,u_i} = g_{x,u_{i-2}}$ for any i implies that $g_e = \beta \in \mathbb{R}$ for $e \in E(V - V_C, V_C)$ since x is arbitrary. When $|V_C|$ is even, we infer $g_{x,u_i} = \beta_1 \in \mathbb{R}$ if i is odd and $g_{x,u_i} = \beta_2 \in \mathbb{R}$ if i is even. Now, we repeat the same procedure using π instead of π^*. Namely, we obtain a new sc-partition π' by switching u_{F_U-2} and u_{F_U+1}. We compare the two equations we obtain by shifting x in π and π' and get $g_{x,u_{F_U-2}} = g_{x,u_{F_U+1}}$, which generalizes into $g_{x,u_i} = g_{x,u_{i-3}}$ since u_{F_U+1} is arbitrary in the sense that we could have placed any three consecutive nodes of V_C into N_2. This shows that $\beta_1 = \beta_2$ when $|V_C|$ is even. In other words, for any $|V_C|$, we have shown $g_e = \beta \in \mathbb{R}$ for $e \in E(V - V_C, V_C)$. Inserting this into (10.3) gives $\alpha = \beta$. But, since we have already set $g_{e'} = 0$ for one $e' \in E(V - V_C)$, we infer $\alpha = \beta = 0$.

We can show $g_e = \gamma \in \mathbb{R}$ for $e \in E_C$ and $g_e = 0$ for $E(V_C) - E_C$ just like we do in Theorem 10.1. □

Theorem 10.3. *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$*. Pick a cycle* $G_C = (V_C, E_C)$ *with* $|V_C| = F_U + 1$*. The cycle inequality (10.1) is facet defining for* $\mathcal{P}^{lu}$ *if and only if* $F_U < 2F_L$*.*

Proof. First assume $F_U < 2F_L$. Let $V_C = \{u_1, u_2, \ldots, u_{F_U+1}\}$ and

$$E_C = \{\{u_1, u_2\}, \ldots, \{u_{F_U}, u_{F_U+1}\}, \{u_{F_U+1}, u_1\}\}.$$

Without loss of generality, suppose $n \notin V_C$. We use the fact that $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$ for setting $g_e = 0$ for $e \in S$ where $S = \{\{1, n\}, \{2, n\}, \ldots, \{n-1, n\}\}$, just like we do in the proof of Theorem 7.5. Pick two arbitrary nodes $x, y \in V - V_C$. Pick an sc-partition $\pi = (N_1, N_2, \ldots, N_k)$ with $|N_i| = F_U$ for $i = 1, 2, \ldots, k$, $N_1 \cap V_C = \{u_1, u_2, \ldots, u_l\}$, $N_2 \cap V_C = \{u_{l+1}, \ldots, u_{F_U+1}\}$, $n \in N_1$ and $\{x, y\} \in N_2$ where $l = \left\lfloor \frac{|V_C|}{2} \right\rfloor$. Switching x and n yields

$$g(x, (N_2 - (V_C \cup x)) + g(x, N_2 \cap V_C) = g(x, N_1 - V_C) + g(x, N_1 \cap V_C). \tag{10.4}$$

Now, switch y and u_1 in π and call this new sc-partition π^*. Switching x and n in π^* yields

$$\begin{aligned} &g(x, N_2 - (V_C \cup x)) - g_{x,y} + g(x, N_2 \cap V_C) + g_{x,u_1} = \\ &g(x, N_1 - V_C) + g_{x,y} + g(x, N_1 \cap V_C) - g_{x,u_1}. \end{aligned}$$

Comparing this equation with (10.4) we obtain $g_{x,y} = g_{x,u_1}$. We can generalize this result to $g_e = \alpha \in \mathbb{R}$ for $e \in E(V - V_C, V)$ since x, y and u_1 are arbitrary.

Next, pick another sc-partition $\tilde{\pi} = (\tilde{N}_1, \tilde{N}_2, \ldots, \tilde{N}_{k+1})$ where $|N_i| = F_L$ for $i = 1, 2, \ldots, k+1$, $N_1 \cap V_C = \{u_1, u_2, \ldots, u_l\}$, $N_2 \cap V_C = \{u_{l+1}, \ldots, u_{F_U+1}\}$ where $l = \left\lfloor \frac{|V_C|}{2} \right\rfloor$. Note that such an sc-partition exists since we assume $F_U < 2F_L$ and $|V_C| = F_U + 1$. Comparing $g(w^{\pi})^T = h$ and $g(w^{\tilde{\pi}})^T = h$, we can infer $\alpha = 0$.

We can show $g_e = \gamma \in \mathbb{R}$ for $e \in E_C$, $g_e = 0$ for $E(V_C) - E_C$ like we do in Theorem 10.1.

Finally, when $F_U \geq 2F_L$, all the points that satisfy the cycle inequality at equality lie on the hyperplanes $w(\delta(u)) = F_U - 1$ for all $u \in V$. □

Theorem 10.4. *Suppose* $dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$*. Pick a cycle* $G_C = (V_C, E_C)$ *with* $|V_C| = F + 1$ *where*

$$F = \begin{cases} F_L, & \text{if } (n \bmod F_L) = 0, \\ F_U, & \text{if } (n \bmod F_U) = 0. \end{cases}$$

The cycle inequality (10.1) is facet defining for $\mathcal{P}^{lu}$*.*

Proof. Let $V_C = \{u_1, u_2, \ldots, u_{F_U+1}\}$ and $E_C = \{\{u_1, u_2\}, \ldots, \{u_{F_U}, u_{F_U+1}\}, \{u_{F_U+1}, u_1\}\}$. Suppose, without loss of generality that $\{n-1, n\} \subset V - V_C$. Like we do in the proof Theorem 8.3, we set $g_e = 0$ for $e \in S = \{\{1, n\}, \{2, n\}, \ldots, \{n-1, n\}, \{1, n-1\}\}$. Then, like in the proof of Theorem 10.3, we can show $g_e = \alpha \in \mathbb{R}$ for $e \in E(V - V_C, V)$. But, of course, combining this with $g_{1,n-1} = 0$, we get $g_e = 0$ for $e \in E(V - V_C, V)$. Finally, we can show $g_e = \gamma \in \mathbb{R}$ for $e \in C$, $g_e = 0$ for $E(V_C) - C$ like we do in Theorem 10.1. □

Mitchell [5] shows that the cycle inequality (10.1) is facet defining for $\mathcal{P}^{equi}(n)$ for even n and $\mathcal{P}^{k-way}(n, k)$. Theorem 10.4 presents an alternative proof for this result.

11 The lower 2-star inequalities

In this section, we introduce the following *lower 2-star inequalities*:

$$(2 + \phi_2^L - \phi_1^L) w_{u,v} + w(\{u,v\}, V - \{u,v\}) \geq \phi_1^L + \phi_2^L - 2 \qquad \forall u, v \in V. \tag{11.1}$$

Let $P_{u,v}$ denote the face defined by the lower 2-star inequality corresponding to the nodes u and v, i.e.

$$P_{u,v} = \{\mathbf{w} \in \mathcal{P}^{lu} : (2 + \phi_2^L - \phi_1^L) w_{u,v} + w(\{u,v\}, V - \{u,v\}) = \phi_1^L + \phi_2^L - 2\}.$$

The following lemma proves that (11.1) is valid for $\mathcal{P}^{lu}$.

Lemma 11.1. *The lower 2-star inequality (11.1) is valid for $\mathcal{P}^{lu}$.*

Proof. Consider two nodes $u, v \in V$ and the corresponding lower 2-star inequality (11.1). Pick an sc-partition $\pi = (N_1, \ldots, N_k)$. First suppose that u and v are packed in the same subset, say N_i, in π. In this case, the left hand side of (11.1) is equal to

$$(2 + \phi_2^L - \phi_1^L) + (2|N_i| - 4), \tag{11.2}$$

which is always larger than or equal to the right hand side of (11.1). Now, suppose that u and v are packed in different subsets, i.e., say $u \in N_i$ and $v \in N_j$. Then, the left hand side of (11.1) is equal to

$$|N_i| + |N_j| - 2, \tag{11.3}$$

which is always larger than or equal to the right hand side of (11.1). □

The next lemma characterizes $P_{u,v}$.

Lemma 11.2. *The characteristic vector w^π of an sc-partition $\pi = (N_1, \ldots, N_k)$ is contained in $P_{u,v}$ if and only if*

- *there exist i^* and j^* such that $|N_{i^*}| = \phi_1^L$ and $|N_{j^*}| = \phi_2^L$,*
- *$|(N_{i^*} \cup N_{j^*}) \cap \{u,v\}| = 2$, and,*
- *$|N_{i^*} \cap \{u,v\}| \geq 1$.*

Proof. Sufficiency is obvious from the proof of Lemma 11.1. For necessity, consider an sc-partition $\pi = (N_1, \ldots, N_k)$. Obviously, if there do not exist i and j such that $|N_i| = \phi_1^L$ and $|N_j| = \phi_2^L$, (11.2) and (11.3) are never equal to the right hand side of (11.1). Similarly, if $|(N_{i^*} \cup N_{j^*}) \cap \{u,v\}| < 2$ or $|N_{i^*} \cap \{u,v\}| = 0$, at least one of (11.2) and (11.3) is different from the right hand side of (11.1). □

As usual, we start by considering the full-dimensional case.

Theorem 11.1. *Suppose that $\mathcal{P}^{lu}$ is full-dimensional. Then the lower 2-star inequalities (11.1) define facets of $\mathcal{P}^{lu}$ if and only if $\phi_2^L \geq \phi_1^L + 2$.*

Proof. *Necessity*: When ϕ_1^L and ϕ_2^L are equal, the lower 2-star inequality (11.1) is obtained by merely adding up the corresponding lower bound inequalities (6.8). When $\phi_2^L = \phi_1^L + 1$, by definition of ϕ_i^L's, we have $\phi_i^L = F_U$ for $i = 3, \ldots, \left\lfloor \frac{n}{F_L} \right\rfloor$. Then,

$$w(E) \leq \binom{\phi_1^L}{2} + \binom{\phi_2^L}{2} + \sum_{i=3}^{k^*} \binom{F_U}{2},$$

where $k^* = \left\lfloor \frac{n}{F_L} \right\rfloor$, is a valid inequality for $\mathcal{P}^{lu}$ and $P_{u,v}$ is contained in the face defined by this valid inequality.

Sufficiency: Pick $x, y, z \in V - \{u, v\}$ arbitrarily; and, pick an sc-partition $\pi = (N_1, \ldots, N_k)$ where $\{v, z\} \subset N_1$, $\{u, x, y\} \subseteq N_2$, $|N_1| = \phi_1^L$ and $|N_2| = \phi_2^L$. Note that $w^\pi \in P_{u,v}$. Obtain a new sc-partition $\pi^* = (N_1^*, N_2^*, \ldots, N_k^*)$ by only switching y and z in π (obviously, $w^{\pi^*} \in P_{u,v}$). Now, shift x from N_2 to N_1 in π to obtain

$$g_{x,u} + g_{x,y} + g(x, N_2 - \{u, x, y\}) = g_{x,z} + g_{x,v} + g(x, N_1 - \{v, z\}). \tag{11.4}$$

Shifting x from N_2^* to N_1^* in π^* yields

$$g_{x,u} + g_{x,z} + g(x, N_2^* - \{u, x, y\}) = g_{x,y} + g_{x,v} + g(x, N_1^* - \{v, z\}).$$

Comparing this with (11.4) gives $g_{x,y} = g_{x,z}$, which implies that $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - \{u, v\})$. Plugging this into (11.4) and rearranging we obtain

$$g_{x,v} - g_{x,u} = (\phi_2^L - \phi_1^L - 1)\alpha. \tag{11.5}$$

Now, consider yet another sc-partition $\tilde{\pi}$ which is obtained by switching u and v in π. Clearly, we still have $w^{\tilde{\pi}} \in P$. If we proceed in the same manner for $\tilde{\pi}$, we get

$$g_{x,u} - g_{x,v} = (\phi_2^L - \phi_1^L - 1)\alpha,$$

which, in comparison with (11.5) yields $\alpha = 0$ since we assume $\phi_2^L - \phi_1^L \geq 2$. Moreover, we also have $g_{x,u} = g_{x,v} = \gamma_x \in \mathbb{R}$ for all $x \in V - \{u, v\}$.

Now, consider $\hat{\pi} = (\hat{N}_1, \hat{N}_2, \ldots, \hat{N}_k)$ with $\{u, v, y\} \subseteq \hat{N}_1$, $x \in \hat{N}_2$, $|\hat{N}_1| = \phi_1^L$ and $|\hat{N}_2| = \phi_2^L$. Clearly, $w^{\hat{\pi}} \in P$. Switching x and y in $\hat{\pi}$ yields

$$g_{x,u} + g_{x,v} = g_{y,u} + g_{y,v}. \tag{11.6}$$

Next, consider yet another sc-partition $\pi' = (N_1', N_2', \ldots, N_k')$ with $\{u, x\} \subset N_1'$, $\{v, y\} \subset N_2'$, $|N_1'| = \phi_1^L$ and $|N_2'| = \phi_2^L$. Clearly, $w^{\pi'} \in P$. Switching x and y in π' yields

$$g_{x,u} + g_{y,v} = g_{x,v} + g_{y,u}. \tag{11.7}$$

Summing up this equation with (11.6) gives $g_{u,x} = g_{u,y}$ and $g_{v,x} = g_{v,y}$, which further implies that $\gamma_x = \gamma_y$ and $g_e = \gamma$ for all $e \in E(\{u, v\}, V - \{u, v\})$.

Finally, switching u and x in $\hat{\pi}$ yields $g_{u,v} = (\phi_2^L - \phi_1^L + 2)\gamma$, which completes the proof. □

Now, we prove the facet defining conditions for the lower 2-star inequalities when $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. In this case, the lower 2-star inequalities are facet defining for $\mathcal{P}^{lu}$ if and only if the lower bound inequalities (6.10) are not facet defining. This complementary facetness property between these two sets of inequalities is a direct consequence of the following theorem and Theorem 6.9.

Theorem 11.2. *Suppose $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. The lower 2-star inequalities (11.1) are facet defining for $\mathcal{P}^{lu}$ if and only if*

- $(n \bmod F_L) = 1$ *and* $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil = 2$, *or,*
- $(n \bmod F_U) = F_U - 1$.

Proof. Using the fact that $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, we set $g_{e'} = 0$ for some $e' \in E(V - \{u, v\})$. Proceeding like in the proof of Theorem 11.1, we can show $g_e = \alpha \in \mathbb{R}$ for all $e \in E(V - \{u, v\})$. But, $g_{e'} = 0$ implies $\alpha = 0$. Now, we can complete the proof proceeding in the same manner as in the proof of Theorem 11.1. When $(n \bmod F_L) = 1$ and $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil \geq 3$, $P_{u,v}$ lies in the corresponding hyperplanes defined by the lower bound inequalities (6.10). □

Now, we state that the lower 2-star inequalities are not facet defining when $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$.

Proposition 11.1. *Suppose $\dim(\mathcal{P}^{lu}) \leq \binom{n}{2} - (n-1)$. The lower 2-star inequalities (11.1) are not facet defining for $\mathcal{P}^{lu}$.*

Proof. When, $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - (n-1)$, $P_{u,v}$ lies in the intersection of the hyperplanes $w(\delta(u)) = F_L - 1$. When $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - n$, the lower 2-star inequalities are just the sum of two hyperplanes from $M(\mathcal{P}^{lu})$. □

12 The upper 2-star inequalities

In this section, we introduce the following *upper 2-star inequalities*:

$$(2 - (\phi_1^U - \phi_2^U))w_{u,v} + w(\{u, v\}, V - \{u, v\}) \leq \phi_1^U + \phi_2^U - 2 \qquad \forall u, v \in V. \tag{12.1}$$

The proofs of the following theorems are very similar to their counterparts in Section 11, so we present them without proofs. Just like in Section 11, when $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$, the upper 2-star inequalities are facets of $\mathcal{P}^{lu}$ if and only if the upper bound inequalities (6.8) are not facet defining. s

Theorem 12.1. *Suppose that $\mathcal{P}^{lu}$ is full-dimensional. Then the upper 2-star inequalities (12.1) define facets of $\mathcal{P}^{lu}$ if and only if $\phi_1^U \geq \phi_2^U + 2$.*

Theorem 12.2. *Suppose $\dim(\mathcal{P}^{lu}) = \binom{n}{2} - 1$. The upper 2-star inequalities (12.1) are facet defining for $\mathcal{P}^{lu}$ if and only if*

- *(n mod F_U) $= F_U - 1$ and* $\left\lfloor \frac{n}{F_L} \right\rfloor = \left\lceil \frac{n}{F_U} \right\rceil = 2$*, or,*

- *(n mod F_L) $= 1$.*

Proposition 12.1. *Suppose* $dim(\mathcal{P}^{lu}) \leq \binom{n}{2} - (n-1)$*. The upper 2-star inequalities (11.1) are not facet defining for* $\mathcal{P}^{lu}$*.*

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