LOWER BOUNDS FOR MEASURABLE CHROMATIC NUMBERS

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ABSTRACT. The Lovász theta function provides a lower bound for the chromatic number of finite graphs based on the solution of a semidefinite program. In this paper we generalize it so that it gives a lower bound for the measurable chromatic number of distance graphs on compact metric spaces.

In particular we consider distance graphs on the unit sphere. There we transform the original infinite semidefinite program into an infinite, two-variable linear program which then turns out to be an extremal question about Jacobi polynomials which we solve explicitly in the limit. As an application we derive new lower bounds for the measurable chromatic number of the Euclidean space in dimensions $10, \ldots, 24$ and we give a new proof that it grows exponentially with the dimension.

1. INTRODUCTION

The *chromatic number of the n-dimensional Euclidean space* is the minimum number of colors needed to color each point of $\mathbb{R}^n$ in such a way that points at distance 1 from each other receive different colors. It is the chromatic number of the graph with vertex set $\mathbb{R}^n$ and in which two vertices are adjacent if they lie at distance 1 from each other. We denote it by $\chi(\mathbb{R}^n)$.

A famous open question is to determine the chromatic number of the plane. In this case, it is only known that $4 \leq \chi(\mathbb{R}^2) \leq 7$, where lower and upper bounds come from simple geometric constructions. In this form the problem was considered, e.g., by E. Nelson, J.R. Isbell, P. Erdős, and H. Hadwiger. For historical remarks and for the best known bounds in other dimensions we refer to L.A. Székely's survey article [20]. The best asymptotic lower bound is due to P. Frankl and R.M. Wilson [9, Theorem 3] and the best asymptotic upper bound is due to D.G. Larman and C.A. Rogers [13]:

$$(1 + o(1))1.2^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n.$$

In this paper we study a variant of the chromatic number of $\mathbb{R}^n$, namely the measurable chromatic number. The *measurable chromatic number* of $\mathbb{R}^n$ is the

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smallest number m such that $\mathbb{R}^n$ can be partitioned into m Lebesgue measurable stable sets. Here we call a set $C \subseteq \mathbb{R}^n$ *stable* if no two points in C lie at distance 1 from each other. In other words, we impose that the sets of points having the same color have to be measurable. We denote the measurable chromatic number of $\mathbb{R}^n$ by $\chi_{\mathrm{m}}(\mathbb{R}^n)$. One reason to study the measurable chromatic number is that then stronger analytic tools are available.

The study of the measurable chromatic number started with K.J. Falconer [8], who proved that $\chi_{\mathrm{m}}(\mathbb{R}^2) \geq 5$. The measurable chromatic number is at least the chromatic number and it is amusing to notice that in case of strict inequality the construction of an optimal coloring necessarily uses the axiom of choice.

Related to the chromatic number of the Euclidean space is the chromatic number of the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : x \cdot x = 1\}$. For $-1 < t < 1$, we consider the graph $G(n,t)$ whose vertices are the points of S^{n-1} and in which two points are adjacent if their inner product $x \cdot y$ equals t. The chromatic number of $G(n,t)$ and its measurable version, denoted by $\chi(G(n,t))$ and $\chi_{\mathrm{m}}(G(n,t))$ respectively, are defined like in the Euclidean case.

The chromatic number of this graph was studied by L. Lovász [15], in particular in the case when t is small. He showed that

$$n \leq \chi(G(n,t)) \quad \text{for } -1 < t < 1,$$
$$\chi(G(n,t)) \leq n+1 \quad \text{for } -1 < t \leq -1/n.$$

P. Frankl and R.M. Wilson [9, Theorem 6] showed that

$$(1 + o(1))(1.13)^n \leq \chi_{\mathrm{m}}(G(n,0)) \leq 2^{n-1}.$$

The (measurable) chromatic number of $G(n,t)$ provides a lower bound for the one of $\mathbb{R}^n$: After appropriate scaling, every proper coloring of $\mathbb{R}^n$ intersected with the unit sphere S^{n-1} gives a proper coloring of the graph $G(n,t)$, and measurability is preserved by the intersection.

In this paper we present a lower bound for the measurable chromatic number of $G(n,t)$. As an application we derive new lower bounds for the measurable chromatic number of the Euclidean space in dimensions $10, \ldots, 24$ and we give a new proof that it grows exponentially with the dimension.

The lower bound is based on a generalization of the Lovász theta function, which L. Lovász introduced in [14] to give upper bounds for the stability number of finite graphs. Using a basic inequality between the stability number and the fractional chromatic number, it also gives a lower bound for the chromatic number. Here we aim at generalizing the theta function to *distance graphs* in compact metric spaces. These are graphs defined on all points of the metric space where the adjacency relation only depends on the distance.

The remaining of the paper is structured as follows: In Section 2 we define the stability number and the fractional measurable chromatic number and give a basic inequality involving them. Then, after reviewing L. Lovász' original formulation of the theta function in Section 3, we give our generalization in Section 4. Like the original theta function for finite graphs, it gives an upper bound for the stability number. Moreover, in the case of the unit sphere, it can be explicitly computed, thanks to classical results on spherical harmonics. The needed material about spherical harmonics is given in Section 5 and an explicit formulation for the theta function of $G(n,t)$ using basic notions from the theory of orthogonal polynomials is given in Section 6.

In Section 7 we choose specific values of t for which we can analytically compute the theta function of $G(n,t)$. This allows us to compute the limit of the theta function for the graph $G(n,t)$ as t goes to 1 in Section 8. This gives improvements on the best known lower bounds for $\chi_{\mathrm{m}}(\mathbb{R}^n)$ in several dimensions. Furthermore this gives a new proof of the fact that $\chi_{\mathrm{m}}(\mathbb{R}^n)$ grows exponentially with n. Although this is an immediate consequence of the result of P. Frankl and R.M. Wilson (and also of a result of P. Frankl and V. Rödl [10]) and our bound of 1.165^n is not an improvement, our result is an easy consequence of the methods we present. Moreover, we think that our proof is of interest because the methods used here are radically different from those of P. Frankl and R.M. Wilson and they can be applied to other metric spaces.

In Section 9 we point out how to apply our generalization to distance graphs in other compact metric spaces, endowed with the continuous action of a compact group. Finally in Section 10 we conclude by showing the relation between our generalization of the theta function and the linear programming bound for spherical codes established by P. Delsarte, J.M. Goethals, and J.J. Seidel [7].

2. THE FRACTIONAL CHROMATIC NUMBER AND THE STABILITY NUMBER

Let $G = (V, E)$ be a finite or infinite graph whose vertex set is equipped with the measure μ. We assume that the measure of V is finite. In this section we define the stability number and the measurable fractional chromatic number of G and derive the basic inequality between these two invariants. In the case of a finite graph one recovers the classical notions if one uses the uniform measure $\mu(C) = |C|$ for $C \subseteq V$.

Let $L^2(V)$ be the Hilbert space of real-valued square-integrable functions defined over V with inner product

$$(f,g) = \int_V f(x)g(x)d\mu(x)$$

for $f, g \in L^2(V)$. The constant function 1 is measurable and its squared norm is the number $(1,1) = \mu(V)$. The characteristic function of a subset C of V we denote by $\chi^C \colon V \to \{0,1\}$.

A subset C of V is called a *measurable stable set* if C is a measurable set and if no two vertices in C are adjacent. The *stability number* of G is

$$\alpha(G) = \sup\{\mu(C) : C \subseteq V \text{ is a measurable stable set}\}.$$

Similar measure-theoretical notions of the stability number have been considered before by other authors for the case in which V is the Euclidean space $\mathbb{R}^n$ or the sphere S^{n-1}. We refer the reader to the survey paper of Székely [20] for more information and further references.

The *fractional measurable chromatic number* of G is denoted by $\chi^*_{\mathrm{m}}(G)$. It is the infimum of $\lambda_1 + \cdots + \lambda_k$ where $k \geq 0$ and $\lambda_1, \ldots, \lambda_k$ are nonnegative real numbers such that there exist measurable stable sets $C_1, \ldots, C_k$ satisfying

$$\lambda_1 \chi^{C_1} + \cdots + \lambda_k \chi^{C_k} = 1.$$

Note that the measurable fractional chromatic number of the graph G is a lower bound for its measurable chromatic number.

Proposition 2.1. *We have the following basic inequality between the stability number and the measurable fractional chromatic number of a graph $G = (V, E)$:*

$$\alpha(G)\chi^*_{\mathrm{m}}(G) \geq \mu(V). \tag{1}$$

So, any upper bound for $\alpha(G)$ provides a lower bound for $\chi^*_{\mathrm{m}}(G)$.

Proof. Let $\lambda_1, \ldots, \lambda_k$ be nonnegative real numbers and $C_1, \ldots, C_k$ be measurable stable sets such that $\lambda_1 \chi^{C_1} + \cdots + \lambda_k \chi^{C_k} = 1$. Since C_i is measurable, its characteristic function χ^{C_i} lies in $L^2(V)$. Hence

$$\begin{aligned}(\lambda_1 + \cdots + \lambda_k)\alpha(G) &\geq \lambda_1 \mu(C_1) + \cdots + \lambda_k \mu(C_k)\\ &= \lambda_1(\chi^{C_1}, 1) + \cdots + \lambda_k(\chi^{C_k}, 1)\\ &= (1, 1)\\ &= \mu(V).\end{aligned}$$

□

3. The Lovász theta function for finite graphs

In the celebrated paper [14] L. Lovász introduced the theta function for finite graphs. It is an upper bound for the stability number which one can efficiently compute using semidefinite programming. In this section we review its definition and properties, which we generalize in Section 4.

The *theta function* of a graph $G = (V, E)$ is defined by

$$\begin{aligned}\vartheta(G) = \max\big\{ \textstyle\sum_{x \in V} \sum_{y \in V} K(x, y) :&\\ &K \in \mathbb{R}^{V \times V} \text{ is positive semidefinite},\\ &\sum_{x \in V} K(x, x) = 1,\\ &K(x, y) = 0 \text{ if } \{x, y\} \in E\big\}.\end{aligned} \tag{2}$$

Theorem 3.1. *For any finite graph G, $\vartheta(G) \geq \alpha(G)$.*

Although this result follows from [14, Lemma 3] and [14, Theorem 4], we give a proof here to stress the analogy between the finite case and the more general case we consider in our generalization of Theorem 4.1.

Proof of Theorem 3.1. Let $C \subseteq V$ be a stable set. Consider the characteristic function $\chi^C \colon V \to \{0, 1\}$ of C and define the matrix $K \in \mathbb{R}^{V \times V}$ by

$$K(x, y) = \frac{1}{|C|}\chi^C(x)\chi^C(y).$$

Notice K satisfies the conditions in (2). Moreover, we have $\sum_{x\in V}\sum_{y\in V} K(x,y) = |C|$, and so $\vartheta(G) \geq |C|$. □

Remark 3.2. *There are many equivalent definitions of the theta function. Possible alternatives are reviewed by D.E. Knuth in* [12]. *We use the one of* [14, Theorem 4].

If the graph G has a nontrivial automorphism group, it is not difficult to see that one can restrict oneself in (2) to the functions K which are invariant under the action of any subgroup Γ of $\mathrm{Aut}(G)$, where $\mathrm{Aut}(G)$ is the *automorphism group* of G, i.e., it is the group of all permutations of V that preserve adjacency. Here we say that K is *invariant under* Γ if $K(\gamma x, \gamma y) = K(x,y)$ holds for all $\gamma \in \Gamma$ and all $x, y \in V$. If moreover Γ acts transitively on G, the second condition $\sum_{x\in V} K(x,x) = 1$ is equivalent to $K(x,x) = 1/|V|$ for all $x \in V$.

4. A GENERALIZATION OF THE LOVÁSZ THETA FUNCTION FOR DISTANCE GRAPHS ON COMPACT METRIC SPACES

We assume that V is a compact metric space with distance function d. We moreover assume that V is equipped with a Borel regular measure μ for which $\mu(V)$ is finite. Let D be a closed subset of the image of d. We define the graph $G(V,D)$ to be the graph with vertex set V and edge set $E = \{\{x,y\} : d(x,y) \in D\}$.

The elements of $L^2(V\times V)$ are called *kernels*. In the following we only consider *symmetric* kernels, i.e., kernels K with $K(x,y) = K(y,x)$ for all $x, y \in V$. We denote by $\mathcal{C}(V\times V)$ the subspace of continuous kernels. A kernel $K \in L^2(V\times V)$ is called *positive* if, for all $f \in L^2(V)$,

$$\int_V \int_V K(x,y) f(x) f(y) d\mu(x) d\mu(y) \geq 0.$$

We are now ready to extend the definition (2) of the Lovász theta function to the graph $G(V,D)$. We define

$$\begin{aligned}\vartheta(G(V,D)) = \sup\Big\{ \int_V\int_V & K(x,y)d\mu(x)d\mu(y) : \\ & K \in \mathcal{C}(V\times V) \text{ is positive}, \\ & \int_V K(x,x)d\mu(x) = 1, \\ & K(x,y) = 0 \text{ if } d(x,y) \in D\Big\}.\end{aligned} \tag{3}$$

Theorem 4.1. *The theta function is an upper bound for the stability number, i.e.,*

$$\vartheta(G(V,D)) \geq \alpha(G(V,D)).$$

Proof. Fix $\varepsilon > 0$ arbitrarily. Let $C \subseteq V$ be a stable set such that $\mu(C) \geq \alpha(G(V,D)) - \varepsilon$. Since μ is regular, we may assume that C is closed, as otherwise we could find a stable set with measure closer to $\alpha(G(V,D))$ and use a suitable inner-approximation of it by a closed set.

Note that, since C is compact and stable, there must exist a number $\beta > 0$ such that $|d(x,y) - \delta| > \beta$ for all $x, y \in C$ and $\delta \in D$. But then, for small enough $\xi > 0$, the set

$$B(C, \xi) = \{x \in V : d(x, C) < \xi\},$$

where $d(x, C)$ is the distance from x to the closed set C, is stable. Moreover, notice that $B(C, \xi)$ is open and that, since it is stable, $\mu(B(C, \xi)) \leq \alpha(G(V, D))$.

Now, the function $f \colon V \to [0, 1]$ given by

$$f(x) = \xi^{-1} \cdot \max\{\xi - d(x, C), 0\}$$

for all $x \in V$ is continuous and such that $f(C) = 1$ and $f(V \setminus B(C, \xi)) = 0$. So the kernel K given by

$$K(x, y) = \frac{1}{(f, f)} f(x) f(y)$$

for all $x, y \in V$ is feasible in (3).

Let us estimate the objective value of K. Since we have

$$(f, f) \leq \mu(B(C, \xi)) \leq \alpha(G(V, D))$$

and

$$\int_V \int_V f(x) f(y) d\mu(x) d\mu(y) \geq \mu(C)^2 \geq (\alpha(G(V, D)) - \varepsilon)^2,$$

we finally have

$$\int_V \int_V K(x, y) d\mu(x) d\mu(y) \geq \frac{(\alpha(G(V, D)) - \varepsilon)^2}{\alpha(G(V, D))}$$

and, since ε is arbitrary, the theorem follows. □

Let us now assume that a compact group Γ acts continuously on V, preserving the distance d. Then, if K is a feasible solution for (3), so is $(x, y) \mapsto K(\gamma x, \gamma y)$ for all $\gamma \in \Gamma$. Averaging on Γ leads to a Γ-invariant feasible solution

$$\overline{K}(x, y) = \int_\Gamma K(\gamma x, \gamma y) d\gamma,$$

where $d\gamma$ denotes the Haar measure on Γ normalized so that Γ has volume 1. Moreover, observe that the objective value of $\overline{K}$ is the same as that of K. Hence we can restrict ourselves in (3) to Γ-invariant kernels. If moreover V is homogeneous under the action of Γ, the second condition in (3) may be replaced by $K(x, x) = 1/\mu(V)$ for all $x \in V$.

We are mostly interested in the case in which V is the unit sphere S^{n-1} endowed with the Euclidean metric of $\mathbb{R}^n$, and in which D is a singleton. If $D = \{\delta\}$ and $\delta^2 = 2 - 2t$, so that $d(x, y) = \delta$ if and only if $x \cdot y = t$, the graph $G(S^{n-1}, D)$ is denoted by $G(n, t)$. Since the unit sphere is homogeneous under the action of the orthogonal group $\mathrm{O}(\mathbb{R}^n)$, the previous remarks apply.

5. Harmonic analysis on the unit sphere

It turns out that the continuous positive kernels on the sphere have a nice description coming from classical results of harmonic analysis reviewed in the this section. This allows us the calculation of $\vartheta(G(n,t))$. For information on spherical harmonics we refer to [1, Chapter 9] and [22].

The unit sphere S^{n-1} is homogeneous under the action of the orthogonal group $\mathrm{O}(\mathbb{R}^n) = \{A \in \mathbb{R}^{n\times n} : A^t A = I_n\}$, where I_n denotes the identity matrix. Moreover, it is two-point homogeneous, meaning that the orbits of $\mathrm{O}(\mathbb{R}^n)$ on pairs of points are characterized by the value of their inner product. The orthogonal group acts on $L^2(S^{n-1})$ by $Af(x) = f(A^{-1}x)$, and $L^2(S^{n-1})$ is equipped with the standard $\mathrm{O}(\mathbb{R}^n)$-invariant inner product

$$(f,g) = \int_{S^{n-1}} f(x)g(x)d\omega(x) \tag{4}$$

for the standard surface measure ω. The surface area of the unit sphere is $\omega_n = (1,1) = 2\pi^{n/2}/\Gamma(n/2)$.

It is a well-known fact (see e.g. [22, Chapter 9.2]) that the space $\mathcal{C}(S^{n-1})$ of continuous functions decomposes under the action of $\mathrm{O}(\mathbb{R}^n)$ into a Hilbert space direct sum as follows:

$$\mathcal{C}(S^{n-1}) = H_0 \perp H_1 \perp H_2 \perp \ldots, \tag{5}$$

where H_k is isomorphic to the $\mathrm{O}(\mathbb{R}^n)$-irreducible space

$$\mathrm{Harm}_k = \left\{ f \in \mathbb{R}[x_1,\ldots,x_n] : f \text{ homogeneous}, \deg f = k, \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f = 0 \right\}$$

of harmonic polynomials in n variables which are homogeneous and have degree k. We set $h_k = \dim(\mathrm{Harm}_k) = \binom{n+k-1}{n-1} - \binom{n+k-3}{n-1}$. The equality in (5) means that every $f \in \mathcal{C}(S^{n-1})$ can be uniquely written in the form $f = \sum_{k=0}^{\infty} p_k$, where $p_k \in H_k$, and where the convergence is absolute and uniform.

The *addition formula* (see e.g. [1, Chapter 9.6]) plays a central role in the characterization of $\mathrm{O}(\mathbb{R}^n)$-invariant kernels: For any orthonormal basis $e_{k,1},\ldots,e_{k,h_k}$ of H_k and for any pair of points $x, y \in S^{n-1}$ we have

$$\sum_{i=1}^{h_k} e_{k,i}(x)e_{k,i}(y) = \frac{h_k}{\omega_n} P_k^{(\alpha,\alpha)}(x\cdot y), \tag{6}$$

where $P_k^{(\alpha,\alpha)}$ is the normalized Jacobi polynomial of degree k with parameters (α,α), with $P_k^{(\alpha,\alpha)}(1) = 1$ and $\alpha = (n-3)/2$. The *Jacobi polynomials* with parameters (α,β) are orthogonal polynomials for the weight function $(1-u)^\alpha(1+u)^\beta$ on the interval $[-1,1]$. We denote by $P_k^{(\alpha,\beta)}$ the normalized Jacobi polynomial of degree k with normalization $P_k^{(\alpha,\beta)}(1) = 1$.

In [17, Theorem 1] I.J. Schoenberg gave a characterization of the continuous kernels which are positive and $\mathrm{O}(\mathbb{R}^n)$-invariant: They are those which lie in the cone spanned by the kernels $(x,y) \mapsto P_k^{(\alpha,\alpha)}(x\cdot y)$. More precisely, a continuous kernel $K \in \mathcal{C}(S^{n-1} \times S^{n-1})$ is $\mathrm{O}(\mathbb{R}^n)$-invariant and positive if and only if there exist nonnegative real numbers $f_0, f_1, \ldots$ such that K can be written as

$$K(x,y) = \sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(x \cdot y), \tag{7}$$

where the convergence is absolute and uniform.

6. The theta function of $G(n,t)$

We obtain from Section 4 in the case $V = S^{n-1}$, $D = \{\sqrt{2-2t}\}$, and $\Gamma = \mathrm{O}(\mathbb{R}^n)$, the following characterization of the theta function of the graph $G(n,t)$:

$$\begin{aligned} \vartheta(G(n,t)) = \max \Big\{ \int_{S^{n-1}} \int_{S^{n-1}} & K(x,y) d\omega(x) d\omega(y) : \\ & K \in \mathcal{C}(S^{n-1} \times S^{n-1}) \text{ is positive}, \\ & K \text{ is invariant under } \mathrm{O}(\mathbb{R}^n), \\ & K(x,x) = 1/\omega_n \text{ for all } x \in S^{n-1}, \\ & K(x,y) = 0 \text{ if } x \cdot y = t \Big\}. \end{aligned} \tag{8}$$

(It will be clear later that the maximum above indeed exists.)

Corollary 6.1. *We have*

$$\omega_n / \vartheta(G(n,t)) \leq \chi_{\mathrm{m}}^*(G(n,t)).$$

Proof. Immediate from Theorem 4.1 and the considerations in Section 2. □

A result of N.G. de Bruijn and P. Erdős [5] implies that the chromatic number of $G(n,t)$ is attained by a finite induced subgraph of it. So one might wonder if computing the theta function for a finite induced subgraph of $G(n,t)$ could give a better bound than the previous corollary. This is not the case as we will show in Section 10.

The theta function for finite graphs has the important property that it can be computed in polynomial time, in the sense that it can be approximated with arbitrary precision using semidefinite programming. In the following two subsections we are concerned with the computability of the generalization (8).

6.1. **Primal formulation.** First, we apply Schoenberg's characterization (7) of the continuous kernels which are $\mathrm{O}(\mathbb{R}^n)$-invariant and positive. This transforms the original formulation (3), which is a semidefinite programming problem in infinitely many variables having infinitely many constraints, into the following linear programming problem with optimization variables f_k:

$$\vartheta(G(n,t)) = \max\bigl\{\omega_n^2 f_0 : \\ f_k \geq 0 \text{ for } k = 0, 1, \ldots, \\ \sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(1) = 1/\omega_n, \\ \sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(t) = 0\bigr\}, \tag{9}$$

where $\alpha = (n-3)/2$.

To obtain (9) we simplified the objective function in the following way. Because of the orthogonal decomposition (5) and because the subspace H_0 contains only the constant functions, we have

$$\int_{S^{n-1}} \int_{S^{n-1}} \sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(x \cdot y) d\omega(x) d\omega(y) = \omega_n^2 f_0.$$

We refer to (9) as the *primal formulation* of $\vartheta(G(n,t))$.

6.2. **Dual formulation.** A *dual formulation* for (9) is the following linear programming problem in two variables z_1 and z_t with infinitely many constraints:

$$\min\bigl\{z_1/\omega_n : \\ z_1 P_0^{(\alpha,\alpha)}(1) + z_t P_0^{(\alpha,\alpha)}(t) \geq \omega_n^2, \\ z_1 P_k^{(\alpha,\alpha)}(1) + z_t P_k^{(\alpha,\alpha)}(t) \geq 0 \text{ for } k = 1, 2, \ldots\bigr\}. \tag{10}$$

It is easy to check that weak duality holds between (9) and (10), so that the value of any feasible solution of the dual formulation gives an upper bound for the value of any feasible solution of the primal formulation and thus an upper bound for $\vartheta(G(n,t))$. Indeed, let $f_0, f_1, \ldots$ be a feasible solution to (9) and z_1, z_t be a feasible solution to (10). Then,

$$\begin{aligned} z_1/\omega_n &= z_1 \sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(1) + z_t \sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(t) \\ &= f_0(z_1 P_0^{(\alpha,\alpha)}(1) + z_t P_0^{(\alpha,\alpha)}(t)) + \sum_{k=1}^{\infty} f_k (z_1 P_k^{(\alpha,\alpha)}(1) + z_t P_k^{(\alpha,\alpha)}(t)) \\ &\geq \omega_n^2 f_0. \end{aligned}$$

Simplifying (10) using $P_0^{(\alpha,\alpha)} = 1$ and $P_k^{(\alpha,\alpha)}(1) = 1$ we finally obtain the equivalent problem

$$\min\bigl\{z_1/\omega_n : \\ z_1 + z_t \geq \omega_n^2, \\ z_1 + z_t P_k^{(\alpha,\alpha)}(t) \geq 0 \text{ for } k = 1, 2, \ldots\bigr\}. \tag{11}$$

Now, the following theorem characterizes the minimum of (11). Moreover, we have strong duality between the primal formulation and the dual formulation.

Theorem 6.2. *Let $m(t)$ be the minimum of $P_k^{(\alpha,\alpha)}(t)$ for $k = 0, 1, \ldots$ Then the optimal values of* (9) *and* (10) *are equal to*

$$\vartheta(G(n,t)) = \omega_n \frac{m(t)}{m(t) - 1}.$$

Proof. We first claim that the minimum $m(t)$ exists and is negative. Indeed, if $P_k^{(\alpha,\alpha)}(t) \geq 0$ for all $k \geq 1$, then $z_1 = 0$ and $z_t = \omega_n^2$ would be a solution for (10) of value 0. But this is a contradiction, since by weak duality any feasible solution of (10) has value at least $\vartheta(G(n,t)) \geq \alpha(G(n,t)) > 0$. So we know that for some $k \geq 1$, $P_k^{(\alpha,\alpha)}(t) < 0$. This, coupled with the fact that $P_k^{(\alpha,\alpha)}(t)$ goes to zero as k goes to infinity (cf. [1, Chapter 6.6] or [19, Chapter 8.22]), proves the claim.

To compute the optimal value of (11), one may use the geometric approach to solve linear programming problems of two variables. The constraints

$$z_1 + z_t P_k^{(\alpha,\alpha)}(t) \geq 0, \ k = 1, 2, \ldots,$$

correspond to half planes through the origin of $\mathbb{R}^2$. The strongest of these constraints is the one for which $P_k^{(\alpha,\alpha)}(t)$ is as small as possible. In this case the optimal solution is given by the intersection of the two lines

$$z_1 + m(t) z_t = 0 \quad \text{and} \quad z_1 + z_t = \omega_n^2,$$

so that the optimal value is $\omega_n \cdot m(t)/(m(t) - 1)$.

Now, let k^* be such that $m(t) = P_{k^*}^{(\alpha,\alpha)}(t)$. Consider the optimization variables f_k of the primal formulation which are zero everywhere with the exceptions $f_0 = m(t)/(\omega_n(m(t) - 1))$ and $f_{k^*} = -1/(\omega_n(m(t) - 1))$. This provides a feasible solution to (9) of value $\omega_n \cdot m(t)/(m(t) - 1)$. From the weak duality relation, it is also an optimal solution. □

Example 6.3. *The minimum of $P_k^{(\alpha,\alpha)}(0.9999)$ for $\alpha = (24 - 3)/2$ is attained at $k = 1131$ and its value is $-0.00059623\ldots$*

We end this section with one remark about the proof of Theorem 6.2. The fact that $m(t)$ is negative can also be seen as a statement about orthogonal polynomials. Then, it is more natural to argue as follows: Let k be the smallest degree so that $P_k^{(\alpha,\alpha)}(t)$ has a zero in the interval $[t, 1]$. The existence of such a k follows from [19, Theorem 6.1.1]. Because of the interlacing property (see Section 7), we have $P_k^{(\alpha,\alpha)}(t) < 0$ in the case $P_k^{(\alpha,\alpha)}(t) \neq 0$, and $P_{k+1}^{(\alpha,\alpha)}(t) < 0$ whenever $P_k^{(\alpha,\alpha)}(t) = 0$. Moreover, the absolute value of $P_k^{(\alpha,\alpha)}(t)$ goes to zero as k tends to infinity (see [1, Chapter 6.6] or [19, Chapter 8.22]). These two facts together imply the existence and negativity of $m(t)$.

7. ANALYTIC SOLUTIONS

In this section we compute the value

$$m(t) = \min\{P_k^{(\alpha,\alpha)}(t) : k = 0, 1, \ldots\}$$

for specific values of t. Namely we choose t to be the largest zero of an appropriate Jacobi polynomial.

Key for the discussion to follow is the *interlacing property* of the zeroes of orthogonal polynomials. It says (cf. [19, Theorem 3.3.2]) that between any pair of consecutive zeroes of $P_k^{(\alpha,\alpha)}$ there is exactly one zero of $P_{k-1}^{(\alpha,\alpha)}$.

We denote the zeros of $P_k^{(\alpha,\beta)}$ by $t_{k,j}^{(\alpha,\beta)}$ with $j = 1, \ldots, k$ and with the increasing ordering $t_{k,j}^{(\alpha,\beta)} < t_{k,j+1}^{(\alpha,\beta)}$. We shall need the following collection of identities:

$$(1-u^2)\frac{d^2P_k^{(\alpha,\alpha)}}{du^2} - (2\alpha+2)u\frac{dP_k^{(\alpha,\alpha)}}{du} + k(k+2\alpha+1)P_k^{(\alpha,\alpha)} = 0, \tag{12}$$

$$(-1)^k P_k^{(\alpha,\alpha)}(-u) = P_k^{(\alpha,\alpha)}(u), \tag{13}$$

$$(-1)^k(\alpha+1)P_k^{(\alpha,\alpha+1)}(-u) = (k+\alpha+1)P_k^{(\alpha+1,\alpha)}(u), \tag{14}$$

$$(2\alpha+2)\frac{dP_k^{(\alpha,\alpha)}}{du} = k(k+2\alpha+1)P_{k-1}^{(\alpha+1,\alpha+1)}, \tag{15}$$

$$(2\alpha+2)P_k^{(\alpha,\alpha+1)} = (k+2\alpha+2)P_k^{(\alpha+1,\alpha+1)} - kP_{k-1}^{(\alpha+1,\alpha+1)}, \tag{16}$$

$$(2k+2\alpha+2)P_k^{(\alpha+1,\alpha)} = (k+2\alpha+2)P_k^{(\alpha+1,\alpha+1)} + kP_{k-1}^{(\alpha+1,\alpha+1)}, \tag{17}$$

$$(k+\alpha+1)P_k^{(\alpha+1,\alpha)} = (\alpha+1)\frac{P_k^{(\alpha,\alpha)} - P_{k+1}^{(\alpha,\alpha)}}{1-u}. \tag{18}$$

They can all be found in [1, Chapter 6], although with different normalization. Formula (12) is [1, (6.3.9)]; (13) and (14) are [1, (6.4.23)]; (15) is [1, (6.3.8)], (16) is [1, (6.4.21)]; (17) follows by the change of variables $u \mapsto -u$ from (16) and (13), (14); (18) is [1, (6.4.20)]. (18) and (13), (14).

Proposition 7.1. *Let $t = t_{k-1,k-1}^{(\alpha+1,\alpha+1)}$ be the largest zero of the Jacobi polynomial $P_{k-1}^{(\alpha+1,\alpha+1)}$. Then, $m(t) = P_k^{(\alpha,\alpha)}(t)$.*

Proof. We start with the following crucial observation: From (15), t is a zero of the derivative of $P_k^{(\alpha,\alpha)}$. Hence it is a minimum of $P_k^{(\alpha,\alpha)}$ because it is the last extremal value in the interval $[-1,1]$ and because $P_k^{(\alpha+1,\alpha+1)}(1) = 1$, whence (using (15)) $P_k^{(\alpha,\alpha)}(u)$ is increasing on $[t,1]$.

Now we prove that $P_k^{(\alpha,\alpha)}(t) < P_j^{(\alpha,\alpha)}(t)$ for all $j \neq k$ where we treat the cases $j < k$ and $j > k$ separately.

It turns out that the sequence $P_j^{(\alpha,\alpha)}(t)$ is decreasing for $j \leq k$. From (18), the sign of $P_j^{(\alpha,\alpha)}(t) - P_{j+1}^{(\alpha,\alpha)}(t)$ equals the sign of $P_j^{(\alpha+1,\alpha)}(t)$. We have the inequalities

$$t_{j,j}^{(\alpha+1,\alpha)} \leq t_{k-1,k-1}^{(\alpha+1,\alpha)} < t_{k-1,k-1}^{(\alpha+1,\alpha+1)} = t.$$

The first one is a consequence of the interlacing property. From (17) one can deduce that $P_{k-1}^{(\alpha+1,\alpha)}$ has exactly one zero in the interval $[t_{k-2,i-1}^{(\alpha+1,\alpha+1)}, t_{k-1,i}^{(\alpha+1,\alpha+1)}]$ since it changes sign at the extreme points of it, and by the same argument $P_{k-1}^{(\alpha+1,\alpha)}$ has a zero left to $t_{k-1,1}^{(\alpha+1,\alpha+1)}$. Thus, $t_{k-1,k-1}^{(\alpha+1,\alpha)} < t_{k-1,k-1}^{(\alpha+1,\alpha+1)} = t$. So t lies right to the largest zero of $P_j^{(\alpha+1,\alpha)}$ and hence $P_j^{(\alpha+1,\alpha)}(t) > 0$ which shows that $P_j^{(\alpha,\alpha)}(t) - P_{j+1}^{(\alpha,\alpha)}(t) > 0$ for $j < k$.

Let us consider the case $j > k$. The inequality [1, (6.4.19)] implies that

$$\text{(19)} \qquad \text{for all } j > k, \quad P_k^{(\alpha,\alpha)}(t_{k-1,k-1}^{(\alpha+1,\alpha+1)}) < P_j^{(\alpha,\alpha)}(t_{j-1,j-1}^{(\alpha+1,\alpha+1)}).$$

The next observation, which finishes the proof of the lemma, is stated in [1, (6.4.24)] only for the case $\alpha = 0$:

$$\text{(20)} \qquad \text{for all } j \geq 0, \quad \min\{P_j^{(\alpha,\alpha)}(u) : u \in [0,1]\} = P_j^{(\alpha,\alpha)}(t_{j-1,j-1}^{(\alpha+1,\alpha+1)}).$$

To prove it consider

$$g(u) = P_j^{(\alpha,\alpha)}(u)^2 + \frac{1-u^2}{j(j+2\alpha+1)}\Big(\frac{dP_j^{(\alpha,\alpha)}}{du}\Big)^2.$$

Applying (12) in the computation of g' shows that

$$g'(u) = \frac{(4\alpha+2)u}{j(j+2\alpha+1)}\Big(\frac{dP_j^{(\alpha,\alpha)}}{du}\Big)^2.$$

The polynomial g' takes positive values on $[0,1]$ and hence g is increasing on this interval. In particular,

$$g(t_{j-1,i-1}^{(\alpha+1,\alpha+1)}) < g(t_{j-1,i}^{(\alpha+1,\alpha+1)}) \quad \text{for all } i \leq j-1 \text{ with } t_{j-1,i-1}^{(\alpha+1,\alpha+1)} \geq 0,$$

which simplifies to

$$P_j^{(\alpha,\alpha)}(t_{j-1,i-1}^{(\alpha+1,\alpha+1)})^2 < P_j^{(\alpha,\alpha)}(t_{j-1,i}^{(\alpha+1,\alpha+1)})^2 \quad .$$

Since $t_{j-1,i}^{(\alpha+1,\alpha+1)}$ are the local extrema of $P_j^{(\alpha,\alpha)}$, we have proved (20). □

8. New lower bounds for the Euclidean space

In this section we give new lower bounds for the measurable chromatic number of the Euclidean space for dimensions $10, \ldots, 24$. This improves on the previous best known lower bounds due to L.A. Székely and N.C. Wormald [21]. Table 8.1 compares the values. Furthermore we give a new proof that the measurable chromatic number grows exponentially with the dimension.

For this we give a closed expression for $\lim_{t \to 1} m(t)$ which involves the Bessel function J_α of the first kind of order $\alpha = (n-3)/2$ (see e.g. [1, Chapter 4]). The appearance of Bessel functions here is due to the fact that the largest zero of the Jacobi polynomial $P_k^{(\alpha,\alpha)}$ behaves like the first positive zero j_α of the Bessel function J_α. More precisely, it is known [1, Theorem 4.14.1] that, for the largest zero $t_{k,k}^{(\alpha+1,\alpha+1)} = \cos\theta_k$ of the polynomial $P_k^{(\alpha+1,\alpha+1)}$,

$$\lim_{k\to+\infty} k\theta_k = j_{\alpha+1} \tag{21}$$

and, with our normalization (cf. [1, Theorem 4.11.6]),

$$\lim_{k\to+\infty} P_k^{(\alpha,\alpha)}\left(\cos\frac{u}{k}\right) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(u)}{x^\alpha}. \tag{22}$$

Theorem 8.1. *We have*

$$\lim_{t\to 1} m(t) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(j_{\alpha+1})}{(j_{\alpha+1})^\alpha}.$$

Proof. The asymptotic formula for the Jacobi polynomials [19, Theorem 8.21.8] converges uniformly in every open interval contained in $[-1,1]$, hence the function $m(t)$ is continuous in the open interval $(-1,1)$ as then locally it can be written as the minimum of finitely many continuous functions. Hence, by Proposition 7.1 and by the fact that the zeros $t_{k-1,k-1}^{(\alpha+1,\alpha+1)}$ tend to 1,

$$\lim_{t\to 1} m(t) = \lim_{k\to\infty} P_k^{(\alpha,\alpha)}(t_{k-1,k-1}^{(\alpha+1,\alpha+1)}).$$

We estimate the difference

$$|P_k^{(\alpha,\alpha)}(t_{k-1,k-1}^{(\alpha+1,\alpha+1)}) - 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(j_{\alpha+1})}{(j_{\alpha+1})^\alpha}|,$$

that we upper bound by

$$\begin{aligned}&\left|P_k^{(\alpha,\alpha)}(t_{k-1,k-1}^{(\alpha+1,\alpha+1)}) - P_k^{(\alpha,\alpha)}\left(\cos\frac{j_{\alpha+1}}{k}\right)\right| \\ &\quad + \left|P_k^{(\alpha,\alpha)}\left(\cos\frac{j_{\alpha+1}}{k}\right) - 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(j_{\alpha+1})}{(j_{\alpha+1})^\alpha}\right|.\end{aligned}$$

The second term tends to 0 from (22). Define θ_{k-1} by $t_{k-1,k-1}^{(\alpha+1,\alpha+1)} = \cos\theta_{k-1}$. By the mean value theorem we have

$$\begin{aligned}&\left|P_k^{(\alpha,\alpha)}(t_{k-1,k-1}^{(\alpha+1,\alpha+1)}) - P_k^{(\alpha,\alpha)}\left(\cos\frac{j_{\alpha+1}}{k}\right)\right| \\ \leq\ & \Big(\max_{u\in[-1,1]}|\frac{dP_k^{(\alpha,\alpha)}}{du}|\Big)|\cos\theta_{k-1} - \cos\frac{j_{\alpha+1}}{k}| \\ \leq\ & \Big(\max_{u\in[-1,1]}|\frac{dP_k^{(\alpha,\alpha)}}{du}|\Big)\big(\max_{\theta\in I_k}|\sin\theta|\big)|\theta_{k-1} - \frac{j_{\alpha+1}}{k}|,\end{aligned}$$

where I_k denotes the interval with extremes θ_{k-1} and $\frac{j_{\alpha+1}}{k}$. Then, with (21),

$$\begin{aligned}\theta_{k-1} - \frac{j_{\alpha+1}}{k} &= \theta_{k-1} - \frac{j_{\alpha+1}}{k-1} + \frac{j_{\alpha+1}}{k(k-1)} \\ &= \frac{1}{k-1}((k-1)\theta_{k-1} - j_{\alpha+1}) + \frac{j_{\alpha+1}}{k(k-1)} = o\left(\frac{1}{k}\right),\end{aligned}$$

and for all $\theta \in I_k$

$$|\sin\theta| \leq |\theta| \leq \frac{j_{\alpha+1}}{k} + \left|\theta_{k-1} - \frac{j_{\alpha+1}}{k}\right| = O\left(\frac{1}{k}\right).$$

From (15),

$$\max_{u\in[-1,1]} \left|\frac{dP_k^{(\alpha,\alpha)}}{du}\right| \sim k^2.$$

Hence we have proved that

$$\lim_{k\to\infty} \left|P_k^{(\alpha,\alpha)}(t_{k-1,k-1}^{(\alpha+1,\alpha+1)}) - P_k^{(\alpha,\alpha)}\left(\cos\frac{j_{\alpha+1}}{k}\right)\right| = 0. \qquad \square$$

Corollary 8.2. *We have*

$$\chi_{\mathrm{m}}(\mathbb{R}^n) \geq 1 + \frac{(j_{\alpha+1})^\alpha}{2^\alpha\Gamma(\alpha+1)|J_\alpha(j_{\alpha+1})|},$$

where $\alpha = (n-3)/2$. $\square$

We use this corollary to derive new lower bounds for $n = 10, \ldots, 24$. We give them in Table 8.1. For $n = 2, \ldots, 8$ our bounds are worse than the existing ones and for $n = 9$ our bound is 35 which is also the best known one.

n	best lower bound previously known for $\chi_{\mathrm{m}}(\mathbb{R}^n)$	new lower bound for $\chi_{\mathrm{m}}(\mathbb{R}^n)$
10	45	48
11	56	64
12	70	85
13	84	113
14	102	147
15	119	191
16	148	248
17	174	319
18	194	408
19	263	521
20	315	662
21	374	839
22	526	1060
23	754	1336
24	933	1679

TABLE 8.1. Lower bounds for $\chi_{\mathrm{m}}(\mathbb{R}^n)$.

We can also use the corollary to show that our bound is exponential in the dimension. To do so we use the inequalities (cf. [1, (4.14.1)] and [23, Section 15.3, p. 485])

$$j_{\alpha+1} > j_\alpha > \alpha$$

and (cf. [1, (4.9.13)])

$$|J_\alpha(x)| \leq 1/\sqrt{2}$$

to obtain

$$\frac{(j_{\alpha+1})^\alpha}{2^\alpha \Gamma(\alpha+1)|J_\alpha(j_{\alpha+1})|} > \sqrt{2}\frac{\alpha^\alpha}{2^\alpha \Gamma(\alpha+1)},$$

and with Stirling's formula $\Gamma(\alpha+1) \sim \alpha^\alpha e^{-\alpha}\sqrt{2\pi\alpha}$ we have that the exponential term is $\left(\frac{e}{2}\right)^\alpha \sim (1.165)^n$.

9. Further generalizations

In this section we want to go back to our generalization (3) of the theta function and discuss its computation in more general situations than the one of the graph $G(n,t)$ encountered in Section 6. We assume that a compact group Γ acts continuously on V. Then, the computation only depends on the orthogonal decomposition of the space of continuous functions (23).

9.1. **Two-point homogeneous spaces.** First, it is worth noticing that all results in Section 6 are valid — one only has to use the appropriate zonal polynomials and appropriate volumes — for distance graphs in infinite, two-point homogeneous, compact metric spaces where edges are given by exactly one distance. If one considers distance graphs in infinite, compact, two-point homogeneous metric spaces with s distances, then the dual formulation is an infinite linear programming problem in dimension $s+1$.

9.2. **Symmetric spaces.** Next we may consider infinite compact metric spaces V which are not two-point homogeneous but symmetric. Since the space of continuous functions $\mathcal{C}(V)$ still has a multiplicity-free orthogonal decomposition one gets a linear programming bound, but with the additional complication that one has to work with multivariate zonal polynomials. The most prominent case of the Grassmann manifold was considered by the first author in [2] in the context of finding upper bounds for finite codes.

9.3. **General homogeneous spaces.** For the most general case one would have multiplicities m_k in the decomposition of $\mathcal{C}(V)$ which is given by the Peter-Weyl Theorem:

$$\mathcal{C}(V) = (H_{0,1} \perp \ldots \perp H_{0,m_0}) \perp (H_{1,1} \perp \ldots \perp H_{1,m_1}) \perp \ldots, \tag{23}$$

where $H_{k,l}$ are Γ-irreducible subspaces which are equivalent whenever their first index coincides. In this case one uses S. Bochner's characterization of the continuous, Γ-invariant, positive kernels given in [4, Section III] to get the primal formulation of ϑ which yields a true semidefinite programming problem.

10. RELATION TO DELSARTE'S LINEAR PROGRAMMING BOUND

That it is possible to treat all two-point homogeneous spaces simultaneously is similar to the linear programming bounds for finite codes which were established by P. Delsarte in [6] and put into the framework of group representations, which we use here, by G.A. Kabatiansky and V.I. Levenshtein in [11]. In this section we devise an explicit connection between these two bounds. The connection between the linear programming bound and the theta function was already observed by R.J. McEliece, E.R. Rodemich, H.C. Rumsey Jr. in [16] and independently by A. Schrijver in [18] in the case of finite graphs.

As we already pointed out in Remark 3.2, there are many alternative ways to define the theta function for finite graphs and it is somewhat mysterious that they all give the same. For our generalization we used the definition given in [14, Theorem 4] to give an upper bound for the measure of any stable set in $G(n,t)$. Now we generalize the definition given in [14, Theorem 3] to give an upper bound for the maximal cardinality of a finite stable set in the complement of the graph $G(n,t)$. This is the graph on the unit sphere where two points are adjacent whenever their inner product is not equal to t. Similarly we could also in the following consider the graph on the unit sphere where the points are adjacent whenever their inner product is at most t. In this way we recover the linear programming bound and shed some light on the connection between these two definitions.

Let $G = (V, E)$ be a finite graph. By taking the automorphism group $\mathrm{Aut}(G)$ into account, definition [14, Theorem 3] becomes

$$\begin{aligned}\overline{\vartheta}(G) = \min\big\{\lambda : \quad & K \in \mathbb{R}^{V\times V} \text{ is positive semidefinite},\\ & K \text{ is invariant under } \mathrm{Aut}(G),\\ & K(x,x) = \lambda - 1 \text{ for all } x \in V,\\ & K(x,y) = -1 \text{ if } \{x,y\} \in E\big\}.\end{aligned} \tag{24}$$

In other words, $\overline{\vartheta}(G)$ is the smallest largest eigenvalue of any positive semidefinite matrix $K \in \mathbb{R}^{V\times V}$ with $K(x,y) = 1$ whenever $x = y$ or $\{x,y\} \in E$. This gives an upper bound for the cardinality of any stable set in the complement of G, i.e., the graph with vertex set V in which two vertices are adjacent whenever they are not adjacent in G.

Now we generalize (24) for the graph $G(n,t)$ by defining

$$\begin{aligned}\overline{\vartheta}(G(n,t)) = \min\big\{\lambda : \quad & K \in \mathcal{C}(S^{n-1}\times S^{n-1}) \text{ is positive},\\ & K \text{ is invariant under } \mathrm{O}(\mathbb{R}^n),\\ & K(x,x) = \lambda - 1 \text{ for all } x \in S^{n-1},\\ & K(x,y) = -1 \text{ if } x\cdot y = t\big\}.\end{aligned} \tag{25}$$

Proposition 10.1. *Let $C \subseteq S^{n-1}$ be a subset of the unit sphere such that every pair of points in C has inner product t. Then its cardinality is at most $\overline{\vartheta}(G(n,t))$.*

Proof. Let K be a kernel satisfying the conditions in (25). Then, by the positivity of the continuous kernel K it follows by [4, Lemma 1] that for any nonnegative integer m, any points $x_1, \dots, x_m \in S^{n-1}$, and any real numbers $u_1, \dots, u_m$, we have $\sum_{i=1}^m \sum_{j=1}^m K(x_i, x_j) u_i u_j \geq 0$. In particular,

$$0 \leq \sum_{(c,c') \in C^2} K(c, c') = \sum_c K(c, c) + \sum_{c \neq c'} K(c, c') \leq |C| K(c, c) - |C|(|C| - 1),$$

so that $|C| - 1 \leq K(c, c)$ and the statement follows. □

The values $\vartheta(G)$ and $\overline{\vartheta}(G)$ are related by the identity $\vartheta(G)\overline{\vartheta}(G) = |V|$ if G is a finite, homogeneous graph ([14, Theorem 8]), and by the inequality $\vartheta(G)\overline{\vartheta}(G) \geq |V|$ if G is an arbitrary finite graph ([14, Corollary 2]). Using the same arguments as in the proof of Theorem 6.2 one shows that $\overline{\vartheta}(G(n, t)) = (m(t) - 1)/m(t)$. So we have

$$\vartheta(G(n, t))\overline{\vartheta}(G(n, t)) = \omega_n.$$

It follows immediately from the definitions that $\overline{\vartheta}(H) \leq \overline{\vartheta}(G(n, t))$ for any finite induced subgraph H of $G(n, t)$, since one can extract a feasible solution for $\overline{\vartheta}(H)$ from any feasible solution for $\overline{\vartheta}(G(n, t))$. Hence, as we noticed after Corollary 6.1, we cannot improve the bound on the measurable chromatic number of $G(n, t)$ given in Corollary 6.1 by computing the theta function of a finite induced subgraph of it.

We finish by showing how the linear programming bound can be obtained from (25). Consider the graph on the unit sphere where two points are not adjacent whenever their inner product lies between -1 and t. Using Schoenberg's characterization (7) the semidefinite programming problem (25) simplifies to the linear programming problem

$$\begin{aligned} \inf \big\{ \lambda : \quad & f_0 \geq 0, f_1 \geq 0, \dots, \\ & \sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(1) = \lambda - 1, \\ & \sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(u)) = -1 \text{ for all } u \in [-1, t] \big\}. \end{aligned}$$

We can strengthen it by requiring $\sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(u) \leq -1$ for all $u \in [-1, t]$. By restricting $f_0 = 0$ the infimum is not effected. Then, after simplification, we get the linear programming bound (cf. [3, (7)])

$$\begin{aligned} \inf \{ 1 + \sum_{k=1}^{\infty} f_k : \quad & f_1 \geq 0, f_2 \geq 0, \dots, \\ & \sum_{k=1}^{\infty} f_k P_k^{(\alpha,\alpha)}(u) \leq -1 \text{ for all } u \in [-1, t] \big\}, \end{aligned}$$

yielding an upper bound for the maximal number of points on the unit sphere with minimal angular distance $\arccos t$.

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