An Algorithm to Determine the Clique Number of a Split Graph

O.Kettani

email :o_ket1@yahoo.fr

Abstract

In this paper, we propose an algorithm to determine the clique number of a split graph.

Introduction

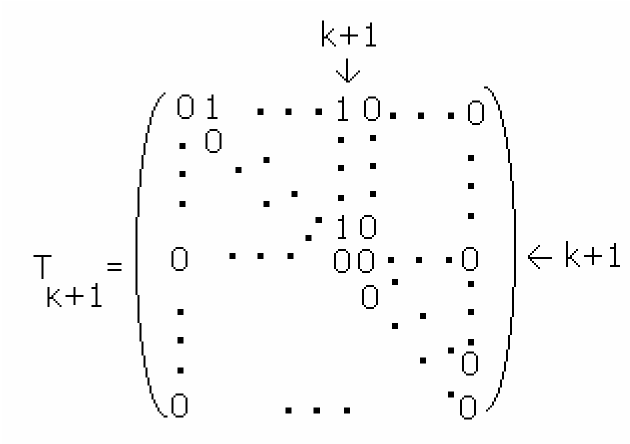
Let G=(V,E) be a split graph on n vertices. A clique C is a subset of V, such that all vertices of C are pairewise adjacent. The clique number $\omega(G)$ is the largest cardinality among all clique of G. Finding the clique number of a given graph is a NP-hard problem[1]. However, this problem can be solved efficiently for some particular class graphs.

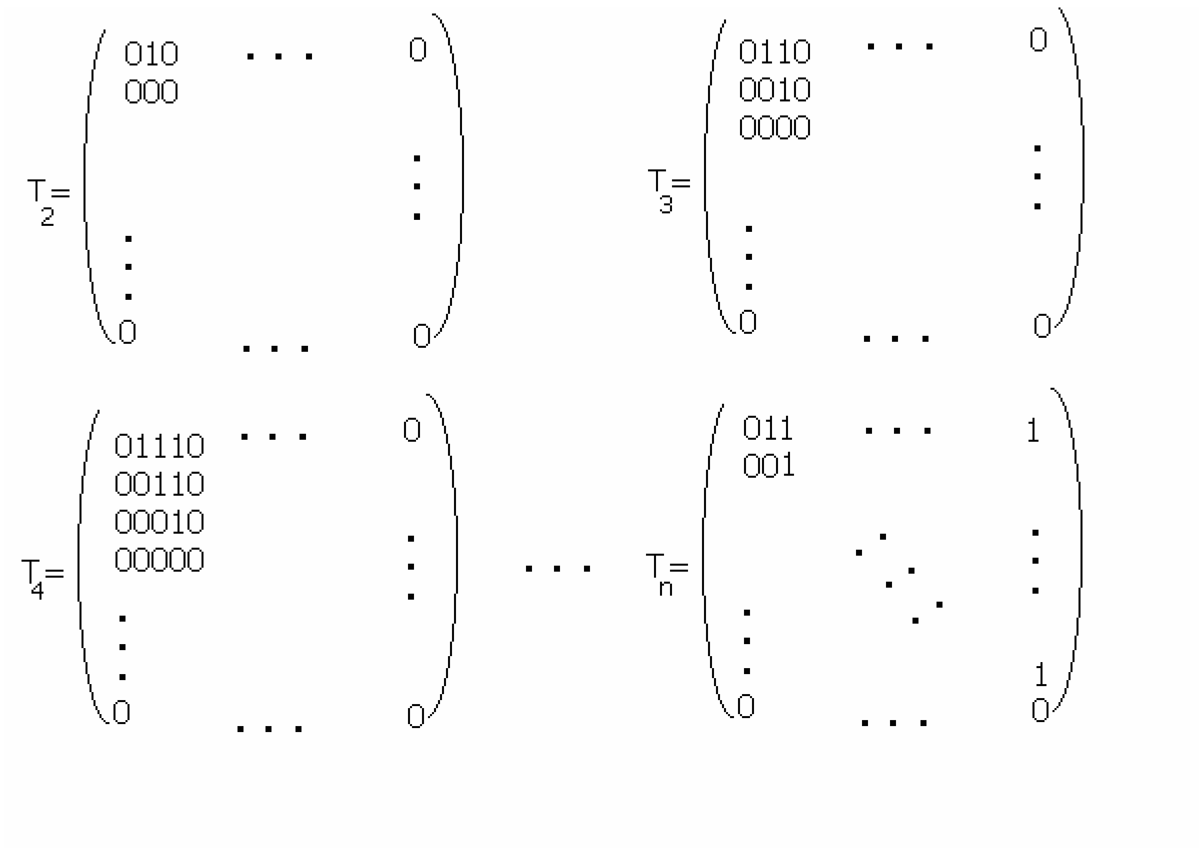
In this paper, we propose an algorithm to determine the clique number of a split graph (graph in which the vertices can be partitioned into a clique and an independent set).

Let n be an integer such $n\geq 3$, for k such $1\leq k\leq n-1$, we define the nxn matrices T_{k+1} by :

$(T_{k+1})_{i,j}=1$ if $1\leq i\leq k$ and $2\leq j\leq k+1$ and $i<j$

$(T_{k+1})_{i,j}=0$ otherwise.





Proposition 1 :

For k such $1 \le k \le n-1$, $(T_{k+1})^{k+1} = \theta$

and $(T_{k+1})^{h} \neq \theta$ if h is such $1 \le h \le k$.

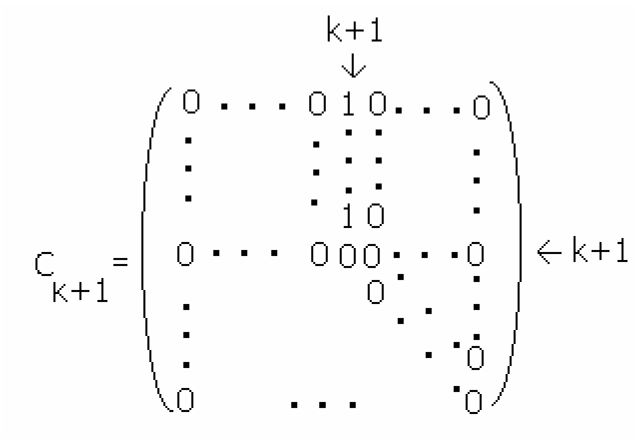
(θ denote the nxn null matrix).

Proof :

For k such $1 \le k \le n-1$, we define the matrix C_{k+1} by :

$(C_{k+1})_{i,j} = 1$ if $j = k+1$ and $1 \le i \le k$

$(C_{k+1})_{i,j} = 0$ otherwise.



For k such $1 \leq k \leq n-1$, we define the set of matrices $\mathcal{M}(k)$ by :

$M \in \mathcal{M}(k)$ iff:

$(M)_{i,j}$ integer>0 if $j=k+1$ and $1 \leq i \leq k$

$(M)_{i,j}=0$ otherwise.

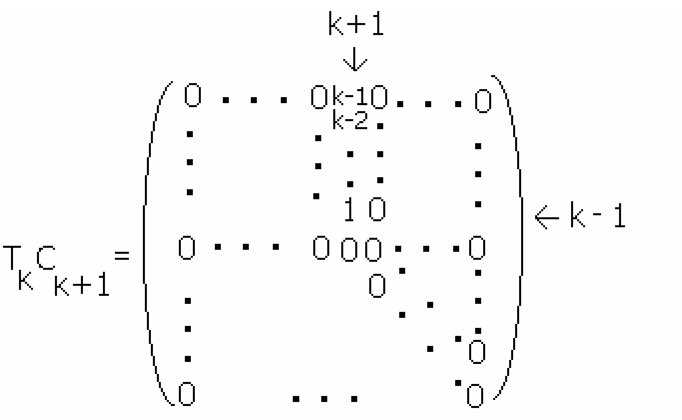
For k such $2 \leq k \leq n-1$, we have :

$T_{k+1}= T_k + C_{k+1}$

then

$(T_{k+1})^2=(T_k)^2+ T_k C_{k+1} + C_{k+1} T_k + (C_{k+1})^2$

since $(C_{k+1})^2=\theta$ and $C_{k+1}T_k=\theta$ and $T_k C_{k+1} \in \mathcal{M}(k-1)$



then

$(T_{k+1})^2=(T_k)^2+ T_k C_{k+1}$ and $T_k C_{k+1} \in \mathcal{M}(k-1)$

$(T_{k+1})^3=(T_k)^3+ T_k(T_k C_{k+1})$

$T_kC_{k+1}\in \mathcal{M}(k-1)\Rightarrow T_k(T_kC_{k+1}) \in \mathcal{M}(k-2)$

then:

$(T_{k+1})^3=(T_k)^3+ (T_k)^2C_{k+1}$ and $(T_k)^2C_{k+1}\in \mathcal{M}(k-2)$

Suppose by induction on p that for p such $k>p\geq 1$:

$(T_{k+1})^p=(T_k)^p+ (T_k)^{p-1}C_{k+1}$ and $(T_k)^{p-1}C_{k+1}\in \mathcal{M}(k-p+1)$

then

$(T_{k+1})^{p+1}=(T_k)^{p+1}+ T_k(T_k)^{p-1} C_{k+1}$ and $T_k(T_k)^{p-1}C_{k+1}\in \mathcal{M}(k-p)$

Then for p such $k>p\geq 1$, we have :

$(T_{k+1})^{p+1}=(T_k)^{p+1}+(T_k)^pC_{k+1}$ and $(T_k)^pC_{k+1}\in \mathcal{M}(k-p)$

for p=k-1 , we have :

$(T_{k+1})^k=(T_k)^k+(T_k)^{k-1}C_{k+1}$ and $(T_k)^{k-1}C_{k+1}\in \mathcal{M}(1)$

since $(T_k)^{k-1}C_{k+1}\in \mathcal{M}(1)\Rightarrow T_k(T_k)^{k-1}C_{k+1}=\theta$

then

$(T_{k+1})^{k+1}=(T_k)^{k+1}+(T_k)^kC_{k+1}$

then

$$(T_{k+1})^{k+1}=(T_k)^{k+1} \qquad (1)$$

$(T_2)^2=\theta$

Suppose by induction on k that : $(T_k)^k=\theta$, then equation (1) implies :

$(T_{k+1})^{k+1}=\theta$

On the other hand, for p such$1\leq p\leq k$, we have:

$(T_{k+1})^p=(T_k)^p+(T_k)^{p-1}C_{k+1}$ and $(T_k)^{p-1}C_{k+1}\in \mathcal{M}(k-p+1)$

since $1\leq p\leq k \Rightarrow k-p+1\geq 1 \Rightarrow (T_k)^{p-1}C_{k+1}\neq\theta$ and $(T_{k+1})^p\neq\theta$ for p such $1\leq p\leq k$.

Then for p such $1\leq p\leq k$, we have:

$(T_{k+1})^p\neq\theta$.

Now, for k such $1\leq k\leq n-2$, we define the matrices B_{k+1} by :

$(B_{k+1})_{i,j} \in\{0,1\}$ if $1\leq i\leq k$ and $k+2\leq j\leq n$

$(B_{k+1})_{i,j}=0$ otherwise.

For k such $1\leq k\leq n-1$, we define the set of matrices $\mathcal{B}(k)$ by :

$M \in \mathcal{B}(k)$ iff:

$(M)_{i,j} \in \{0,1\}$ if $k+1<j\leq n$ and $1\leq i\leq k$

$(M)_{i,j}=0$ otherwise.

Then, we define the matrices S_{k+1} :

$S_{k+1}= T_{k+1}+ B_{k+1}$

Proposition 2 :

For k such $1\leq k\leq n-1$, $(S_{k+1})^{k+1}=\theta$

And $(S_{k+1})^{h} \neq \theta$ if h is such $1\leq h\leq k$.

Proof :

Indeed :

$(S_{k+1})^2= (T_{k+1})^2+ (B_{k+1})^2+ B_{k+1}T_{k+1} +T_{k+1}B_{k+1}$

Since $(B_{k+1})^2=\theta$ and $B_{k+1}T_{k+1}=\theta$ and $T_{k+1}B_{k+1} \in \mathcal{B}(k-1)$

then

$(S_{k+1})^2= (T_{k+1})^2+T_{k+1}B_{k+1}$ and $T_{k+1}B_{k+1} \in \mathcal{B}(k-1)$

Suppose by induction on p such $1\leq p\leq k-1$, that :

$(S_{k+1})^p= (T_{k+1})^p +(T_{k+1})^{p-1}B_{k+1}$ and $(T_{k+1})^{p-1}B_{k+1} \in \mathcal{B}(k-p+1)$

then

$(S_{k+1})^{p+1}=(T_{k+1})^p(T_{k+1}+ B_{k+1})= (T_{k+1})^{p+1} + (T_{k+1})^pB_{k+1}$ and $(T_{k+1})^pB_{k+1} \in \mathcal{B}(k-p)$

Because $(T_{k+1})^{p-1}B_{k+1} \in \mathcal{B}(k-p+1) \Rightarrow (T_{k+1})^pB_{k+1} \in \mathcal{B}(k-p)$.

For p=k-1, we have :

$(S_{k+1})^k=(T_{k+1})^k+ (T_{k+1})^{k-1}B_{k+1}$ and $(T_{k+1})^{k-1}B_{k+1} \in \mathcal{B}(1)$

And

$(T_{k+1})^{k-1}B_{k+1} \in \mathcal{B}(1) \Rightarrow (T_{k+1})^kB_{k+1}=\theta$

Then

$(S_{k+1})^{k+1}=(T_{k+1})^{k+1}=\theta$

And

$(S_{k+1})^p = (T_{k+1})^p + (T_{k+1})^{p-1}B_{k+1}$ and $(T_{k+1})^{p-1}B_{k+1} \in$ B(k-p+1) for p such $1 \le p \le k$,

Since, by proposition 1, $(T_{k+1})^p \neq \theta$ for p such $1 \le p \le k$, then

$(S_{k+1})^p \neq \theta$ for p such $1 \le p \le k$,

we deduce that for k such $1 \le k \le n-1$, we have:

$(S_{k+1})^{k+1} = \theta$ and

$(S_{k+1})^h \neq \theta$ if h is such that $1 \le h \le k$.

Proposition 3:

Let G=(V,E) be a split graph on n vertices. Let U denote the upper triangular matrix extracted from the adjacency matrix of G. Then :

$$\omega(G) = \min_{1 \le h \le n} \{h, \text{ such } (U)^h = \theta\}$$

Proof :

If G contains a maximum clique of size k+1 , then U is congruent to a matrix S_{k+1} such :

$U = P^T S_{k+1} P$

with P is a permutation matrix and P^T is the transpose matrix of P.

then

$(U)^{k+1} = P^T (S_{k+1})^{k+1} P$

Then

$(U)^{k+1} = P^T \theta P = \theta$, by proposition 2.

and

$(U)^{h+1} = P^T (S_{k+1})^{h+1} P \neq \theta$ if h is such that $1 \le h < k$, by proposition 2.

Then :

$$k+1 = \min_{1 \le h \le n} \{h, \text{ such } (U)^h = \theta\}$$

We deduce that:

$$\omega(G) = \min_{1 \le h \le n} \{h, \text{ such } (U)^h = \theta\}$$

Proposition 5 :

Let G=(V,E) be a split graph on n vertices. Let U denote the upper triangular matrix extracted from the adjacency matrix of G. Then $\omega(G)$ can be computed by the following algorithm:

Input : U

1 $k \leftarrow 1$ et $T \leftarrow I$

2 $T \leftarrow TU$

if $T=\theta$ output(k) else $k \leftarrow k+1$ goto step 2.

Example 1 :

G is the complete graph K_n then $U=T_n$ and $\omega(G)=n$.

Example 2 :

G is the graph $K_{1,n}$ then $\omega(G)=2$.

$$U = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & \vdots \\ & & & 1 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

Reference:

[1] M. R. Garey, D. S. Johnson, "Computers and intractability. A guide to the theory of NP-completness".1979