Computational study of a chance constrained portfolio selection problem

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Abstract

We study approximations of chance constrained problems. In particular, we consider the *Sample Average Approximation* (SAA) approach and discuss convergence properties of the resulting problem. A method for constructing bounds for the optimal value of the considered problem is discussed and we suggest how one should tune the underlying parameters to obtain a good approximation of the true problem. We apply these methods to a linear portfolio selection problem with returns following a multivariate lognormal distribution. In addition to the SAA, we also analyze the *Scenario Approximation* approach, which can be regarded as a special case of the SAA method. Our computational results indicate the scenario approximation method gives gives a conservative approximation to the original problem. Interpreting the chance constraint as a Value-at-Risk constraint, we consider another approximation replacing it by the Conditional Value-at-Risk constraint. Finally, we discuss a method to approximate a sum of lognormals that allows us to find a closed expression for the chance constrained problem and compute an efficient frontier for the lognormal case.

Key words: Chance Constraints, Sample Average Approximation, Portfolio Selection, Conditional Value at Risk

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1 Introduction

We consider chance constrainted problems of the form

$$\begin{array}{ll} \operatorname*{Min}_{x\in X} & f(x) \\ \text{s.t.} & \mathsf{Prob}\big\{G(x,\xi)\le 0\big\}\ge 1-\alpha. \end{array} \tag{1.1}$$

Here $X \subset \mathbb{R}^n$, ξ is a random vector[1] with probability distribution P supported on a set $\Xi \subset \mathbb{R}^d$, $\alpha \in (0,1)$, $f : \mathbb{R}^n \to \mathbb{R}$ is a real valued convex function and $G : \mathbb{R}^n \times \Xi \to \mathbb{R}^m$. Chance constrained problems were introduced in Charnes, Cooper and Symmonds [8] and have been extensively studied since. For a theoretical background we may refer to Prékopa [23] where an extensive list of references can be found. Applications of chance constrained programming include, e.g., soil management [29], water management [12] and optimization of chemical processes [14],[15].

Although chance constraints were introduced almost 50 years ago, little progress was made until recently. Even for simple functions $G(\cdot,\xi)$, e.g., linear, problem (1.1) may be extremely difficult to solve numerically. One of the reasons is that for a given $x \in X$ the quantity $\mathsf{Prob}\{G(x,\xi) \le 0\}$ is hard to compute since it requires a multi-dimensional integration. Thus, it may happen that the only way to check feasibility, of a given point $x \in X$, is by Monte-Carlo simulation. In addition, the feasible set of problem (1.1) can be nonconvex even if the set X is convex and the function $G(x,\xi)$ is convex in x. Therefore the development went into two somewhat different directions. One is to discretize the probability distribution P and consequently to solve the obtained combinatorial problem (see, e.g., Dentcheva, Prékopa and Ruszczyński [11], Luedtke, Ahmed and Nemhauser [19]). Another approach is to employ convex approximations of chance constraints (cf., Nemirovski and Shapiro [22]).

In this paper we discuss the *Sample Average Approximation* (SAA) approach to chance constrained problems. Such an approximation is obtained by replacing the actual distribution in chance constraint by an empirical distribution corresponding to a random sample. This approach is well known for stochastic programs with expected values objectives [27]. SAA methods for chance constrained problems have been investigated in [4] and [18]. In this work, we investigate several approximation schemes for a chance constrained portfolio problem with random returns. When the vector of returns follows a multivariate normal distribution, the optimization problem

[1] We use the same notation ξ to denote a random vector and its particular realization. Which of these two meanings will be used in a particular situation will be clear from the context.

becomes a second-order conic program (SOCP) and can be efficiently solved by the available software. However, assuming normality may not be very realistic. We also analyze the case when the returns follow a multivariate lognormal distribution. Since for such distribution no closed form is available, we apply the SAA approach and other methods to approximate the solution.

The remaining of the paper is organized as follows. In Section 2 we provide theoretical background for the SAA approach, showing the convergence of the optimal value of the approximation to the optimal value of the true problem. In addition, following [22] we describe how to construct bounds for the optimal value of chance constrained problems of the form (1.1). In Section 3, we present a chance constrained portfolio selection problem and apply the SAA to obtain upper bounds as well as candidate solutions to the problem. In addition, we present two distinct approximation methods, based on the Conditional Value-at-Risk (CVaR) and on Fenton's approximation. In Section 4 we present the numerical results obtained by applying the approximation schemes to the portfolio selection problem. Section 5 concludes the paper and suggest directions for future research.

We use the following notation throughout the paper. The integer part of number $a \in \mathbb{R}$ is denoted by $\lfloor a \rfloor$. By $\Phi(z)$ we denote the cumulative distribution function of standard normal random variable and by z_α the corresponding critical value, i.e., $\Phi(z_\alpha) = 1 - \alpha$, for $\alpha \in (0, 1)$,

$$B(k; p, N) := \textstyle\sum_{i=0}^{k} \binom{N}{i} p^i (1-p)^{N-i}, \quad k = 0, ..., N, \tag{1.2}$$

denotes the cdf of binomial distribution. For sets $A, B \subset \mathbb{R}^n$ we denote by

$$\mathbb{D}(A, B) := \sup_{x \in A} \operatorname{dist}(x, B) \tag{1.3}$$

the *deviation* of set A from set B.

2 Theoretical background

In order to simplify the presentation we assume in this section that the constraint function $G : \mathbb{R}^n \times \Xi \to \mathbb{R}$ is real valued. Of course, a number of constraints $G_i(x, \xi) \leq 0$, $i = 1, \ldots, m$, can be equivalently replaced by one constraint $G(x, \xi) := \max_{1 \leq i \leq m} G_i(x, \xi) \leq 0$. Such operation of taking maximum preserves convexity of functions $G_i(\cdot, \xi)$. We assume that the set X is *closed*, the function $f(x)$ is *continuous* and the function $G(x, \xi)$ is a *Carathéodory function*, i.e., $G(x, \cdot)$ is measurable for every $x \in \mathbb{R}^n$ and $G(\cdot, \xi)$ continuous for a.e. $\xi \in \Xi$.

Problem (1.1) can be written in the following equivalent form

$$\operatorname*{Min}_{x \in X} f(x) \;\; \text{s.t.}\; p(x) \le \alpha, \tag{2.1}$$

where

$$p(x) := \mathsf{Prob}_P\{G(x,\xi) > 0\}.$$

Now let $\xi^1, \dots, \xi^N$ be an *independent identically distributed* (iid) sample of N realizations of random vector ξ and $P_N := N^{-1}\sum_{j=1}^N \Delta(\xi^j)$ be the respective empirical measure. Here $\Delta(\xi)$ denotes measure of mass one at point ξ, and hence P_N is a discrete measure assigning probability $1/N$ to each point ξ^j, $j = 1, \dots, N$. The sample average approximation $\hat{p}_N(x)$ of function $p(x)$ is obtained by replacing the 'true' distribution P by the empirical measure P_N. That is, $\hat{p}_N(x) := \mathsf{Prob}_{P_N}\{G(x,\xi) > 0\}$. Let $\mathbb{1}_{(0,\infty)} : \mathbb{R} \to \mathbb{R}$ be the indicator function of $(0,\infty)$, i.e.,

$$\mathbb{1}_{(0,\infty)}(t) := \begin{cases} 1, & \text{if } t > 0, \\ 0, & \text{if } t \le 0. \end{cases}$$

Then we can write that $p(x) = \mathbb{E}_P[\mathbb{1}_{(0,\infty)}(G(x,\xi))]$ and

$$\hat{p}_N(x) = \mathbb{E}_{P_N}[\mathbb{1}_{(0,\infty)}(G(x,\xi))] = \frac{1}{N}\sum_{j=1}^N \mathbb{1}_{(0,\infty)}\left(G(x,\xi^j)\right).$$

That is, $\hat{p}_N(x)$ is equal to the proportion of times that $G(x,\xi^j) > 0$. The problem, associated with the generated sample $\xi^1, \dots, \xi^N$, is

$$\operatorname*{Min}_{x \in X} f(x) \;\; \text{s.t.}\; \hat{p}_N(x) \le \gamma. \tag{2.2}$$

We refer to problems (2.1) and (2.2) as the true and SAA problems, respectively, at the respective significance levels α and γ. Note that, following [18], we allow the significance level $\gamma \ge 0$ of the SAA problem to be different from the significance level α of the true problem. Next we discuss the convergence of a solution of the SAA problem (2.2) to that of the true problem (2.1) with respect to the sample size N and the significance level γ. A convergence analysis has been carried out in [18], and here we present similar results under slightly different assumptions.

Recall that a sequence $f_k(x)$ of extended real valued functions is said to *epiconverge* to a function $f(x)$, written $f_k \xrightarrow{e} f$, if for any point x the following two conditions hold: (i) for any sequence x_k converging to x one has

$$\liminf_{k \to \infty} f_k(x_k) \ge f(x), \tag{2.3}$$

(ii) there exists a sequence x_k converging to x such that

$$\limsup_{k\to\infty} f_k(x_k) \le f(x). \tag{2.4}$$

By the (strong) Law of Large Numbers (LLN) we have that for any x, $\hat{p}_N(x)$ converges w.p.1 to $p(x)$.

Proposition 1 *Let $G(x,\xi)$ be a Carathéodory function. Then the functions $p(x)$ and $\hat{p}_N(x)$ are lower semicontinuous, and $\hat{p}_N(x) \xrightarrow{e} p$ w.p.1. Moreover, suppose that for every $\bar{x} \in X$ the set $\{\xi \in \Xi : G(\bar{x},\xi) = 0\}$ has P-measure zero, i.e., $G(\bar{x},\xi) \neq 0$ w.p.1. Then the function $p(x)$ is continuous at every $x \in X$ and $\hat{p}_N(x)$ converges to $p(x)$ w.p.1 uniformly on any compact set $C \subset X$, i.e.,*

$$\sup_{x\in C} |\hat{p}_N(x) - p(x)| \to 0 \ \text{ w.p.1 as } \ N \to \infty. \tag{2.5}$$

Proof. Consider function $\psi(x,\xi) := \mathbb{1}_{(0,\infty)}\big(G(x,\xi)\big)$. Recall that $p(x) = \mathbb{E}_P[\psi(x,\xi)]$ and $\hat{p}_N(x) = \mathbb{E}_{P_N}[\psi(x,\xi)]$. Since the function $\mathbb{1}_{(0,\infty)}(\cdot)$ is lower semicontinuous and $G(\cdot,\xi)$ is a Carathéodory function, it follows that the function $\psi(x,\xi)$ is random lower semicontinuous[2] (see, e.g., [25, Proposition 14.45]). Then by Fatou's lemma we have for any $\bar{x} \in \mathbb{R}^n$,

$$\begin{aligned} \liminf_{x\to\bar{x}} p(x) &= \liminf_{x\to\bar{x}} \int_\Xi \psi(x,\xi)dP(\xi) \\ &\ge \int_\Xi \liminf_{x\to\bar{x}} \psi(x,\xi)dP(\xi) \ge \int_\Xi \psi(\bar{x},\xi)dP(\xi) = p(\bar{x}). \end{aligned}$$

This shows lower semicontinuity of $p(x)$. Lower semicontinuity of $\hat{p}_N(x)$ can be shown in the same way.

The epiconvergence $\hat{p}_N(x) \xrightarrow{e} p$ w.p.1 is a direct implication of Artstein and Wets [2, Theorem 2.3]. Note that, of course, $|\psi(x,\xi)|$ is dominated by an integrable function since $|\psi(x,\xi)| \le 1$.

Suppose, further, that for every $\bar{x} \in X$, $G(\bar{x},\xi) \neq 0$ w.p.1, which implies that $\psi(\cdot,\xi)$ is continuous at $\bar{x}$ w.p.1. Then by the Lebesgue Dominated Convergence Theorem we have for any $\bar{x} \in X$,

$$\begin{aligned} \lim_{x\to\bar{x}} p(x) &= \lim_{x\to\bar{x}} \int_\Xi \psi(x,\xi)dP(\xi) \\ &= \int_\Xi \lim_{x\to\bar{x}} \psi(x,\xi)dP(\xi) = \int_\Xi \psi(\bar{x},\xi)dP(\xi) = p(\bar{x}). \end{aligned}$$

This shows that $p(x)$ is continuous at $x = \bar{x}$. Finally, the uniform convergence (2.5) follows by a version of the uniform Law of Large Numbers (see,

[2]Random lower semicontinuous functions are called normal integrands in [25].

e.g., [27, Proposition 7, p.363]). ∎

By lower semicontinuity of $p(x)$ and $\hat{p}_N(x)$ we have that the feasible sets of the 'true' problem (2.1) and its SAA counterpart (2.2) are closed sets. Therefore, if the set X is bounded (i.e., compact), then problems (2.1) and (2.2) have nonempty sets of optimal solutions denoted, respectively, as S and $\hat{S}_N$, provided that these problems have nonempty feasible sets. We also denote by ϑ^* and $\hat{\vartheta}_N$ the optimal values of the true and the SAA problems, respectively. The following result shows that for $\gamma = \alpha$, under mild regularity conditions, $\hat{\vartheta}_N$ and $\hat{S}_N$ converge w.p.1 to their counterparts of the true problem.

We make the following assumption.

(A) There is an optimal solution $\bar{x}$ of the true problem (2.1) such that for any $\varepsilon > 0$ there is $x \in X$ with $\|x - \bar{x}\| \le \varepsilon$ and $p(x) < \alpha$.

In other words the above condition (A) assumes existence of a sequence $\{x_k\} \subset X$ converging to an optimal solution $\bar{x} \in S$ such that $p(x_k) < \alpha$ for all k, i.e., $\bar{x}$ is an accumulation point of the set $\{x \in X : p(x) < \alpha\}$.

Proposition 2 *Suppose that the significance levels of the true and SAA problems are the same, i.e., $\gamma = \alpha$, the set X is compact, the function $f(x)$ is continuous, $G(x, \xi)$ is a Carathéodory function, and condition* (A) *holds. Then $\hat{\vartheta}_N \to \vartheta^*$ and $\mathbb{D}(\hat{S}_N, S) \to 0$ w.p.1 as $N \to \infty$.*

Proof. By the condition (A), the set S is nonempty and there is $x \in X$ such that $p(x) < \alpha$. We have that $\hat{p}_N(x)$ converges to $p(x)$ w.p.1. Consequently $\hat{p}_N(x) < \alpha$, and hence the SAA problem has a feasible solution, w.p.1 for N large enough. Since $\hat{p}_N(\cdot)$ is lower semicontinuous, the feasible set of SAA problem is compact, and hence $\hat{S}_N$ is nonempty w.p.1 for N large enough. Of course, if x is a feasible solution of an SAA problem, then $f(x) \ge \hat{\vartheta}_N$. Since we can take such point x arbitrary close to $\bar{x}$ and $f(\cdot)$ is continuous, we obtain that

$$\limsup_{N\to\infty} \hat{\vartheta}_N \le f(\bar{x}) = \vartheta^* \text{ w.p.1.} \tag{2.6}$$

Now let $\hat{x}_N \in \hat{S}_N$, i.e., $\hat{x}_N \in X$, $\hat{p}_N(\hat{x}_N) \le \alpha$ and $\hat{\vartheta}_N = f(\hat{x}_N)$. Since the set X is compact, we can assume by passing to a subsequence if necessary that $\hat{x}_N$ converges to a point $\bar{x} \in X$ w.p.1. Also we have that $\hat{p}_N(x) \xrightarrow{e} p$ w.p.1, and hence

$$\liminf_{N\to\infty} \hat{p}_N(\hat{x}_N) \ge p(\bar{x}) \text{ w.p.1.}$$

It follows that $p(\bar{x}) \le \alpha$ and hence $\bar{x}$ is a feasible point of the true problem, and thus $f(\bar{x}) \ge \vartheta^*$. Also $f(\hat{x}_N) \to f(\bar{x})$ w.p.1, and hence

$$\liminf_{N\to\infty} \hat{\vartheta}_N \ge \vartheta^* \ \text{ w.p.1.} \tag{2.7}$$

It follows from (2.6) and (2.7) that $\hat{\vartheta}_N \to \vartheta^*$ w.p.1. It also follows that the point $\bar{x}$ is an optimal solution of the true problem and consequently we obtain that $\mathbb{D}(\hat{S}_N, S) \to 0$ w.p.1. ∎

Condition (A) is essential for the consistency of $\hat{\vartheta}_N$ and $\hat{S}_N$. Think, for example, about a situation where the constraint $p(x) \le \alpha$ defines just one feasible point $\bar{x}$ such that $p(\bar{x}) = \alpha$. Then arbitrary small changes in the constraint $\hat{p}_N(x) \le \alpha$ may result in that the feasible set of the corresponding SAA problem becomes empty. Note also that condition (A) was not used in the proof of inequality (2.7). Verification of condition (A) can be done by ad hoc methods.

Suppose now that $\gamma > \alpha$. Then by Proposition 2 we may expect that with increase of the sample size N, an optimal solution of the SAA problem will approach an optimal solution of the true problem with the significance level γ rather than α. Of course, increasing the significance level leads to enlarging the feasible set of the true problem, which in turn may result in decreasing of the optimal value of the true problem. For a point $\bar{x} \in X$ we have that $\hat{p}_N(\bar{x}) \le \gamma$, i.e., $\bar{x}$ is a feasible point of the SAA problem, iff no more than γN times the event "$G(\bar{x}, \xi^j) > 0$" happens in N trials. Since probability of the event "$G(\bar{x}, \xi^j) > 0$" is $p(\bar{x})$, it follows that

$$\mathsf{Prob}\big\{\hat{p}_N(\bar{x}) \le \gamma\big\} = B\big(\lfloor \gamma N \rfloor; p(\bar{x}), N\big). \tag{2.8}$$

Recall that by Chernoff inequality ([9]) for $k > Np$,

$$B(k; p, N) \ge 1 - \exp\left\{-N(k/N - p)^2/(2p)\right\}.$$

It follows that if $p(\bar{x}) \le \alpha$ and $\gamma > \alpha$, then $1 - \mathsf{Prob}\big\{\hat{p}_N(\bar{x}) \le \gamma\big\}$ approaches zero at a rate of $\exp(-\kappa N)$, where $\kappa := (\gamma - \alpha)^2/(2\alpha)$. Of course, if $\bar{x}$ is an optimal solution of the true problem and $\bar{x}$ is a feasible point of the SAA problem, then $\hat{\vartheta}_N \le \vartheta^*$. That is, if $\gamma > \alpha$, then the probability of the event "$\hat{\vartheta}_N \le \vartheta^*$" approaches one exponentially fast. By similar analysis we have that if $p(\bar{x}) = \alpha$ and $\gamma < \alpha$, then probability that $\bar{x}$ is a feasible point of the corresponding SAA problem approaches zero exponentially fast (see [18]).

The above is a qualitative analysis. For a given candidate point $\bar{x} \in X$, say obtained as a solution of an SAA problem, we would like to validate its quality as a solution of the true problem. This involves two questions, namely whether $\bar{x}$ is a feasible point of the true problem, and if yes, then what is the optimality gap $f(\bar{x}) - \vartheta^*$. Of course, if $\bar{x}$ is a feasible point of the true problem, then $f(\bar{x}) - \vartheta^*$ is nonnegative and is zero iff $\bar{x}$ is an optimal solution of the true problem.

Let us start with verification of feasibility of $\bar{x}$. For that we need to estimate the probability $p(\bar{x})$. We proceed by employing again the Monte Carlo sampling techniques. Generate an iid sample $\xi^1, ..., \xi^N$ and estimate $p(\bar{x})$ by $\hat{p}_N(\bar{x})$. Note that this random sample should be generated independently of a random procedure which produced the candidate solution $\bar{x}$, and that we can use a very large sample since we do not need to solve any optimization problem here. The estimator $\hat{p}_N(\bar{x})$ of $p(\bar{x})$ is unbiased and for large N and not "too small" $p(\bar{x})$ its distribution can be approximated reasonably well by the normal distribution with mean $p(\bar{x})$ and variance $p(\bar{x})(1 - p(\bar{x}))/N$. This leads to the following approximate $(1 - \beta)$-confidence upper bound on $p(\bar{x})$:

$$U_{\beta,N}(\bar{x}) := \hat{p}_N(\bar{x}) + z_\beta \sqrt{\hat{p}_N(\bar{x})(1 - \hat{p}_N(\bar{x}))/N}. \tag{2.9}$$

A more accurate $(1 - \beta)$-confidence upper bound is given by (cf., [22]):

$$U^*_{\beta,N}(\bar{x}) := \sup_{\rho \in [0,1]} \{\rho : B(k; \rho, N) \geq \beta\}, \tag{2.10}$$

where $k := N\hat{p}_N(\bar{x}) = \sum_{j=1}^N \mathbb{1}_{(0,\infty)}\left(G(\bar{x}, \xi^j)\right)$.

In order to get a lower bound for the optimal value ϑ^* we proceed as follows. Let us choose two positive integers M and N, and let

$$\theta_N := B\big(\lfloor \gamma N \rfloor; \alpha, N\big)$$

and L be the *largest* integer such that

$$B(L - 1; \theta_N, M) \leq \beta. \tag{2.11}$$

Next generate M independent samples $\xi^{1,m}, \dots, \xi^{N,m}$, $m = 1, \dots, M$, each of size N, of random vector ξ. For each sample solve the associated optimization problem

$$\operatorname*{Min}_{x \in X} f(x) \ \text{ subject to } \ \sum_{j=1}^N \mathbb{1}_{(0,\infty)}\left(G(x, \xi^{j,m})\right) \leq \gamma N, \tag{2.12}$$

and hence calculate its optimal value $\hat{\vartheta}_N^m$, $m = 1, \dots, M$. That is, solve M times the corresponding SAA problem at the significance level γ. It may happen that problem (2.12) is either infeasible or unbounded from below, in which case we assign its optimal value as $+\infty$ or $-\infty$, respectively. We can view $\hat{\vartheta}_N^m$, $m = 1, \dots, M$, as an iid sample of the random variable $\hat{\vartheta}_N$, where $\hat{\vartheta}_N$ is the optimal value of the respective SAA problem at significance level γ. Next we rearrange the calculated optimal values in the nondecreasing order as follows $\hat{\vartheta}_N^{(1)} \leq \cdots \leq \hat{\vartheta}_N^{(M)}$, i.e., $\hat{\vartheta}_N^{(1)}$ is the smallest, $\hat{\vartheta}_N^{(2)}$ is the second smallest etc, among the values $\hat{\vartheta}_N^m$, $m = 1, \dots, M$. We use the random quantity $\hat{\vartheta}_N^{(L)}$ as a lower bound of the true optimal value ϑ^*. It is possible to show that with probability at least $1 - \beta$, the random quantity $\hat{\vartheta}_N^{(L)}$ is below the true optimal value ϑ^*, i.e., $\hat{\vartheta}_N^{(L)}$ is indeed a lower bound of the true optimal value with confidence at least $1 - \beta$ (see[3] [22]). We will discuss later how to choose the constants M, N and γ.

3 A chance constrained portfolio problem

We start the present section with the formulation of the portfolio chance constrained problem we will work with. We also discuss two possible distributions for the vector of random returns, namely the multivariate normal and multivariate lognormal distribution. In addition, we describe how one can apply the SAA to obtain upper bounds (it is a maximization problem) on the objective function value of the problem and how to generate good candidate solution using several different approaches.

3.1 The model

In the remainder of the paper we study a linear portfolio selection problem, which is a particular case of the chance constrainted problem (1.1). Mathematically, the problem can be formulated as follows:

$$\operatorname*{Max}_{x \in X} \mathbb{E}\left[r^T x\right] \quad \text{subject to} \quad \mathsf{Prob}\left\{r^T x \geq v\right\} \geq 1 - \alpha. \tag{3.13}$$

Here $x \in \mathbb{R}^n$ is vector of decision variables, $r \in \mathbb{R}^n$ is a random (data) vector (with known probability distribution), $v \in \mathbb{R}$ and $\alpha \in (0, 1)$ are constants, e is a vector whose components are all equal to 1 and

$$X := \{x \in \mathbb{R}^n \,:\, e^T x = 1,\, x \geq 0\}.$$

[3]In [22] this lower bound was derived for $\gamma = 0$. It is straightforward to extend the derivations to the case of $\gamma > 0$.

Note that, of course, $\mathbb{E}\left[r^T x\right] = \overline{r}^T x$, where $\overline{r} := \mathbb{E}[r]$ is the corresponding mean vector. That is, the objective function of problem (3.13) is linear and deterministic.

The motivation for studying (3.13) is the portfolio selection problem going back to Markowitz [20]. The vector x represents the percentage of a total wealth of one dollar invested in each of n available assets, r is the vector of random returns of these assets and the decision agent wants to maximize the mean return subject to having a return greater or equal to a desired level v, with probability at least $1-\alpha$.

Our goal is to compare different approximation schemes for chance constrained problems using problem (3.13) as an example. A similar version of the problem was analyzed in [29], where the authors obtained important information about the different policies available in soil management as well as the trade off between net returns and soil loss. We also would like to mention that due to flexibility of stochastic programming modeling, it is possible to impose more constraints on (3.13) in order to make it more realistic, but this is out of the scope of this work.

We will consider two distinct situations, namely when vector of random returns r follows multivariate normal and multivariate lognormal distributions. The two cases are very distinct; on the former one can solve explicitly the chance constraint, while in the latter no explicit formula is known. Under normality, we can compare the quality of the approximations with the true optimal value, while in the lognormal case we have to rely on approximations.

Assume that r follows a multivariate normal distribution with mean vector $\overline{r}$ and covariance matrix Σ, written $r \sim \mathcal{N}(\overline{r}, \Sigma)$. In that case $r^T x \sim \mathcal{N}\left(\overline{r}^T x, x^T \Sigma x\right)$, and hence (as it is well known) the chance constraint in (3.13) can be written as a convex second order conic constraint (SOCC) in the form

$$v - \overline{r}^T x + z_\alpha \sqrt{x^T \Sigma x} \leq 0. \tag{3.14}$$

Using the explicit form (3.14) of the chance constraint, one can efficiently solve the convex problem (3.13) for different values of α. An *efficient frontier* of portfolios can be constructed in an objective function value versus confidence level plot, that is, for every confidence level α we associate the optimal value of problem (3.13). For the portfolio model the efficient frontier approach was already used in Markowitz [20]. A discussion of the subject can be found, e.g., in [10].

3.2 Upper bounds

In [22], the authors develop a method to compute lower bounds of chance constrained problems of the form (1.1). We summarized their procedure at the end of Section 2 leaving the question of how to choose the constants L, M and N. Given β, M and N, it is straightforward to specify L: it is the largest integer that satisfies (2.11). For a given N, the larger M the better because we are approximating the L-th order statistic of the random variable $\hat{\vartheta}_N$. However, note that M represents the number of problems to be solved and this value is often constrained by computational limitations.

In [22] an indication of how N should be chosen is not given. It is possible to gain some insight on the magnitude of N by doing some algebra in inequality (2.11). With $\gamma = 0$, the first term ($i = 0$) of the sum (2.11) is

$$\left[1 - (1-\alpha)^N\right]^M \approx \left[1 - e^{-N\alpha}\right]^M . \tag{3.15}$$

Approximation (3.15) suggests that for small values of α we should take N of order $O(\alpha^{-1})$. If N is much bigger than $1/\alpha$ then we would have to choose a very large M in order to honor inequality (2.11). For instance if $\alpha = 0.10, \beta = 0.01$ and $N = 100$ instead of $N = 1/\alpha = 10$ or $N = 2/\alpha = 20$, we need M to be greater than 100 000 in order to satisfy (2.11), which can be computationally intractable for some problems. If $N = 200$ then M has to be grater then 10^9, which is impractical for most applications.

In [18], the authors applied the same technique to generate bounds on probabilistic versions of the Set Cover Problem and the Transportation Problem. To construct the bounds they varied N and used $M = 10$ and $L = 1$. For many instances they obtained lower bounds slightly smaller (less than 2%) or even equal to the best optimal values generated by the SAA. We will se later in Section 4 that in the portfolio problem, the choice $L = 1$ generated poor upper bounds. Better results were obtained by choosing $N \approx \mathrm{O}(\frac{1}{\alpha})$.

3.3 Candidate solutions

3.3.1 Sample Average Approximation

The *Sample Average Approximation* (SAA) of problem (3.13) can be stated as

$$\begin{array}{ll} \displaystyle\operatorname*{Max}_{x \in X} & \bar{r}^T x \\ \text{subject to} & \hat{p}_N(x) \leq \gamma, \end{array} \tag{3.16}$$

where $\hat{p}_N(x) := N^{-1} \sum_{i=1}^{N} \mathbb{1}_{(0,\infty)}(v - r_i^T x)$ and $\gamma \in [0, 1)$. The reason we use γ instead of α is to suggest that for a fixed α, a different choice of the parameter γ in (3.16) might be suitable. For instance if $\gamma = 0$ we have the so-called *Scenario Approximation*, which in this case is a linear program.

$$\begin{array}{ll} \underset{x \in X}{\text{Max}} & \bar{r}^T x \\ \text{subject to} & r_i^T x \geq v, \;\; i = 1, \ldots, N. \end{array} \tag{3.17}$$

A recent paper by Campi and Garatti [7], building on the work of Calafiore and Campi [6], provides an expression for the probability of an optimal solution $\hat{x}_N$ of the SAA problem (2.2), with $\gamma = 0$, to be infeasible for the true problem (2.1). That is, under the assumptions that the set X and functions $f(\cdot)$ and $G(\cdot, \xi)$, $\xi \in \Xi$, are convex and that w.p.1 the SAA problem attains unique optimal solution, we have that for $N \geq n$,

$$\mathsf{Prob}\,\{p(\hat{x}_N) > \alpha\} \leq B(n-1; \alpha, N), \tag{3.18}$$

and the above bound is tight. We apply this bound to the considered portfolio selection problem to conclude that for a confidence parameter $\beta \in (0, 1)$ and a sample size N^* such that

$$B(n-1; \alpha, N^*) \leq \beta, \tag{3.19}$$

the optimal solution of problem (3.17) is feasible for the corresponding true problem (3.13) with probability at least $1 - \beta$.

For $\gamma \neq 0$, problem (3.16) becomes a mixed-integer linear program, with one binary variable for each sample N.

$$\begin{array}{ll} \underset{x \in X}{\text{Max}} & \bar{r}^T x \\ \text{subject to} & r_i^T x + v Z_i \geq v, \\ & \sum_{i=1}^{N} Z_i \leq N\gamma, \\ & Z_i \in \{0, 1\}. \end{array} \tag{3.20}$$

Given a fixed α in (3.13), it is not clear what is the the best choice of γ for approximation 3.20. It probably is problem-dependent and we suggest solving for a few different values such as $\gamma = \alpha$ and $\gamma = \alpha/2$.

3.3.2 Conditional Value-at-Risk

Given a random variable Y having cumulative distribution function (cdf) $H(\cdot)$, the Value-at-Risk of Y, at confidence level $\alpha \in (0, 1)$, is defined as

$$\mathsf{VaR}_\alpha(Y) := H^{-1}(1 - \alpha).$$

That is, $\mathsf{VaR}_\alpha(Y)$ is the $(1-\alpha)$-quantile of Y. It is immediate to see that the chance constraint of problem (3.13) is equivalent to

$$\mathrm{VaR}_\alpha(v - r^T x) \leq 0. \tag{3.21}$$

The Value-at-Risk is a popular risk measure, but it has been criticized, among other reasons, for not being coherent according to [3]. A risk measure, that in recent years gained increasing popularity, is the Conditional Value-At-Risk measure

$$\mathsf{CVaR}_\alpha(Y) := \inf_{t\in\mathbb{R}} \left\{ t + \alpha^{-1}\mathbb{E}[Y - t]_+ \right\}. \tag{3.22}$$

Unlike the Value-at-Risk, the Conditional Value-At-Risk is a coherent risk measure and the corresponding optimization problem

$$\begin{array}{ll} \underset{x\in X,\, t\in\mathbb{R}}{\text{Max}} & \overline{r}^T x \\ \text{subject to} & \mathsf{CVaR}_\delta(v - r^T x) \leq 0 \end{array} \tag{3.23}$$

is convex and much more tractable than problem (3.13). Note that the significance level δ of the above problem (3.23) is allowed to be different from α and can be tuned. It is well know that $\mathsf{CVaR}_\alpha(Y) \geq \mathsf{VaR}_\alpha(Y)$, thus for $\delta = \alpha$ problem (3.23) provides a conservative approximation for the chance constrained problem (3.13).

For a random vector having continuous distribution, the computation of its CVaR involves a multi-dimensional integral that is usually impossible to compute. Following [24], we take a sample approximation of this integral and reduce the task of computing the CVaR to that of solving a linear program. Applying this methodology to our problem for a confidence level δ and a sample size M, we end up with a conservative approximation to the original problem (3.13):

$$\begin{array}{ll} \underset{x\in X,\, t\in\mathbb{R}}{\text{Max}} & \overline{r}^T x \\ \text{subject to} & t + \frac{1}{M\delta}\sum_{i=1}^M u_i \leq 0, \\ & u_i \geq v - r_i^T x - t, \;\; i = 1, \ldots, M, \\ & u_i \geq 0, \;\; i = 1, \ldots, M. \end{array} \tag{3.24}$$

Under the multivariate normal assumption, closed expressions can be obtained for both VaR and CVaR and we can find an α such that the VaR_α and the CVaR_δ constraints are equivalent. That is, if r has a multivariate normal distribution, then the constraints

$$\mathsf{VaR}_\alpha(v - r^T x) \leq 0 \;\text{ and }\; \mathsf{CVaR}_\delta(v - r^T x) \leq 0$$

are equivalent if we choose δ such that

$$z_\alpha = \frac{1}{\delta\sqrt{2\pi}} e^{-z_\delta^2/2}. \quad (3.25)$$

3.3.3 Fenton's moment matching approach

When the vector of returns r follows a multivariate lognormal distribution, one cannot obtain a closed expression for the chance constraint in problem (3.13) since the distribution of a sum of dependent lognormals is not known. In actuarial and financial theory, the problem of approximating the sum of lognormals is of great interest because several practical situations can be modelled by such a sum. For a discussion on the subject and several examples, see for example [28].

Electrical engineers have been studying the problem for more than 40 years. In many important communication systems (e.g., FM, mobile phone), interference of different signals using the same frequency channel (frequency reuse) is modelled as a sum of lognormals. Several methods ([13], [26], [16], [21]) have been developed and they all assume that the sum of lognormals is approximately a lognormal itself and employ different techniques to tune the parameters of the resulting approximation.

We are going to use an extension of Fenton's method [13] developed in [1]. Despite its simplicity, the method is of similar accuracy compared to more involved methods. The basic idea of Fenton's method is to assume that a sum of n lognormal random variables can be well approximated by another lognormal, that is,

$$L = e^{Y_1} + e^{Y_2} + \cdots + e^{Y_n} \cong e^W, \quad (3.26)$$

where $Y_i \sim \mathcal{N}(\mu_{y_i}, \sigma_{y_i}^2)$ and $W \sim \mathcal{N}(\mu_w, \sigma_w^2)$. We need to determine the parameters μ_w and σ_w^2 to completely characterize the lognormal random variable e^W. Fenton proposed a moment-matching method that allows us to solve a system for the desired parameters. Since we need to estimate only two parameters, it suffices to match the first two moments of the sum and then solve for μ_w and σ_w^2. The original paper [13] only considered the independent case. A complete description on how to estimate those parameters for the dependent case can be found in [1].

In the portfolio problem (3.13), we have a slightly different situation that can be readily converted on an expression similar to (3.26). If the prices of the assets follow a multivariate lognormal distribution $r \sim (e^{Q_1}, e^{Q_2}, \ldots, e^{Q_n})$ with $Q = (Q_1, Q_2, \ldots, Q_n)$ following a multivariate normal distribution with covariance matrix Σ_Q, then the return of a portfolio x is given by

$$r^T x = r_1 x_1 + r_2 x_2 + \cdots + r_n x_n = x_1 e^{Q_1} + x_2 e^{Q_2} + \cdots + x_n e^{Q_n}, \quad (3.27)$$

where $Q_t \sim \mathcal{N}(\mu_{q_i}, \sigma^2_{q_i})$. Observe that $x_i e^{Q_i}$ is a lognormal random variable $e^{Y_i} = e^{Q_i + \ln x_i}$ with parameters $\ln x_i + \mu_{q_i}$ and $\sigma^2_{q_i}$, as long as $x_i > 0$. If $x_i = 0$ then the i-th term of the sum (3.27) vanishes. We can write

$$r^T x = e^{Y_1} + e^{Y_2} + \cdots + e^{Y_n}, \quad (3.28)$$

which is analogous to (3.26). Going back to the portfolio chance constrained problem (3.13), we can apply Fenton's approximation to convert it into an explicit constraint in terms of x as follows.

$$\begin{aligned} &\mathsf{Prob}\left\{r^T x \geq v\right\} \geq 1-\alpha \Leftrightarrow \mathsf{Prob}\left\{e^W \geq v\right\} \geq 1-\alpha \Leftrightarrow \\ &\mathsf{Prob}\left\{W \geq \ln v\right\} \geq 1-\alpha \Leftrightarrow 1-\Phi\left(\frac{\ln v - \mu_w}{\sigma_w}\right) \geq 1-\alpha \Leftrightarrow \\ &\mu_w - \ln v + \sigma_w \Phi^{-1}(\alpha) \geq 0, \end{aligned}$$

where the first equivalence makes use of Fenton's approximation. The resulting problem can be expressed as

$$\begin{array}{ll} \underset{x \in X}{\text{Max}} & \bar{r}^T x \\ \text{subject to} & \mu_w - \ln v + \sigma_w \Phi^{-1}(\alpha) \geq 0, \end{array} \quad (3.29)$$

Let us take a closer look on the constraint of problem (3.29). Following [1], we have

$$2\ln\left(\sum_{i=1}^{t} k_i x_i\right) - 1/2\ln\left(\sum_{i=1}^{t} c_i x_i^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} d_i x_i x_j\right) - \ln(v) +$$
$$\sqrt{\ln\left(\sum_{i=1}^{t} c_i x_i^2 + 2\sum_{i=1}^{t-1}\sum_{j=i+1}^{t} d_i x_i x_j\right) - 2\ln\left(\sum_{i=1}^{t} k_i x_i\right)}\,\Phi^{-1}(\alpha) \geq 0, \quad (3.30)$$

where k_i, c_i and d_i are positive constants. To our knowledge, it is not clear if this constraint is convex, concave or has some special structure. However, numerical experiments indicate that problem (3.29) has a unique solution. We solved several instances with different starting points in the simplex X and they all led to same solution for a given α.

4 Numerical experiments

First we performed numerical experiments for the *Scenario Approximation* problem (3.17) assuming that $r \sim \mathcal{N}(\bar{r}, \Sigma)$. We considered 10 assets ($n = 10$) and the data for the estimation of the parameters was taken from historical monthly returns of 10 US major companies (JP Morgan, Oracle, Intel, Exxon, Wal-Mart, Apple, Sun Microsystems, Microsoft, Yahoo and Procter & Gamble). The sample was generated by the *Triangular Factorization Method* [5]. We wrote the codes in GAMS and solved the linear and binary problems using CPLEX 9.0. The computer was a PC with an Intel Core 2 processor and 2GB of RAM.

Let us fix $\alpha = 0.10$ and $\beta = 0.01$. For these values, the sample size $N^* = 183$ according to (3.19). We ran 20 independent replications of (3.17) for each of the sample sizes $N = 30, 40, \ldots, 200$ and for $N^* = 183$. We also build an *efficient frontier* of optimal portfolios in a objective value versus $\mathsf{Prob}\{r^T x_\alpha \geq v\}$ plot, where x_α is the optimal solution of problem (3.13) for a given α. We show in the same plot (Figure 1) the corresponding objective function values and $\mathsf{Prob}\{r^T \hat{x}_N \geq v\}$ for each optimal solution $\hat{x}_N$ found for the *Scenario Approximation* (3.17). To identify each point with a sample size, we used a gray scale that attributes light tones of gray to smaller sample sizes and darker ones to larger samples. The efficient frontier curve is calculated for $\alpha = 0.8, 0.81, \ldots, 0.99$ and then connected by lines. The vertical and horizontal lines are for reference only: they represent the optimal value for problem (3.13) with $\alpha = 0.10$ and the 90% reliability level, respectively. Figure 1 shows interesting features of the *Scenario Approximation* (3.17). Although larger sample sizes always generate feasible points, the value of the objective function is in general considerably small if compared with the optimal value 1.004311 of problem (3.13) with $\alpha = 0.10$. We also observe the absence of a convergence property: if we increase the sample size, the feasible region of problem (3.17) gets smaller and the approximation becomes more and more conservative and therefore suboptimal. The reason is that for a large sample size we enforce $r_i^T x \geq v$ for several realizations of r.

We performed similar experiments for the lognormal case. For each point obtained in the *Scenario Approximation*, we estimated the probability by Monte-Carlo. To generate samples from a multivariate lognormal distribution, the reader should consult [17] for detailed instructions. Since in the lognormal case one cannot compute the efficient frontier, we also included in Figure 2 the upper bounds for $\alpha = 0.02, \ldots, 0.20$, calculated according to (2.11).

We calculate the upper bounds fixing $\beta = 0.01$ for all cases and by choosing three different values for the constants L, M and N. First we fixed $L = 1$ and $N = \lceil 1/\alpha \rceil$ (upper bound 1 in Figure 2). The constant M was chosen to satisfy the inequality (2.11). The results were not satisfactory, mainly because M ended up being to small. Since the constant M defines the number of samples from $\hat{v}_N$ and since our problem is a linear one, we decided to fix $M = 1\,000$. Then we chose $N = \lceil 1/\alpha \rceil$ (upper bound 2) and $\lceil 2/\alpha \rceil$ (upper bound 3) in the next two experiments. The constant L was chosen to be the largest integer such that (2.11) is satisfied. Figure 2 the generated points for the *Scenario Approximation* along with the three upper bounds. In order to find candidate solutions for problem (3.13) with $\gamma \neq 0$, we need to solve problem (3.16) which is a combinatorial problem usually harder to solve if compared to the *Scenario Approximation*. Since our portfolio problem is a linear one, we still can solve problem (3.16) efficiently for small (e.g. 100 constraints) instances. We performed tests for problem (3.16) fixing $\gamma = 0.05$ and 0.10 and changing N as in the sample approximation case. The results are in Figures 3 and 4.

It is harder to construct upper bounds with $\gamma \neq 0$. The difficulty lies on the appropriate choice of the parameters since we cannot have very large values of M or N when solving binary programs. Based on experiments, a good choice for this problem is $M = 500, N = 100$ and $\gamma = \alpha/2 = 0.05$, which provided a tighter upper bound (upper bound 4 in Figures 3 and 4) than the ones obtained for $\gamma = 0$. However, in many situations one wants an upper bound without much computational effort. If that is the case, it might be appropriate to use equation (2.11) for $\gamma = 0$ only since the corresponding problems could be easier to solve.

For the sake of comparison, we included upper bound 3 obtained for $\gamma = 0$ in Figures 3 and 4. Note that the upper bound for $\gamma = \alpha/2$ is tighter, specially for values of α close to 0.10. The best candidate solutions to problem (3.13) were obtained by choosing $\gamma = 0.05$. Although several points are infeasible to the original problem (1.1), we observe in Figure 3 that whenever a point is feasible it is close to the upper bound. For $\gamma = 0.10$, Figure 4 shows us that almost every generated point is infeasible.

We also run these experiments for the multivariate normal case to confirm these empirical findings, since we have the optimal solutions available in this case. We observe a similar pattern, that is, feasible solutions for $\gamma = 0.05$ are nearly optimal and almost all points generated for $\gamma = 0.10$ are infeasible to the original problem (3.13) as can be seen in Figures 5 and 6. To further justify this claim, note that among the feasible points in Figure 5, more than 70% of them are within 0.2% of the true optimal value 1.004311 of problem (3.13) with $\alpha = 0.10$ If we relax the tolerance to 0.3%, then more than 93% of the points are no more than 0.3% away from the optimal value.

For the CVaR approach, we performed experiments for the multivariate lognormal case fixing $M = 5\,000$ and letting $\delta = 0.03, 0.04, \ldots, 0.20$. Since we do not have an efficient frontier in this case, it is hard to say something about the quality of the solutions. Nevertheless, for $\delta \leq 0.10$ Figure 7 shows several points close to upper bound 4 obtained for $\gamma = 0.05$, which suggests that good candidates can be obtained by employing the CVaR approximation. Following Fenton's approximation, we solved problem (3.29) for $\alpha = 0.01, \ldots, 0.20$. For every optimal solution x_α obtained, we estimated the quantity $p(x) = \mathsf{Prob}\{r^T x \geq v\}$ by Monte-Carlo sampling. Since there is no optimization problem involved in the estimate, we used 50 000 Monte-Carlo samples. As mentioned in Section 2, one can construct upper bounds on $p(x)$ but in this case it seems unnecessary due to the large sample size we used. Figure 8 shows the efficient frontier of (3.29) and the Monte-Carlo estimates for the probability. We included the results of the sample approximation for $\gamma = 0.95$ to test the quality of the solutions obtained by Fenton's approximation. We see that Fenton's method yields a conservative approximation for problem (3.13), as seen by the large number of points above the approximate efficient frontier.

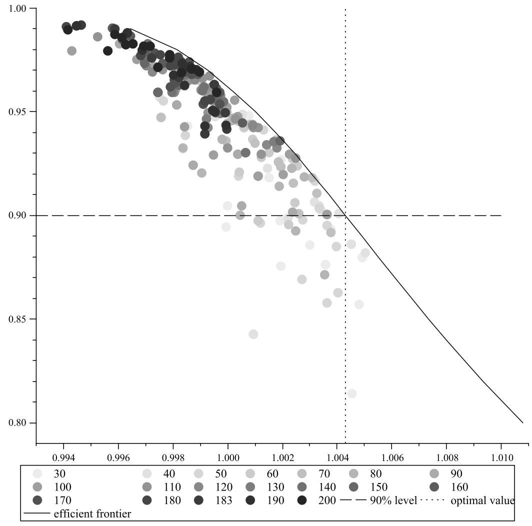

Figure 1: SAA with $\gamma = 0$ and efficient frontier.

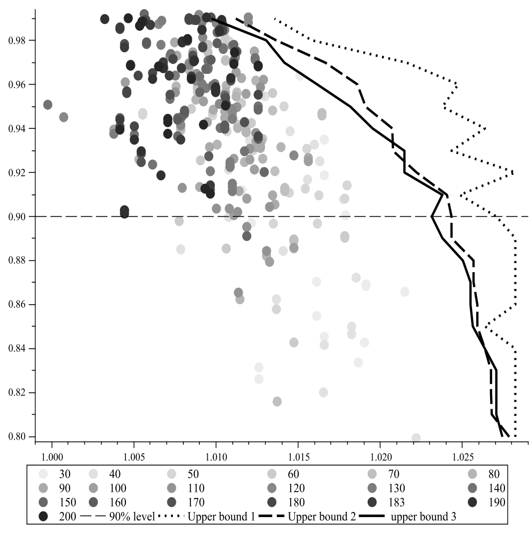

Figure 2: SAA with $\gamma = 0$ for the lognormal model.

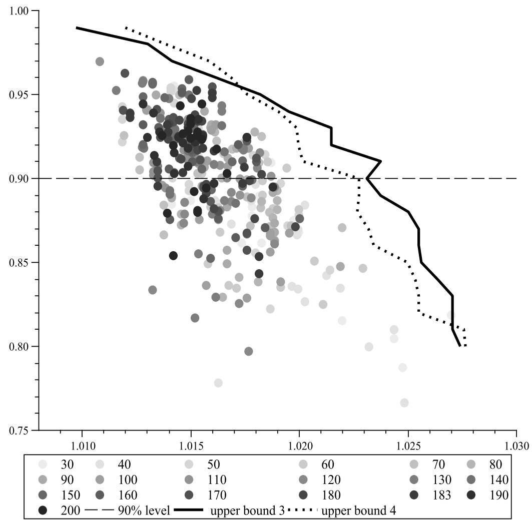

Figure 3: SAA with lognormal returns for $\gamma = 0.05$.

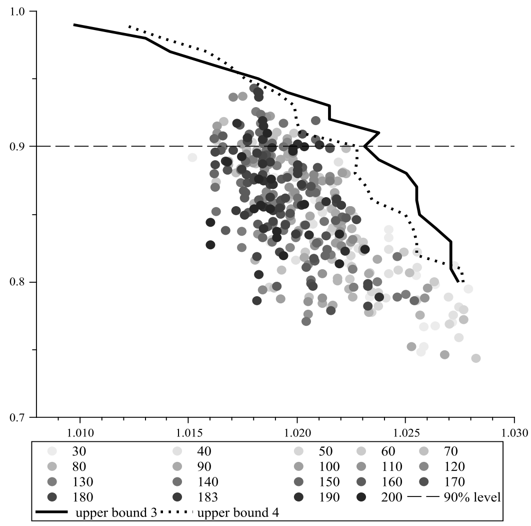

Figure 4: SAA with lognormal returns for $\gamma = 0.10$.

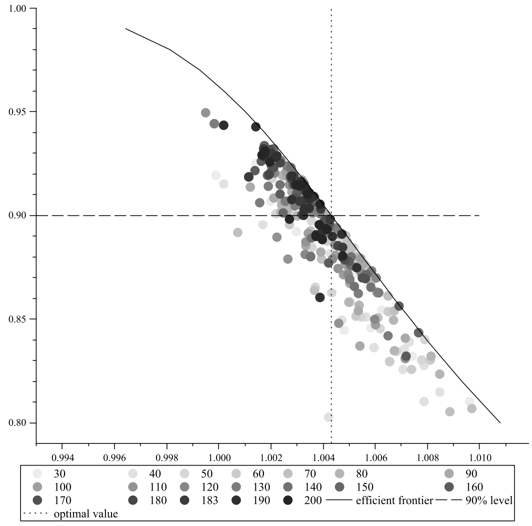

Figure 5: SAA with normal returns for $\gamma = 0.05$.

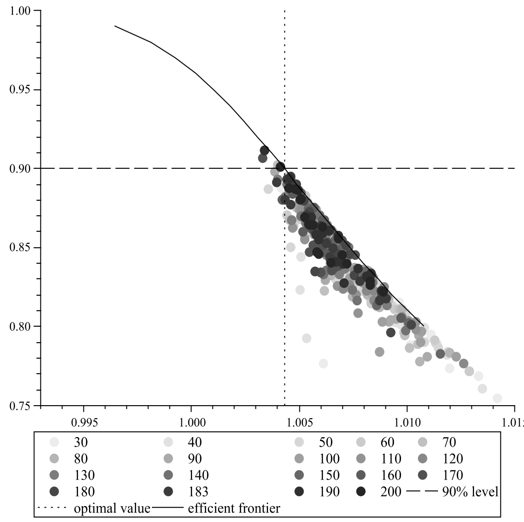

Figure 6: SAA with normal returns for $\gamma = 0.10$

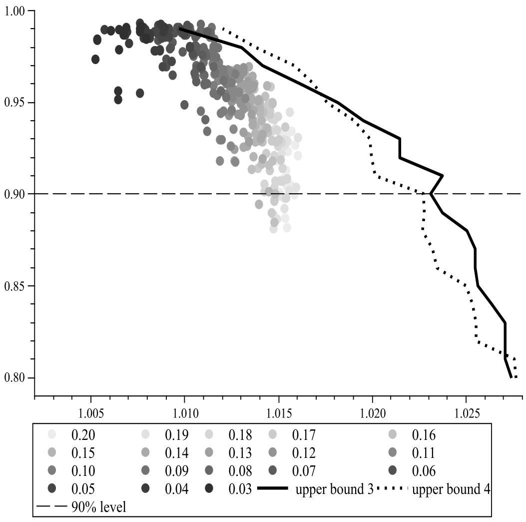

Figure 7: CVaR approximation.

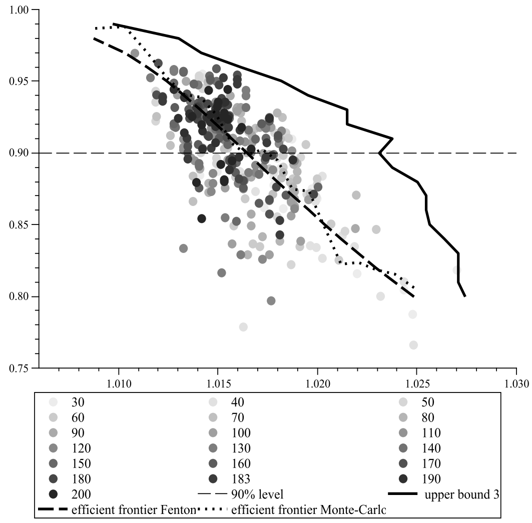

Figure 8: Fenton's approximation.

5 Conclusions

We have discussed chance constrained problems with a single constraint and proved convergence results about the SAA. We applied the SAA to a portfolio chance constrained problem with random returns. In the normal case, we can compute the *efficient frontier* and thus we used it as a benchmark solution. Fixing $\alpha = 0.10$ and $\gamma = 0$ in the SAA, we concluded that the sample size suggested by [7] was too conservative for our problem: a much smaller sample can yield feasible solutions. We observed that the quality of the solutions obtained was poor. Similar results were obtained for the lognormal case, where upper bounds were computed using a method developed in [22]

With $\gamma = 0.05$, much better solutions were obtained, although we had more infeasibility compared to the case $\gamma = 0$. Upper bounds were also constructed for $\gamma = 0.05$ and they were tighter than the ones with $\gamma = 0$. Nevertheless, the process is more computationally demanding and unless very precise upper bounds are needed, the one obtained with $\gamma = 0$ should be satisfactory.

In addition we introduced the CVaR, which leads to a conservative approximation of our original problem (3.13). Although the candidate solutions were relatively far from the upper bound, one need only to solve a linear program to obtain these candidates based on the CVaR. Finally, we presented Fenton's moment-matching method, a technique used to approximate the sum of lognormals by another lognormal. Application of Fenton's approach allowed us to find a deterministic expression for the chance constraint in the lognormal case and to compute an approximate *efficient frontier* for this situation.

Future work in this area will include stronger dependence concepts between the assets of the model, such as comonotonicity and the use of copulas. We are also interested in the application of the SAA to dynamic/multistage chance constrained problems, where the decision maker has to come with a feasible *policy*, that is, with a set of decisions for each period of time based on the information available until the present.

References

[1] A.A. Abu-Dayya and N.C. Beaulieu. Outage probabilities in the presence of correlated lognormal interferers. *IEEE Transactions on Vehicular Technology*, 43(1):164173, 1994.

[2] Z. Artstein and R.J.B. Wets. Consistency of minimizers and the slln for stochastic programs. *Journal of Convex Analysis*, 2:1–17, 1996.

[3] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 9:203–228, 1999.

[4] J. Atlason, M. A. Epelman, and S. G. Henderson. Optimizing call center staffing using simulation and analytic center cutting plane methods, 2008.

[5] D.R. Barr and N.L. Slezak. A comparison of multivariate normal generators. *Communications of the ACM*, 15(12):1048–1049, 1972.

[6] G. Calafiore and M.C. Campi. The scenario approach to robust control design. *IEEE Transactions on Automatic Control*, 51:742–753, 2006.

[7] M.C. Campi and S. Garatti. The exact feasibility of randomized solutions of robust convex programs. *Optimization online (www.optimization-online.org)*, 2007.

[8] A. Charnes, Cooper W. W., and G.H. Symmonds. Cost horizons and certainty equivalents: an approach to stochastic programming of heating oil. *Management Science*, 4:235–263, 1958.

[9] H. Chernoff. A measure of asymptotic efficiency for tests of a hypothesis based on the sum observations. *Annals Math. Statistics*, 23:493–507, 1952.

[10] G. Constantinides and A.G. Malliaris. Portfolio theory. In R.A. Jarrow, V. Maksimovic, and W.T. Ziemba, editors, *Finance*, volume 9 of *Handbooks in OR & MS*, pages 1–30. North-Holland Publishing Company, 1995.

[11] D. Dentcheva, A. Prékopa, and A. Ruszczyński. Concavity and efficient points of discrete distributions in probabilistic programming. *Mathematical Programming*, 89:55–77, 2000.

[12] J. Dupačová, A Gaivoronski, Z. Kos, and T. Szántai. Stochastic programming in water management: a case study and a comparison of solution techniques. *European Journal of Operational Research*, 52:28–44, 1991.

[13] L.F. Fenton. The sum of log-normal probability distributions in scatter transmission systems. *IEEE Transactions on Communications Systems*, CS-8:57–67, 1960.

[14] R. Henrion, P. Li, A. Möller, M.C. Steinbach, M. Wendt, and G. Wozny. Stochastic optimization for operating chemical processes under uncertainty. In M. Grtschel, S. Krunke, and J. Rambau, editors, *Online optimization of large scale systems*, pages 457–478. Springer, 2001.

[15] R. Henrion and A. Möller. Optimization of a continuous distillation process under random inflow rate. *Computers and mathematics with applications*, 45:247–262, 2003.

[16] C.-L. Joshua Lam and T. Le-Ngoc. Estimation of typical sum of lognormal random variables using log shifted gamma approximation. *IEEE Communications Letters*, 10:234–235, 2006.

[17] A. Law and W.D. Kelton. *Simulation Modeling and Analysis*. Industrial Engineering and Management Science Series. McGraw-Hill Science/Engineering/Math, 3 edition, 1999.

[18] J. Luedtke and S. Ahmed. A sample approximation approach for optimization with probabilistic constraints. Submitted for publication, 2007.

[19] J. Luedtke, S. Ahmed, and G.L. Nemhauser. An integer programming approach for linear programs with probabilistic constraints. Submitted for publication, 2007.

[20] H. Markowitz. Portfolio selection. *Journal of Finance*, 7:77–97, 1952.

[21] N.B. Mehta, A.F. Molisch, J. Wu, and J. Zhang. Approximating the sum of correlated lognormal or lognormal-rice radom variables. technical report TR2006-010, Mitsubishi Electric Research Labs (MERL), 2006.

[22] A. Nemirovski and A. Shapiro. Convex approximations of chance constrained programs. *SIAM Journal of Optimization*, 17(4):969–996, 2006.

[23] A. Prékopa. *Stochastic Programming.* Kluwer Acad. Publ., Dordrecht, Boston, 1995.

[24] R.T. Rockafellar and S. Uryasev. Optimization of conditional value-at-risk. *The Journal of Risk*, 2(3):21–41, 2000.

[25] R.T. Rockafellar and R.J.-B. Wets. *Variational Analysis.* Springer, Berlin, 1998.

[26] S.C. Schwartz and Y.S. Yeh. On the distribution function and moments of power sums with lognormal components. *Bell Systems Technical Journal*, 61:1441–1462, 1982.

[27] A. Shapiro. Monte carlo sampling methods. In A. Ruszczyński and A. Shapiro, editors, *Stochastic Programming*, volume 10 of *Handbooks in OR & MS*, pages 353–425. North-Holland Publishing Company, 2003.

[28] S. Vanduffel. *Comonotonicity: from risk measurement to risk management.* PhD thesis, University of Amsterdam, 2005.

[29] M. Zhu, D. B. Taylor, S. C. Sarin, and R. A. Kramer. Chance constrained programming models for risk-based economic and policy analysis of soil conservation. *Agricultural and Resource Economics Review*, 23:58–65, 1994.