Pricing with uncertain customer valuations

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Abstract

Uncertain demand in pricing problems is often modeled using the sum of a linear price-response function and a zero-mean random variable. In this paper, we argue that the presence of uncertainty motivates the introduction of non-linearities in the demand as a function of price, both in the risk-neutral and risk-sensitive models. We motivate our analysis by investigating the impact of uncertainty on the individual customers' valuations. We derive a family of price-response functions, parametrized by a risk sensitivity coefficient, which includes the special case of risk neutrality.

1 Introduction

Traditional models of revenue management under uncertainty assume the precise knowledge of the underlying probabilities for the random demand. However, it is difficult to estimate such probabilities accurately, and over-reliance on wrong guesses might yield dire outcomes for companies. For instance, Honda announced in June 2007 that it would discontinue its Accord Hybrid model after only three years of commercialization and cited management's failure to anticipate the importance of gas mileage, as opposed to horsepower, for hybrids-buying motorists as the main reason for the car's weak sales [8]; in the video game industry "Nintendo's Wii is hard to find in stores because the company miscalculated demand" [7]. Even well-established product lines face significant sources of randomness: according to industry analysts, Dell's global market share fell by 8.7% to 13.9% in 2006 while Hewlett-Packard's rose by 23.9% to 17.4%, because Dell anticipated neither the move of U.S. customers from desktops to notebooks, nor the flagging sales of business computers as companies evaluated Microsoft Vista ([1], [2]). Managers incorrectly estimated the value customers would attach to product features, overpricing the Accord Hybrid at Honda and desktops at Dell, and underpricing the Wii game console at Nintendo, which resulted in stockpiles of unsold inventory in the former case and significant backlogs in the latter.

From a methodological perspective, it is a common practice in revenue management to model uncertain demand using an additive model of randomness coupled with a linear nominal price-response function (see Phillips [3] and Talluri and van Ryzin [6]). In that framework, the expected revenue of the risk-neutral decision-maker faced with uncertainty is equal to the revenue in the deterministic world for the same price. In this paper, we question the validity of such a model by investigating how uncertainty on the individual customers' valuations affects the structure of the price-response function. Specifically, we argue that uncertainty introduces nonlinear terms due to boundary effects for the customers with the lowest and highest valuations; incorporating these effects in industry pricing models would help practitioners gain a better understanding of the impact of uncertainty on the optimal price, and select a strategy with deeper insights into its implications. Our contributions can be summarized as follows:

- We incorporate uncertainty into the *individual customer valuations* rather than the aggregate demand for the item and describe uncertainty using a deterministic framework based on *range forecasts*.

- We take into account the decision-maker's *sensitivity to risk* through a single parameter that captures in some sense the likelihood, from the manager's perspective, of uncertain customers buying the item.

- We obtain *closed-form solutions* in the risk-neutral model and *simple equations* in the risk-sensitive case that highlight the impact of uncertainty on the optimal price.

Range forecasts have been identified by practitioners and researchers alike as an important way to model uncertainty in real-life operations management (Simchi-Levi et. al. [5], Sheffi [4]). To the best of our knowledge, this is the first work that addresses uncertainty at the customer valuation level. We present the risk-neutral framework in Section 2 and extend our results to the risk-sensitive case in Section 3. Section 4 illustrates the methodology on an example.

2 Risk-Neutral Pricing

2.1 Generalities

To introduce the model, we assume that there are n potential buyers and buyer i, $i = 1, \ldots, n$, has uncertain valuation v_i for the product (or service) on sale. There is neither competition nor substitution, i.e., we only have one seller and one type of product. Although the seller does not know the v_i's precisely, he has established that valuation v_i of customer i, $i = 1, \ldots, n$, belongs to the interval $[\underline{v}_i, \overline{v}_i]$, with $\underline{v}_i < \overline{v}_i$. These lower and upper bounds allow the decision-maker to compute a pessimistic demand curve $\underline{d}$, based on the $\underline{v}_i$, and an optimistic demand curve $\overline{d}$, based on the $\overline{v}_i$:

$$\underline{d}(p) = \sum_{i=1}^{n} 1_{\{p \leq \underline{v}_i\}} \text{ and } \overline{d}(p) = \sum_{i=1}^{n} 1_{\{p \leq \overline{v}_i\}} \ \forall p \geq 0. \tag{1}$$

At any given price p, $\overline{d}(p) - \underline{d}(p)$ is the number of customers whose behavior is not known. Intuitively, as this number increases, the likelihood that at least a few of these buyers will be willing to buy the item increases as well: the true demand function might coincide with the pessimistic demand curve when uncertainty on customer behavior is low (the reaction of *only a few buyers* to the posted price is random, so it is quite possible for all of them to have a low valuation of the item), but will move closer toward the optimistic demand curve when uncertainty is large. In the spirit of the law of large numbers, it is reasonable to expect some customers' low valuation to be counterbalanced by others' higher willingness to pay when the pool of undecided buyers is large. (Throughout the paper, we use the term "undecided buyers" as a shortcut to describe buyers whose behavior is not known exactly by the seller.)

To quantify this insight in mathematical terms, we describe customer i's willingness to pay $(i = 1, \ldots, n)$ as a function w_i of price with the following properties:

(i) w_i takes values in $[0, 1]$,

(ii) if $w_i(p) = 1$, customer i is certain to buy the item at price p,

(iii) if $w_i(p) = 0$, customer i is certain to decline buying the item at price p, and

(iv) if $0 < w_i(p) < 1$, the behavior of customer i is not known precisely by the seller.

Since the information available to the seller is limited to the range forecast $[\underline{v}_i, \overline{v}_i]$ for each i, we describe customer i's willingness to pay using the following piecewise linear function w_i:

$$w_i(p) = \begin{cases} 1, & \text{if } p \leq \underline{v}_i, \\ \dfrac{\overline{v}_i - p}{\overline{v}_i - \underline{v}_i}, & \text{if } \underline{v}_i < p < \overline{v}_i, \\ 0 & \text{if } p \geq \overline{v}_i. \end{cases} \tag{2}$$

The appeal of this function lies in the fact that it does not require the knowledge of any parameters beyond $\underline{v}_i$ and $\overline{v}_i$ for all i. The demand under uncertainty, $\widetilde{d}(p)$, which is defined by $\widetilde{d}(p) =$

$\sum_{i=1}^{n} w_i(p)$, can then be formulated as:

$$\widetilde{d}(p) = \underline{d}(p) + \sum_{i|\underline{v}_i < p < \overline{v}_i} \left(\frac{\overline{v}_i - p}{\overline{v}_i - \underline{v}_i} \right). \tag{3}$$

The first term in the right-hand side of Equation (3) represents the deterministic part of the demand, while the second term represents the uncertain part.

2.2 Problem Setup Under Additive Uncertainty

In this paper, we will focus on the case of additive uncertainty; our results extend to multiplicative uncertainty as well. We assume that both $\underline{v}_i$ and $\overline{v}_i$ can be written under the form: $\underline{v}_i = v_i - \widehat{v}$ and $\overline{v}_i = v_i + \widehat{v}$ for all i and some $\widehat{v} > 0$. The parameter $\widehat{v}$ quantifies the amount of uncertainty on the valuations. All proofs are given in the appendix.

Lemma 2.1

(i) Worst-case and best-case demands defined in Equation (1) satisfy:

$$\underline{d}(p) = d(p + \widehat{v}), \tag{4}$$

$$\overline{d}(p) = d(p - \widehat{v}), \tag{5}$$

where d is the nominal demand when the customer valuations are equal to their nominal values v_i, $i = 1, \ldots, n$.

(ii) The demand under uncertainty defined in Equation (3) becomes:

$$\widetilde{d}(p) = d(p + \widehat{v}) + \frac{1}{2\widehat{v}} \left\{ \sum_{i|p-\widehat{v} \le v_i \le p+\widehat{v}} v_i + [d(p - \widehat{v}) - d(p + \widehat{v})]\ (\widehat{v} - p) \right\}. \tag{6}$$

The key issue is to express $\sum_{i|p-\widehat{v} \le v_i \le p+\widehat{v}} v_i$ in a tractable manner. To do this, we will consider a continuous model to approximate the demand, as is commonly the case in the literature (see Phillips [3] for a review of pricing formulations).

Let $V : [0,1] \to \mathcal{R}^+$ the continuous valuation function, which satisfies:

$$v_i = V\left(\frac{i}{n}\right), \ \forall i = 1, \ldots, n. \tag{7}$$

V is the inverse of $p \to \int_p^\infty WTP(q)\, dq$ with WTP the willingness to pay. We use the following approximation:

$$\sum_{i|p-\widehat{v}\le v_i \le p+\widehat{v}} v_i \approx \int_{j\in[0,n]} V\left(\frac{j}{n}\right)\, dj \approx n \int_{x\in[0,1]} V(x)\, dx. \tag{8}$$

In the continuous model, we will denote by d_{max} rather than n the size of the population, as there is no longer a finite set of customers. In order to derive meaningful insights, we need to specify V; we will follow the literature in associating a linear nominal demand model to the additive uncertainty framework. Theorem 2.2 presents the reformulated pricing problem that we will analyze in the remainder of this section; the model follows immediately from the derivations above. We assume that $\widehat{v} \le v_{min}$, so that customer valuations always remain nonnegative. v_{min} and v_{max} are the minimum and maximum customer valuations for the product in the nominal model, respectively. In this work, we also assume that production and ordering costs are sunk and that capacity is infinite.

Theorem 2.2 *The risk-neutral pricing problem is:*

$$\begin{aligned} \max \quad & p\, \widetilde{d}(p) \\ s.t. \quad & p,\, \widetilde{d}(p) \ge 0, \end{aligned} \tag{9}$$

with:

$$\widetilde{d}(p) = d(p+\widehat{v}) + \frac{1}{2\widehat{v}} \left\{ d_{max} \int_{V(x)\in[p-\widehat{v},p+\widehat{v}]} V(x)\, dx + \left[d(p-\widehat{v}) - d(p+\widehat{v})\right]\, (\widehat{v} - p) \right\}, \tag{10}$$

and:

$$V(x) = \begin{cases} 1, & \text{if } x < v_{min}, \\ v_{max} - x(v_{max} - v_{min}), & \text{if } v_{min} \le x \le v_{max}, \\ 0, & \text{if } x > v_{max}. \end{cases} \tag{11}$$

In applications, the parameters $v_{\min}$, $v_{\max}$, $\widehat{v}$ could be estimated using focus groups, whose participants would be asked the minimum and maximum amount they would be willing to pay for the item. Because answers to such questions are always imprecise, $\widehat{v}$ for instance would be computed using the average half-range of reported answers, if such answers indicate that the additive model is appropriate for the product considered (e.g., if valuations are spaced approximately evenly). $v_{\min}$ and $v_{\max}$ would be computed in the same fashion. Note also that the only complexity added onto the nominal framework is the calibration of $\widehat{v}$; estimating $v_{\min}$ and $v_{\max}$ is a necessity of any model, traditional and otherwise. Another way to estimate these parameters would be to analyze the actual demand data collected for similar products. The use of uncertainty might even yield greater accuracy for the pricing model, as respondents will no longer feel that they have to come up with one precise valuation threshold and are encouraged to give a range of values instead.

Remarks:

- The demand in Equation (10) is the sum of the certain demand and a term that depends on the difference between the price p and the average maximum valuation of undecided customers.
- Using Equation (11), we have:

$$\int_{V(x)\in[p-\widehat{v},p+\widehat{v}]} V(x)\,dx = \int_{\max\left(0,\frac{v_{max}-(p+\widehat{v})}{v_{max}-v_{min}}\right)}^{\min\left(1,\frac{v_{max}-(p-\widehat{v})}{v_{max}-v_{min}}\right)} [v_{max} - x(v_{max} - v_{min})]. \tag{12}$$

Two cases arise:

(i) $\widehat{v} \le \frac{v_{max} - v_{min}}{2}$, i.e., the extreme valuation intervals $[v_{min} - \widehat{v},\, v_{min} + \widehat{v}]$ and $[v_{max} - \widehat{v},\, v_{max} + \widehat{v}]$ do not overlap, which we refer to as *low uncertainty*,

(ii) $\widehat{v} > \frac{v_{max} - v_{min}}{2}$, which is the case with *high uncertainty*.

We will see that the price-response function does exhibit a linear behavior over the middle range, which is defined as $[v_{min} + \widehat{v},\, v_{max} - \widehat{v}]$ for low uncertainty and $[v_{max} - \widehat{v},\, v_{min} + \widehat{v}]$ for high uncertainty, with nonlinearities capturing end effects outside these ranges. It is obvious that

$\widetilde{d}(p) = d_{max}$ for $p \leq v_{min} - \widehat{v}$ and $\widetilde{d}(p) = 0$ for $p \geq v_{max} + \widehat{v}$; hence, the optimal price is found on $[v_{min} - \widehat{v},\, v_{max} + \widehat{v}]$, which will be the focus of our analysis.

2.3 Optimal Solution

Lemma 2.3 (Demand)

(i) If $\widehat{v} \leq \frac{v_{max} - v_{min}}{2}$*, the demand under uncertain customer valuations is given by:*

$$\widetilde{d}(p) = \begin{cases} d_{max}\left[1 - \frac{1}{4\,\widehat{v}}\,\frac{1}{v_{max} - v_{min}}\,(p + \widehat{v} - v_{min})^2\right], & \text{if } v_{min} - \widehat{v} < p \leq v_{min} + \widehat{v}, \\ \frac{d_{max}}{v_{max} - v_{min}}\,(v_{max} - p)\,, & \text{if } v_{min} + \widehat{v} < p \leq v_{max} - \widehat{v}, \\ \frac{1}{4\,\widehat{v}}\,\frac{d_{max}}{v_{max} - v_{min}}\,(v_{max} + \widehat{v} - p)^2, & \text{if } v_{max} - \widehat{v} < p \leq v_{max} + \widehat{v}. \end{cases} \tag{13}$$

(ii) If $\widehat{v} > \frac{v_{max} - v_{min}}{2}$*, the demand under uncertain customer valuations is given by:*

$$\widetilde{d}(p) = \begin{cases} d_{max}\left[1 - \frac{(p + \widehat{v} - v_{min})^2}{4\,\widehat{v}\,(v_{max} - v_{min})}\right], & \text{if } v_{min} - \widehat{v} < p \leq v_{max} - \widehat{v}, \\ \frac{d_{max}}{2\,\widehat{v}}\left[\frac{v_{max} + v_{min}}{2} + \widehat{v} - p\right], & \text{if } v_{max} - \widehat{v} < p \leq v_{min} + \widehat{v}, \\ \frac{d_{max}}{4\,\widehat{v}}\,\frac{(v_{max} + \widehat{v} - p)^2}{v_{max} - v_{min}}, & \text{if } v_{min} + \widehat{v} < p \leq v_{max} + \widehat{v}. \end{cases} \tag{14}$$

Remark: Over $[v_{min} - \widehat{v},\, v_{max} + \widehat{v}]$, the demand always consists of three parts: a *linear* one (similar to the price-response function in the deterministic model) between two *quadratic* pieces, which capture boundary effects. These boundary effects are due to the smaller number of undecided customers at the two extremes of the valuation range.

Theorem 2.4 (Price) *The optimal price is given by:*

$$p^* = \begin{cases} \frac{v_{max}}{2}, & \text{if } \widehat{v} \leq \frac{v_{max}}{2} - v_{min}, \\ \frac{2}{3}\,(v_{min} - \widehat{v}) + \frac{1}{3}\,\sqrt{(v_{min} - \widehat{v})^2 + 12\,\widehat{v}\,(v_{max} - v_{min})}, & \text{if } \frac{v_{max}}{2} - v_{min} < \widehat{v} \leq \frac{v_{max} - v_{min}/3}{2}, \\ \frac{1}{2}\left[\frac{v_{max} + v_{min}}{2} + \widehat{v}\right], & \text{if } \widehat{v} > \frac{v_{max} - v_{min}/3}{2}. \end{cases} \tag{15}$$

Remark: We observe three main phases for the price as uncertainty increases:

1. The price is *constant*, equal to its nominal price, for $\widehat{v} \leq \frac{v_{max}}{2} - v_{min}$. (In particular, the nominal price is never optimal if $v_{max} < 2\, v_{min}$.)

2. The price is *nonlinear* for $\frac{v_{max}}{2} - v_{min} < \widehat{v} \leq \frac{v_{max}}{2} - \frac{v_{min}}{6}$.

3. The price increases *linearly* in $\widehat{v}$ for $\widehat{v} > \frac{v_{max}}{2} - \frac{v_{min}}{6}$.

3 Pricing with Risk Sensitivity

3.1 Evaluating the Demand

We now consider the case where the decision-maker is risk-sensitive. We use the following model for his demand forecast when there are n customers:

$$\widetilde{d}(p) = \underline{d}(p) + \sum_{i \,|\, \underline{v}_i < p < \overline{v}_i} \left(\frac{\overline{v}_i - p}{\overline{v}_i - \underline{v}_i} \right)^{\alpha}, \tag{16}$$

with $\alpha \geq 0$. Equation (16) is the counterpart of Equation (3), with $\alpha = 1$, respectively $\alpha > 1$, $\alpha < 1$ corresponding to a risk-neutral manager, respectively risk-averse, risk-taking. For instance, $\alpha \to \infty$ corresponds to the most risk-averse case (because $0 < \frac{\overline{v}_i - p}{\overline{v}_i - \underline{v}_i} < 1$ for all i such that $\underline{v}_i < p < \overline{v}_i$, so the demand tends towards $\underline{d}(p)$ as α grows large). Similarly, $\alpha = 0$ corresponds to the most risk-taking decision-maker. The parameter α is specified by the user.

With additive uncertainty, Equation (16) can be rewritten in the continuous case as:

$$\widetilde{d}(p) = d(p + \widehat{v}) + \int_{V(x) \in [p - \widehat{v}, p + \widehat{v}]} \left(\frac{V(x) + \widehat{v} - p}{2\widehat{v}} \right)^{\alpha} d_{max}\, dx. \tag{17}$$

Injecting the linear nominal demand model, we obtain after a change of variables ($y = v_{max} + \widehat{v} - p - x\,(v_{max} - v_{min})$) and using that $\int_a^b y^{\alpha}\, dy = \frac{1}{\alpha+1}[b^{\alpha+1} - a^{\alpha+1}]$:

$$\widetilde{d}(p) = d(p+\widehat{v}) + \frac{d_{max}}{(2\,\widehat{v})^{\alpha}\,(1+\alpha)} \frac{1}{v_{max} - v_{min}} \left\{ [\min(2\,\widehat{v},\, v_{max} + \widehat{v} - p)]^{1+\alpha} - [\max(0,\, v_{min} + \widehat{v} - p)]^{1+\alpha} \right\}. \tag{18}$$

3.2 Optimal Solution

Lemma 3.1 (Demand)

(i) If $\widehat{v} \leq \frac{v_{max} - v_{min}}{2}$*, the demand under uncertain customer valuations is given by:*

$$\widetilde{d}(p) = \begin{cases} \frac{d_{max}}{v_{max} - v_{min}} \left[v_{max} - (p + \widehat{v}) + \frac{(2\widehat{v})^{1+\alpha} - (v_{min} + \widehat{v} - p)^{1+\alpha}}{(1+\alpha)\,(2\,\widehat{v})^{\alpha}} \right], & \text{if } v_{min} - \widehat{v} < p \leq v_{min} + \widehat{v}, \\ \frac{d_{max}}{v_{max} - v_{min}} \left[v_{max} - p + \frac{1-\alpha}{1+\alpha}\,\widehat{v} \right], & \text{if } v_{min} + \widehat{v} < p \leq v_{max} - \widehat{v}, \\ \frac{d_{max}}{v_{max} - v_{min}} \frac{1}{(2\,\widehat{v})^{\alpha}\,(1+\alpha)} (v_{max} + \widehat{v} - p)^{1+\alpha}, & \text{if } v_{max} - \widehat{v} < p \leq v_{max} + \widehat{v}. \end{cases} \tag{19}$$

(ii) If $\widehat{v} > \frac{v_{max} - v_{min}}{2}$*, the demand under uncertain customer valuations is given by:*

$$\widetilde{d}(p) = \begin{cases} \frac{d_{max}}{v_{max} - v_{min}} \left[v_{max} - (p + \widehat{v}) + \frac{(2\,\widehat{v})^{1+\alpha} - (v_{\min} + \widehat{v} - p)^{1+\alpha}}{(2\,\widehat{v})^{\alpha}(1+\alpha)} \right], & \text{if } v_{min} - \widehat{v} < p \leq v_{max} - \widehat{v}, \\ \frac{d_{max}}{v_{max} - v_{min}} \frac{(v_{max} + \widehat{v} - p)^{1+\alpha} - (v_{min} + \widehat{v} - p)^{1+\alpha}}{(2\,\widehat{v})^{\alpha}(1+\alpha)}, & \text{if } v_{max} - \widehat{v} < p \leq v_{min} + \widehat{v}, \\ \frac{d_{max}}{v_{max} - v_{min}} \frac{(v_{max} + \widehat{v} - p)^{1+\alpha}}{(2\,\widehat{v})^{\alpha}(1+\alpha)}, & \text{if } v_{min} + \widehat{v} < p \leq v_{max} + \widehat{v}. \end{cases} \tag{20}$$

Theorem 3.2 (Price)

If $\mathbf{\widehat{v} \leq \frac{v_{max} - v_{min}}{2}}$**:**

(i) If $\widehat{v} \leq \frac{1+\alpha}{1+3\,\alpha} [v_{max} - 2\,v_{min}]$*, the optimal price is given by:*

$$p^* = \frac{1}{2} \left[v_{max} + \frac{1-\alpha}{1+\alpha}\,\widehat{v} \right]. \tag{21}$$

(ii) Otherwise, the optimal price is the unique solution over $[v_{min} - \widehat{v},\, v_{min} + \widehat{v}]$ *of:*

$$(1+\alpha)\,(v_{max} - \widehat{v} - 2\,p) + 2\,\widehat{v} + \frac{(v_{min} + \widehat{v} - p)^{\alpha}}{(2\,\widehat{v})^{\alpha}} \,[(2+\alpha)\,p - v_{min} - \widehat{v}] = 0. \tag{22}$$

If $\mathbf{\widehat{v} > \frac{v_{max} - v_{min}}{2}}$**:**

Let $\widehat{v}^*$ *be the unique solution for* $\widehat{v}$ *in* $\left[\frac{v_{max} - v_{min}}{2}, \infty\right)$ *of:*

$$\left(1 - \frac{v_{max} - v_{min}}{2\,\widehat{v}}\right)^{\alpha} = 1 + \frac{v_{max} - v_{min}}{v_{min} - (2+\alpha)\,v_{max} + (3+\alpha)\,\widehat{v}}. \tag{23}$$

(i) If $\widehat{v} \leq \widehat{v}^*$, the optimal price is the unique solution in $[v_{min} - \widehat{v},\, v_{max} - \widehat{v}]$ of Equation (22).

(ii) Otherwise, the optimal price p^* is the unique solution over $[v_{max} - \widehat{v},\, v_{min} + \widehat{v}]$ of:

$$\left(\frac{v_{max} + \widehat{v} - p}{v_{min} + \widehat{v} - p}\right)^{\alpha} = \frac{v_{min} + \widehat{v} - (2+\alpha)\, p}{v_{max} + \widehat{v} - (2+\alpha)\, p}. \tag{24}$$

Remark: We observe again three phases for the optimal price, with the first one (for very small uncertainty) exhibiting the closest connections with the deterministic model. Risk sensitivity with low uncertainty decreases the range of $\widehat{v}$ for which the nominal price (21) is optimal; note that for $\alpha > 1$ (the manager is risk-averse), the optimal price given by Equation (21) decreases with the measure of uncertainty $\widehat{v}$, while for $\alpha < 1$ (the manager is risk-taking), the optimal price increases with $\widehat{v}$. This matches our intuition.

4 Numerical Example

We now implement the approach on an example with $v_{min} = 100$, $v_{max} = 250$, and $\widehat{v}$ varying between 0 and 100. We consider 111 customers, where the valuation interval of customer i, $i = 1, \ldots, 111$, is $[99 + i - \widehat{v},\, 99 + i + \widehat{v}]$. The actual valuation of customer i obeys a distribution that is Uniform, Gaussian or Bernoulli over that interval. (In the case of the Gaussian distribution we take the standard deviation equal to $\widehat{v}/3$.) In Figure 1, we compare the revenues obtained with the optimal price in the deterministic model, $\overline{p} = 155$, and the optimal price obtained in the approach we propose here. We observe that our approach slightly outperforms (by 2-3%) the traditional framework. Figure 2 shows the impact of risk sensitivity on the optimal price when $\widehat{v} = 75$; in particular, for the numerical values considered, the price is convex in the parameter α. This matches the intuition as α is the exponent of a term between 0 and 1 in the demand function: increasing α has diminishing impact once that value comes close to zero. Figure 3 illustrates the scaled risk-return trade-off for the three distributions considered; interestingly, the model succeeds in increasing return and decreasing risk when the distribution is Gaussian. For the other two distributions, lowering risk requires lowering return.

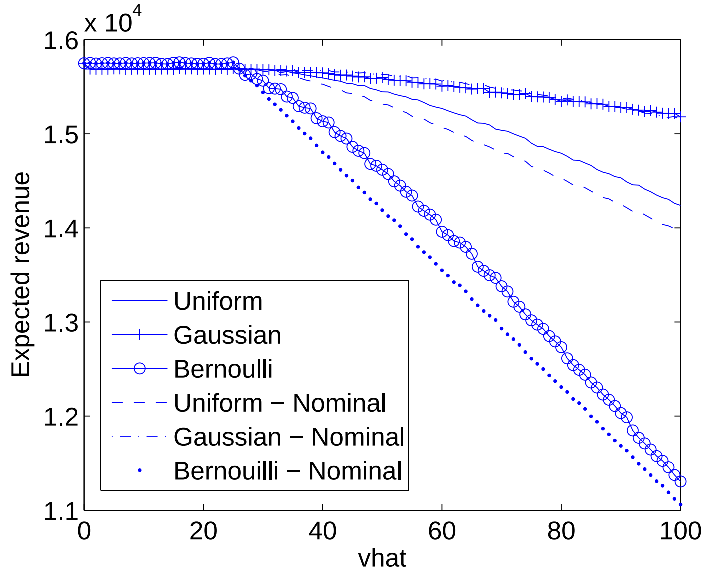

Figure 1: Expected revenue as a function of $\widehat{v}$.

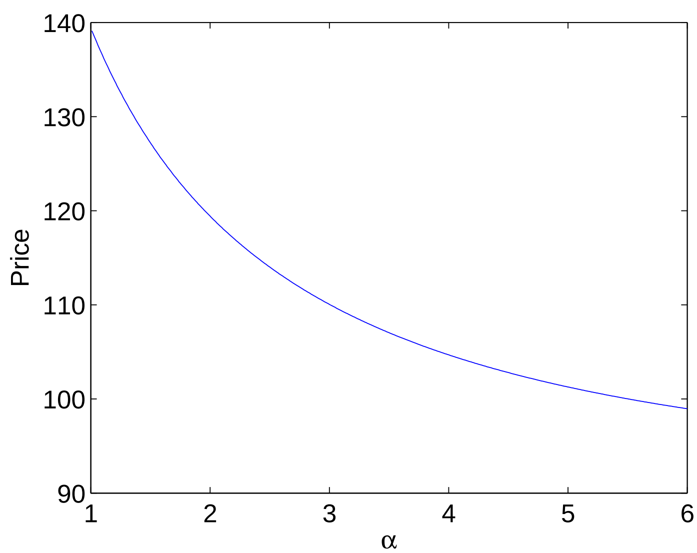

Figure 2: Optimal price as a function of α for $\widehat{v} = 75$.

5 Conclusions

We have analyzed a pricing model with uncertain customer valuations using range forecasts. The main motivation for our study was to gain a better understanding of price-response functions when the randomness is incorporated at the customer level; specifically, we have questioned the validity of considering a linear model in price with an additive zero-mean random variable

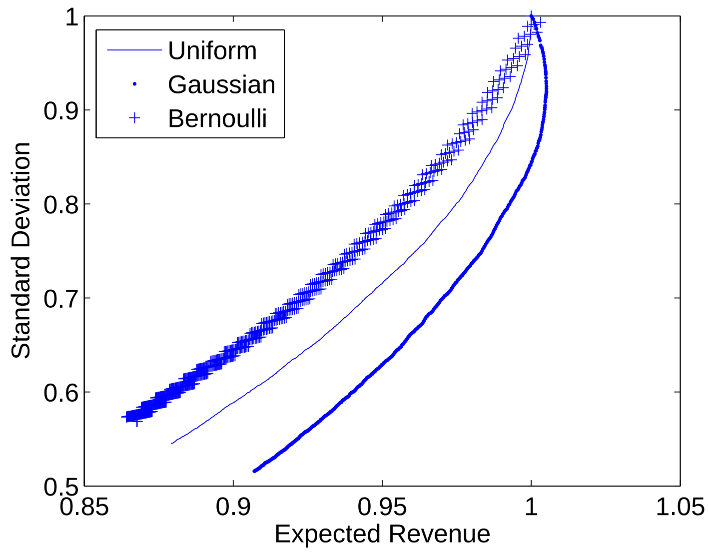

Figure 3: Scaled risk-return tradeoff curves for $\widehat{v} = 75$.

to capture demand fluctuations. The use of range forecasts was motivated by the difficulty in estimating probabilities to characterize the value given to the item by each individual customer: range forecasts provide an elegant alternative when little information is available. Our results indicate important nonlinear effects at the boundaries of the valuation range that are not taken into account by the traditional methodologies; these effects are quadratic in the risk-neutral model and obey a power law in the risk-sensitive model. Intuitively, such boundary effects arise because the impact of customers with lowest and highest valuations cannot be counterbalanced by customers with even lower or higher willingness to pay. We have provided threshold values of the uncertainty, measured by the half-length of the range forecast, that determine when the optimal price in the deterministic model is truly optimal in the risk-neutral framework, and have presented its counterpart in the risk-sensitive model, where we have linked the optimal price to the sole risk sensitivity parameter required in the approach in a simple, intuitive way. We have also extended the analysis to larger amounts of uncertainty, where the optimal solution becomes disconnected from its deterministic counterpart. The risk-neutral demand function in the proposed framework exhibits similarities in its overall shape with the logit model that has emerged in the literature as an alternative price-response function when there is no uncertainty, and in that respect can be viewed as a bridge between the linear and the logit models. From a practical perspective, the present work indicates that, while it is appropriate to use linear price-response functions for a range of customer valuations away from both high and low extremes, it is also important for a realistic model of customer behavior to take into account boundary effects through quadratic or power-law models, depending on the decision-maker's risk sensitivity, to capture the smaller impact on demand of price changes at the boundaries. Incorporating such effects will help revenue managers develop more precise characterizations of aggregate customer response in pricing their products. Future research directions include the analysis of competition and substitution effects.

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A Proofs for Risk-Neutral Model

Proof of Lemma 2.1

(i) By definition and using the assumption of additive uncertainty,

$$\underline{d}(p) = \sum_{i|p\le v_i-\widehat{v}} 1 = \sum_{i|p+\widehat{v}\le v_i} 1. \tag{25}$$

The proof for $\overline{d}$ is similar. (ii) We use Equation (4) and inject $\dfrac{\overline{v}_i - p}{\overline{v}_i - \underline{v}_i} = \dfrac{v_i + \widehat{v} - p}{2\widehat{v}}$ for all i. □

Proof of Lemma 2.3

We will only prove the result when $\widehat{v} \le \frac{v_{max} - v_{min}}{2}$. The case where $\widehat{v} > \frac{v_{max} - v_{min}}{2}$ is similar and left to the reader.

If $p \le v_{min} - \widehat{v}$ or $p \ge v_{max} + \widehat{v}$, there is no uncertainty and the demand is equal to d_{max} or 0, respectively.

If $v_{min} - \widehat{v} < p \le v_{min} + \widehat{v}$, we have $d(p - \widehat{v}) = d_{max}$ because $p - \widehat{v} \le v_{min}$, and $d(p + \widehat{v}) = d_{max}\dfrac{v_{max} - (p + \widehat{v})}{v_{max} - v_{min}}$ because $v_{min} \le p + \widehat{v} \le v_{max}$. Hence,

$$\begin{aligned}
\widetilde{d}(p) &= d_{max}\frac{v_{max} - (p+\widehat{v})}{v_{max} - v_{min}} + \frac{1}{2\widehat{v}}\left\{(\widehat{v} - p)\, d_{max}\left[1 - \frac{v_{max} - (p+\widehat{v})}{v_{max} - v_{min}}\right] + \right. \\
&\qquad \left. d_{max}\int_{\frac{v_{max}-(p+\widehat{v})}{v_{max}-v_{min}}}^{1} \left[v_{max} - x\,(v_{max} - v_{min})\right] dx \right\}, && (26)\\
&= \frac{d_{max}}{v_{max} - v_{min}}\left[v_{max} - (p+\widehat{v}) + \frac{\widehat{v} - p}{2\widehat{v}}\,(p + \widehat{v} - v_{min}) + \frac{1}{2\widehat{v}}\,\frac{(p+\widehat{v})^2 - v_{min}^2}{2}\right], && (27)\\
&= \frac{d_{max}}{v_{max} - v_{min}}\left[v_{max} - p - \widehat{v} + \frac{1}{4\widehat{v}}(p + \widehat{v} - v_{min})\,(3\widehat{v} + v_{min} - p)\right], && (28)\\
&= d_{max} - \frac{d_{max}}{v_{max} - v_{min}}\,\frac{(p + \widehat{v} - v_{min})^2}{4\,\widehat{v}}. && (29)
\end{aligned}$$

The derivations for the cases $v_{min} + \widehat{v} < p \leq v_{max} - \widehat{v}$ and $v_{max} - \widehat{v} < p \leq v_{max} + \widehat{v}$ are similar. For $v_{min} + \widehat{v} < p \leq v_{max} - \widehat{v}$, we use that:

$$d(p + \widehat{v}) = d_{max} \frac{v_{max} - (p + \widehat{v})}{v_{max} - v_{min}} \text{ and } d(p - \widehat{v}) = d_{max} \frac{v_{max} - (p - \widehat{v})}{v_{max} - v_{min}}. \tag{30}$$

For $v_{max} - \widehat{v} < p \leq v_{max} + \widehat{v}$, we use that:

$$d(p + \widehat{v}) = 0 \text{ and } d(p - \widehat{v}) = d_{max} \frac{v_{max} - (p - \widehat{v})}{v_{max} - v_{min}}. \tag{31}$$

For $v_{min} - \widehat{v} < p \leq v_{max} - \widehat{v}$, we use that $d(p - \widehat{v}) = d_{max}$ and $d(p + \widehat{v}) = d_{max} \frac{v_{max} - (p + \widehat{v})}{v_{max} - v_{min}}$. The lower and upper bounds of the integral in Equation (12) are $\frac{v_{max} - (p + \widehat{v})}{v_{max} - v_{min}}$ and 1, respectively.

For $v_{max} - \widehat{v} \leq p \leq v_{min} + \widehat{v}$, we use that $d(p - \widehat{v}) = d_{max}$ and $d(p + \widehat{v}) = 0$. The lower and upper bounds of the integral in Equation (12) are 0 and 1, respectively.

For $v_{min} + \widehat{v} \leq p \leq v_{max} + \widehat{v}$, we use that $d(p - \widehat{v}) = d_{max} \frac{v_{max} - (p - \widehat{v})}{v_{max} - v_{min}}$ and $d(p + \widehat{v}) = 0$. The lower and upper bounds of the integral in Equation (12) are 0 and $\frac{v_{max} - (p - \widehat{v})}{v_{max} - v_{min}}$, respectively.

□ □

Proof of Theorem 2.4

Again, we will only prove the result when $\widehat{v} \leq \frac{v_{max} - v_{min}}{2}$. The case where $\widehat{v} > \frac{v_{max} - v_{min}}{2}$ is similar and left to the reader.

Since demand is equal to d_{max} for $p \leq v_{min} - \widehat{v}$ and 0 for $p \geq v_{max} + \widehat{v}$, the optimal price is reached on $[v_{min} - \widehat{v}, v_{max} + \widehat{v}]$.

(i) For $p \geq v_{max} - \widehat{v}$, the objective function is $p\,(v_{max} + \widehat{v} - p)^2$, which has an unconstrained maximum at $p = \frac{v_{max} + \widehat{v}}{3}$. But $\frac{v_{max} + \widehat{v}}{3} < v_{max} - \widehat{v}$ since $\widehat{v} \leq \frac{v_{max} - v_{min}}{2} < \frac{v_{max}}{2}$, so the revenue is maximized for $p \geq v_{max} - \widehat{v}$ at $v_{max} - \widehat{v}$.

(ii) For $v_{min} + \widehat{v} \leq p < v_{max} - \widehat{v}$, the unconstrained maximum of $\frac{d_{max}}{v_{max} - v_{min}}\, p\,(v_{max} - p)$ is reached at $\frac{v_{max}}{2}$, which belongs to $[v_{min} + \widehat{v}, v_{max} - \widehat{v}]$ if and only if $\widehat{v} \leq \frac{v_{max}}{2} - v_{min}$. Otherwise, the optimum over $[v_{min} + \widehat{v}, v_{max} - \widehat{v}]$ is reached at $v_{min} + \widehat{v}$.

(iii) For $p < v_{min} + \widehat{v}$, the objective function is $p\left(1 - \frac{(p+\widehat{v}-v_{min})^2}{4\,\widehat{v}\,(v_{max}-v_{min})}\right)$. The only nonnegative stationary point for the unconstrained problem is:

$$p = \frac{1}{3}\left[2\,(v_{min} - \widehat{v}) + \sqrt{4\,(\widehat{v} - v_{min})^2 - 3\,[(v_{min} + \widehat{v})^2 - 4\widehat{v}\,v_{max}]}\right]. \tag{32}$$

At $p = 0$ the slope is positive; hence, the revenue increases up to the stationary point and then decreases, and the stationary point is a global maximum over $p \geq 0$. It can be shown after some calculations that it belongs to $[v_{min} - \widehat{v},\, v_{min} + \widehat{v}]$ if and only if $\widehat{v} \geq \frac{v_{max}}{2} - v_{min}$. □

B Proofs for Risk-Sensitive Model

Proof of Lemma 3.1

The proof is similar to that of Lemma 2.3 and left to the reader. □

Proof of Theorem 3.2

If $\widehat{\mathbf{v}} \leq \frac{\mathbf{v_{max}} - \mathbf{v_{min}}}{\mathbf{2}}$:

The optimum price is, again, reached over $[v_{min} - \widehat{v},\, v_{max} + \widehat{v}]$.

For $v_{min} + \widehat{v} < p \leq v_{max} - \widehat{v}$, the revenue is proportional to $p\,(v_{max} - p + \frac{1-\alpha}{1+\alpha}\widehat{v})$ and is maximum at $\frac{1}{2}\left[v_{max} + \frac{1-\alpha}{1+\alpha}\,\widehat{v}\right]$, which belongs to $[v_{min} + \widehat{v},\, v_{max} - \widehat{v}]$ for $\widehat{v} \leq \frac{1+\alpha}{1+3\,\alpha}\,[v_{max} - 2\,v_{min}]$. Otherwise, the maximum revenue over $[v_{min} + \widehat{v},\, v_{max} - \widehat{v}]$ is reached at $v_{min} + \widehat{v}$.

For $v_{max} - \widehat{v} < p \leq v_{max} + \widehat{v}$, the revenue is proportional to $p\,(v_{max} + \widehat{v} - p)^{1+\alpha}$ and is maximum at $\frac{v_{max} + \widehat{v}}{2+\alpha}$, which belongs to $[v_{max} - \widehat{v},\, v_{max} + \widehat{v}]$ if and only if $\widehat{v} \geq v_{max}\,\frac{1+\alpha}{3+\alpha}$. Since $\widehat{v} \leq \frac{v_{max} - v_{min}}{2}$ and $\frac{1+\alpha}{3+\alpha} > \frac{1}{3}$ for all $\alpha > 0$, having $\widehat{v} \geq v_{max}\,\frac{1+\alpha}{3+\alpha}$ here would require $v_{max} \geq 3\,v_{min}$, but then $\widehat{v} \geq 3\,v_{min}\frac{1+\alpha}{3+\alpha} > v_{min}$ for $\alpha > 0$, which is impossible because $\widehat{v} \leq v_{min}$ by assumption. So $\widehat{v} \geq v_{max}\,\frac{1+\alpha}{3+\alpha}$ is never satisfied on this range and the maximum revenue is reached at $v_{max} - \widehat{v}$.

For $v_{min} - \widehat{v} \leq p \leq v_{min} + \widehat{v}$, the revenue is proportional to:

$$r(p) = p\left[v_{max} - (p + \widehat{v}) + \frac{(2\widehat{v})^{1+\alpha} - (v_{min} + \widehat{v} - p)^{1+\alpha}}{(1+\alpha)\,(2\,\widehat{v})^{\alpha}}\right]. \tag{33}$$

The stationary points are the solution of $r'(p) = 0$, i.e.:

$$(v_{max} - \widehat{v} - 2\,p) + \frac{1}{1+\alpha}\left\{2\,\widehat{v} + \frac{(v_{min} + \widehat{v} - p)^{\alpha}}{(2\,\widehat{v})^{\alpha}}\,\left[(2+\alpha)\,p - v_{min} - \widehat{v}\right]\right\} = 0. \tag{34}$$

We have:

$$r^{''}(p) = -2 + \frac{1}{(2\,\widehat{v})^{\alpha}}\,(v_{min} + \widehat{v} - p)^{\alpha-1}\,[2\,(v_{min} + \widehat{v}) - (2+\alpha)\,p]. \tag{35}$$

$r^{''}(p) < 0$ for all p in $[v_{min} - \widehat{v},\, v_{min} + \widehat{v}]$ because $2 - \alpha\,p/(v_{min} + \widehat{v} - p) < 2$ for all $\alpha > 0$ and $(v_{min} + \widehat{v} - p)/(2\,\widehat{v}) < 1$ for all $p > v_{min} - \widehat{v}$. Therefore, the revenue is concave over $[v_{min} - \widehat{v},\, v_{min} + \widehat{v}]$ and $r'(p) = 0$ has at most one solution in that interval. Furthermore, $r'(v_{min} - \widehat{v}) = v_{max} - v_{min} > 0$, and $r'(v_{min} + \widehat{v}) = v_{max} - 2\,v_{min} - \widehat{v}\,\dfrac{1+3\,\alpha}{1+\alpha}$. We conclude that the maximum revenue over $[v_{min} - \widehat{v},\, v_{min} + \widehat{v}]$ is achieved at $v_{min} + \widehat{v}$ for $v_{max} - 2\,v_{min} \geq \widehat{v}\,\dfrac{1+3\,\alpha}{1+\alpha}$ and at the unique solution of $r'(p) = 0$ on $[v_{min} - \widehat{v},\, v_{min} + \widehat{v}]$ otherwise.

If $\widehat{\mathbf{v}} > \frac{\mathbf{v_{max}} - \mathbf{v_{min}}}{\mathbf{2}}$:

We know that the optimal price is in $[v_{min} - \widehat{v},\, v_{max} + \widehat{v}]$.

For $v_{min} - \widehat{v} \leq p \leq v_{max} - \widehat{v}$, the revenue is proportional to $p\left[v_{max} - (p + \widehat{v}) + \dfrac{(2\,\widehat{v})^{1+\alpha} - (v_{\min} + \widehat{v} - p)^{1+\alpha}}{(2\,\widehat{v})^{\alpha}(1+\alpha)}\right]$. We have studied this function in Theorem 3.2; in particular we have seen that it was concave over $[v_{\min} - \widehat{v},\, v_{\min} + \widehat{v}]$, and hence here over $[v_{\min} - \widehat{v},\, v_{\max} - \widehat{v}]$. Therefore, Equation (34), which sets the slope to zero, has exactly one solution in $[v_{\min} - \widehat{v},\, v_{\max} - \widehat{v}]$ if and only of $r'(v_{\max} - \widehat{v}) < 0$. Otherwise the maximum over that interval is reached at $v_{\max} - \widehat{v}$. Rearranging the terms of $r'(v_{\max} - \widehat{v}) = 0$ and studying unicity yields the critical value of $\widehat{v}$ in Equation (23) and the fact that $r'(v_{\max} - \widehat{v}) < 0$ if and only if $\widehat{v} > \widehat{v}^{*}$.

For $v_{max} - \widehat{v} \leq p \leq v_{min} + \widehat{v}$, the revenue is proportional to $p\left[(v_{max} + \widehat{v} - p)^{1+\alpha} - (v_{min} + \widehat{v} - p)^{1+\alpha}\right]$

and stationary points on $[v_{max} - \widehat{v},\, v_{min} + \widehat{v}]$ are the solutions of, after re-arranging the terms of the first derivative:

$$\frac{v_{max} + \widehat{v} - (2+\alpha)\, p}{v_{min} + \widehat{v} - (2+\alpha)\, p} = \left(\frac{v_{min} + \widehat{v} - p}{v_{max} + \widehat{v} - p}\right)^{\alpha}. \tag{36}$$

Due to monotonicity (one is increasing and the other is decreasing in p), the functions on each side of Equation (36) intersect only once over $[\frac{v_{max} + \widehat{v}}{2+\alpha},\, v_{min} + \widehat{v}]$. (They cannot intersect for $p < \frac{v_{max} + \widehat{v}}{2+\alpha}$ as one term is then less than 1 and the other is greater than 1.) To check whether the intersection occurs on $[v_{max} - \widehat{v},\, v_{min} + \widehat{v}]$, we check the sign of the first derivative at $v_{max} - \widehat{v}$, which yields Equation (23) again as critical value of $\widehat{v}$. This time the intersection occurs on $[v_{max} - \widehat{v},\, v_{min} + \widehat{v}]$ if and only if $\widehat{v} \le \widehat{v}^{*}$. Finally, the unique stationary point over $[v_{max} - \widehat{v},\, v_{min} + \widehat{v}]$ is a local and hence global maximum because the first derivative is positive at $v_{max} - \widehat{v}$ for $\widehat{v} \le \widehat{v}^{*}$.

For $v_{min} + \widehat{v} \le p \le v_{max} + \widehat{v}$, the revenue is proportional to $p\,(v_{max} + \widehat{v} - p)^{1+\alpha}$ and has an unconstrained maximum at $\frac{v_{max} + \widehat{v}}{2+\alpha}$, which is at most $\frac{v_{max} + \widehat{v}}{2}$ for $\alpha \ge 0$. We have seen that, because $2\,\widehat{v} > v_{max} - v_{min}$, this number can never exceed $v_{min} + \widehat{v}$. Therefore, the maximum of the revenue over $[v_{min} + \widehat{v},\, v_{max} + \widehat{v}]$ is reached at $v_{min} + \widehat{v}$. □