Asymptotic almost-equivalence of abstract evolution systems [★]

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Abstract

We study the asymptotic behavior of almost-orbits of abstract evolution systems in Banach spaces with or without a Lipschitz assumption. In particular, we establish convergence, convergence in average and almost-convergence of almost-orbits both for the weak and the strong topologies based on the behavior of the orbits. We also analyze the set of almost-stationary points.

Key words: Evolution systems, almost-orbits, asymptotic analysis.

1 Introduction and preliminaries

Roughly speaking, a dynamical system in discrete (resp. continuous) time is a rule that determines a sequence (resp. trajectory) departing from certain initial data and which evolves in an either finite or infinite dimensional space. In this sense, any iterative algorithm may be considered as a discrete evolution system. If it is possible to find a continuous-in-time version for the discrete procedure, it is then natural to expect that some of the properties of the former are

close to the similar properties of the latter. Of course, in general such an inheritance of properties is not true without additional conditions, in particular on the parameters of the algorithm. The dynamical approach to iterative methods has certain advantages: a continuous-in-time evolution system satisfying nice qualitative properties may suggest new iterative methods, and sometimes the techniques used to investigate the continuous case can be adapted to obtain results for the discrete algorithm. On the other hand, one may be concerned with different aspects of the trajectories of a given continuous-in-time dynamical system, namely: existence, exact or approximate computation, regularity, long-term behavior, stability, numerical integration. Of special interest is the effect of certain perturbations of the original system on the qualitative properties of the corresponding trajectories. In this context, this paper deals with some of the asymptotic properties that are common to systems which can be considered equivalent in a sense to be made precise later on.

Let C be a nonempty Borel subset of a Banach space $(X, \| \cdot \|)$. An *evolution system* (ES for short) on C is a two-parameter family $U = \{U(t,s) \mid t \geq s \geq 0\}$ of possibly non-linear maps from C into itself satisfying:

i) $\forall t \geq 0$, $\forall x \in C$, $U(t,t)x = x$; and
ii) $\forall t \geq s \geq r \geq 0$, $\forall x \in C$, $U(t,s)U(s,r)x = U(t,r)x$.

A *M-Lipschitz evolution system* (M-LES) is an ES U such that $\|U(t,s)x - U(t,s)y\| \leq M\|x - y\|$ for some $M > 0$ and all $t \geq s \geq 0$, $x, y \in C$. A *contracting evolution system* (CES) is a 1-LES.

An ES U is *autonomous* if for all $t, s \geq 0$ we have $U(t,0) = U(t+s,s)$. For such an ES, the family $T = \{T(t) := U(t,0) \mid t \geq 0\}$ defines a *semigroup*, that is $T(0)x = x$ and $T(t)T(s)x = T(t+s)x$ for all $t, s \geq 0$, $x \in C$.

Example 1 Let F be a (possibly multivalued) function from $[t_0, \infty) \times C$ to C. Suppose that for every $s \geq t_0$ and $x \in C$ the differential inclusion $u'(t) \in F(t, u(t))$, with initial condition $u(s) = x$, has a unique solution $u_{s,x} : [s, \infty) \mapsto C$. The family U defined by $U(t,s)x = u_{s,x}(t)$ is an evolution system on C. If X is Hilbert space and $F(t,x) = -A_t x$, where $\{A_t\}$ is a family of maximal monotone operators, then the corresponding U is a CES. □

Example 2 Take a strictly increasing unbounded sequence $\{\sigma_n\}$ of positive numbers and set $\nu(t) = \max\{n \in \mathbb{N} \mid \sigma_n \leq t\}$. Consider a family $\{F_n\}$ of functions from C into C and define $U(t,s) = \prod_{n=\nu(s)+1}^{\nu(t)} F_n$, the product representing composition of functions. Then U is an ES. If each F_n is M_n-Lipschitz and the product $\prod_{n=1}^{\infty} M_n$ is bounded from above by M, then U is an M-LES. For instance, if $F_n = (I + A_n)^{-1}$, where $\{A_n\}$ is a family of m-accretive operators on C, then the piecewise constant interpolation of infinite products of resolvents defines a CES. □

If U is an ES on C, an *orbit* of U is a function $u : [0, \infty) \to C$ such that for some $t_0 \geq 0$ and $x_0 \in C$, $u(t) = U(t, t_0)x_0$ for all $t \geq t_0$. Throughout this paper, all orbits are assumed to be measurable and locally bounded, hence locally integrable on $[0, \infty)$. More generally, we say that a function $u \in L^\infty_{\mathrm{loc}}(0, \infty; C)$ is an *almost-orbit* of U if

$$\lim_{t \to \infty} \sup_{h \geq 0} \|u(t + h) - U(t + h, t)u(t)\| = 0. \tag{1}$$

Clearly, orbits are almost-orbits. Two ES are *asymptotically almost-equivalent* (AAE) if every orbit of each one is an almost-orbit of the other.

If U is an autonomous CES and V is AAE to U, then V is an *asymptotic semigroup* as defined in [14]. In that case, every orbit of U converges strongly (or weakly) if, and only if, every orbit of V does (see Proposition 9 below).

Remark 3 Suppose U is a M-LES on C. If u is an almost-orbit of U, then so is any function $v \in L^\infty_{\mathrm{loc}}(0, \infty; C)$ satisfying $\lim_{t \to \infty} \|v(t) - u(t)\| = 0$. $\qquad\square$

Remark 4 Suppose that for each $r > 0$ there is $G_r : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that $\lim_{t \to \infty} \sup_{h \geq 0} G_r(t+h, t) = 0$ and $\|U(t, s)x - V(t, s)x\| \leq G_r(t, s)$ for all $x \in B(0, r)$ and $t \geq s \geq 0$. Every bounded orbit of U is an almost-orbit of V and viceversa. If the same $G_r \equiv G$, the boundedness assumption is unnecessary. $\qquad\square$.

The term "almost-orbit" was introduced in [12] for a *continuous* function satisfying (1). Later, in [11], the author gives a weaker definition, just requiring $\lim_{t, h \to \infty} \|u(t + h) - U(t + h, t)u(t)\| = 0$, but still for continuous functions. The latter is slightly weaker than (1) for practical purposes. In fact, the example provided in [11] to motivate the interest of studying almost-orbits also satisfies (1). In both cited works the authors give criteria that can be applied to an almost-orbit in order to ensure certain asymptotic behavior. The same approach is used in [19]. In [16,15], the authors carry out a similar analysis for *uniformly asymptotically almost nonexpansive curves* (a concept that includes almost-orbits *of almost nonexpansive semigroups*) in Hilbert space. Other results on the asymptotic behavior of almost-orbits of nonexpansive semigroups can be found in [8] (see also the references therein). Notice that [12,11] contain versions of Proposition 13 below in different settings. Our intention is to show how to derive many asymptotic properties of almost-orbits by studying only the orbits. In that sense our work is different but complementary to [12,11].

Example 5 Let A be a m-accretive operator on X and let U be the autonomous CES defined by the inclusion $-\dot{u} \in Au$ as in Example 1.

(1) Take $f \in L^1(0, \infty; X)$ and let V be the CES defined by the integral solutions of $-\dot{u} \in Au + f$. The orbits of V are almost-orbits of U (see [12]). This result is generalized in [1].

(2) For a sequence $\{\lambda_n\}$ in $(0, \infty)$ set $\sigma_n = \sum_{k=0}^{n} \lambda_k$ and $\nu(t)$ as in Example 2. Define the CES $W(t, s) = \prod_{n=\nu(s)+1}^{\nu(t)} (I + \lambda_n A)^{-1}$. If $\{\lambda_n\} \in \ell^2 \setminus \ell^1$ then U and W are AAE; see [9], although the fact that the orbits of W are almost-orbits of U had already been proved in [12]. This was shown earlier in [14] by assuming A to be single-valued and Lipschitz.

(3) If A is the subdifferential of a proper, closed and convex function in a Hilbert space, U and W are AAE if $\{\lambda_n\} \notin \ell^1$ (see [7]). $\qquad\square$

We shall see that orbits and almost-orbits of an ES have the same asymptotic behavior in terms of boundedness, convergence and other related properties. A few results of this kind can be found in [14,12,9,7] for differential inclusions of the type $-u'(t) \in Au(t)$, where A is m-accretive. A first attempt to deal with nonautonomous and non-Lipschitz systems can be found in [11].

The paper is organized as follows: In section 2 we focus on M-LES. We show basic properties of their almost-orbits, exploiting the dissipativity behind the uniform Lipschitz constant. We also state and prove some asymptotic equivalence results. Despite the surprising fact that the main convergence results hold for arbitrary ES almost as stated, we prefer to present this easier case first for the sake of a clear exposition. Section 3 contains further results on the special class of strongly contracting ES and the structure of the set of almost-stationary points of an M-LES. The asymptotic properties for general ES without any assumptions on the space-dependence are given in section 4. Neither Lipschitz continuity nor asymptotic nonexpansiveness is imposed. We present some additional results on uniform continuity and cluster points in section 5 and some remarks on the applicability of this theory in section 6.

2 Lipschitz evolution systems

The following is similar to [12, Lemma 3.1]:

Proposition 6 *Let U be a M-LES and u_1, u_2 two almost-orbits of U. Then*

i) $\limsup_{t\to\infty} \|u_1(t) - u_2(t)\| \leq M \liminf_{t\to\infty} \|u_1(t) - u_2(t)\| < \infty$.

ii) If one almost-orbit of U is bounded, then every almost-orbit of U is.

iii) If 0 is a cluster point of $\|u_1(t) - u_2(t)\|$ then $\lim_{t\to\infty} \|u_1(t) - u_2(t)\| = 0$.

iv) If U is a CES, the limit $\lim_{t\to\infty} \|u_1(t) - u_2(t)\|$ always exists.

Proof. We just prove $i)$. For $i = 1, 2$ let $\psi_i(t) = \sup_{h \geq 0} \|u_i(t+h) - U(t+h, t)u_i(t)\|$. Then $\|u_1(t+h) - u_2(t+h)\| \leq \psi_1(t) + \psi_2(t) + M\|u_1(t) - u_2(t)\|$ for every $h \geq 0$. Hence $\limsup_{h\to\infty} \|u_1(h) - u_2(h)\| \leq \psi_1(t) + \psi_2(t) + M\|u_1(t) - u_2(t)\| < \infty$

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and finally $\limsup_{h\to\infty} \|u_1(h) - u_2(h)\| \le M \liminf_{t\to\infty} \|u_1(t) - u_2(t)\|$. ∎

Remark 7 Let U, V be M-LES which are AAE. If **one** almost-orbit of U **or** V is bounded, **every** almost-orbit of U **and** V is so. □

Remark 8 Let u be a bounded almost-orbit of an M-LES U so that $\|u\|_\infty = \sup_t \|u(t)\| < \infty$. Since u is an almost-orbit, there exists $p_0 \ge 0$ such that for all $p \ge p_0$, we have $\|u(p + h) - U(p + h, p)u(p)\| \le 1$ for all $h \ge 0$. Hence, for all $p \ge p_0$ and $h \ge 0$ we get $\|U(p + h, p)u(p)\| \le 1 + \|u\|_\infty$. □

The following result and its proof are inspired by [14, Lemma 1], where the author studies two special cases: when U is an autonomous CES, and when the almost-orbits are in fact the orbits of a semigroup of contractions. This result had already been presented by the authors for arbitrary CES in [1]. Here we give a shorter proof in a more general context.

Proposition 9 *Let U be an M-LES. If every orbit of U converges strongly (weakly) as time goes to infinity, so does every almost-orbit of U.*

Proof. Let τ denote the hypothesized topology. And suppose that the $\tau-$limit of $U(t, s)x$ as $t \to \infty$ exists for all x and s. Let u be an almost-orbit of U. Take $p \ge 0$ and set $\zeta(p) = \tau - \lim_{t\to\infty} U(t, p)u(p)$. We have

$$\zeta(p + h) - \zeta(p) = \tau - \lim_{t\to\infty} \{U(t, p + h)u(p + h) - U(t, p)u(p)\}.$$

But for all $t \ge p + h$ the quantity $\|U(t, p + h)u(p + h) - U(t, p)u(p)\|$ can be bounded above by $M\|u(p+h) - U(p+h, p)u(p)\|$ and by τ-lower semicontinuity of the norm we get $\|\zeta(p+h) - \zeta(p)\| \le M\|u(p+h) - U(p+h, p)u(p)\|$. Since u is an almost-orbit of U, the right-hand side tends to zero as $p \to \infty$ uniformly in $h \ge 0$. Therefore $\{\zeta(p) : p \to \infty\}$ is a Cauchy net that converges strongly to a limit ζ_∞. Finally, we can express $u(p + h) - \zeta_\infty$, for all $p, h \ge 0$, as $u(p+h) - \zeta_\infty = [u(p+h) - U(p+h, p)u(p)] + [U(p+h, p)u(p) - \zeta(p)] + [\zeta(p) - \zeta_\infty]$. Given $\varepsilon > 0$ we can choose p large enough so that the first and third terms on the right-hand side are less than ε in norm, uniformly in h for the first term. Next for such a fixed p, we let $h \to \infty$ so that the second term $\tau-$converges to zero. Hence $u(t)$ is $\tau-$convergent to ζ_∞ as $t \to \infty$. ∎

Remark 10 Consider the following more general setting: Let (X, d) be a complete metric space (not even the linear structure is necessary). The Lipschitz condition in the definition of M-LES reads $d(U(t, s)x, U(t, s)y) \le Md(x, y)$. The definition of almost-orbit can be rephrased as $\lim_{t\to\infty} \sup_{h\ge 0} d(u(t+h), U(t+h, t)u(t)) = 0$. It is easy to see that Proposition 6 and the statement in Proposition 9 concerning the strong topology are still true. □

In [13] the authors proved strong convergence of the orbits of some semigroups. More than two decades later the result was extended in [19] for the almost-

orbits. This extension is straightforward using Proposition 9. An extension to more general spaces can be found in [6].

Given $v \in L^\infty_{\text{loc}}(0, \infty; X)$, define $\overline{v}(t) = \frac{1}{t} \int_0^t v(\xi) \, d\xi$. We say v converges strongly (weakly) *in average* if $\overline{v}(t)$ has a strong (weak) limit as $t \to \infty$. Convergence in average is also inherited by almost-orbits.

Remark 11 Given $v \in L^\infty_{\text{loc}}(0, \infty; X)$, $h, t \geq 0$, define $v_h(t) = v(h + t)$. Since $\overline{v_h}(t) = \frac{1}{t} \int_0^t v(h + \xi) d\xi = \left(\frac{t+h}{t}\right) \frac{1}{t+h} \int_0^{t+h} v(\eta) d\eta - \frac{1}{t} \int_0^h v(\xi) d\xi$, if v converges strongly (weakly) in average to L, the same holds for v_h, for each $h \geq 0$.   □

Proposition 12 *Let U be an M-LES. If every orbit of U converges strongly (weakly) in average, so does every almost-orbit.*

Proof. Let u be an almost-orbit of U. For $p, h \geq 0$ and t sufficiently large, define $\sigma_h(t, p) = \frac{1}{t} \int_0^t U(p + h + \xi, p) u(p) \, d\xi$ and set $\zeta(p) = \tau - \lim_{t \to \infty} \sigma_0(t, p)$, where τ stands for either the strong or the weak topology according to the hypothesis. Notice that

$$[\sigma_0(t, p+h) - \sigma_0(t+h, p)] - [\sigma_h(t, p) - \sigma_0(t+h, p)] = [\sigma_0(t, p+h) - \sigma_h(t, p)]. \tag{2}$$

By virtue of Remark 11, $\tau - \lim_{t \to \infty} \sigma_h(t, p) = \tau - \lim_{t \to \infty} \sigma_0(t, p) = \tau - \lim_{t \to \infty} \sigma_0(t+h, p)$ for each $h \geq 0$. We let $t \to \infty$ in equation (2) and use the weak lower-semicontinuity of the norm to obtain $\|\zeta(p + h) - \zeta(p)\| \leq \|\sigma_0(t, p + h) - \sigma_h(t, p)\| \leq M \|u(p + h) - U(p + h, p)u(p)\|$, which in turn tends to zero as $p \to \infty$ uniformly in $h \geq 0$. As a consequence, $\zeta(p)$ converges strongly to some ζ_∞ as $p \to \infty$. Finally, for any $p, h \geq 0$ we write

$$\overline{u}(p + h) - \zeta_\infty = \frac{1}{p + h} \int_0^p u(\xi) \, d\xi + \left[\frac{h}{p + h} \sigma(h, p) - \zeta(p)\right] + [\zeta(p) - \zeta_\infty]$$
$$+ \frac{1}{p + h} \int_0^h [u(p + \xi) - U(p + \xi, p)u(p)] \, d\xi.$$

The second term is bounded by $\sup_{k \geq 0} \|u(p + k) - U(p + k, p)u(p)\|$, which is independent of h and tends to zero as $p \to \infty$. The last term converges strongly to zero as $p \to \infty$. Thus, given any $\varepsilon > 0$, we can choose p_ε large enough so that the second and forth terms are both less than ε. Having fixed p_ε, the first term converges strongly to zero as $h \to \infty$ while the third term τ-converges to zero. As a consequence $\overline{u}(t)$ is τ-convergent to ζ_∞ as $t \to \infty$. ■

A function $v \in L^\infty_{\text{loc}}(0, \infty; X)$ is strongly (weakly) *almost-convergent* (in the sense of Lorentz, [10]) if there is $y \in X$ such that $\overline{v_h}(t)$ converges strongly (weakly) to y as $t \to \infty$ uniformly in $h \geq 0$. Almost-convergence implies convergence in average. Conversely, according to Remark 11, if v converges in average then $\overline{v_h}$ converges for each $h \geq 0$, so the uniformity in $h \geq 0$ is what makes the difference. Almost-convergence is interesting because a trajectory

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$v(t)$ is convergent if, and only if, it is almost-convergent and *asymptotically regular* (the difference $v(t + h) - v(t)$ converges to zero as $t \to \infty$ for each $h \geq 0$) for the corresponding topology (see [10]). This fact $-$ or method of proof $-$ has been used, for instance, in [4]. The following result shows that almost-convergence of the orbits is also inherited by the almost-orbits.

Proposition 13 *Let U be an M-LES. If every bounded orbit of U is strongly (weakly) almost-convergent, so is every bounded almost-orbit of U.*

Proof. Define $\sigma_h(t, p) = \frac{1}{t} \int_0^t U(p + h + \xi, p) u(p) \, d\xi$, where u is a bounded almost-orbit of U. According to Remark 8, there exists $p_0 \geq 0$ such that for all $p \geq p_0$ and $h \geq 0$ we have $\|U(p + h, p) u(p)\| \leq 1 + \|u\|_\infty$. Therefore, by virtue of the hypothesis, for every $p \geq p_0$ there exists $\zeta(p) \in X$ such that for all $h \geq 0$, $\zeta(p) = \tau - \lim_{t \to \infty} \sigma_h(t, p)$, and the convergence is uniform in $h \geq 0$. Next, we prove that $\{\zeta(p) : p \geq 0\}$ is a Cauchy net. For every $p, h \geq 0$ and $t \geq p + h$ we have $\|\sigma_0(t, p + h) - \sigma_h(t, p)\| \leq M \|u(p + h) - U(p + h, p) u(p)\|$. Let $t \to \infty$ to get $\|\zeta(p + h) - \zeta(p)\| \leq M \|u(p + h) - U(p + h, p) u(p)\|$, which tends to 0 as $p \to \infty$ uniformly in $h \geq 0$. Hence $\zeta(p) \to \zeta_\infty$ as $p \to \infty$, for some ζ_∞. For any $p, h, k \geq 0$ we write

$$\overline{u_k}(p + h) - \zeta_\infty = \frac{1}{p + h} \int_0^p u(k + \xi) \, d\xi + \left[\frac{h}{p + h} \sigma_k(h, p) - \zeta(p) \right]$$
$$+ \frac{1}{p + h} \int_0^h [u(p + k + \xi) - U(p + k + \xi, p) u(p)] \, d\xi$$
$$+ [\zeta(p) - \zeta_\infty].$$

The first term on the right-hand side is bounded by $p/(p + h) \|u\|_\infty$, independently of k. The second term is bounded by $\sup_{q \geq 0} \|u(p + q) - U(p + q, p) u(p)\|$, which is independent of h and k, and tends to zero as $p \to \infty$. The last term converges strongly to zero as $p \to \infty$. Thus, given any $\varepsilon > 0$, we can choose p_ε large enough so that the second and forth terms are both less than ε. Then, for such p_ε, the first term converges strongly to zero as $h \to \infty$ while the third term τ-converges to zero, both uniformly in k. As a consequence $\overline{u_k}(t)$ is τ-convergent to ζ_∞ as $t \to \infty$ uniformly in k. ∎

Remark 14 Proposition 13 was proved in [12] under additional assumptions: i) U is an autonomous and strongly continuous CES, ii) the set of stationary points is nonempty, and iii) for the weak topology, the space X is assumed to be *weakly complete*, which means that every weak Cauchy net converges weakly to an element in X. The spaces ℓ^1, L^1 and all reflexive Banach spaces have this property. It is not the case if X contains c_0, though. □

Remark 15 Compared with Propositions 9 and 12, the hypotheses and conclusion in Proposition 13 are weaker. According to Remark 7, the two formulations are equivalent whenever the ES has bounded almost-orbits. For Proposition 13 to be useful, one must prove that the system has at least one

bounded almost-orbit. In practice, this step tends to be useful for proving that the orbits are convergent. In many applications one has to do it anyway. □

Let us introduce some general notions of convergence with respect to time-dependent probability measures in order to unify and summarize the results of the previous section.

Let μ be a probability measure on $[0, \infty)$. A function $v \in L^\infty_{\mathrm{loc}}(0, \infty; X)$ is μ-integrable if the μ-mean of v on $[0, \infty)$, $\mu(v) = \int_0^\infty v(\xi) d\mu(\xi)$ exists. Given a family $\{\mu_t\}_{t \geq 0}$ of probability measures on $[0, \infty)$, a function $v \in L^\infty_{\mathrm{loc}}(0, \infty; X)$ is $\{\mu_t\}$-integrable if $\mu_t(v)$ exists for all $t \geq 0$. We say v converges to y in μ_t-mean for the topology τ if $y = \tau - \lim_{t \to \infty} \mu_t(v)$.

Example 16 Let $v \in L^\infty_{\mathrm{loc}}(0, \infty; X)$. If $\mu_t = \delta_t$ is the Dirac mass at t, then $\mu_t(v) = v(t)$ and convergence in μ_t-mean is standard convergence. If $d\mu_t(\xi) = \frac{1}{t}\chi_{[0,t]}(\xi)d\xi$, where χ_A is the characteristic function of the set A, then $\mu_t(v) = \frac{1}{t}\int_0^t v(\xi)\, d\xi = \overline{v}(t)$ and convergence in μ_t-mean is convergence in average. □

In the rest of this section, $\{\mu_t\}_{t \geq 0}$ is a family of probability measures such that $\mu_t([0, p]) \to 0$ as $t \to \infty$ for all $p \geq 0$. Under this assumption, Propositions 17 and 19 below can be proved by a direct adaptation of the proofs of Propositions 12 and 13, respectively. We leave the details to the reader. We restate them as Theorem 20 in terms of almost-equivalent evolution systems.

Proposition 17 *If the μ_t-mean of every orbit of an M-LES U converges strongly (weakly) as $t \to \infty$, so does the μ_t-mean of every almost-orbit.*

Given $v \in L^\infty_{\mathrm{loc}}(0, \infty; X)$ and $h \geq 0$, we set $v_h(t) = v(h + t)$ for $t \geq 0$. If there is $y \in X$ such that $y = \tau - \lim_{t \to \infty} \mu_t(v_h) = \tau - \lim_{t \to \infty} \int_0^\infty v(h + \xi)\, d\mu_t(\xi)$ uniformly in $h \geq 0$, for τ the strong (weak) topology, we say v converges strongly (weakly) to y *in μ_t-mean, uniformly with respect to translations.*

Example 18 If μ_t is the Dirac mass at t, then $\mu_t(v_h) = v(t + h)$ and convergence in μ_t-mean recovers standard convergence, which is automatically uniform with respect to translations. If $d\mu_t(\xi) = \frac{1}{t}\chi_{[0,t]}(\xi)d\xi$, then $\mu_t(v_h) = \frac{1}{t}\int_0^t v(h + \xi)\, d\xi$. In this case, convergence in μ_t-mean uniformly with respect to translations is almost-convergence. □

Proposition 19 *If every bounded orbit of an M-LES U converges strongly (weakly) in μ_t-mean uniformly with respect to translations, so does every bounded almost-orbit.*

Theorem 20 *Let U and V be two M-LES which are AAE. If every orbit of U converges strongly (weakly) in μ_t-mean, so does every orbit of V. If the convergence is uniform with respect to translations for all bounded orbits of U, the same holds for the bounded orbits of V.*

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3 Further results on Lipschitz evolution systems

If U is an ES on C, let $SP(U) := \{x \in C | U(t,s)x = x, \forall t \geq s\}$ be its (possibly empty) set of *stationary points*. Similarly, denote by $ASP(U)$ its set of *almost-stationary points*: $ASP(U) = \{x \in C | \lim_{t \to \infty} \sup_{h \geq 0} \|U(t+h,t)x - x\| = 0\}$. Clearly $SP(U) \subseteq ASP(U)$ and if U is autonomous, then $ASP(U) = SP(U)$. This is not the case in general even for a CES (take $U(t,s)x = x + e^{-s} - e^{-t}$, $x \in \mathbb{R}$, for which $ASP(U) = \mathbb{R}$ and $SP(U) = \emptyset$).

Remark 21 A nice characterization of $SP(T)$ is given in [17] when T is an autonomous CES on a weakly compact subset of a Banach space with the Opial property: $z \in SP(U)$ if, and only if, there is $t_n \to \infty$ such that w-$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} T(s)z\,dz = z$. A similar result using strong limits is given in [18]. A challenging task is to extend this characterization to nonautonomous ES. $\square$

If $x^* \in ASP(U)$, the constant function $u(t) \equiv x^*$ is a bounded almost-orbit of U. According to Remark 7, if $ASP(U) \neq \emptyset$ every almost-orbit of U is bounded. We now turn our attention to closedness and convexity of $ASP(U)$.

Lemma 22 *Suppose C is closed for the strong topology. If U is an M-LES on C then $ASP(U)$ is closed for the strong topology.*

Proof. Let $\{x_n\} \in ASP(U)$ converge to x. Since $x \in C$, $\|U(t+h,t)x - x\| \leq (M+1)\|x_n - x\| + \|U(t+h,t)x_n - x_n\|$. Hence $\limsup_{t \to \infty} \sup_{h \geq 0} \|U(t+h,t)x - x\| \leq (M+1)\|x_n - x\|$. Letting $n \to \infty$, we conclude that $x \in ASP(U)$. $\blacksquare$

By Remark 3, if U is an M-LES, $ASP(U)$ contains all the limits of the strongly convergent almost-orbits if there are any. For weak limits we have the following:

Corollary 23 *Let C be strongly closed. If the weak limits of orbits of an M-LES U lie in $ASP(U)$, the same holds for weak limits of almost-orbits of U.*

Proof. With the notation introduced in the proof of Proposition 9, as $t \to \infty$, $u(t)$ converges to ζ_∞, which is the strong limit of weak limits of orbits of U. $\blacksquare$

Remark 24 If U is an autonomous M-LES whose orbits converge weakly to points in $SP(U)$, the weak limits of almost-orbits are also in $SP(U)$. $\square$

Let $K \subset X$ be nonempty and convex. Let $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous and strictly increasing with $\gamma(0) = 0$. A function $F : K \to X$ is *of type γ* if $\gamma(\|F(\lambda x + (1-\lambda)y) - \lambda F(x) - (1-\lambda)F(y)\|) \leq \|x - y\| - \|F(x) - F(y)\|$ for all $x, y \in K$ and $\lambda \in [0,1]$.

Proposition 25 *Let U be a CES on a convex set C and suppose that there exists γ such that for each $t \geq s$, $U(t,s)$ is of type γ on a convex set K*

containing $ASP(U)$. Then $ASP(U)$ is convex.

Proof. Take $x_1, x_2 \in ASP(U)$ and define $\psi_i(t) = \sup_{h \geq 0} \|U(t+h,t)x_i - x_i\|$ for $i = 1, 2$. Now take $\lambda \in (0,1)$ and set $z = \lambda x_1 + (1 - \lambda)x_2$. We have

$$
\begin{aligned}
\|U(t+h,t)z - z\| &\leq \|U(t+h,t)z - \lambda U(t+h,t)x_1 - (1-\lambda)U(t+h,t)x_2\| \\
&\quad + \lambda \psi_1(t) + (1 - \lambda)\psi_2(t) \\
&\leq \gamma^{-1}\left(\|x_1 - x_2\| - \|U(t+h,t)x_1 - U(t+h,t)x_2\|\right) \\
&\quad + \lambda \psi_1(t) + (1 - \lambda)\psi_2(t) \\
&\leq \gamma^{-1}\left(\psi_1(t) + \psi_2(t)\right) + \lambda \psi_1(t) + (1 - \lambda)\psi_2(t)
\end{aligned}
$$

Letting $t \to \infty$ we get the result. ■

If X is uniformly convex and K is bounded and convex, there is γ such that every nonexpansive function $F : K \to X$ is of type γ (see [5, Lemma 1.1]). If U is a CES on a convex set C then for every bounded $A \subset C$, there is γ such that $U(t,s)$ is of type γ on $\mathrm{co}(A)$ for each $t \geq s$. We deduce the following:

Corollary 26 *Let U be a CES on a convex subset C of a uniformly convex Banach space X. Then $ASP(U)$ is convex.*

Proof. Let $x_1, x_2 \in ASP(U)$. The line segment K joining x_1 and x_2 is bounded and convex, so there is a function γ such that the restriction of $U(t,s)$ to K is of type γ for all $t \geq s \geq 0$. The rest follows as in Proposition 25. ■

Remark 27 Proposition 25 is valid if we replace $ASP(U)$ with the set of all almost-orbits. The proof is quite similar and uses Proposition 6 iv) for concluding. As a consequence, if X is uniformly convex and U is a CES on a bounded convex set C, then the set of all almost-orbits is convex. □

Let $\{M(t,s)\}_{t \geq s \geq 0}$ be a family of positive numbers satisfying $\lim_{t \to \infty} M(t,s) = 0$ for each s. A *strongly contracting evolution system* (SCES) on C is an ES U such that $\|U(t,s)x - U(t,s)y\| \leq M(t,s)\|x - y\|$ for all $x, y \in C$ and $t \geq s \geq 0$.

Proposition 28 *Let U be a SCES. We have the following:*

i) If u_1 and u_2 are almost-orbits of U then $\lim_{t \to \infty} \|u_1(t) - u_2(t)\| = 0$;

ii) The set $ASP(U)$ has at most one element; and

iii) If $ASP(U) \neq \emptyset$, then every almost-orbit of U converges strongly to the unique $x^* \in ASP(U)$.

Proof. We have $\|u_1(t+s) - u_2(t+s)\| \leq \psi_1(t) + \psi_2(t) + M(t+s,t)\|u_1(t) - u_2(t)\|$ as in part i) of Proposition 6. But $\limsup_{s \to \infty} \|u_1(s) - u_2(s)\| \leq \psi_1(t) + \psi_2(t)$, so $\lim_{s \to \infty} \|u_1(s) - u_2(s)\| = 0$. Parts ii) and iii) are a trivial consequence. ■

4 Asymptotic equivalence without the Lipschitz condition

Not all the results in the previous section are true without the Lipschitz assumption on the ES. For instance, having a bounded almost-orbit does not imply that all the almost-orbits are bounded (take $U(t,s)x = e^{(t-s)}x$). We discuss on some properties of the orbits that do hold for the almost-orbits. The first work that contains equivalence results for non-contracting ES seems to be [11], where they study strongly continuous semigroups which are "asymptotically nonexpansive in the intermediate sense". We shall not go into the details but just mention that the existing results on asymptotic equivalence require additional (and strong!) regularity with respect both to time and space.

Let $\{\mu_t\}_{t\geq 0}$ be a family of probability measures on $[0,\infty)$.

Hypothesis **H**: For each $\{\mu_t\}$-integrable g with $\lim\limits_{t\to\infty} \int_0^\infty g(\xi)\,d\mu_t(\xi) = L$, each $\varepsilon > 0$ and $K > 0$ there exists $T > 0$ such that for all $t \geq T$ one has $\left\| \int_0^\infty g(\xi)\,d\mu_t(\xi + K) - L \right\| < \varepsilon$.

The families described in Example 16 do satisfy Hypothesis **H**: This is trivial if μ_t is the Dirac mass at t. If $d\mu_t(\xi) = \frac{1}{t}\chi_{[0,t]}(\xi)$, then for t large enough $\int_0^\infty g(\xi)d\mu_t(\xi + K) = \left(\frac{t-K}{t}\right)\frac{1}{t-K}\int_0^{t-K} g(\xi)d\xi$, which tends to L as $t \to \infty$. The fact that $\lim\limits_{t\to\infty}\mu_t(B) = 0$ for each bounded set B does not imply that Hypothesis **H** will hold:

Example 29 Define $n(\xi) = \sum_{k\geq 0}\chi_{[2k,2k+1)}(\xi)$ and $\hat{n}(\xi) = n(\xi + 1)$ so that $n^2 \equiv n$ and $n\hat{n} \equiv 0$. Let $d\mu_t(\xi) = \alpha^{-1}(t)n(\xi)\chi_{[0,t]}(\xi)d\xi$, where $\alpha(t) = \int_0^t n(\xi)d\xi$. Then $\mu_t(B) \to 0$ for every bounded set B (this is obvious) but does not fulfill Hypothesis **H**. To see this, simply notice that $\int_0^\infty n(\xi)d\mu_t(\xi) = 1$ while $\int_0^\infty n(\xi)d\mu_t(\xi + 1) = \alpha^{-1}\int_1^{t-1}\hat{n}(\xi)n(\xi)d\xi = 0$ for all t. □

Theorem 30 *Let U be an ES and let $\{\mu_t\}$ satisfy Hypothesis **H**. If each orbit converges strongly in μ_t-mean, so does every $\{\mu_t\}$-integrable almost-orbit.*

Proof. Suppose u is a $\{\mu_t\}$-integrable almost-orbit of U and let $\varepsilon > 0$. Choose $S > 0$ such that $\sup_{h\geq 0} \|u(t+h) - U(t+h,t)u(t)\| < \varepsilon/6$ for all $t \geq S$. Define $\zeta(S) = \lim\limits_{t\to\infty}\int_0^\infty U(S + \xi, S)u(S)\,d\mu_t(\xi)$. By hypothesis, there is T_1 such that $\|\zeta(S) - \int_0^\infty U(S + \xi, S)u(S)\,d\mu_t(\xi)\| < \varepsilon/6$ for all $t \geq T_1$. We have
$$\|\mu_t(u) - \zeta(S)\| \leq \int_0^S \|u(\xi)\|\,d\mu_t(\xi) + \int_S^\infty \|u(\xi) - U(\xi, S)u(S)\|\,d\mu_t(\xi)$$
$$+ \|\zeta(S) - \int_0^\infty U(S + \xi, S)u(S)\,d\mu_t(\xi + S)\|.$$
For the first term, since $\lim\limits_{t\to\infty}\mu_t([0,S]) = 0$, we can take T_2 such that $\mu_t([0,S]) < \varepsilon/6C$ for all $t \geq T_2$, where $C = \sup_{0\leq\xi\leq S}\|u(\xi)\|$. The second term is less than $\varepsilon/6$. By Hypothesis **H** there is T_3 such that the last term is less than $\varepsilon/6$ whenever $t \geq T_3$. Hence if $t \geq T = \max\{\,T_1,\,T_2,\,T_3\,\}$, we have $\|\mu_t(u) - \zeta(S)\| < \varepsilon/2$ for all $h \geq 0$. We have found $T > 0$ such that $\|\mu_t(u) - \mu_s(u)\| < \varepsilon$

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for all $t, s \geq T$ and therefore $\mu_t(u)$ converges to some y as $t \to \infty$. ∎

If $\mu_t = \delta_t$ the argument above gives an alternative proof of the assertion for the strong topology in Proposition 9 without the Lipschitz assumption. Moreover, this proof is simpler because we do not perform the intermediate step of proving the convergence of $\zeta(p)$. This argument fails when dealing with the weak topology. To overcome this problem, consider the following hypothesis, also satisfied by the families described in Example 16:

Hypothesis **w-H**: For each $\{\mu_t\}$-integrable g with w-$\lim_{t\to\infty} \int_0^\infty g(\xi)\, d\mu_t(\xi) = L$, each $\varepsilon > 0$, $K > 0$ and $f \in X^*$ there exists $T > 0$ such that for all $t \geq T$ one has $\left| \langle \int_0^\infty g(\xi)\, d\mu_t(\xi + K) - L, f \rangle \right| < \varepsilon$.

Under Hypothesis **w-H** the argument above shows that if the orbits of U converge weakly in μ_t-mean, then $\mu_t(u)$ has the Cauchy property for the weak topology whenever u is an almost-orbit of U (i.e. $\lim_{t,s\to\infty} \langle \mu_t(u) - \mu_s(u), \phi \rangle = 0$ for each $\phi \in X^*$). If X is weakly complete (see Remark 14) the net $\{\mu_t(u)\}$ converges weakly. This is a version of Theorem 30 for the weak topology.

Recall that a $\{\mu_t\}$-integrable function v τ-converges to $y \in X$ in μ_t-mean, uniformly with respect to translations if $\mu_t(v_h)$ τ-converges to y as $t \to \infty$ uniformly in $h \geq 0$. This notion includes standard convergence and almost-convergence for the families of measures in Example 16. The uniformity in $h \geq 0$ requires a slightly stronger assumption on $\{\mu_t\}$ (that still hold for the families mentioned above) in order to prove the equivalence results:

Hypothesis $\mathbf{H_u}$: For each $\{\mu_t\}$-integrable g with $\lim_{t\to\infty} \int_0^\infty g(\xi)\, d\mu_t(\xi) = L$, each $\varepsilon > 0$ and $K > 0$ there exists $T > 0$ such that for all $t \geq T$ and $k \in [0, K]$ one has $\left\| \int_0^\infty g(\xi)\, d\mu_t(\xi + k) - L \right\| < \varepsilon$.

Theorem 31 *Let U be an ES and assume $\{\mu_t\}$ satisfies Hypothesis $\mathbf{H_u}$. If $U(t, s)x$ converges strongly in μ_t-mean, uniformly with respect to translations for all x and s, then so does every $\{\mu_t\}$-integrable almost-orbit.*

Proof. Suppose u is a $\{\mu_t\}$-integrable almost-orbit of U and let $\varepsilon > 0$. Choose $S > 0$ such that $\sup_{h\geq 0} \|u(t + h) - U(t + h, t)u(t)\| < \varepsilon/6$ for all $t \geq S$. Define $\zeta(S) = \lim_{t\to\infty} \int_0^\infty U(S + \xi, S)u(S)\, d\mu_t(\xi)$. By hypothesis, there is T_1 such that $\|\zeta(S) - \int_0^\infty U(S + h + \xi, S)u(S)\, d\mu_t(\xi)\| < \varepsilon/6$ for all $t \geq T_1$ and $h \geq 0$ (the convergence is uniform in $h \geq 0$). We divide the rest of the proof in two parts:

$\underline{0 \leq h \leq S}$: As in the proof of Theorem 30 we have
$$\|\mu_t(u_h) - \zeta(S)\| \leq \int_0^{S-h} \|u(h + \xi)\|\, d\mu_t(\xi) + \int_{S-h}^\infty \|u(h + \xi) - U(h + \xi, S)u(S)\|\, d\mu_t(\xi)$$
$$+ \|\zeta(S) - \int_0^\infty U(S + \xi, S)u(S)\, d\mu_t(\xi + (S - h))\|.$$
For the first term, since $\mu_t([0, S]) \to 0$ as $t \to \infty$, we can take T_2 such that $\mu_t([0, S]) < \varepsilon/6C$ for all $t \geq T_2$, where $C = \sup_{0\leq\xi\leq S} \|u(\xi)\|$. The second term

is always less than $\varepsilon/6$. Finally, use Hypothesis $\mathbf{H_u}$ to find T_3 such that the last term is less than $\varepsilon/6$ whenever $t \geq T_3$. Hence if $t \geq T = \max\{\, T_1,\ T_2,\ T_3\,\}$, we have $\|\mu_t(u_h) - \zeta(S)\| < \varepsilon/2$ for all $h \geq 0$.

$\underline{h \geq S}:\ \ \|\mu_t(u_h) - \zeta(S)\| \leq \int\limits_0^\infty \|u(h+\xi) - U(h+\xi,S)u(S)\|\ d\mu_t(\xi)$
$$+ \|\zeta(S) - \int_0^\infty U(h+\xi,S)u(S)\ d\mu_t(\xi)\|,$$
whenever $t \geq T_1$. Each term is less than $\varepsilon/6$, so $\|\mu_t(u_h) - \zeta(S)\| < \varepsilon/3 < \varepsilon/2$ for all $t \geq T_1$ and $h \geq S$. Finally, $\|\mu_t(u_h) - \zeta(S)\| < \varepsilon/2$ for all $t \geq T$ and $h \geq 0$. This implies $\|\mu_t(u_h) - \mu_s(u_k)\| < \varepsilon$ for all $t, s \geq T$ and $h, k \geq 0$ and so u is strongly convergent in μ_t-mean, uniformly with respect to translations. ∎

We obtain the corresponding result for the weak topology (essentially with the same proof if X is weakly complete) if we replace Hypothesis $\mathbf{H_u}$ by:

Hypothesis $\mathbf{w\text{-}H_u}$: For each $\{\mu_t\}$-integrable g with $\text{w-}\lim\limits_{t\to\infty} \int_0^\infty g(\xi)\ d\mu_t(\xi) = L$,
   each $\varepsilon > 0$, $K > 0$ and $f \in X^*$ there exists $T > 0$ such that for all
   $t \geq T$ and $k \in [0, K]$ one has $\left| \langle\ \int_0^\infty g(\xi)\ d\mu_t(\xi + k) - L, f\ \rangle \right| < \varepsilon$.

5   A couple of additional results

For an autonomous CES, every orbit is uniformly continuous. According to [12], so is every continuous almost-orbit. Their proof uses the contracting property, which we show to be unnecessary. The key lies on the time-dependence.

Proposition 32 *Let U be an evolution system. If every orbit is uniformly continuous, so is every continuous almost-orbit.*

Proof. Let u be a continuous almost-orbit of U and $\varepsilon > 0$. First, take $T > 0$ such that $\|u(\tau) - U(\tau, T)u(T)\| < \varepsilon/3$ for all $\tau \geq T$. Since u is continuous, it is uniformly continuous on $[0, T+1]$. Hence there is $\delta_1 > 0$ such that for every $t, s \in [0, T+1]$ satisfying $|t - s| < \delta_1$ one has $\|u(t) - u(s)\| < \varepsilon$. Now consider the function $\tau \mapsto U(\tau, T)u(T)$ defined for $\tau \geq T$. By hypothesis it is uniformly continuous, so there is $\delta_2 > 0$ such that for every $t, s \geq T$ such that $|t - s| < \delta_2$ one has $\|v(t, T)u(T) - U(s, T)u(T)\| < \varepsilon/3$. Therefore, if $t, s \geq T$ and $|t-s| < \delta_2$, the quantity $\|u(t) - u(s)\|$ is bounded by $\|u(t) - U(t, T)u(T)\| + \|U(s, T)u(T) - u(s)\| + \|U(t, T)u(T) - U(s, T)u(T)\|$. For $t, s \in [0, \infty)$ with $|t - s| < \min\{\delta_1, \delta_2, 1\}$ we have $\|u(t) - u(s)\| < \varepsilon$. ∎

The ω-limits of almost-orbits have some kind of invariance under U:

Proposition 33 *Let u be an almost-orbit of an ES U and suppose $\{s_n\}$ is strictly increasing with $\lim\limits_{n\to\infty} s_n = \infty$ and $\tau\text{-}\lim\limits_{n\to\infty} u(s_n) = x^*$.*

i) There exist a sequence $\{t_n\}$ of positive numbers and a sequence $\{x_n\}$ in C

such that $\tau\text{-}\lim\limits_{n\to\infty} x_n = x^*$, $t_n > s_n$ *and* $\tau\text{-}\lim\limits_{n\to\infty} U(t_n, s_n)x_n = x^*$. *The sequence* $\{t_n\}$ *can be chosen such that* $\lim\limits_{n\to\infty}(t_n - s_n) = \infty$.

ii) If U is an M-LES and τ is the strong topology, then $\lim\limits_{n\to\infty} U(t_n, s_n)x^* = x^*$.

Proof. For the first part, let $\varphi : \mathbb{N} \to \mathbb{N}$ be any positive function and set $h_n = s_{n+\varphi(n)} - s_n$. Write $t_n = s_n + h_n$ and $x_n = u(s_n)$. Since u is an almost orbit of V, for every $\varepsilon > 0$ there is $N \geq 0$ such that $\|u(s_n + h_n) - U(s_n + h_n, s_n)u(s_n)\| < \varepsilon$ for all $n \geq N$. Therefore $U(t_n, s_n)x_n - x^* = U(s_n + h_n, s_n)u(s_n) - u(s_n + h_n) + u(s_{n+\varphi(n)}) - x^*$ tends to zero for the topology τ. Clearly φ can be chosen so that $s_{n+\varphi(n)} - s_n$ tends to ∞ as $n \to \infty$. For the second part, just notice that $\|U(t_n, s_n)x^* - x^*\| \leq M\|u(s_n) - x^*\| + \|U(t_n, s_n)u(s_n) - u(t_n)\| + \|u(t_n) - x^*\|$, which tends to zero as $n \to \infty$. ∎

6 Concluding remarks

The tools developed here are potentially useful in different scenarios, namely: in general *asymptotic analysis*, information on the asymptotic behavior of a system can be derived from the study of one that is AAE (as in [14] and [7]). In *numerical analysis*, to determine whether a discretization has the same asymptotic properties as the continuous-time model. For instance, it would be possible to know *a priori* if one must take averages in order to approximate the solution of a problem. In *perturbations theory*, to know how much a system can be perturbed without changing its asymptotic behavior. This could help predict or control the effect of errors and noises. For *ill-posed problems*, to get an idea of what kind of perturbations can force a system to converge when it does not. For example, in some optimization problems it is known that a viscosity term can force a nonconverging system to converge (see [2] or [3]). Several applications in optimization, fixed-point theory, games theory and parabolic equations will be presented in a forthcoming paper in preparation.

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