An LPCC Approach to Nonconvex Quadratic Programs

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Abstract

Filling a gap in nonconvex quadratic programming, this paper shows that the global resolution of a feasible quadratic program (QP), which is not known *a priori* to be bounded or unbounded below, can be accomplished in finite time by solving a linear program with linear complementarity constraints, i.e., an LPCC. Alternatively, this task can be divided into two LPCCs: the first one confirms whether or not the QP is bounded below on the feasible set and computes a feasible ray on which the QP is unbounded if such a ray exists; the second LPCC computes a globally optimal solution if it exists, by identifying a stationary point that yields the best quadratic objective value. In turn, the global resolution of these LPCCs can be accomplished by a parameter-free, mixed integer-programming based, finitely terminating algorithm developed recently by the authors, which can be enhanced in this context by a new kind of valid cuts derived from the second-order conditions of the QP and by exploiting the special structure of the LPCCs. Throughout, our treatment makes no boundedness assumption of the QP; this is a significant departure from much of the existing literature which consistently employs the boundedness of the feasible set as a blanket assumption. The general theory is illustrated by 3 classes of indefinite problems: QPs with simple upper and lower bounds (existence of optimal solutions is guaranteed); same QPs with an additional inequality constraint (extending the case of simple bound constraints); and nonnegatively constrained copositive QPs (no guarantee of the existence of an optimal solution).

1 Introduction

Quadratic programming is among the most important subjects in mathematical programming, having a central role to play in all aspects of the field. Yet, the global resolution of nonconvex quadratic programs (QPs) remains a daunting task to date, especially when it is not known in advance whether or not the problems are bounded below. The survey [8] presents an excellent overview of these nonconvex programs, summarizing in particular the fundamental properties of this challenging class of optimization problems and describing some of the most successful algorithms for special subclasses, including problems with simple bounds and with concave objectives (for minimization).

The present work is inspired by two recent developments in the global resolution of certain nonconvex optimization problems. On one hand, the papers [4, 5, 24, 25] studied in detail an LPCC (for Linear Program with Linear Complementarity Constraints) approach for finding a global minimum of a nonconvex quadratic program (QP) with a finite optimum solution, in particular, a box-constrained QP. On the

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other hand, extending a paper [22] that pertains to a special LPCC arising from the minimization of the value-at-risk, the paper [16] develops a finite, parameter-free mixed integer-programming based algorithm for the global resolution of a general LPCC, without any assumption on the problem. In turn, the key idea behind this algorithm is closely related to the logic-based Benders decomposition for solving linear disjunctive programs [6, 12, 13]. These developments raise a question: is the LPCC approach applicable to a general quadratic program which is not known in advance to have a finite optimal solution? Besides its theoretical interest, a positive answer to this question has several practical implications. Namely, it provides a constructive method that will effectively determine in finite time if a quadratic program is: infeasible, feasible with an unbounded objective, or optimally solvable, with each such outcome being supported by a provably valid certificate. More importantly, this answer fills a long-standing gap in the literature of nonconvex quadratic programming wherein the boundedness of the feasible set has so far been an essential assumption. An exception is the paper [1] that deals with a bilinear program with product constraints on the variables, which is a special nonconvex quadratic program. In this reference, the authors were able to reduce the unbounded case to the bounded case via some auxiliary problems with bounded regions. Another noteworthy point is that the asymptotic convergence of all iterative descent methods for computing a stationary point of a nonconvex QP requires that the QP objective be bounded below on the feasible set, thereby ensuring the existence of a global minimum. Such an iterative method cannot be used to determine the existence or non-existence of a stationary point in finite time.

To place the contribution of this work in a proper context, we recall the well-known Frank-Wolfe theorem [9] which states that a feasible QP has a finite minimum solution if and only if the objective of the QP is bounded below on its feasible set. This fundamental result was subsequently sharpened by Eaves [7] who proved, using Lemke's complementary pivot algorithm applied to the first-order Karush-Kuhn-Tucker (KKT) conditions of the QP formulated as a linear complementarity problem (LCP), that the same necessary and sufficient condition holds with "feasible set" replaced by "feasible rays". These two results beg a question that does not seem to have been fully addressed in the vast literature of quadratic programming; namely, is there a finite procedure to determine if a feasible QP attains a finite minimum solution? A partial answer to this question was provided by Giannessi and Tomasin [10] who showed that if the QP is known to have a finite optimal solution, then a global solution to the QP can be obtained by solving an LPCC obtained from the minimization of an appropriate linear objective function over the set of stationary solutions of the QP. The related paper [2] describes a "constraint activating approach" that effectively solves an LPCC. The finite termination of the algorithm is established under the assumption of the existence of a finite optimum solution to the QP.

Most recently, when the third author of this paper posed the above question to Paul Tseng of the University of Washington in a private communication, Tseng provided a finite procedure to answer the question positively that is based on a complete enumeration of the extreme points and rays of the feasible set of the QP and which requires solving a finite number of nonconvex QPs defined by these extreme points and rays. Our work provides a new LPCC formulation for a general nonconvex QP that fully answers the question in the affirmative. More importantly, employing the parameter-free IP procedure developed in [16], our LPCC approach bypasses the enumeration of extreme points and rays, which is computationally an impossible task. Yet another positive answer to the same question is provided by Adrian Lewis in a private communication who reminded us that using the Tarski-Seidenberg algorithm in mathematical logic, one could answer the question in finite time, by recognizing that the set of quadratic objective values on the feasible region is "semi-algebraic". Nevertheless, this algorithm is at best conceptual in nature and cannot be used for computational purposes in practice.

For ease of reference, we present the LPCC in general form. Given vectors and matrices: $c \in \mathbb{R}^n$, $d \in \mathbb{R}^m$, $e \in \mathbb{R}^m$, $f \in \mathbb{R}^k$, $A \in \mathbb{R}^{k\times n}$, $B \in \mathbb{R}^{k\times m}$, and $C \in \mathbb{R}^{k\times m}$, the LPCC is to find a triple

$(x, y, w) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ in order to globally

$$\begin{array}{ll} \underset{(x,y,w)}{\text{minimize}} & c^T x + d^T y + e^T w \\ \text{subject to} & Ax + By + Cw \geq f \\ \text{and} & 0 \leq y \perp w \geq 0. \end{array} \tag{1}$$

The global resolution of this problem means the generation of a certificate showing that the problem is in one of its 3 possible states: (a) it is infeasible, (b) it is feasible but unbounded below, or (c) it attains a finite optimal solution. Needless to say, linear equations (in addition to linear inequalities as stated above) connecting the variables (x, y, w) are allowed in the constraints of the LPCC; for convenience of presentation, such equality constraints are omitted.

The organization of the remainder of this paper is as follows. The next section presents a preliminary discussion of a general quadratic program, reviewing some known facts to motivate the LPCC formulation that will be introduced and proved in Section 3. In Section 4, we summarize an algorithm, which is based on the one introduced in [16], for globally resolving the LPCC (1). Section 5 discusses the specialization of the key steps in this algorithm to the case of QPs with finite optima. Section 6 focus on 3 classes of indefinite QPs that we use to illustrate the developed methodology; numerical results are presented in Section 7. The paper ends with some concluding remarks in Section 8.

2 Preliminary Discussion

Consider the quadratic program:

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & q(x) \equiv \frac{1}{2} x^T Q x + c^T x \\ \text{subject to} & Ax \leq b, \end{array} \tag{2}$$

where Q is a symmetric, but not necessarily positive semidefinite, matrix of order n, A is an $m \times n$ matrix and c and b are n- and m-vectors, respectively. Without loss of generality, we assume throughout the paper that (2) is feasible (this can easily be decided by solving a linear program). Let X denote the feasible set of (2). Since (2) is obviously equivalent to

$$\begin{array}{ll} \underset{x^\pm \in \mathbb{R}^{2n}}{\text{minimize}} & \frac{1}{2} (x^+ - x^-)^T Q (x^+ - x^-) + c^T (x^+ - x^-) \\ \text{subject to} & A(x^+ - x^-) \leq b \\ \text{and} & x^\pm \geq 0, \end{array}$$

we may assume, to simplify the notation throughout, that the recession cone $\mathcal{D} \equiv \{ d \in \mathbb{R}^n : Ad \leq 0 \}$ of X is contained in the nonnegative orthant $\mathbb{R}^n_+$; this assumption will facilitate the truncation of the recession cone into compact subsets with the addition of the single constraint $\mathbf{1}_n^T d \leq \rho$, for a positive scalar ρ, where $\mathbf{1}_n$ is the n-vector of all ones. A set of the form $\{ x + \tau d : \tau \geq 0 \}$, where $(x, d) \in X \times \mathcal{D}$, constitutes a feasible ray of X.

The Karush-Kuhn-Tucker (KKT) conditions of (2) are given by:

$$\begin{array}{rcl} 0 & = & c + Qx + A^T \xi \\ 0 & \leq & \xi \perp b - Ax \geq 0. \end{array} \tag{3}$$

A feasible vector x of (2) is a KKT point, or equivalently, a stationary point if there exists a multiplier ξ such that the pair (x, ξ) satisfies the above KKT conditions. The tangent cone of X at $x \in X$, denoted by $\mathcal{T}(X; x)$ is equal to $\{ d \in \mathbb{R}^n : A_{i \bullet} d \leq 0 \ \forall \, i \in \mathcal{A}(x) \}$, where $A_{i \bullet}$ denotes the i-th row of A and $\mathcal{A}(x)$

is the index set of binding constraints at x; i.e., $\mathcal{A}(x) \equiv \{i : A_{i\bullet}x = b_i\}$. The critical cone of X at $x \in X$ is by definition the cone $\mathcal{C}_{\text{qp}}(x) \equiv \mathcal{T}(X;x) \cap (c + Qx)^{\perp}$, where $v^{\perp}$ denotes the linear subspace of vectors orthogonal to v. If x is a KKT point, then $\mathcal{C}_{\text{qp}}(x)$ can be represented in terms of any multiplier ξ satisfying the KKT conditions (3); namely:

$$\mathcal{C}_{\text{qp}}(x) = \left\{ \begin{array}{ll} v \in \mathbb{R}^n : & A_{j\bullet}v = 0, \quad \forall\, j \in \alpha(x,\xi) \\ & A_{j\bullet}v \leq 0, \quad \forall\, j \in \beta(x,\xi) \end{array} \right\}$$

where

$$\begin{aligned} \alpha(x,\xi) &\equiv \{\, j : b_j - A_{j\bullet}x = 0 < \xi_j \,\} \\ \beta(x,\xi) &\equiv \{\, j : b_j - A_{j\bullet}x = 0 = \xi_j \,\} \\ \gamma(x,\xi) &\equiv \{\, j : b_j - A_{j\bullet}x > 0 < \xi_j \,\} \end{aligned}$$

are the 3 fundamental index sets associated with the KKT pair (x,ξ). Note that $\alpha(x,\xi) \cup \beta(x,\xi) = \mathcal{A}(x)$ and $\mathcal{C}_{\text{qp}}(x) \cup \mathcal{D} \subseteq \mathcal{T}(X;x)$. The 3 cones, $\mathcal{C}_{\text{qp}}(x)$, $\mathcal{D}$, and $\mathcal{T}(X;x)$ all play significant roles in the QP (2). Notice that while the three index sets $\alpha(x,\xi)$, $\beta(x,\xi)$, and $\gamma(x,\xi)$ depend on both x and ξ, the critical cone $\mathcal{C}_{\text{qp}}(x)$ depends on x only. From the above representation of $\mathcal{C}_{\text{qp}}(x)$, it follows that the lineality space of $\mathcal{C}_{\text{qp}}(x)$ is equal to

$$\mathcal{C}_{\text{qp}}(x) \cap (\,-\mathcal{C}_{\text{qp}}(x)\,) = \bigcap_{i \in \mathcal{A}(x)} \{\, v \in \Re^n : A_{i\bullet}v = 0 \,\}.$$

We recall that a matrix M is copositive on a cone C if $x^T M x \geq 0$ for all $x \in C$; M is strictly copositive on C if $x^T M x > 0$ for all nonzero $x \in C$.

Proposition 1 below summarizes various known facts about the QP (2). Part (a) provides necessary and sufficient conditions for a feasible vector to be a (strict) local minimum; part (b) provides necessary and sufficient conditions for the QP to have a finite optimal solution; part (c) asserts that a quadratic program has only finitely many (possibly zero) stationary values — i.e., values of the objective function at the stationary points, one of which must be the minimum objective value of the QP, provided that the latter is finite. The significance of part (c) is that while the set of stationary points of a QP is in general a continuum, the set of its stationary values is finite. Proofs of these results can be found in the cited references.

Proposition 1. Suppose that the QP (2) is feasible.

(a) [19] A feasible vector $x \in X$ is a (strict) local minimum of (2) if and only if x is a KKT point and Q is (strictly) copositive on $\mathcal{C}_{\text{qp}}(x)$.

(b) [7] The QP (2) attains a global minimum solution if and only if its objective function is bounded below on X, or equivalently, on the feasible rays of X; furthermore, this holds if and only if (a) Q is copositive on $\mathcal{D}$, and (b) $(c + Qx)^T d \geq 0$ for all $(x,d) \in X \times \mathcal{D}$ satisfying $d^T Q d = 0$.

(c) [18] The quadratic objective function attains finitely many values on the set of stationary points of (2).

(d) [10] If the QP (2) has a finite optimal solution, then the minimum objective value is equal to the minimum stationary value. □

Thus, if (2) has an optimal solution, then such a solution can be computed by solving the LPCC:

$$\begin{array}{ll} \underset{(x,\xi)\in\mathbb{R}^{n+m}}{\text{minimize}} & c^T x - b^T \xi \\ \text{subject to} & 0 = c + Qx + A^T \xi \\ & 0 \leq \xi \perp b - Ax \geq 0. \end{array} \tag{4}$$

Nevertheless, the above LPCC alone does not provide the needed information to determine if the QP attains a finite minimum; see the example (5) below. The goal of this paper is to propose a finite procedure for filling this gap, via the introduction of an augmented LPCC whose status will provide a certificate for the finite solvability of the QP. Furthermore, we will discuss how the practical implementation of the procedure can be enhanced by the generation of certain new kinds of cutting planes that must be satisfied by an optimal solution of the QP, if the latter exists.

2.1 Some insights

If the QP (2) is unbounded below, a feasible ray on which the objective function is unbounded need not emanate from any of the stationary points or their convex hull. A simple example is the following QP, which also shows that the finiteness assumption cannot be dropped in part (d) of Proposition 1:

$$\begin{array}{ll} \underset{(x_1,x_2)\in\mathbb{R}^2}{\text{minimize}} & (\,x_1 - 1\,)\,(\,x_2 - 1\,) \\ \text{subject to} & x_1, x_2 \,\geq 0. \end{array} \tag{5}$$

This QP has a unique stationary point, namely, $(1,1)$. The recession cone of the feasible region is $\mathbb{R}^2_+$. It is clear that the objective function is bounded below on any feasible ray starting at the stationary point. Yet the same function is unbounded below on the ray $\{(0,\tau) : \tau \geq 0\}$ emanating from the origin.

The above example illustrates a noteworthy property of a QP. To explain this property, note that if $d \in \mathcal{D}$ is such that $d^TQd > (<)0$, then for any vector x, $q(x+\tau d) \to \infty(-\infty)$ as $\tau \to \infty$. Thus, as far as the boundedness of the quadratic objective function on feasible rays is concerned, the directions of most interest are those recession directions d for which $d^TQd = 0$. Let $\mathcal{D}_0$ be the set of these directions, which we call the *essential recession directions*. Also, let $\mathcal{S}$ be the convex hull of stationary points of the QP (2). The result below states that the objective of (2) is always bounded below on all feasible rays emanating from any convex combination of stationary points along any essential recession direction.

Proposition 2. For any $(x,d) \in \mathcal{S} \times \mathcal{D}_0$, $\liminf_{\tau\to\infty} q(x+\tau d) > -\infty$.

Proof. Since $(c+Q\bar{x})^Td \geq 0$ for all stationary points $\bar{x}$ and all recession directions d, it follows that $(c+Qx)^Td \geq 0$ for all $(x,d) \in \mathcal{S} \times \mathcal{D}_0$. Thus, we have $q(x+\tau d) = q(x) + \tau(c+Qx)^Td + \frac{1}{2}\tau^2 d^TQd \geq q(x)$ for all $\tau \geq 0$. □

Thus we cannot simply focus on the stationary points or their convex combinations to check the (un)boundedness of the QP (2). Instead, we need to broaden the search to points outside the convex hull of stationary points in order to identify unbounded rays.

As a further motivation for the definition of the key LPCC, we note that the truncated QP:

$$\begin{array}{ll} \underset{x\in\mathbb{R}^n}{\text{minimize}} & q(x) \,\equiv\, \frac{1}{2}\,x^TQx + c^Tx \\ \text{subject to} & Ax \,\leq\, b \\ \text{and} & \mathbf{1}_n^Tx \,\leq \rho \end{array} \tag{6}$$

must have a nonempty, bounded feasible set for all $\rho > 0$ sufficiently large, due to our assumption that the recession cone $\mathcal{D} \subset \mathbb{R}^n_+$. Therefore, an optimal solution, say x^ρ, exists, which must satisfy, along with multipliers (ξ^ρ, t_ρ), the following KKT conditions:

$$\begin{array}{rcl} 0 &=& c + Qx^\rho + A^T\xi^\rho + t_\rho\,\mathbf{1}_n \\ 0 &\leq& \xi^\rho \,\perp\, b - Ax^\rho \,\geq\, 0 \\ 0 &\leq& t_\rho \,\perp\, \rho - \mathbf{1}_n^Tx^\rho \,\geq\, 0; \end{array}$$

moreover, the triple $(x^\rho, \xi^\rho, t_\rho)$ can be obtained by solving the LPCC:

$$\begin{array}{ll} \underset{(x,\xi,t)\in\mathbb{R}^{n+m+1}}{\text{minimize}} & c^T x - b^T \xi - t\,\rho \\ \text{subject to} & 0 \,=\, c + Qx + A^T\xi + t\,\mathbf{1}_n \\ & 0 \,\leq\, \xi \,\perp\, b - Ax \,\geq\, 0 \\ & 0 \,\leq\, t \,\perp\, \rho - \mathbf{1}_n^T x \,\geq\, 0. \end{array} \tag{7}$$

This follows from part (d) of Proposition 1, due to the fact that on the set of stationary points of (6), $q(x) = c^T x - b^T \xi - t\,\rho$ for all triples (x, ξ, t) satisfying (7).

If the QP (2) is not known to be bounded below, we need another LPCC to resolve this issue. In turn, this is motivated by an LPCC characterization of the vectors in the cone $\mathcal{D}_0$. Specifically, by considering the copositivity of Q on $\mathcal{D}$, we are led to the following truncated QP:

$$\begin{array}{ll} \underset{d\in\mathbb{R}^n}{\text{minimize}} & \frac{1}{2}\, d^T Q d \\ \text{subject to} & Ad \,\leq\, 0 \\ \text{and} & \mathbf{1}_n^T d \,=\, 1, \end{array} \tag{8}$$

which is equivalent to the following LPCC:

$$\begin{array}{ll} \underset{(d,\lambda,s)\in\mathbb{R}^{n+m+1}}{\text{minimize}} & -s \\ \text{subject to} & 0 \,=\, Qd + A^T\lambda + s\,\mathbf{1}_n \\ & 0 \,\leq\, \lambda \,\perp\, -Ad \,\geq\, 0 \\ & s \text{ free}, \quad 1 - \mathbf{1}_n^T d \,=\, 0. \end{array} \tag{9}$$

Specifically, Q is copositive on $\mathcal{D}$ if and only if the LPCC (9) has a nonnegative optimal objective value. By part (b) of Proposition 1, the copositivity of Q on $\mathcal{D}$ is a necessary condition for the existence of an optimal solution to the QP.

3 The Equivalent LPCC

By combining the two LPCCs (7) and (9) into a single LPCC, we can completely resolve the boundedness of the original QP (2). The key thing in such a combination is to treat the large parameter ρ in (7) implicitly. This leads to the objective function of the following LPCC:

$$\begin{array}{ll} \underset{(x,d,\xi,\lambda,\mu,t,s)\in\mathbb{R}^{2n+3m+2}}{\text{minimize}} & -t \\ \text{subject to} & 0 \,=\, c + Qx + A^T\xi + t\,\mathbf{1}_n \\ & 0 \,=\, Qd + A^T\lambda - A^T\mu + s\,\mathbf{1}_n \\ & 0 \,\leq\, \xi \,\perp\, b - Ax \,\geq\, 0 \\ & 0 \,\leq\, \mu \,\perp\, b - Ax \,\geq\, 0 \\ & 0 \,\leq\, \lambda \,\perp\, -Ad \,\geq\, 0 \\ & 0 \,\leq\, \xi \,\perp\, -Ad \,\geq\, 0 \\ & 0 \,\leq\, \mu \,\perp\, -Ad \,\geq\, 0 \\ & 0 \,\leq\, s, \quad \mathbf{1}_n^T d \,\geq\, 1. \end{array} \tag{10}$$

There are several additional complementarity conditions:

$$0 \leq \mu \perp b - Ax \geq 0, \quad 0 \leq \xi \perp -Ad \geq 0 \quad \text{and} \quad 0 \leq \mu \perp -Ad \geq 0$$

and the additional term $-A^T\mu$ in the second constraint of (10). Roughly speaking, these are imposed to ensure that the feasible ray $x + \tau d$ for $\tau \geq 0$ lies in one face of the region X. This will become clear when we prove the main result of the section. The inequality $\mathbf{1}_n^T d \geq 1$ is a scaling constraint to ensure that d is not equal to zero.

Theorem 3. Suppose the QP (2) is feasible. This QP is unbounded below if and only if the LPCC (10) has a feasible solution with a negative objective value.

Proof. Suppose that QP is unbounded below. Let $\{\rho_k\}$ be a sequence of increasing scalars tending to ∞ such that with x^k being an optimal solution of the truncated QP (6) with $\rho = \rho_k$, we have

$$\lim_{k\to\infty} q(x^k) = -\infty.$$

Let (ξ^k, t_k) be a pair of KKT multipliers corresponding to x^k such that

$$\begin{array}{lll}(x^k, \xi^k, t_k) \in & \underset{(x,\xi,t)\in\mathbb{R}^{n+m+1}}{\text{argmin}} & c^T x - b^T \xi - t\,\rho_k \\ & \text{subject to} & 0 = c + Qx + A^T\xi + t\,\mathbf{1}_n \\ & & 0 \leq \xi \perp b - Ax \geq 0 \\ & \text{and} & 0 \leq t \perp \rho_k - \mathbf{1}_n^T x \geq 0. \end{array}$$

If $t_k = 0$, then x^k is a stationary point of the original QP (2). Since the quadratic function $q(x)$ attains only finitely many values on the set of such stationary points, it follows that except for finitely many k's, we must have $t_k > 0$. Consequently, without loss of generality, we may assume that $t_k > 0$ for all k, implying that $\rho_k = \mathbf{1}_n^T x^k$ for all k; thus the sequence $\{x^k\}$ is unbounded. Let α be an index set (possible empty) with complement $\bar{\alpha}$ such that $(Ax^k - b)_\alpha = 0$ and $(Ax^k - b)_{\bar{\alpha}} < 0$ for infinitely many k. By working with the corresponding subsequence of $\{x^k\}$, we may assume without loss of generality that

$$x^k \in \widehat{X} \equiv \{ x \in X : (Ax - b)_\alpha = 0 \} \text{ and } (Ax^k - b)_{\bar{\alpha}} < 0, \quad \forall\, k.$$

Thus, by complementarity, $\xi^k_{\bar{\alpha}} = 0$ for all k. Let $\widehat{X}^e$ be the convex hull of the extreme points of $\widehat{X}$ and $\widehat{\mathcal{D}}$ be the recession cone of $\widehat{X}$; note that

$$\widehat{D} = \{ d \in \mathcal{D} : (Ad)_\alpha = 0 \} \subset \mathbb{R}^n_+.$$

It follows that for each k, we can write

$$x^k \equiv \widehat{x}^k + \tau_k\, d^k,$$

for some $\widehat{x}^k \in \widehat{X}^e$, τ_k is a nonnegative scalar, and $d^k \in \widehat{\mathcal{D}}$ satisfying $\mathbf{1}_n^T d^k = 1$. Since $\{x^k\}$ is unbounded, we may assume, by working once more with a subsequence if necessary, that $\tau_k > 0$ for all k; since each d^k has unit norm, it follows that $\lim_{k\to\infty} \tau_k = \infty$. Without loss of generality, we may assume that the sequence $\{d^k\}$ converges to a vector d^∞, which must necessarily belong to $\widehat{D}$ and satisfy $\mathbf{1}_n^T d^\infty = 1$. Since

$$q(x^k) = q(\widehat{x}^k) + \tau_k\,(c + Q\widehat{x}^k)^T d^k + \frac{\tau_k^2}{2}\,(d^k)^T Q d^k,$$

$q(x^k) \to -\infty$, and $\tau_k \to \infty$, it follows that $(d^\infty)^T Q d^\infty \leq 0$. Hence the quadratic program:

$$\begin{array}{ll} \underset{d}{\text{minimize}} & \frac{1}{2}\, d^T Q d \\ \text{subject to} & d \in \widehat{\mathcal{D}} = \{ d : Ad \leq 0,\ (Ad)_\alpha \geq 0 \} \\ \text{and} & \mathbf{1}^T d \leq 1 \end{array} \tag{11}$$

has a optimal solution $\widehat{d}$ satisfying $\widehat{d}^T Q \widehat{d} \leq 0$. Without loss of generality, we may assume that $\widehat{d} \neq 0$; otherwise, we may use d^∞ to play the role of $\widehat{d}$. Together with suitable multipliers $\widehat{\lambda}$, $\widehat{\mu}$ with $\widehat{\mu}_{\bar{\alpha}} = 0$, and $\widehat{s}$, the vector $\widehat{d}$ satisfies the following KKT conditions of (11):

$$\begin{aligned}
0 &= Qd + A^T\lambda - A^T\mu + s\mathbf{1}_n \\
0 &\leq \lambda \perp -Ad \geq 0 \\
0 &\leq \mu \perp -Ad \geq 0 \\
0 &\leq s \perp 1 - \mathbf{1}_n^T d \geq 0.
\end{aligned}$$

Clearly $\xi^k \perp -A\widehat{d}$ and $\widehat{\mu} \perp b - Ax^k$. Consequently, the tuple $\left(x^k, \sigma\widehat{d}, \xi^k, \sigma\widehat{\lambda}, \sigma\widehat{\mu}, t_k, \sigma\widehat{s}\right)$, where $\sigma \equiv 1/\mathbf{1}_n^T\widehat{d}$, is feasible to the LPCC (10) with $t_k > 0$. This completes the "only if" part of the theorem.

Conversely, suppose that the LPCC (10) has a feasible tuple $(x, d, \xi, \lambda, \mu, t, s)$ with $t > 0$. Then

$$0 = d^T(\,Qd + A^T\lambda - A^T\mu + s\,\mathbf{1}_n\,) = d^TQd + s\mathbf{1}_n^T d,$$

implying that $d^TQd \leq 0$. Moreover,

$$0 = d^T(\,c + Qx + A^T\xi + t\,\mathbf{1}_n\,) = d^T(\,c + Qx\,) + t\,\mathbf{1}_n^T d,$$

which implies $d^T(c + Qx) = -t\,\mathbf{1}_n^T d < 0$. Consequently, $q(x + \tau d) \to -\infty$ as $\tau \to \infty$, showing that the QP (2) is unbounded. □

Comparing the constraints in (9) and (10), we note that we need an extra variable μ to restrict the recession directions d; this is due to the fact that we have worked with the cone $\widehat{\mathcal{D}}$ in the above proof, instead of the full recession cone $\mathcal{D}$, and more importantly, to the possibility that the limiting vector d^∞ is not necessarily an optimal solution of (11) or of (8). Nevertheless, if Q is known to be copositive on $\mathcal{D}$, then we can resort to the latter simplified QP and drop the variable μ. This special case is summarized in the following corollary.

Corollary 4. Suppose the QP (2) is feasible and Q is copositive on $\mathcal{D}$. This QP is unbounded below if and only if the LPCC:

$$\begin{array}{ll}
\underset{(x,d,\xi,\lambda,t)\in\mathbb{R}^{2(n+m)+1}}{\text{minimize}} & -t \\
\text{subject to} & 0 = c + Qx + A^T\xi + t\,\mathbf{1}_n \\
& 0 = Qd + A^T\lambda \\
& 0 \leq \xi \perp b - Ax \geq 0 \\
& 0 \leq \lambda \perp -Ad \geq 0 \\
& 0 \leq \xi \perp -Ad \geq 0 \\
& 1 \leq \mathbf{1}_n^T d
\end{array} \tag{12}$$

has a feasible solution with a negative objective value.

Proof. Since every feasible solution of (12), together with $\mu = 0$ and $s = 0$, must be feasible to (10), the "if" statement follows from Theorem 3. Conversely, under the copositivity assumption of Q on $\mathcal{D}$, the vector d^∞ in the proof of the theorem must be an optimal solution to the following homogeneous QP:

$$\begin{array}{ll}
\underset{d}{\text{minimize}} & \frac{1}{2}\,d^TQd \\
\text{subject to} & Ad \leq 0.
\end{array}$$

By writing down the optimality conditions of the latter QP and scaling d^∞ if necessary, the proof of the corollary is completed similarly to the proof of Theorem 3. $\square$

Summarizing the derivations, we conclude that the complete resolution of the QP (2) can be accomplished by solving two LPCCs: the first one (10), which can be written in the following compact form: find $(x, d, \xi, \lambda, \mu, y, w, \zeta, \varphi, t, s)$ in $\mathbb{R}^{2n+7m+2}$ to

$$\begin{array}{ll}
\text{minimize} & -t \\
\text{subject to} & (\, x, d\,) \text{ free}, \quad (\, \xi, \lambda, \mu, s\,) \geq 0, \quad \mathbf{1}_m^T d \geq 1
\end{array}$$

$$\begin{pmatrix} c \\ 0 \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{bmatrix} Q & 0 & A^T & 0 & 0 & \mathbf{1}_n & 0 \\ 0 & Q & 0 & A^T & -A^T & 0 & \mathbf{1}_n \\ -A & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_m & 0 & \mathbf{I}_m & 0 & 0 \\ 0 & 0 & \mathbf{I}_m & \mathbf{I}_m & \mathbf{I}_m & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ d \\ \xi \\ \lambda \\ \mu \\ t \\ s \end{pmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -\mathbf{I}_m & 0 \\ 0 & -\mathbf{I}_m \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} y \\ w \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -\mathbf{I}_m & 0 \\ 0 & -\mathbf{I}_m \end{bmatrix} \begin{pmatrix} \zeta \\ \varphi \end{pmatrix} = 0$$

$$\text{and} \qquad 0 \leq \begin{pmatrix} y \\ w \end{pmatrix} \perp \begin{pmatrix} \zeta \\ \varphi \end{pmatrix} \geq 0$$

to determine if the QP (2) has a finite optimal solution; and, once the existence of an optimal solution is affirmatively determined, the second one (4) to compute such a solution.

4 Solving the General LPCC

To pave the way for the solution of the LPCCs derived from the QPs, we summarize the key computational steps of the logical Benders method for solving the LPCC in general form (1); details of the method and a proof of finite termination with the asserted conclusions are presented in [16]. To begin, we consider a mixed integer program (MIP) formulation of (1) with a conceptually very large parameter $\theta > 0$:

$$\begin{array}{llcll}
\underset{(x,y,w,z)}{\text{minimize}} & c^T x + d^T y + e^T w & & & \\
\text{subject to} & Ax + By + Cw & \geq & f & (\, u\,) \\
& -w & \geq & -\theta\, z & (\, v^+\,) \\
& -y & \geq & -\theta\,(\, \mathbf{1} - z\,) & (\, v^-\,) \\
& w, y & \geq & 0; \quad \text{and} \quad z \in \{\, 0, 1\,\}^m &
\end{array}$$

where the variables u and $v^\pm$ written in the parentheses are the dual variables of the respective constraints. A main difficulty in dealing with the above MIP is that the scalar θ is not explicitly available; indeed, it may not even exist!

Associated with the above MIP, we define the value function:

$$\begin{array}{rlll} \{\pm\infty\} \cup \mathbb{R}^n \ni \varphi(z) \equiv & \underset{u,v^\pm}{\text{maximum}} & f^T u & \\ & \text{subject to} & A^T u = c & \\ & & B^T u - v^- \leq d & \\ & & C^T u - v^+ \leq e & \\ & & u, v^\pm \geq 0 & \\ & \text{and} & z^T v^+ + (\mathbf{1} - z)^T v^- \leq 0. & \end{array}$$

Letting $\alpha \equiv \{i : z_i = 1\}$ denote the *support* of z and $\bar{\alpha}$ be the complement of α in $\{1, \cdots, m\}$, and noticing that any feasible solution $(u, v^\pm)$ to the above LP must satisfy $v^+_{\bar{\alpha}} = 0$ and $v^-_{\alpha} = 0$, we arrive at the following simplified expression for the above value function:

$$\begin{array}{rlll} \varphi(z) \overset{\alpha=\text{supp}(z)}{=\!=\!=\!=} & \underset{u \geq 0}{\text{maximum}} & f^T u & \\ & \text{subject to} & A^T u = c & \\ & & (B_{\bullet\bar{\alpha}})^T u \leq d_{\bar{\alpha}} & \\ & \text{and} & (C_{\bullet\alpha})^T u \leq e_\alpha, & \end{array} \tag{13}$$

where $B_{\bullet\bar{\alpha}}$ and $C_{\bullet\alpha}$ denote the columns of B and C indexed by $\bar{\alpha}$ and α, respectively, and $d_{\bar{\alpha}}$ and e_α denote the subvectors of d and e indexed by these index sets, respectively. [The omitted constraints $(B_{\bullet\alpha})^T u - v^+_\alpha \leq d_\alpha$ and $(C_{\bullet\bar{\alpha}})^T u - v^-_{\bar{\alpha}} \leq e_{\bar{\alpha}}$ can always be satisfied by choosing v^+_α and $v^-_{\bar{\alpha}}$ sufficiently large.] It is easily seen that the maximization problem (13) is the dual of the linear program (LP) piece of the LPCC (1) corresponding to the binary variable z, or equivalently, the index set α:

$$\begin{array}{ll} \underset{(x,y,w_\alpha,y_{\bar{\alpha}})}{\text{minimize}} & c^T x + \sum_{i \notin \alpha} d_i y_i + \sum_{i \in \alpha} e_i w_i \\ \text{subject to} & Ax + B_{\bullet\bar{\alpha}} y_{\bar{\alpha}} + C_{\bullet\alpha} w_\alpha \geq f \\ & w_\alpha \geq 0, \quad \text{and} \quad y_{\bar{\alpha}} \geq 0, \end{array} \tag{14}$$

where we have dropped the variables $y_\alpha = 0$ and $w_{\bar{\alpha}} = 0$. Associated with a given binary vector z, we also define the homogeneous value function:

$$\begin{array}{rlll} \{0, \infty\} \ni \varphi_0(z) \overset{\alpha=\text{supp}(z)}{=\!=\!=\!=} & \underset{u \geq 0}{\text{maximum}} & f^T u & \\ & \text{subject to} & A^T u = 0 & \\ & & (B_{\bullet\bar{\alpha}})^T u \leq 0 & \\ & \text{and} & (C_{\bullet\alpha})^T u \leq 0 & \end{array} \tag{15}$$

and the set:

$$\mathcal{Z} \equiv \{\{0,1\}^m : \varphi_0(z) = 0\},$$

whose elements are the *feasibility descriptors* of the LP pieces of the LPCC (1). Specifically, $z \in \mathcal{Z}$ if and only if the LP (14) is feasible. The following result gives the fundamental connection between the LPCC (1) and the minimization problem

$$\text{minimize} \quad \varphi(z) : z \in \mathcal{Z}. \tag{16}$$

In this result, we let $\text{LPCC}_{\min} \in \{\pm\infty\} \cup \mathbb{R}$ denote the extended optimal objective value of (1). A proof of the theorem can be found in [16].

Theorem 5. The following three statements hold:

(a) the LPCC (1) is infeasible if and only if $\min_{z \in \mathcal{Z}} \varphi(z) = \infty$ (i.e., $\mathcal{Z} = \emptyset$);

(b) the LPCC (1) is feasible and has an unbounded objective value if and only if $\min_{z \in \mathcal{Z}} \varphi(z) = -\infty$ (i.e., $z \in \mathcal{Z}$ exists such that $\varphi(z) = -\infty$);

(c) the LPCC (1) attains a finite optimal objective value if and only if $-\infty < \min_{z \in \mathcal{Z}} \varphi(z) < \infty$.

In all cases, $\mathrm{LPCC}_{\min} = \min_{z \in \mathcal{Z}} \varphi(z)$; moreover, for any $z \in \{0,1\}^m$ for which $\varphi(z) > -\infty$, $\mathrm{LPCC}_{\min} \leq \varphi(z)$. □

At the beginning of an iteration, a pool $\widehat{\mathcal{Z}}$ of satisfiability constraints in the binary variable z, each of the form:

$$\sum_{i \in \mathcal{I}} z_i + \sum_{j \in \mathcal{J}} (\, 1 - z_j \,) \, \geq \, 1, \tag{17}$$

for two disjoint index subsets $\mathcal{I}$ and $\mathcal{J}$ of $\{1, \cdots, m\}$, and an upper bound, denoted $\mathrm{LPCC}_{\mathrm{ub}}$, to $\mathrm{LPCC}_{\min}$ are given. [Initially, $\mathrm{LPCC}_{\mathrm{ub}} = \infty$ and $\widehat{\mathcal{Z}}$ is empty.] Select a binary vector $\widehat{z} \in \{0,1\}^m$ satisfying the constraints in $\widehat{\mathcal{Z}}$ by a specialized algorithm for solving satisfiability problems, such as the MAX-SAT method described in [17, 3]. If no such $\widehat{z}$ exists, then either (1) is infeasible or a globally optimal solution to the LPCC (1) is on hand; in turn this will depend on whether $\widehat{\mathcal{Z}}$ contains point cuts (see below for definition). Otherwise, with $\widehat{z}$ computed, we consider the LP $\varphi(\widehat{z})$, which is either: (a) infeasible; (b) feasible and unbounded; or (c) feasible with a finite optimum solution. In case (a), we consider the homogeneous problem $\varphi_0(\widehat{z})$. If $\varphi_0(z) = \infty$, then a nonzero ray u^r of (15) is obtained, which defines the *ray cut*:

$$\sum_{i \in \alpha : (\, B^T u^r \,)_i > 0} (\, 1 - z_i \,) + \sum_{i \in \bar{\alpha} : (\, C^T u^r \,)_i > 0} z_i \, \geq \, 1. \tag{18}$$

A sparsification of this cut (see below) will be added to the pool $\widehat{\mathcal{Z}}$ of generated cuts. If (15) has zero as its optimal objective value, then the LPCC is feasible and unbounded. In case (b), an extreme ray u^r of (13) is obtained and a similar cut is formed. In case (c), an optimal extreme point solution u^p of (13) is obtained; this point defines a *point cut* of the form:

$$\sum_{i \in \alpha : (\, B^T u^p - d \,)_i > 0} (\, 1 - z_i \,) + \sum_{i \in \bar{\alpha} : (\, C^T u^p - e \,)_i > 0} z_i \, \geq \, 1. \tag{19}$$

A sparsification of this cut will be added to the pool $\widehat{\mathcal{Z}}$ of generated cuts. The iteration ends with a new pool $\widehat{\mathcal{Z}}$ if the algorithm is not yet terminated with a bounded homogeneous problem (15). As observed and explained in detail in [16], sparsification of the satisfiability inequalities is a very important step in the practical success of the overall algorithm.

To end this section, we summarize the key steps in a finite algorithm for globally resolving the LPCC.

Sketch of an algorithm

Step 0. Generate some initial cuts by a pre-processing procedure.

Step 1. Solve a satisfiability feasibility system to determine a binary vector z and let $\alpha \equiv \mathrm{supp}(z)$.

Step 2. Solve the homogeneous LP (15). If $\varphi_0(z) = \infty$, then a ray cut is obtained. Otherwise, solve either the primal LP (14) or its dual (13) to obtain either a point cut or an unboundedness certificate for the LPCC.

Step 3. Apply a problem-specific procedure to sparsify the obtained cut(s), by solving tight LP relaxations of the LPCC restricted by the sparsified cut under testing. Add the sparsified cuts to update the satisfiability system. Return to Step 1.

5 Specialization to Solvable QPs

The details of the algorithm sketched above lie in the generation of the cuts and the sparsification step. This section presents these details for the LPCC (4), whose conceptual MIP formulation is:

$$\begin{array}{lll} \underset{(x,\xi,z)\in\mathbb{R}^{n+3m}}{\text{minimize}} & c^T x - b^T \xi & \\ \text{subject to} & 0 \,=\, c + Qx + A^T\xi & \\ & 0 \,\le\, b - Ax \,\le\, \theta\, z & (20) \\ & 0 \,\le\, \xi \,\le\, \theta\,(\,\mathbf{1}_m - z\,) & \\ \text{and} & z \,\in\, \{\,0,1\,\}^m. & \end{array}$$

By focusing on this LPCC, we are restricting the discussion to the solvable QP (2) that we assume to attain a finite optimal solution. In Subsection 6.3, we very briefly mention a nonnegatively constrained copositive QP that is not necessarily bounded.

By sparsification of the satisfiability inequality (17), we mean testing if the *sparser* inequality is valid:

$$\sum_{i\in\mathcal{I}_1} z_i + \sum_{j\in\mathcal{J}_1} (\,1 - z_j\,) \,\ge\, 1, \tag{21}$$

where $\mathcal{I}_1$ and $\mathcal{J}_1$ are (proper) subsets of $\mathcal{I}$ and $\mathcal{J}$, respectively. Whereas the goal of the sparsification step is to obtain a valid satisfiability inequality with as few terms as possible, one must in general balance the work required in this step (see below) with the strength of the resulting sparsified inequality. In particular, a judicious choice of the subsets $\mathcal{I}_1$ and $\mathcal{J}_1$ is important for the overall efficiency of the algorithm; such a choice remains an open task that deserves further investigation.

To test the validity of the constraint (21), we set $z_i = 0$ for all $i \in \mathcal{I}_1$ and $z_j = 1$ for all $j \in \mathcal{J}_1$ in (20) and restore the complementarity formulation of the resulting restricted IP:

$$\begin{array}{lll} \underset{(x,\xi)\in\mathbb{R}^{n+2m}}{\text{minimize}} & c^T x - b^T \xi & \\ \text{subject to} & 0 \,=\, c + Qx + A^T\xi & \\ & 0 \,\le\, b - Ax \,\perp\, \xi \,\ge\, 0 & (22) \\ & \xi_{\mathcal{J}_1} \,=\, 0 \quad \text{and} \quad (\,b - Ax\,)_{\mathcal{I}_1} \,=\, 0, & \end{array}$$

which remains an LPCC. We relax the complementarity constraints by lifting certain products of variables as follows. Specifically, defining

$$\zeta_{ik} \,\equiv\, x_i\xi_k \quad \text{for all } i,k \text{ such that } a_{ki} \ne 0 \text{ and } k \notin \mathcal{J}_1, \tag{23}$$

we have, for all $k \notin (\mathcal{J}_1 \cup \mathcal{I}_1)$,

$$0 \,=\, \xi_k w_k \,=\, b_k\xi_k - \sum_{i\,:\,a_{ki}\ne 0} a_{ki}\,\zeta_{ik}.$$

A (very loose) LP relaxation of (22) is

$$\begin{array}{ll} \underset{(x,\xi,y,\zeta)}{\text{minimize}} & c^T x - b^T \xi \\ \text{subject to} & 0 \,=\, c + Qx + \displaystyle\sum_{j\notin\mathcal{J}_1} (\,A_{j\bullet}\,)^T\xi_j \\ & 0 \,=\, (\,b - Ax\,)_i, \quad i \,\in\, \mathcal{I}_1 \\ & 0 \,\le\, (\,b - Ax\,)_i, \quad i \,\notin\, \mathcal{I}_1 \\ & 0 \,=\, b_k\xi_k - \displaystyle\sum_{i\,:\,a_{ki}\ne 0} a_{ki}\,\zeta_{ik}, \quad k \,\notin\, \mathcal{J}_1 \\ \text{and} & \xi_j \,\ge\, 0, \quad j \,\notin\, \mathcal{J}_1, \end{array}$$

wherein the nonlinear definitions of the variables ζ_{ik} are dropped. Needless to say, such a preliminary relaxation cannot be expected to be tight without further restricting these auxiliary variables. Instead of discussing this issue in a general setting, for which the reader can consult [23] for some bounding and enveloping techniques for handling products of variables, we will specialize the discussion to problems with bounded variables.

5.1 Valid cuts from second-order necessary conditions

The second-order necessary optimality condition for the QP (2) stipulates that if x is a local minimum, then the matrix Q must be copositive on $\mathcal{C}_{\rm qp}(x)$. Consequently, we have the following corollary.

Proposition 6. If x is a local minimum of the QP (2), then Q is positive semidefinite on the lineality space of $\mathcal{C}_{\rm qp}(x)$. □

Proposition 6 motivates some valid inequalities for the MIP (20). To introduce these inequalities, we define the family of index sets:

$$\mathcal{J} \equiv \{ J \subseteq \{1, \cdots, m\} : Q \text{ is not positive definite on the kernel of the matrix } A_{J\bullet} \}$$

and the set:

$$\mathcal{Z}_2 \equiv \left\{ z \in \mathcal{Z} : \sum_{j \notin J} (1 - z_j) \geq 1, \quad \forall J \in \mathcal{J} \right\}.$$

We call each inequality in $\mathcal{Z}_2$ corresponding to a $J \in \mathcal{J}$ a *2nd-order cut* of the mixed IP (20). Roughly speaking, the proposition below asserts that if the QP (2) has an optimal solution, then it must have one such that a corresponding binary descriptor of that solution belongs to the set $\mathcal{Z}_2$, or equivalently, that for every index set $J \in \mathcal{J}$, at least one constraint corresponding to an index $j \notin J$ must be satisfied as an equality by that optimal solution.

Proposition 7. If the QP (2) has a finite optimal solution, then $\text{QP}_{\min} = \min\limits_{z \in \mathcal{Z}_2} \varphi(z)$.

Proof. It suffices to show $\text{QP}_{\min} \geq \min\limits_{z \in \mathcal{Z}_2} \varphi(z)$; in turn, it suffices to identify a binary vector $z \in \mathcal{Z}_2$ such that $\text{QP}_{\min} = \varphi(z)$. For this purpose, take any optimal solution x^* of (2); let ξ^* be a KKT multiplier associated with x^*. Let $z^0 \in \mathcal{Z}$ be such that $\text{supp}(z^0) \equiv \{i : (b - Ax^*)_i > 0\} \equiv \alpha_0$. Let $\bar{\alpha}_0$ be the complement of α_0 in $\{1, \cdots, m\}$. The pair (x^*, ξ^*) must be an optimal solution to the LP piece corresponding to α_0; i.e., $\text{QP}_{\min} = \varphi(z^0)$, where

$$\begin{array}{rll} \varphi(z^0) = & \underset{(x,\xi) \in \mathbb{R}^{n+m}}{\text{minimize}} & c^T x - b^T \xi \\ & \text{subject to} & 0 = c + Qx + A^T \xi \\ & & 0 = (b - Ax)_{\bar{\alpha}_0} \\ & & 0 \leq (b - Ax)_{\alpha_0} \\ & & 0 = \xi_{\alpha_0}, \quad \text{and} \quad 0 \leq \xi_{\bar{\alpha}_0}. \end{array}$$

Suppose that $\sum\limits_{j \notin J_0} (1 - z_j^0) = 0$ for some $J_0 \in \mathcal{J}$. We then have $z_j^0 = 1$ for all $j \notin J_0$. Thus $\xi_j^* = 0$ for all $j \notin J_0$. If $(b - Ax^*)_{\widehat{j}} = 0$ for some index $\widehat{j} \notin J_0$, then defining the binary vector z^* such that $z_k^* = z_k^0$ for all $k \neq \widehat{j}$ and $z_{\widehat{j}}^* = 0$, we see that $\varphi(z^*) = \text{QP}_{\min}$ and $z^* \in \mathcal{Z}_2$. Thus, without loss of generality, we may assume that $(b - Ax^*)_j > 0$ for all $j \notin J_0$. Since $J_0 \in \mathcal{J}$, there exists a vector d with at least one negative component satisfying $A_{j\bullet} d = 0$ for all $j \in J_0$ and $d^T Q d \leq 0$. It then follows that $x^* + \tau d$ remains feasible to the QP (2) for all $\tau > 0$ sufficiently small; moreover, with $\tau > 0$ properly chosen, $b_j = A_{j\bullet}(x^* + \tau d)$ for at least one $j \notin J_0$. Furthermore, we have

$$0 = (c + Qx^*)^T d + \sum_{j=1}^{m} \xi_j^* (A_{j\bullet} d) = (c + Qx^*)^T d.$$

Consequently, it follows that

$$\begin{aligned} c^T(x^* + \tau d) + \tfrac{1}{2}(x^* + \tau d)^T Q(x^* + \tau d) &= c^T x^* + \tfrac{1}{2}(x^*)^T Q x^* + \tau (c + Qx^*)^T d + \frac{\tau^2}{2} d^T Q d \\ &\leq c^T x^* + \tfrac{1}{2}(x^*)^T Q x^* = \mathrm{QP}_{\min}. \end{aligned}$$

Hence for an appropriate $\tau > 0$ sufficiently small, $x^* + \tau d$ remains an optimal solution to the QP (2) and satisfies at least one more constraint as an equality than x^*. Proceeding in this manner, we arrive at either an optimal solution of the QP with a corresponding binary descriptor belonging to $\mathcal{Z}_2$, or an optimal solution that satisfies all the constraints as binding. In either case, the proposition follows. $\square$

In principle, we could enlarge the family of second-order cuts by employing the original second-order condition that involves the copositivity of Q on critical cones. The reason for restricting to the above family $\mathcal{J}$ is that checking positive definiteness is much simpler than checking copositivity.

6 Simply Constrained QPs

In this section, we further specialize the discussion to three special subclasses of indefinite QPs: (a) bounded-variable problems, (b) bounded-variable problems with one additional inequality constraint, and (c) nonnegatively constrained, possibly unbounded problems. We restrict our discussion to these problems for the following reasons:

• Bounded-variable indefinite QPs have been studied extensively; our goal is to demonstrate, via computational results and comparisons with the recent work [25], that the global resolution methods for solving LPCCs enhanced by their special structures lead to promising new algorithms for solving these QPs.

• The second class of QPs provides credible evidence establishing the viability of the LPCC approach for solving QPs with finite optima. Admittedly, one-constraint problems are very special; yet our discussion sheds light on generalizations which we contend would require deeper investigation that goes beyond the scope of this paper.

• The third class of QPs provides supporting evidence showing that the LPCC approach is capable of positively detecting unbounded problems, a task that no practical algorithm is known to be able to accomplish thus far.

• Most importantly, whereas the primary goal of this paper is to provide compelling evidence to support the LPCC approach to indefinite QPs, both theoretically and computationally, our contention is that to cover this approach in full detail cannot be accomplished in a single work. Thus, with the theoretical results in Section 3 and the preliminary computational results reported subsequently, we hope to convince the readers that the LPCC is of fundamental importance in the treatment of general indefinite QPs and that a further investigation of this approach is warranted in future research.

6.1 Bounded-variable QPs

Consider the following QP with simple upper and lower bounds:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \tfrac{1}{2} x^T Q x + c^T x \\ \text{subject to} & 0 \leq x \leq \mathbf{1}_n, \end{array} \tag{24}$$

where Q is symmetric. By our approach, the MIP formulation for this QP is: for $\theta > 0$ sufficiently large,

$$
\begin{array}{lll}
\underset{(x,y,z,\lambda)}{\text{minimize}} & c^T x - \mathbf{1}_n^T y & \\
\text{subject to} & 0 \leq c + Qx + y \leq \theta\, z & \\
& 0 \leq \mathbf{1}_n - x \leq \theta\, \lambda & \\
& 0 \leq x \leq \theta\,(\mathbf{1}_n - z\,) & \qquad (25)\\
& 0 \leq y \leq \theta\,(\mathbf{1}_n - \lambda\,) & \\
\text{and} & z, \lambda \in \{\,0, 1\,\}^n. &
\end{array}
$$

A remark is in order. While a suitable scalar θ can easily be computed for this problem, the computational results in [24] suggest that a specialized branch-and-cut method proposed by the authors of the reference outperforms default MINTO [21] settings. Subsequently, we will compare our results with the branch-and-cut results; for this purpose, we continue to use the above MIP without fixing θ.

Valid cuts

Since x cannot equal 0 and 1 simultaneously, it follows that the inequality

$$(\,\mathbf{1}_n - z\,) + \lambda \geq \mathbf{1}_n. \quad \text{or equivalently,} \quad \lambda \geq z, \tag{26}$$

must be valid for all binary pairs (z, λ) satisfying (25). Next we discuss certain second-order cuts determined by the low-order principal submatrices of Q. Being consequences of Proposition 7, these cuts are sparse, easy to generate, and will be placed in the pre-processing step of the algorithm. In applying this proposition, we note that in the notation of (20), the matrix $A = \begin{bmatrix} \mathbf{I}_n \\ -\mathbf{I}_n \end{bmatrix}$; the following cuts are obtained by taking the matrix $A_{J\bullet}$ with the index set J in the family $\mathcal{J}$ to consist of all rows of A except for a few pairs of rows of the identity matrix and its negative.

• Second-order cuts of order 1. Suppose that a diagonal entry of Q, say q_{jj}, is non-positive. Proposition 7 yields the cut $z_j + (\,1 - \lambda_j\,) \geq 1$; thus for such an index j, we must have $z_j = \lambda_j$, by combining the latter inequality with (26). Incidentally, these order 1-cuts have been recognized as early as in the work [11] and also used recently in [24, 25]. Nevertheless, the second-order cuts of higher order described below are introduced here for the first time.

• Second-order cuts of order 2. Suppose q_{ii} and q_{jj} are two positive diagonal entries of Q such that

$$\det \begin{bmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{bmatrix} \leq 0.$$

Again, it follows from the same proposition that the following inequality must be valid:

$$z_i + z_j + (\,1 - \lambda_i\,) + (\,1 - \lambda_j\,) \geq 1.$$

• Second-order cuts of order 3. Suppose (i, j, k) are 3 distinct indices such that the following three 2×2 matrices:

$$\begin{bmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{bmatrix}, \quad \begin{bmatrix} q_{ii} & q_{ik} \\ q_{ki} & q_{kk} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} q_{jj} & q_{jk} \\ q_{kj} & q_{kk} \end{bmatrix}$$

are all positive definite but

$$\det \begin{bmatrix} q_{ii} & q_{ij} & q_{ik} \\ q_{ji} & q_{jj} & q_{jk} \\ q_{ki} & q_{kj} & q_{kk} \end{bmatrix} \leq 0.$$

Similarly to the cuts of order 2, it follows that the cut: $z_i + z_j + z_k + (1-\lambda_i) + (1-\lambda_j) + (1-\lambda_k) \geq 1$ must be valid.

• Second-order cuts of order ≥ 4. These are not generated for several reasons: they contain more terms, thus increasing the complexity of finding a binary vector obeying the satisfiability system and diminishing their strength; the generation of these cuts requires the evaluation of determinants of higher-order matrices and there are too many of them.

Summarizing, we obtain the following valid cuts that we can add permanently to the MIP (25):

Valid cuts for (25)

(A) $\lambda_j + (1 - z_j) \geq 1$ for all j;

(B) $z_j = \lambda_j$ for all j such that $q_{jj} \leq 0$;

(C) $z_i + z_j + (1-\lambda_i) + (1-\lambda_j) \geq 1$ for all $i \neq j$ such that $\min(q_{ii}, q_{jj}) > 0$ and $q_{ij}^2 \geq q_{ii}q_{jj}$;

(D) $z_i + z_j + z_k + (1-\lambda_i) + (1-\lambda_j) + (1-\lambda_k) \geq 1$ for all triples of distinct indices (i,j,k) such that $\min(q_{ii}, q_{jj}, q_{kk}) > 0$, $q_{ij}^2 < q_{ii}q_{jj}$, $q_{ik}^2 < q_{ii}q_{kk}$, $q_{jk}^2 < q_{jj}q_{kk}$, and $q_{ii}q_{jj}q_{kk} + 2q_{ij}q_{jk}q_{ki} - q_{ik}^2 q_{jj} - q_{ij}^2 q_{kk} - q_{jk}^2 q_{ii} \leq 0$.

• Point and ray cuts. These are derived by specializing the procedure sketched in the last section for the general LPCC. Introducing dual variables for the linear constraints, we may rewrite the MIP (25) as follows:

$$\begin{array}{llrcll}
\underset{(x,y,z,\lambda)\in\mathbb{R}^{4n}}{\text{minimize}} & c^T x - \mathbf{1}_n^T y & & & & \\
\text{subject to} & & Qx + y & \geq & -c & (\,u^+\,) \\
& & -Qx - y & \geq & c - \theta\, z & (\,u^-\,) \\
& & -x & \geq & -\mathbf{1}_n & (\,s^+\,) \\
& & x & \geq & \mathbf{1}_n - \theta\,\lambda, & (\,s^-\,) \\
& & -x & \geq & -\theta\,(\,\mathbf{1}_n - z\,) & (\,v^+\,) \\
& & -y & \geq & -\theta\,(\,\mathbf{1}_n - \lambda\,) & (\,v^-\,) \\
& & (\,x, y\,) & \geq & 0 & \\
& & (\,\mathbf{1}_n - z\,) + \lambda & \geq & \mathbf{1}_n & \\
\text{and} & & (\,z, \lambda\,) & \in & \{\,0, 1\,\}^{2n}. &
\end{array}$$

For a given binary pair $(z, \lambda) \in \{0,1\}^{2n}$ satisfying $\lambda \geq z$, we define the value function

$$\begin{array}{rll}
\mathbb{R} \cup \{\,\infty\,\} \ni \varphi(z,\lambda) \equiv & \underset{(u^\pm, v^\pm, s^\pm)\in\mathbb{R}^{4n}}{\text{maximum}} & c^T(\,u^- - u^+\,) + \mathbf{1}_n^T(\,s^- - s^+\,) \\
& \text{subject to} & Q(\,u^+ - u^-\,) - (\,s^+ - s^-\,) - v^+ \,\leq\, c \\
& & u^+ - u^- - v^- \,\leq\, -\mathbf{1}_n \\
& & (\,u^\pm, v^\pm, s^\pm\,) \,\geq\, 0 \\
& \text{and} & z^T u^- + \lambda^T s^- + (\,\mathbf{1}_n - z\,)^T v^+ + (\,\mathbf{1}_n - \lambda\,)^T v^- \,\leq\, 0.
\end{array}$$

Let $\alpha \equiv \text{supp}(z) \subseteq \text{supp}(\lambda) \equiv \gamma$ and let $\beta \equiv \gamma \setminus \alpha$. Let $\bar{\alpha}$ and $\bar{\gamma}$ denote the complements of α and γ in $\{1, \cdots, n\}$, respectively. Thus, the triplet of index sets $(\alpha, \beta, \bar{\gamma})$ partitions $\{1, \cdots, n\}$. These index sets are indicators of the three possible states of the variable x: namely, α is the candidate set of indices i for which $x_i = 0$, β is the candidate set of indices i for which $x_i \in (0,1)$, and $\bar{\gamma}$ is the candidate set of indices i for which $x_i = 1$. In terms of these index sets and their complements, any feasible $(u^\pm, v^\pm, s^\pm)$ to $\varphi(z,\lambda)$ must satisfy $u^-_\alpha = 0$, $s^-_\gamma = 0$, $v^+_{\bar\alpha} = 0$, and $v^-_{\bar\gamma} = 0$. Hence

$$\begin{array}{rll} \varphi(z,\lambda) \equiv & \text{maximum} & -c^T u^+ + c^T_\beta u^-_\beta + c^T_{\bar\gamma} u^-_{\bar\gamma} - \mathbf{1}^T_n s^+ + \mathbf{1}^T_{|\bar\gamma|} s^-_{\bar\gamma} \\ & \text{subject to} & Q_{\alpha\bullet}\, u^+ - Q_{\alpha\beta}\, u^-_\beta - Q_{\alpha\bar\gamma}\, u^-_{\bar\gamma} - s^+_\alpha - v^+_\alpha \,\leq\, c_\alpha \\ & & Q_{\beta\bullet}\, u^+ - Q_{\beta\beta}\, u^-_\beta - Q_{\beta\bar\gamma}\, u^-_{\bar\gamma} - s^+_\beta \,\leq\, c_\beta \\ & & Q_{\bar\gamma\bullet}\, u^+ - Q_{\bar\gamma\beta}\, u^-_\beta - Q_{\bar\gamma\bar\gamma}\, u^-_{\bar\gamma} - s^+_{\bar\gamma} + s^-_{\bar\gamma} \,\leq\, c_{\bar\gamma} \\ & & u^+_\alpha - v^-_\alpha \,\leq\, -\mathbf{1}_{|\alpha|} \\ & & u^+_\beta - u^-_\beta - v^-_\beta \,\leq\, -\mathbf{1}_{|\beta|} \\ & & u^+_{\bar\gamma} - u^-_{\bar\gamma} \,\leq\, -\mathbf{1}_{|\bar\gamma|} \\ & & (\, u^+, u^-_\beta, u^-_{\bar\gamma}, v^+_\alpha, v^-_\alpha, v^-_\beta, s^+, s^-_{\bar\gamma}\,) \,\geq\, 0, \end{array}$$

which can easily be seen to be feasible. Moreover, we have

$$\begin{array}{rll} \mathbb{R}\cup\{\,\infty\,\} \ni \varphi(z,\lambda) \equiv & \text{maximum} & -c^T u^+ + c^T_\beta u^-_\beta + c^T_{\bar\gamma} u^-_{\bar\gamma} - \mathbf{1}^T_{|\beta|} s^+_\beta - \mathbf{1}^T_{|\bar\gamma|} s^+_{\bar\gamma} + \mathbf{1}^T_{|\bar\gamma|} s^-_{\bar\gamma} \\ & \text{subject to} & Q_{\beta\bullet}\, u^+ - Q_{\beta\beta}\, u^-_\beta - Q_{\beta\bar\gamma}\, u^-_{\bar\gamma} - s^+_\beta \,\leq\, c_\beta \\ & & Q_{\bar\gamma\bullet}\, u^+ - Q_{\bar\gamma\beta}\, u^-_\beta - Q_{\bar\gamma\bar\gamma}\, u^-_{\bar\gamma} - s^+_{\bar\gamma} + s^-_{\bar\gamma} \,\leq\, c_{\bar\gamma} \\ & & u^+_{\bar\gamma} - u^-_{\bar\gamma} \,\leq\, -\mathbf{1}_{|\bar\gamma|} \\ & & (\, u^+, u^-_\beta, u^-_{\bar\gamma}, s^+_\beta, s^+_{\bar\gamma}, s^-_{\bar\gamma}\,) \,\geq\, 0, \end{array}$$

from which we can recover an optimal $(s^+_\alpha, v^+_\alpha, v^-_\alpha, v^-_\beta)$ by letting $s^+_\alpha = 0$,

$$\begin{array}{rcl} v^+_\alpha & \equiv & \max\left\{\, 0, Q_{\alpha\bullet}\, u^+ - Q_{\alpha\beta}\, u^-_\beta - Q_{\alpha\bar\gamma}\, u^-_{\bar\gamma} - c_\alpha \,\right\} \\ v^-_\alpha & \equiv & u^+_\alpha + \mathbf{1}_{|\alpha|} \\ v^-_\beta & \equiv & \max\left\{\, 0, u^+_\beta - u^-_\beta + \mathbf{1}_{|\beta|} \,\right\}. \end{array}$$

In turn, by letting

$$\xi_{\bar\gamma} \,\equiv\, -\mathbf{1}_{|\bar\gamma|} - u^+_{\bar\gamma} + u^-_{\bar\gamma} \,\geq\, 0,$$

and

$$\begin{array}{rcl} s^-_{\bar\gamma} - s^+_{\bar\gamma} & \equiv & c_{\bar\gamma} - Q_{\bar\gamma\bullet}\, u^+ + Q_{\bar\gamma\beta}\, u^-_\beta + Q_{\bar\gamma\bar\gamma}\, u^-_{\bar\gamma} \\ & = & c_{\bar\gamma} + Q_{\bar\gamma\bar\gamma}\mathbf{1}_{|\bar\gamma|} - Q_{\bar\gamma\alpha}\, u^+_\alpha + Q_{\bar\gamma\beta}\,(\, u^-_\beta - u^+_\beta\,) + Q_{\bar\gamma\bar\gamma}\,\xi_{\bar\gamma} \\ & = & \bar c_{\bar\gamma} - Q_{\bar\gamma\alpha}\, u^+_\alpha + Q_{\bar\gamma\beta}\,(\, u^-_\beta - u^+_\beta\,) + Q_{\bar\gamma\bar\gamma}\,\xi_{\bar\gamma}. \end{array}$$

where $\bar c \equiv c + Q_{\bullet\bar\gamma}\mathbf{1}_{|\bar\gamma|}$, we have

$$v^+_\alpha \,\equiv\, \max\left\{\, 0, -\bar c_\alpha + Q_{\alpha\alpha}\, u^+_\alpha + Q_{\alpha\beta}\,(\, u^+_\beta - u^-_\beta\,) - Q_{\alpha\bar\gamma}\,\xi_{\bar\gamma} \,\right\}$$

and

$$
\begin{array}{rcl}
\varphi(z,\lambda) & \equiv & 2\,\mathbf{1}_{|\bar\gamma|}^T c_{\bar\gamma} + \mathbf{1}_{|\bar\gamma|}^T Q_{\bar\gamma\bar\gamma}\mathbf{1}_{|\bar\gamma|} + \\
& & \begin{array}{ll}
\text{maximum} & -\bar c_\alpha^T u_\alpha^+ + \bar c_\beta^T (\, u_\beta^- - u_\beta^+ \,) + \bar c_{\bar\gamma}^T \xi_{\bar\gamma} - \mathbf{1}_{|\beta|}^T s_\beta^+ \\
\text{subject to} & Q_{\beta\alpha}\, u_\alpha^+ - Q_{\beta\beta}\,(\, u_\beta^- - u_\beta^+ \,) - Q_{\beta\bar\gamma}\,\xi_{\bar\gamma} - s_\beta^+ \,\leq\, \bar c_\beta \\
\text{and} & (\, u_\alpha^+, u_\beta^\pm, \xi_{\bar\gamma}, s_\beta^+ \,) \,\geq\, 0
\end{array} \\
& = & 2\,\mathbf{1}_{|\bar\gamma|}^T c_{\bar\gamma} + \mathbf{1}_{|\bar\gamma|}^T Q_{\bar\gamma\bar\gamma}\mathbf{1}_{|\bar\gamma|} + \\
& & \begin{array}{lll}
\text{minimum} & \bar c_\beta^T \widehat{x}_\beta & \\
\text{subject to} & \bar c_\alpha + Q_{\alpha\beta}\,\widehat{x}_\beta \,\geq\, 0, & (\, u_\alpha^+ \,) \\
& \bar c_\beta + Q_{\beta\beta}\,\widehat{x}_\beta \,=\, 0, & (\, \xi_\beta \,) \\
& \bar c_{\bar\gamma} + Q_{\bar\gamma\beta}\,\widehat{x}_\beta \,\leq\, 0, & (\, \xi_{\bar\gamma} \,) \\
\text{and} & \mathbf{1}_{|\beta|} \,\geq\, \widehat{x}_\beta \,\geq\, 0. &
\end{array}
\end{array}
$$

In reduced variables and constraints, the linear program

$$
\begin{array}{ll}
\text{maximize} & -\bar c_\alpha^T u_\alpha^+ + \bar c_\beta^T (\, u_\beta^- - u_\beta^+ \,) + \bar c_{\bar\gamma}^T \xi_{\bar\gamma} - \mathbf{1}_{|\beta|}^T s_\beta^+ \\
\text{subject to} & Q_{\beta\alpha}\, u_\alpha^+ - Q_{\beta\beta}\,(\, u_\beta^- - u_\beta^+ \,) - Q_{\beta\bar\gamma}\,\xi_{\bar\gamma} - s_\beta^+ \,\leq\, \bar c_\beta \\
\text{and} & (\, u_\alpha^+, u_\beta^\pm, \xi_{\bar\gamma}, s_\beta^+ \,) \,\geq\, 0,
\end{array}
\tag{27}
$$

or its dual, is a workhorse to generate the valid point cuts, as explained below. Valid ray cuts are obtained by solving the homogenizations of (27). There are two cases associated with (27): either it has a finite optimal solution; or it is unbounded. In the former case, we can generate a point cut as follows. Let $(\widehat{u}_\alpha^{p,+}, \widehat{u}_\beta^{p,\pm}, \widehat{\xi}_{\bar\gamma}^{p}, \widehat{s}_\beta^{p,+})$ be an optimal extreme point solution of (27). Define

$$
\begin{array}{rcl}
\widehat{s}_{\bar\gamma}^{p,-} & \equiv & \max\left\{ 0,\, \bar c_{\bar\gamma} - Q_{\bar\gamma\alpha}\,\widehat{u}_\alpha^{p,+} + Q_{\bar\gamma\beta}\,(\,\widehat{u}_\beta^{p,-} - \widehat{u}_\beta^{p,+}\,) + Q_{\bar\gamma\bar\gamma}\,\widehat{\xi}_{\bar\gamma}^{p} \right\} \\
\widehat{v}_\alpha^{p,+} & \equiv & \max\left\{ 0, -\bar c_\alpha + Q_{\alpha\alpha}\,\widehat{u}_\alpha^{p,+} - Q_{\alpha\beta}\,(\,\widehat{u}_\beta^{p,-} - \widehat{u}_\beta^{p,+}\,) - Q_{\alpha\bar\gamma}\,\widehat{\xi}_{\bar\gamma}^{p} \right\} \\
\widehat{v}_\beta^{p,-} & \equiv & \max\left\{ 0, \widehat{u}_\beta^{p,+} - \widehat{u}_\beta^{p,-} + \mathbf{1}_{|\beta|} \right\}.
\end{array}
$$

The point cut is then:

$$
\sum_{i\notin\gamma} z_i + \sum_{i\in\beta:\widehat{u}_i^{p,-}>0} z_i + \sum_{i\notin\gamma:\widehat{s}_i^{p,-}>0} \lambda_i + \sum_{i\in\alpha:\widehat{v}_i^{p,+}>0} (\,1 - z_i\,) + \sum_{i\in\beta:\widehat{v}_i^{p,-}>0} (\,1-\lambda_i\,) + \sum_{i\in\alpha} (\,1-\lambda_i\,) \,\geq\, 1.
$$

In the latter case, we let $(\widehat{u}_\alpha^{r,+}, \widehat{u}_\beta^{r,\pm}, \widehat{\xi}_{\bar\gamma}^{r}, \widehat{s}_\beta^{r,+})$ be an extreme ray of (27) on which the objective tends to ∞. Define

$$
\begin{array}{rcl}
\widehat{s}_{\bar\gamma}^{r,-} & \equiv & \max\left\{ 0,\, -Q_{\bar\gamma\alpha}\,\widehat{u}_\alpha^{r,+} + Q_{\bar\gamma\beta}\,(\,\widehat{u}_\beta^{r,-} - \widehat{u}_\beta^{r,+}\,) + Q_{\bar\gamma\bar\gamma}\,\widehat{\xi}_{\bar\gamma}^{r} \right\} \\
\widehat{v}_\alpha^{r,+} & \equiv & \max\left\{ 0, Q_{\alpha\alpha}\,\widehat{u}_\alpha^{r,+} - Q_{\alpha\beta}\,(\,\widehat{u}_\beta^{r,-} - \widehat{u}_\beta^{r,+}\,) - Q_{\alpha\bar\gamma}\,\widehat{\xi}_{\bar\gamma}^{r} \right\} \\
\widehat{u}_{\bar\gamma}^{r,-} & \equiv & \widehat{\xi}_{\bar\gamma}^{r}.
\end{array}
$$

The ray cut is then:

$$
\sum_{i\notin\alpha:\widehat{u}_i^{r,-}>0} z_i + \sum_{i\notin\gamma:\widehat{s}_i^{r,-}>0} \lambda_i + \sum_{i\in\alpha:\widehat{v}_i^{r,+}>0} (\,1 - z_i\,) + \sum_{i\in\gamma:\widehat{u}_i^{r,+}>0} (\,1-\lambda_i\,) \,\geq\, 1.
$$

Sparsification of cuts

Suppose that we want to test the validity of the satisfiability inequality:

$$\sum_{i\in\mathcal{I}_1} z_i + \sum_{i\in\mathcal{I}_2} (1 - z_i) + \sum_{i\in\mathcal{J}_1} \lambda_i + \sum_{i\in\mathcal{J}_2} (1 - \lambda_i) \geq 1. \tag{28}$$

Since $\lambda \geq z$, it suffices to consider the case where $\mathcal{I}_2 \cap \mathcal{J}_1 = \emptyset$. To accomplish this test, we assume the contrary, i.e., assume

$$\sum_{i\in\mathcal{I}_1} z_i + \sum_{i\in\mathcal{I}_2} (1 - z_i) + \sum_{i\in\mathcal{J}_1} \lambda_i + \sum_{i\in\mathcal{J}_2} (1 - \lambda_i) \leq 0, \tag{29}$$

and consider the LPCC resulting from setting the corresponding variables in the respective index sets equal to zero, i.e., $z_i = 0$ for all $i \in \mathcal{I}_1$, $z_i = 1$ for all $i \in \mathcal{I}_2$; $\lambda_i = 0$ for all $i \in \mathcal{J}_1$, and $\lambda_i = 1$ for all $i \in \mathcal{J}_2$:

$$\begin{array}{ll} \underset{(x,y)}{\text{minimize}} & c^T x - \mathbf{1}_n^T y \\ \text{subject to} & 0 \leq c + Qx + y \perp x \geq 0 \\ & 0 \leq \mathbf{1}_n - x \perp y \geq 0 \\ & (c + Qx + y)_i = 0, \quad \forall\, i \in \mathcal{I}_1 \\ & 1 - x_i = 0, \quad \forall\, i \in \mathcal{J}_1 \\ & x_i = 0, \quad \forall\, i \in \mathcal{I}_2 \\ & y_i = 0, \quad \forall\, i \in \mathcal{J}_2 \end{array} \tag{30}$$

With some of the x-variables fixed, the complementarity condition could be used to fix further variables, using the following implications:

$$[\, x_i = 1 \Rightarrow (c + Qx + y)_i = 0 \,] \quad \text{and} \quad [\, x_i = 0 \Rightarrow y_i = 0 \,].$$

Thus (30) is equivalent to:

$$\begin{array}{ll} \underset{(x,y)}{\text{minimize}} & c^T x - \mathbf{1}_n^T y \\ \text{subject to} & 0 \leq c + Qx + y \perp x \geq 0 \\ & 0 \leq \mathbf{1}_n - x \perp y \geq 0 \\ & (c + Qx + y)_i = 0, \quad \forall\, i \in \mathcal{I}_1 \\ & 1 - x_i = 0 = (c + Qx + y)_i, \quad \forall\, i \in \mathcal{J}_1 \\ & x_i = 0 = y_i, \quad \forall\, i \in \mathcal{I}_2 \\ & y_i = 0, \quad \forall\, i \in \mathcal{J}_2 \end{array} \tag{31}$$

Further implications and fixings of variables can be derived by noting that

$$\begin{aligned} c_i + \sum_{j=1}^n q_{ij}\, x_j &= c_i + \sum_{j\in\mathcal{J}_1} q_{ij} + \sum_{\substack{j\notin(\mathcal{I}_2\cup\mathcal{J}_1) \\ q_{ij}\neq 0}} q_{ij}\, x_j \\ &\geq c_i + \sum_{j\in\mathcal{J}_1} q_{ij} + \sum_{\substack{j\notin(\mathcal{I}_2\cup\mathcal{J}_1) \\ q_{ij}< 0}} q_{ij} \equiv \sigma_i^+. \end{aligned}$$

Similarly,

$$c_i + \sum_{j=1}^{n} q_{ij}\, x_j \;\leq\; \sum_{j \in \mathcal{J}_1} q_{ij} + \sum_{\substack{j \notin (\mathcal{I}_2 \cup \mathcal{J}_1) \\ q_{ij} > 0}} q_{ij} \;\equiv\; \sigma_i^- .$$

The implications below must be valid for the problem (31):

$$\begin{array}{ll} \left[\, \sigma_i^+ \,=\, 0 \;\Rightarrow\; y_i \,=\, 0 \,\right], & \left[\, \sigma_i^+ \,>\, 0 \;\Rightarrow\; x_i \,=\, 0 \,=\, y_i \,\right] \\ \left[\, \sigma_i^- \,=\, 0 \;\Rightarrow\; (\, c + Qx + y\,)_i \,=\, 0 \,\right], & \left[\, \sigma_i^- \,<\, 0 \;\Rightarrow\; x_i \,=\, 1 \,\right] \end{array}$$

Yet a third venue to fix more variables as a result of the constraint (29) is via the satisfiability inequalities already shown to be valid. To illustrate, suppose that the inequality $z_1 + (1 - z_2) + \lambda_3 \geq 1$ is known to be valid and we are testing the inequality $z_1 + (1 - z_2) + (1 - \lambda_4) \geq 1$. Setting $z_1 = 1 - z_2 = 1 - \lambda_4 = 0$ yields, by the former valid inequality, $\lambda_3 = 0$, meaning that we can set $x_3 = 1$.

We next consider a relaxation of the complementarity constraint for an index i for which neither x_i nor $(c + Qx + y)_i$ are fixed yet:

$$\begin{array}{rcl} 0 & = & x_i\, (\, c + Qx + y\,)_i \\ & = & \left(c_i + \displaystyle\sum_{j \in \mathcal{J}_1} q_{ij} \right) x_i + \displaystyle\sum_{\substack{q_{ij} \neq 0 \\ j : x_j \text{ not yet fixed}}} q_{ij}\, w_{ij} + y_i, \end{array}$$

where $w_{ij} \equiv x_i x_j$ for all i and j such that x_i and x_j are not yet fixed and such that $q_{ij} \neq 0$. We now relax the nonlinear equation $w_{ij} \equiv x_i x_j$ and replace it by the following linear constraints:

$$\max(\, 0, x_i + x_j - 1\,) \;\leq\; w_{ij} \;\leq\; \min(\, x_i, x_j\,).$$

Additional restrictions on the variables w_{ij} can be imposed if both w_{ii} (if $q_{ii} \neq 0$) and w_{jj} (if $q_{jj} \neq 0$) are well defined. Namely, $w_{ii} + w_{jj} \geq 2w_{ij}$ and $(x_i + x_j)^2 \leq w_{ii} + 2w_{ij} + w_{jj}$ are two valid constraints. While the latter is a nonlinear constraint, it is convex and of the second-order cone (SOC) type, which can be effectively handled by state-of-the-art SOC computer softwares. One last constraint that can be added is based on the implication:

$$q_{ii} \;<\; 0 \;\Rightarrow\; [\, x_i \;=\; 0 \text{ or } 1\,] \;\Rightarrow\; w_{ii} \;=\; x_i.$$

Collecting all the above implied variable fixings and linear constraints on the auxiliar variables w_{ij}, we arrive at the following LP relaxation of (31):

$$\begin{array}{ll} \underset{(x,y,w)}{\text{minimize}} & c^T x - \mathbf{1}_n^T y \\ \text{subject to} & 0 \;\leq\; c + Qx + y, \quad x \;\geq\; 0 \\ & 0 \;\leq\; \mathbf{1}_n - x, \quad y \geq 0 \\ & 0 \;=\; \left(c_i + \displaystyle\sum_{j \in \mathcal{J}_1} q_{ij} \right) x_i + \displaystyle\sum_{\substack{q_{ij} \neq 0 \\ j : x_j \text{ not fixed}}} q_{ij}\, w_{ij} + y_i, \quad \forall\, i \text{ with } x_i \text{ not fixed} \\ \text{plus} & \text{fixed complementarities; cf. (30)} \\ \text{and} & \text{all implied variable fixings and linear constraints on } w_{ij} \text{ as described above.} \end{array} \tag{32}$$

If the optimal objective value of the latter LP exceeds a valid upper bound of $\mathrm{QP}_{\min}$, then the satisfiability constraint (29) cannot be valid; hence (28) is valid and can be added to the overall satisfiability system to determine the next LP piece to continue the search.

Local search

In solving the LP (32), it is possible to obtain an optimal solution $(\widetilde{x}, \widetilde{y})$ that fails to satisfy the complementarity conditions: $x^T(c + Qx + y) = y^T(\mathbf{1}_n - x) = 0$, because some of these conditions have been relaxed. The so-obtained $(\widetilde{x}, \widetilde{y})$ is therefore not a KKT pair of the QP (24). When this happens, we apply a local search method, such as a simple gradient projection iteration initiated at $\widetilde{x}$, to recover a KKT pair. This local search is the *feasibility recovery step* described in [16] for the general LPCC, which in this case can be implemented by a standard local method for QPs. A local search is also employed to obtain an initial KKT pair at the beginning of the algorithm.

6.2 Bounded-variable QPs with 1 constraint

Consider the following QP with simple upper and lower bounds and one additional inequality constraint:

$$\begin{array}{ll} \displaystyle\operatorname*{minimize}_{x} & \frac{1}{2}\,x^TQx + c^Tx \\ \text{subject to} & 0 \,\leq\, x \,\leq\, \mathbf{1}_n \\ \text{and} & \mathbf{1}_n^T x \,\leq\, f \end{array} \tag{33}$$

where Q is symmetric and f is a positive scalar. The MIP formulation for this QP is: for $\theta > 0$ sufficiently large,

$$\begin{array}{ll} \displaystyle\operatorname*{minimize}_{(x,y,z,\lambda)} & c^Tx - \mathbf{1}_n^T y \\ \text{subject to} & 0 \,\leq\, c + Qx + y + \mathbf{1}_n\,\eta \,\leq\, \theta\,z \\ & 0 \,\leq\, \mathbf{1}_n - x \,\leq\, \theta\,\lambda \\ & 0 \,\leq\, f - \mathbf{1}_n^T x \,\leq\, \theta\,\lambda_{n+1} \\ & 0 \,\leq\, x \,\leq\, \theta\,(\,\mathbf{1}_n - z\,) \\ & 0 \,\leq\, y \,\leq\, \theta\,(\,\mathbf{1}_n - \lambda\,) \\ & 0 \,\leq\, \eta \,\leq\, \theta\,(\,1 - \lambda_{n+1}\,) \\ \text{and} & z, \lambda \,\in\, \{\,0,1\,\}^n, \quad \lambda_{n+1} \,\in\, \{\,0,1\,\}. \end{array} \tag{34}$$

Similar to the bounded-variable problem with no other constraint, we can establish valid cuts for (34) by noticing that in the notation of (20), the matrix $A = \begin{bmatrix} \mathbf{I}_n \\ -\mathbf{I}_n \\ -\mathbf{1}_n^T \end{bmatrix}$. The only difference between these cuts and those in the previous case is that the extra term $1 - \lambda_{n+1}$ is needed in each of the cases (B–D). In principle, cuts without this extra term can be added under more restrictive conditions; for instance, if $\min(q_{ii}, q_{jj}) > 0$ and $q_{ii} + q_{jj} \leq 2q_{ij}$, with the latter condition being equivalent, under the former condition, to the condition that the 2×2 matrix

$$\begin{bmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{bmatrix}$$

is not positive definite on the null space $x_i + x_j = 0$, then the constraint $z_i + z_j + (1 - \lambda_i) + (1 - \lambda_j) \geq 1$ is valid. Similarly, suppose (i, j, k) are 3 distinct indices such that the following three 2×2 matrices:

$$\begin{bmatrix} q_{ii} & q_{ij} \\ q_{ji} & q_{jj} \end{bmatrix}, \quad \begin{bmatrix} q_{ii} & q_{ik} \\ q_{ki} & q_{kk} \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} q_{jj} & q_{jk} \\ q_{kj} & q_{kk} \end{bmatrix}$$

are all positive definite but

$$\begin{bmatrix} q_{ij} & q_{ij} & q_{ik} \\ q_{ji} & q_{jj} & q_{jk} \\ q_{ki} & q_{kj} & q_{kk} \end{bmatrix}$$

is not positive definite on the subspace: $x_i + x_j + x_k = 0$, or equivalently, if the matrix

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} q_{ii} & q_{ij} & q_{ik} \\ q_{ji} & q_{jj} & q_{jk} \\ q_{ki} & q_{kj} & q_{kk} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} q_{ii} - 2q_{ij} + q_{jj} & q_{ij} - q_{ik} - q_{jj} + q_{jk} \\ q_{ji} - q_{ki} - q_{jj} + q_{kj} & q_{jj} - 2q_{jk} + q_{kk} \end{bmatrix}$$

is not positive definite, then the constraint $z_i + z_j + z_k + (1 - \lambda_i) + (1 - \lambda_j) + (1 - \lambda_k) \geq 1$ is valid.

Valid cuts for (34)

(A) $\lambda_j + (1 - z_j) \geq 1$ for all j;

(B) $z_j + (1 - \lambda_j) + (1 - \lambda_{n+1}) \geq 1$ for all j such that $q_{jj} \leq 0$;

(C) $z_i + z_j + (1 - \lambda_i) + (1 - \lambda_j) + (1 - \lambda_{n+1}) \geq 1$ for all $i \neq j$ such that $\min(q_{ii}, q_{jj}) > 0$ and $q_{ij}^2 \geq q_{ii}q_{jj}$;

(D) $z_i + z_j + z_k + (1 - \lambda_i) + (1 - \lambda_j) + (1 - \lambda_k) + (1 - \lambda_{n+1}) \geq 1$ for all triples of distinct indices (i, j, k) such that $\min(q_{ii}, q_{jj}, q_{kk}) > 0$, $q_{ij}^2 < q_{ii}q_{jj}$, $q_{ik}^2 < q_{ii}q_{kk}$, $q_{jk}^2 < q_{jj}q_{kk}$, and $q_{ii}q_{jj}q_{kk} + 2q_{ij}q_{jk}q_{ki} - q_{ik}^2 q_{jj} - q_{ij}^2 q_{kk} - q_{jk}^2 q_{ii} \leq 0$.

(E) Additional cuts corresponding to other choices of the index set J in the family $\mathcal{J}$ may be added, including those similar to (C) and (D) but without the $(1 - \lambda_{n+1})$ term.

The sparsification of cuts is accomplished by solving LPs similar to (32). The only difference is that terms in the satisfiability constraints involving z_{n+1} or λ_{n+1} are always kept in the sparsified constraints in the procedure. The details are not repeated.

6.3 Nonnegatively constrained, copositive QPs

Consider the following nonnegatively constrained QP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \frac{1}{2}\, x^T Q x + c^T x \\ \text{subject to} & x \geq 0, \end{array} \tag{35}$$

where Q is symmetric and copositive. This problem either attains a finite optimal solution or is unbounded below. By Corollary 4, the resolution of this dichotomy can be determined by solving the LPCC:

$$\begin{array}{ll} \underset{(x,d,t) \in \mathbb{R}^{2n+1}}{\text{minimize}} & -t \\ \text{subject to} & \mathbf{1}_n^T d \geq 1 \\ & 0 \leq c + Qx + t\,\mathbf{1}_n \perp x \geq 0 \\ & 0 \leq Qd \perp d \geq 0 \\ \text{and} & c + Qx + t\,\mathbf{1}_n \perp d. \end{array} \tag{36}$$

whose MIP formulation is:

$$\begin{array}{ll}
\underset{(x,d,z,\lambda,t)\in\mathbb{R}^{4n+1}}{\text{minimize}} & -t \\
\text{subject to} & \mathbf{1}_n^T d \,\geq\, 1 \\
& 0 \,\leq\, c + Qx + t\,\mathbf{1}_n \,\leq\, \theta\, z \\
& 0 \,\leq\, x \,\leq\, \theta\,(\,\mathbf{1}_n - z\,) \\
& 0 \,\leq\, Qd \,\leq\, \theta\,\lambda \\
& 0 \,\leq\, d \,\leq\, \theta\,(\,\mathbf{1}_n - \lambda\,) \\
& d \,\leq\, \theta\,(\,\mathbf{1}_n - z\,) \\
\text{and} & (\,z, \lambda\,) \,\in\, \{\,0, 1\,\}^{2n}.
\end{array} \tag{37}$$

Note that if $z_i = 1$, then $d_i = 0$; hence we may take $\lambda_i = 1$. Thus we can impose the constraint $\lambda \geq z$ without loss of generality. While it may be possible to derive other cuts for this MIP, we have not investigated this problem as thoroughly as the bounded-variable problem. In the numerical results, we solved the LPCC (36) using the general algorithm sketched in Section 4.

7 Computational Results

We have implemented three algorithms coded in a C-environment for solving the special classes of QPs. Details of these algorithms can be found in the doctoral thesis of the first author [15]. We used CPLEX 10.0 to solve the LPs and zChaff 3.12 to solve the satisfiability problems. Implementing the Chaff algorithm [20], zChaff is designed for solving the boolean satisfiability problem and free for non-commercial use downloadable from the website: `http://www.princeton.edu/~chaff/zchaff.html`. The experiments were run on a Dell desktop computer with a Core Duo CPU, 2.33 GHz processor, and 1.95 GB of RAM. The data for the bounded-variable QPs are the same as those in [25]; in particular, the entries of Q and c are randomly generated integers between -50 and 50. The experiments in this reference were run on a SUN Ultra-80 with 2x450-MHz UltraSPARC-II processors and 1-GB Memory. The same data was used for the bound-variable QPs with one additional inequality constraint with the right-hand constant f in this constraint equal to $n/2$; cf. (33). We solved two sets of the nonnegatively constrained, copositive QPs; the data for the first set of these problems are generated as follows. The 2×2 leading principal submatrix of the matrix Q in each problem is $\begin{bmatrix} 25 & -25 \\ -25 & 25 \end{bmatrix}$. The remaining entries are nonnegative integers between 0 and 50. So the matrix Q is copositive on the nonnegative orthant, and not necessarily positive semidefinite. The first two entries of the vector c are -2 and 1; the rest of the entries of c are randomly generated integers between -20 and 20. Therefore, this group of QPs is unbounded on the ray $(1, 1, 0 \cdots 0)$, and may or may not have stationary solutions. In order to test a related class of nonnegatively constrained, copositive QPs that are guaranteed to have a stationary solution but remain unbounded, we modified the data of the first set of such QPs as follows. The matrix Q has

$$Q_3 \,\equiv\, \begin{bmatrix} 25 & -25 & 50 \\ -25 & 25 & -25 \\ 50 & -25 & 25 \end{bmatrix}$$

as its leading 3×3 principal submatrix and the remaining entries of Q are generated in the same way as before; the vector $c = -Qx^0$, where x^0 is the vector with the first 3 components equal to 1 and the rests equal zero. Thus x^0 is a stationary solution of the QP (35). Since the first 3 components of c are -50, 25, and 50, this QP remains unbounded on the ray $(1,\,1,\,0\cdots0)$. The matrix Q_3 is copositive because

$$\left(\begin{array}{ccc} d_1 & d_2 & d_3 \end{array}\right) Q_3 \left(\begin{array}{c} d_1 \\ d_2 \\ d_3 \end{array}\right) \,=\, 25\,(\,d_1 - d_2 + d_3\,)^2 + 50\,d_1 d_3 \,\geq\, 0$$

for all $(d_1, d_3) \geq 0$.

Tables 1–4 report the computational results for the bounded-variable QPs with 30, 40, 50, and 60 variables, respectively. The computational results for problems of various sizes and density are reported in the same order as in [25]. Each table contains results corresponding to various densities of the matrix Q. Since our approach is quite different from the branch-and-cut algorithm employed in the reference, it is difficult to compare the performance of the two approaches directly. To at least give some idea of the performance of our approach relative to that of the reference, we highlight two columns in the tables: one giving the total number of LPs solved in each instance and the other giving the computational time for that instance. The results from the reference are labeled by "VN". In our case, the LPs solved are of 3 types: those used to generate the ray cuts ($\varphi_0(z)$) or point cuts ($\varphi(z)$), and those (32) used in the sparsification step. The numbers of these LPs solved are reported in the columns labeled by cnt-dual, cnt-M, and cnt-rlx, respectively. The first column in Tables 1–4 lists the number of satisfiability problems solved in each run. The last column "Gtime" reports the time in seconds when a global optimal solution is obtained, but its global optimality is not yet verified. This optimal solution is usually obtained in the local search procedure at the beginning of the runs, but its global optimality is verified only when the algorithm terminates. Comparing the second and last column, we note that the bulk of computations lies in the latter verification step. Notice that among this set of problems, there is only one in the last row of Table 2 that could not be solved/verified to global optimality after 10000 iterations; this problem is also not solved by the branch-and-cut approach in [25].

Since the LPs in our approach and the VN approach are quite different, considering the numbers of these LPs being solved is not a completely fair comparison. Nevertheless, setting aside this difference, we can draw some informative observations about the two approaches. First, the numbers of LPs solved in our approach are quite reasonable and indeed less than the VN numbers in many cases. To be fair, we should note that for several problems that require significantly more LPs (e.g., the 7th problem in Table 1, the 10th, 16th, 19th, and 20th problem in Table 2, etc.), our numbers can be much higher; this suggests that there is room for improvement in our approach. An interesting observation from these tables is that as the QP sizes increase, there are more problems that solve less LPs in our approach than the VN approach. Specifically, on problems with 30 variables (Table 1), there are 8 out of 15 problems where our approach solve less LPs; on problems with 40 variables (Table 2), there are 14 out of 24 problems to our favor; on problems with 50 variables, this fraction increases to 5 out of 6; and finally, on the last set of 3 problems with 60 variables, our approach solves less LPs than the VN approach.

Tables 5–7 report the computational results for box-constrained QPs (33) with one additional inequality. These tables have the same columns as the previous tables, except for the absence of comparisons. In these tables, the starred problems have the last constraint not binding. The rest of the problems have the inequality constraint binding. Our LPCC approach is able to solve all of the problems presented in the tables to global optimality, which includes two tasks: obtain the optimal solution and verify its global optimality. By comparing the results for the same problems with and without the last constraint, we see that the computational effort increases for the problems with one additional constraint. We suspect two possible reasons for this increased effort: one is that unlike the previous set of problems where we have included many implied fixings of variables in tightening the relaxed LPs (32) in the sparsification step, we have not attempted similar fixings in this set of problems. The other reason is that the derived cuts are probably not as strong as they should be. Of course, the binding or not binding of the additional constraint doubles the number of disjunctions, thereby increasing the complexity of the problem.

Tables 8 and 9 contain the results for the first set of nonnegatively constrained, copositive, unbounded

iter	time		LPcnt		cnt-rx	cnt-dual	cnt-M	Gtime
	ours	VN	ours	VN				
233	20.14	22.86	2102	1772	1211	867	24	0.14
3	1.69	1.07	115	55	112	1	2	0.06
83	7.72	15.39	688	1185	487	189	12	0.52
739	77.19	109.41	5617	6676	3078	2477	62	0.17
1	6.55	4.97	304	315	300	0	4	0.16
9	5.53	5.33	282	367	263	15	4	2.53
593	120.55	85.10	6500	5151	3393	3091	16	0.38
1	2.19	2.01	105	113	104	0	1	0.01
1	3.80	2.32	143	121	141	0	2	0.05
45	14.83	41.31	706	2327	516	188	2	0.02
98	38.83	19.12	1430	1156	912	516	2	8.08
24	15.47	14.83	603	804	512	89	2	0.02
369	139.23	77.30	4525	4373	2366	2155	4	0.17
633	172.08	80.18	6458	4259	3348	3084	26	0.08
12	16.19	22.92	453	1278	417	31	5	0.50

Table 1: Box constrained QPs with $Q \in R^{30\times 30}$

QPs. We solved two groups of problems with 40 and 50 variables, respectively, and 10 randomly generated problems in each group. The algorithm implemented for these QPs does not have the sparsification procedure. So the number of relaxed LPs is not reported in these two tables. The number of the LPs reported in the third column is the sum of the fourth and the fifth columns. Since the algorithm is stopped whenever a negative objective value is obtained, we omitted the column "Gtime" in these tables too. From the results, we can tell that for 19 out of 20 problems, our approach is able to identify the unboundedness of the QP. Only one problem, marked with an asterisk sign, does not terminate with a certificate of unboundedness after 20,000 iterations. Tables 10 and 11 contain the results for the second set of nonnegatively constrained, copositive, unbounded QPs that are guaranteed to have stationary solutions. Interestingly, this second group of QPs seems to be easier to solve than the first group of QPs where stationary solutions are not guaranteed to exist.

As a final comment, we note that in most of the problems solved in the experiments, the numbers (cnt-dual) of ray cuts are significantly more than the numbers (cnt-M) of point cuts. This suggests that there are many infeasible LP pieces in these QPs.

8 Concluding Remarks

In this paper, we have investigated an LPCC approach to the global resolution of indefinite, possibly unbounded quadratic programs. Our main contributions are as follows:

- we have introduced an LPCC whose global resolution will certify whether or not the QP is bounded below (Theorem 3 and Corollary 4);
- we have identified some valid inequalities for the MIP formulation of the LPCC derived from a solvable QP that are motivated by its second-order optimality conditions (Proposition 7);
- we have described a new algorithm for solving bounded-variable QPs (Subection 6.1);
- computational results provide evidence supporting the promise of the LPCC approach to indefinite QPs (Section 7).

While contending that these are all positive contributions to the study of indefinite QPs, we admit that this is only the first step in developing a general algorithm for the global resolution of the non-

iter	time		LPcnt		cnt-rx	cnt-dual	cnt-M	Gtime
	ours	VN	ours	VN				
1	0.69	3.30	100	186	97	0	3	0.11
1	0.97	2.80	102	201	98	0	4	0.28
1	0.78	2.63	89	139	85	0	4	0.22
548	64.05	143.61	4793	6833	2612	2107	74	0.08
1	4.75	13.55	298	668	296	0	2	0.00
534	66.92	140.51	5105	5861	2791	2259	55	0.27
390	68.27	131.60	4073	5414	2297	1750	26	0.17
195	31.53	70.22	2136	3078	1270	850	16	0.25
193	46.67	55.31	2558	2254	1495	1048	15	2.39
2107	1030.47	983.32	27766	20590	14182	13532	52	0.09
11	11.80	14.42	439	568	409	25	5	0.06
1	7.64	10.09	155	350	153	0	2	0.06
321	241.66	229.41	5312	7622	2860	2448	4	0.03
154	73.84	138.00	2389	4490	1432	950	7	0.08
80	41.42	26.86	1300	1081	859	437	4	0.34
1499	1539.22	359.03	23108	9729	11822	11266	20	0.01
347	277.70	515.81	5987	12690	3152	2830	5	0.11
54	51.59	117.31	928	3687	728	195	5	1.58
751	822.59	199.48	12447	6246	6299	6135	13	0.08
834	833.92	241.00	12558	7202	6407	6126	25	0.33
164	171.83	73.32	2961	2589	1662	1296	3	0.02
1016	2551.34	263.61	21542	7190	10806	10732	4	0.17
3517	6941.48	2169.80	57002	27928	28448	28538	16	0.02
10001*	32206.74	58.84%	184690	36207	91458	93218	14	−−

Table 2: Box constrained QPs with $Q \in R^{40\times 40}$.
The problem with the asterisk sign cannot be solved within 10000 iterations.

iter	time		LPcnt		cnt-rx	cnt-dual	cnt-M	Gtime
	ours	VN	ours	VN				
1	4.11	13.28	263	434	260	0	3	0.23
221	27.63	127.07	2065	4285	1286	731	48	1.92
146	18.73	87.91	1375	2827	936	405	34	0.05
1436	703.14	464.57	19833	11356	10250	9523	60	0.02
669	276.28	455.61	8803	10561	4734	4027	42	0.38
288	163.03	263.06	5464	6464	2977	2480	7	0.02

Table 3: Box constrained QPs with $Q \in R^{50\times 50}$

iter	time		LPcnt		cnt-rx	cnt-dual	cnt-M	Gtime
	ours	VN	ours	VN				
311	16.53	101.89	1599	2781	1014	446	139	0.11
1	2.13	18.04	109	490	106	0	3	0.08
652	53.50	141.46	3526	3876	1988	1286	252	0.59

Table 4: Box constrained QPs with $Q \in R^{60\times 60}$

iter	time	LPcnt	cnt-rx	cnt-dual	cnt-M	Gtime
1072	241.95	11898	6073	5799	26	1.95
347	47.61	3715	1944	1765	6	0.01
967	370.36	13419	6703	6709	7	0.01
2071*	576.13	21420	10788	10561	71	3.84
859	244.27	11355	5607	5741	7	0.02
941	306.75	12847	6415	6424	8	0.08
2063	790.72	24464	12091	12350	23	0.17
331	77.31	3551	1944	1592	15	4.05
1169	333.17	12925	6447	6453	25	2.25
819	342.28	10541	5367	5165	9	0.01
2561	1307.06	33406	16251	17128	27	0.25
894	418.70	12068	6107	5952	9	0.01
1958	1177.31	25854	12671	13165	18	0.05
3383	2181.01	38592	18961	19583	48	4.91
924	456.47	11535	5806	5717	12	448.20

Table 5: Box constrained QPs with one additional inequality constraint; $Q \in R^{30\times 30}$. The starred problem has the last constraint NOT binding.

iter	time	LPcnt	cnt-rx	cnt-dual	cnt-M	Gtime
1*	2.42	136	132	0	4	0.05
1700	535.91	19931	10002	9902	27	18.33
30	9.36	712	509	202	1	0.00
2309*	1034.20	28918	14603	14256	59	0.20
5619	4563.66	70422	34623	35741	58	1.05
5486	6878.78	78040	38870	39144	26	6233.50
3100	1933.02	41100	20673	20383	44	1831.11
2757	2719.78	40506	20146	20342	18	2.23
4319	7410.64	60421	30228	30167	26	0.11

Table 6: Box constrained QPs with one additional inequality constraint; $Q \in R^{40\times 40}$ with density 30%, 40% and 50%. The starred problems have the last constraint NOT binding.

iter	time	LPcnt	cnt-rx	cnt-dual	cnt-M	Gtime
1	11.55	405	405	0	0	0.05
5514	8064.56	81402	41081	40286	35	0.02
2975	4295.75	48139	24276	23855	8	0.31

Table 7: Box constrained QPs with one additional inequality constraint; $Q \in R^{50\times 50}$ with density 30%.

iter	time	LPcnt	cnt-dual	cnt-M
10455	4989.36	106623	106568	55
11883	7693.13	147875	147849	26
8588	4531.78	115280	115271	9
1361	83.95	15038	15026	12
8569	4343.67	104614	104605	9
11209	6554.17	127020	126927	93
1235	77.69	15910	15910	0
472	15.73	5603	5602	1
302	7.89	3856	3856	0
15605	15064.80	185690	185674	16

Table 8: Copositive QPs: $Q \in R^{40\times 40}$, density 0.25, stationary solution not guaranteed.

iter	time	LPcnt	cnt-dual	cnt-M
17091	17047.94	207155	207101	54
12067	6951.44	137896	137836	60
13483	6991.08	146942	146775	167
7679	3657.99	112001	112001	0
5211	1319.44	67054	67039	15
9664	4845.31	118386	118347	39
14364	12076.95	181824	181809	15
20001*	32815.06	267400	267398	2
1170	81.70	18649	18649	0
468	14.03	6885	6885	0

Table 9: Copositive QPs: $Q \in R^{50\times 50}$, density 0.15, stationary solution not guaranteed. The problem with the asterisk sign cannot be solved within 20000 iterations.

iter	time	LPcnt	cnt-dual	cnt-M
829	7.22	2339	2317	22
3854	206.17	16007	15973	34
2931	91.52	9936	9865	71
2115	60.73	10463	10414	49
1207	13.61	3965	3836	129
15059	1844.61	48711	48662	49
656	6.38	2463	2428	35
2768	111.47	12027	11996	31
663	6.78	2533	2517	16
9333	1008.80	39765	39722	43

Table 10: Copositive QPs: $Q \in R^{40\times 40}$, density 0.25, stationary solution guaranteed

iter	time	LPcnt	cnt-dual	cnt-M
2736	118.99	16242	16162	80
9805	2566.11	68836	68781	55
814	14.06	5019	4862	157
10168	1892.77	59116	59063	53
16618	3877.95	70189	70115	74
10832	2413.30	68483	68373	110
4285	402.17	30691	30628	63
7024	917.91	43321	43262	59
2285	126.81	14319	14298	21
5925	619.39	31346	31270	76

Table 11: Copositive QPs: $Q \in R^{50\times 50}$, density 0.25, stationarity guaranteed
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convex QP; in particular, further study is needed to understand more about the LPCC (10) and its MIP formulation, in order to derive deeper cuts and sharper LP relaxations in the sparsification of these cuts.

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