On Newton(like) inequalities for multivariate homogeneous polynomials

Leonid Gurvits *

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Abstract

Let $p(x_1, ..., x_m) = \sum_{r_1+\cdots+r_m=n} a_{r_1,...,r_m} \prod_{1 \leq i \leq m} x_i^{r_i}$ be a homogeneous polynomial of degree n in m variables. We call such polynomial **H-Stable** if $p(z_1, ..., z_m) \neq 0$ provided that the real parts $Re(z_i) > 0 : 1 \leq i \leq m$. It can be assumed WLOG that the coefficients $a_{r_1,...,r_m} := a_R \geq 0$.
This notion from *Control Theory* is closely related to the notion of *Hyperbolicity* intensively used in the *PDE* theory.
Let $R_0; R_1, ..., R_k$ are integer vectors and $R_0 = \sum_{1 \leq j \leq k} a_j R_j$, where the real numbers $a_j \geq 0 : 1 \leq i \leq k$ and $\sum_{1 \leq j \leq k} a_j = 1$. We define, for an integer vector $R = (r_1, ..., r_m)$, $R! =: \prod_{1 \leq i \leq m} r_i!$.
We prove that $\log(a_R R!) \geq \sum_{1 \leq j \leq k} a_j \log(a_R R_j!) - n\alpha_n$, where $\frac{1}{2}\log(2) \leq \alpha_n \leq \log(\frac{n^n}{n!})$ and get better bounds on α_n for sparse polynomials. We relax a notion of **H-Stability** by introducing two classes of homogeneous polynomials: Alexandrov-Fenchel polynomials and Strongly Log-Concave polynomials, prove analogous inequalities for those classes and use them to prove L-convexity of the supports of polynomials from those classes.
We also present a new view on the standard, i.e. when $m = 2$, Newton inequalities and pose some open problems. Our results provide new necessary conditions for **H-Stability** and can be used for the identification of multivariate stable linear system, i.e. for the interpolation of **H-Stable** polynomials.

1 Standard Newton Inequalities

Definition 1.1: We define the following closed subset of R^{n+1} :

$$LC = \{(d_0, ..., d_n) : d_i \geq 0, 0 \leq i \leq n; d_i^2 \geq d_{i-1}d_{i+1}, 1 \leq i \leq n-1.$$

We also define a weighted shift operator $Shift_{\mathbf{c}} : R^{n+1} \longrightarrow R^{n+1}$,

$$Shift_{\mathbf{c}}((x_0, ..., x_n)^T) = (c_0 x_1, ..., c_{n-1} x_n, 0)^T.$$

*`gurvits@lanl.gov`. Los Alamos National Laboratory, Los Alamos, NM.

If c is the vector of all ones, then $Shift_{\mathbf{c}} =: Shift$.
A positive sequence $(b_0, ..., b_n)$ is called **propagatable** if the following implication holds:

$$(p^{(0)}(0)b_0, ..., p^{(n)}(0)b_n) \in LC \Longrightarrow (p^{(t)}(0)b_0, ..., p^{(n)}(t)b_n) \in LC, t \geq 0'$$

where p is a polynomial of degree at most n. ∎

Proposition 1.2: *Let $c_0, ..., c_{n-1}$ be a nonnegative sequence. Then $exp(tShift_{\mathbf{c}})(LC) \subset LC$ for all $t \geq 0$ if and only if*

$$2c_i \geq c_{i+1} + c_{i-1}, 1 \leq i \leq n-1; 2c_{n-1} \geq c_{n-2}.$$

(In other words, the infinite sequence $(c_0, ..., c_{n-1}, 0, ...)$ is concave.)

Proof:

1. *The "only if" part:* Consider the linear system of differential equations :

   $$X'(t) = Shift_{\mathbf{c}} X(t) : X(0) = (1, 1, ..., 1), X(t) = (X_0(t), ..., X_n(t))^T.$$

   Suppose that $exp(tShift_{\mathbf{c}})(LC) \subset LC, t \geq 0$, i.e $X(t) \in LC : t \geq 0$.
   Define the following smooth functions: $r_i(t) = (X_i(t))^2 - X_{i+1}(t)X_{i-1}(t) : 1 \leq i \leq n-1$.
   It follows that $r_i(0) = 0$ and $r_i(t) \geq 0, t \geq 0$. Therefore $r_i'(0) \geq 0$. But

   $$r_i'(0) = 2c_i - c_{i+1} + c_{i-1}, 1 \leq i \leq n-1; r_{n-1} = 2c_{n-1} - c_{n-2}.$$

2. *The "if" part:* As $exp(A) = \lim_{n\to\infty}(I + \frac{A}{n})^n$, thus it is sufficient to prove that $(I + tShift_{\mathbf{c}})(LC) \subset LC$ for all $t \geq 0$, which is done by straigthforward derivations.
   (The obsevation that $(I + Shift)(LC) \subset LC$ is probably well known, we have learned it from Julius Borcea.)

∎

Theorem 1.3: *Let $(b_0, ..., b_k)$ be a positive sequence. Define $c_i = \frac{b_i}{b_{i+1}}, 0 \leq i \leq k-1$. The sequence $(b_0, ..., b_k)$ is* **propagatable** *iff the infinite sequence $(c_0, ..., c_{k-1}, 0, ...)$ is concave.*

Proof: Define a vector function $Mom_b(t) = (b_0 p^{(0)}(t), ..., b_n p^{(n)}(t))^T$. It follows that $Mom_b(t)' = Shift_{\mathbf{c}}(Mom_b(t))$. Therefore $(b_0, ..., b_n)$ is **propagatable** iff $exp(tShift_{\mathbf{c}})(LC) \subset LC$ for all $t \geq 0$. The result now follows from Proposition (1.2). ∎

Example 1.4: A polynomial $p(t) = \sum_{0 \leq i \leq k} a_i t^i$ with nonnegative coefficients is called n-Newton, where $n \geq k$, if

$$d_i^2 \geq d_{i-1}d_{i+1} : 1 \leq i \leq k-1, d_i =: \frac{a_i}{\binom{n}{i}}.$$

Or, in other words, the vector $(p^{(0)}(0)b_0, ..., p^{(k)}(0)b_k) \in LC$, where $b_i = (n-i)!$.
As $c_i = \frac{b_i}{b_{i+1}} = n - i : 0 \leq i \leq k-1$ it follows from Theorem (1.3) that $(p^{(t)}(0)b_0, ..., p^{(k)}(t)b_k) \in LC : t \geq 0$. Equivalently,

$$(p^{(i+1)}(t))^2 \geq \frac{n-i}{n-i-1} p(i)(t) p^{(i+2)}(t) : t \geq 0, i \leq k-2,$$

which means that the functions $\sqrt[n-i]{p^{(i)}} : 0 \leq i \leq k$ are concave on R_+.

Proposition 1.5: *A polynomial p with nonnegative coefficients is n-Newton, where $n \geq deg(p)$, iff the functions $\sqrt[n-i]{p^{(i)}} : 0 \leq i \leq k$ are concave on R_+.*

Remark 1.6: The standard Newton Inequalities correspond to the case $n = deg(p)$ and hold if the roots of p are real. It was proved in [19] by G. C. Shephard that a polynomial p is n-newton iff $p(t) = Vol_n(tK_1 + K_2)$ for some convex compact subsets(simplexes) $K_1, K_2 \subset R^n$. We used this remarkable result in [12] for alternative (very short and non-computational) proofs of Proposition (1.5) and recent Liggett's convolution theorem, which states that pq is $m+n$-newton provided that p is n-newton and p is m-newton. ∎

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1.1 Multivariate Case

We start with the following definitions.

Definition 1.7:

1. We define integer simplex as $IS(m,n) = \{(r_1, ..., r_m) \in Z_+^m : \sum_{1 \leq i \leq m} r_i = n\}$, where Z_+^m is a set of integer m-dimensional vectors with non-negative coordinates; and the real simplex $Sim_m = \{(a_1, ..., a_m) \in R^m : \sum_{1 \leq i \leq m} r_i = 1\}$

2. We denote as $Hom_R(m,n)(Hom_C(m,n))$ a linear space of homogeneous polynomials with real(complex) coefficients of degree n and in m variables. We denote as $Hom_+(m,n)(Hom_{++}(n,m))$ a convex cone of polynomials $p \in Hom_R(m,n)$ with non-negative (positive) coefficients.

3. For a polynomial $p \in Hom_+(n,n)$ we define its **Capacity** as

$$Cap(p) = \inf_{x_i > 0, \prod_{1 \leq i \leq n} x_i = 1} p(x_1, ..., x_n) = \inf_{x_i > 0} \frac{p(x_1, ..., x_n)}{\prod_{1 \leq i \leq n} x_i} \tag{1}$$

   More generally, we define for vectors $Y = (y_1, ..., y_n) : y_i \geq 0, \sum_{1 \leq i \leq n} y_i = n$

$$C_p(y_1, ..., y_n) = \log \left(\inf_{x_i > 0} \frac{p(x_1, ..., x_n)}{\prod_{1 \leq i \leq n} (\frac{x_i}{y_i})^{y_i}} \right) \tag{2}$$

   Notice that $C_p(Y) = -(G^*(Y) - \sum_{1 \leq i \leq n} y_i \log(y_i))$, where G^* is the **Legendre-Fenchel** transformation of a convex functional $G(X) = \log(p(e^{x_1}, ..., e^{x_m}))$; $\log(Cap(p)) = C_p(1, 1, ..., 1)$.

Example 1.8: If $p \in Hom_+(n,n)$ then $(\partial x_1)\cdots(\partial x_n)p(0) \le Cap(p)$.
Consider an integer vector $r = (r_1, ..., r_n) \in R_+^n : r_1 + \cdots + r_n = n$. Assume WLOG that $r = (r_1, ..., r_k, 0, ..., 0) : r_i > 0, 1 \le i \le k; k \le n$. Define the polynomial $p \in Hom_C(n,n)$, $p_{(r)}(x_1, ..., x_n) = p(e_1(x_1 + ... + x_{r_1}) + \cdots + e_k(x_{r_1 + ... + r_{k-1}+1}))$, where $\{e_1, ..., e_n\}$ is the standard basis in C^n. Then

$$(\partial x_1)^{r_1} ... (\partial x_m)^{r_m} p(0) = (\partial y_1) ... (\partial y_n) p_{(r)}(0)$$

If $p \in Hom_+(m,n)$ then $C_p(r_1, ..., r_n) = \log(Cap(p_{(r)}))$.
Note that $C_p(y_1, ..., y_n) \ge C$ iff the next inequality holds:

$$\log(p(y_1 x_1, ..., y_n x_n)) \ge C + \sum_{1 \le i \le n} y_i \log(x_i) : x_i > 0, 1 \le i \le n \tag{3}$$

∎

4. Let $p \in Hom_C(m,n)$,

$$p(x_1, ..., x_m) = \sum_{(r_1, ..., r_m)} a_{r_1, ..., r_m} \prod_{1 \le i \le m} x_i^{r_i}$$

For a polynomial $p \in Hom_C(m,n)$ and a subset $S \subset \{1, ..., n\}$ we define $Rank_p(S)$ as the maximal joint degree attained on the set S:

$$Rank_p(S) = \max_{a_{r_1, ..., r_n} \ne 0} \sum_{j \in S} r_j \tag{4}$$

If $S = \{i\}$ is a singleton, we define $deg_p(i) = Rank_p(S)$.
We define $supp(p)$, the **support** of a polynomial p, as

$$supp(p) = \{(r_1, ..., r_m) \in Z_+^m : a_{r_1, ..., r_m} \ne 0\}$$

∎

Definition 1.9:

1. A polynomial $p \in Hom_+(m,n)$ is called AF(short for Alexandrov-Fenchel) if the polynomials $\sqrt[n-i]{(\partial x_1)^{c_1} ... (\partial x_m)^{c_m} p}$ are concave on R_+^m, where $(c_1, ..., c_m) \in Z_+^m, \sum_{1 \le j \le m} c_j + i \le n - 1$. A set of AF-polynomials is denoted as $AFP(m,n)$
2. A polynomial $p \in Hom_+(m,n)$ is called **Strongly Log-Concave** if the polynomial $(\partial x_1)^{c_1} ... (\partial x_m)^{c_m} p$ is either zero or $\log((\partial x_1)^{c_1} ... (\partial x_m)^{c_m} p)$ is concave on R_+^m. A set of **Strongly Log-Concave** polynomials is denoted as $SLCP(m,n)$.
3. A polynomial $p \in Hom_C(m,n)$ is called **H-Stable** if $p(Z) \ne 0$ provided $Re(Z) > 0$. We denote a set of **H-Stable** polynomials as $HSP(m,n)$.

4. Let $X \subset IS(m,n)$. We say that X is **L-Convex** if $X = IS(m,n) \cap CO(X)$, where $CO(X)$ is the convex hull of X.
   Consider a functional $f : IS(m,n) \rightarrow R \cup \{-\infty\}$. We define its measure of concavity as

$$MC(f) = \inf_{(a,\dots,a_k) \in Sim_k} (f(\sum_{1 \leq i \leq k < \infty} a_i Y_i) - \sum_{1 \leq i \leq k < \infty} a_i f(Y_i)),$$

   where $Y_1, Y_2, \dots \in IS(m,n); \sum_{1 \leq i < \infty} a_i Y_i \in IS(m,n)$.

5. For a polynomial $p \in Hom_+(m,n)$ we define a map $LC_p : IS(m,n) \rightarrow R \cup \{-\infty\}$, $LC_p(r_1, \dots, r_m) = \log(a_{r_1,\dots,r_m} \prod_{1 \leq i \leq m} r_i!)$.
   Note that $LC_p(r_1, \dots, r_m) = \log\left((\partial x_1)^{r_1} \dots (\partial x_m)^{r_m} p(0)\right)$.

▮

Fact 1.10:

1. If a polynomial $p \in HSP(m,n)$ is **H-Stable** then $\frac{p}{p(t_1,\dots,t_m)} \in Hom_+(m,n)$ for any vector $(t_1, \dots, t_m) \in R^m_{++}$ with real positive coordinates.

2. (Newton-Alexandrov inequalities for **H-Stable** polynomials, they follow from [15] using the connection in Part 1 of Example(1.11))
   Consider $p \in HSP(m,n) \cap Hom_+(m,n), p(x_1, \dots, x_m) = \sum_{(r_1,\dots,r_m)} a_{r_1,\dots,r_m} \prod_{1 \leq i \leq m} x_i^{r_i}$.
   Define $G(R) = a_{r_1,\dots,r_m} \prod_{1 \leq i \leq m} (r_i)!$. Let $\Delta = (1, -1, 0, \dots, 0) \in Z^m$.
   If three vectors $R, R + \Delta, R - \Delta \in IS(n,m)$ then

$$(G(R))^2 \geq G(R + \Delta) G(R - \Delta) \tag{5}$$

3. The following inclusions hold: $(HSP(m,n) \cap Hom_+(m,n)) \subset SLCP(m,n) \subset AFP(m,n)$.

Example 1.11:

1. Let $p \in Hom_C(2,n) : p(z_1, z_2) = (z_2)^n \sum_{0 \leq i \leq n} a_i (\frac{z_1}{z_2})^i$, and $R(t) = \sum_{0 \leq i \leq n} a_i t^i$. Such polynomial p is **H-Stable** iff the univariate polynomial R does not have roots in the set $\{\frac{z_1}{z_2} : Re(z_1), Re(z_2) > 0\} = \{x \in R : x \leq 0\}$.
   In other words, iff all the roots of R are real non-positive numbers.
   In the very same way, $p \in HSP(m,n)$ iff for all real positive vectors Y_1, Y_2 the roots of the equation $p(tY_1 + Y_2) = 0$ are real and negative. This observation connects **H-Stable** polynomials with the fundamental notion of hyperbolicity(see more on this in [7], [8], [11]).

2. A sufficiently smooth map $f : R^m_+ \rightarrow R_+$ is log-concave iff the symmetric matrix

$$grad(f)(Y)(grad(f)(Y))^T - f(Y) Hess(f)(Y) \succeq 0, Y \in R^m_+$$

   is positive semidenite.
   The map $f^{\frac{1}{k}}, k > 1$ is concave iff

$$grad(f)(Y)(grad(f)(Y))^T - \frac{k}{k-1} f(Y) Hess(f)(Y) \succeq 0.$$

Consider a non-zero polynomial $p \in Hom_+(m,2)$, i.e. $p(Y) =< AY, Y >$, where A is a non-zero symmetric matrix with nonnegative entries.
It is well known that in this case p is **H-Stable** iff the matrix A is hyperbolic, which means that its signature is $(+,-,...,-)$. In this quadratic case the condition of log-concavity reads as

$$2(AY)(AY)^T - < AY, Y > A \succeq 0, Y \in R_+^m. \tag{6}$$

It is easy to see that the condition (6) implies that the matrix A is hyperbolic.
Therefore

$$(HSP(m,n) \cap Hom_+(m,2)) = SLCP(m,2) = AFP(m,2).$$

To get an example of AF-polynomial, which is not **H-Stable**, take an arbitrary n-Newton polynomial $p, deg(p) = n \geq 3$ with some complex roots and homogenize it: $q(x,y) =: y^n p(\frac{x}{y})$. This example has very deep connections to the theory of mixed volumes of convex sets [1],[19].

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Our main result is the next Theorem.

Theorem 1.12:

1.

$$-\frac{1}{2}\log(2)n \geq \inf_{p \in SLCP(m,n)} MC(LC_p) \geq -n \tag{7}$$

2.

$$-\frac{1}{2}\log(2)n \geq \inf_{p \in AFP(m,n)} MC(LC_p) \geq \log(\frac{n!}{n^n}) \tag{8}$$

In the case of sparse **H-Stable** polynomials, we get sharper inequality:

Theorem 1.13: *Let $p \in (HSP(m,n) \cap Hom_+(m,n))$ and $deg_p(i) \leq k, 2 \leq k \leq n : 1 \leq i \leq m$. Then $MC(LC_p) \geq (n-k)(k-1)\log(\frac{k-1}{k}) + \log(\frac{k!}{k^k}) \geq n(k-1)\log(\frac{k-1}{k})$.*

Corollary 1.14: *The supports $supp(p)$ of* **Strongly Log-Concave** *polynomials are* **L-Convex**.

Example 1.15: Consider the a polynomial $p(x_1,...,x_{2n}) = (x_1+x_2)(x_2+x_3)...(x_{2n-1}+x_{2n})(x_{2n}+x_1)$. Clearly, $p \in HSP(2n,2n); deg_p(i) = 2 : 1 \leq i \leq 2n$. Consider three vectors: $R_0 = (1,...,1), R_1 = (2,0,2,...,0,2), R_2 = (0,2,...,0,2); 2R_0 = R_1 + R_2$. By direct inspection, $G(R_0) = 2, G(R_1) = G(R_2) = 2^n$. Therefore $MC(LC_p) \leq -\frac{1}{2}\log(2)deg(p)$.
This gives the left inequality in (8). ∎

1.2 Our proof strategy

1. STEP 1.$C_p(y_1, ..., y_m)$ is log-concave if $p \in Hom_+(m, n)$ is log-concave on R_+^m.
This log-concavity statement follows from the inequality (3.

2. STEP 2. Since $C_p(y_1, ..., y_m)$ is log-concave hence the right inequalities in Theorem (1.12) would follow from the following bounds

$$1 \geq \frac{(\partial x_1)^{r_1} ... (\partial x_m)^{r_m} p(0)}{exp(C_p(r_1, ..., r_n))} \geq a(n), \tag{9}$$

where $a(n) = exp(-n)$ if $p \in SLCP(m, n)$ and $a(n) = \frac{n!}{n^n}$ if $p \in AFP(m, n)$

3. STEP 3. Consider a polynomial $p \in Hom_C(m, n)$ and an integer vector $r = (r_1, ..., r_m) \in IS(m, n)$.
Assume WLOG that $r = (r_1, ..., r_k, 0, ..., 0) : r_i > 0, 1 \leq i \leq k; k \leq n$. Define the polynomial $p \in Hom_C(n, n)$,

$$p_{(r)}(y_1, ..., y_n) = p(e_1(y_1 + ... + y_{r_1}) + ... + e_k(x_{r_1 + ... + r_{k-1} + 1} + ... x_n,$$

where $\{e_1, ..., e_m\}$ is the standard basis in C^m. Then

$$(\partial x_1)^{r_1} ... (\partial x_m)^{r_m} p(0) = (\partial y_1) ... (\partial y_n) p_{(r)}(0)$$

If $p \in Hom_+(m, n)$ then $C_p(r_1, ..., r_n) = \log(Cap(p_{(r)}))$.
Therefore, it is sufficient to prove inequalities (9) for $r = (1, 1, ..., 1)$ and polynomials $p \in SLCP(n, n)$, $p \in AFP(n, n)$. Notice that if a polynomial p is **H-Stable**(Alexandrov-Fenchel,**Strongly Log-Concave**) then also the polynomial $p_{(r)}$ is. The inequality $Cap(p) \geq (\partial x_1) ... (\partial x_n) p(0)$ holds for all polynomials $p \in Hom_+(n, n)$.

2 Sketch of the proofs

. We need the next elementary result:

Lemma 2.1: *Consider a function $f : R_+ \rightarrow R_+$ such that the derivative $f'(0)$ exists.*

1. *If $f^{\frac{1}{k}}$ is concave on R_+ for $k > 1$ then $f'(0) \geq (\frac{k-1}{k})^{k-1} \inf_{t>0} \frac{f(t)}{t}$.*

2. *If $f^{\frac{1}{k}}$ is log-concave on R_+ then $f'(0) \geq \frac{1}{e} \inf_{t>0} \frac{f(t)}{t}$.*

3. *Let $R(t) = a_0 + ... + a_n t^n$ be a strongly log-concave on R_+ univariate polynomial with nonnegative coefficients:*
$G(i)^2 \geq G(i-1)G(i+1) : 1 \leq i \leq n-1, G(i) = a_i i!$.
Then $f'(0) \geq L(n) \inf_{t>0} \frac{f(t)}{t}$, where $L(n) = (\inf_{t>0} \frac{exp_n(t)}{t})^{-1}$ and the truncated exponential $exp_n(t) = 1 + ... + \frac{1}{n!} t^n$. (Notice that exp_n is strongly log-concave on R_+.)

Corollary 2.2: *Let $p \in Hom_+(n,n)$ and $g_n(x_1,...,x_n) = (\partial x_{n+1})p(x_1,...,x_n,0)$. If $p^{\frac{1}{k}}$ is concave on R_+ for $k > 1$ then*

$$Cap(q_{n+1}) \geq g(k)(\frac{k-1}{k})^{k-1}Cap(p), \text{ where } g(k) =: (\frac{k-1}{k})^{k-1}. \tag{10}$$

Proof: We need to prove that $(\partial x_{n+1})p(x_1,...,x_n,0) \geq (\frac{k-1}{k})^{k-1}Cap(p)$,
where $\prod_{1\leq i\leq k} x_i = 1, x_i > 0$. Define an univariate polynomial
$f(t) = p(x_1,...,x_n,t)$. Then
$f(t) \geq Cap(p)t : t \geq 0$ and $f'(0) = (\partial x_{n+1})p(x_1,...,x_n,0)$. It follows from Lemma(2.1) that $(\partial x_{n+1})p(x_1,...,x_n,0) \geq (\frac{k-1}{k})^{k-1}Cap(p)$. ∎

As it was explained in STEP 3 above, we need to prove the lower bound

$$(\partial x_1)...(\partial x_n)p(0) \geq aCap(p), \tag{11}$$

where $a = \frac{n!}{n^n}$ for $p \in AFP(n,n)$ and $a = e^{-n}$ for $p \in SLCP(n,n)$. We will prove here the $AFP(n,n)$ case as the $SLCP(n,n)$ case has very similar proof.
Our proof is by induction:
Define the following polynomials $q_i \in Hom_+(i,i)$:
$q_n = p$, $q_i(x_1,...,x_i) = \frac{\partial^{n-i}}{\partial x_{i+1}...\partial x_n}p(x_1,...,x_i,0,...,0)$. *Notice that* $q_1(x_1) = \frac{\partial^n}{\partial x_1...\partial x_n}p(0)x_1$.
Therefore, $Cap(q_1) = \frac{\partial^n}{\partial x_1...\partial x_n}p(0)$.

For what follows, we need polynomials $(q_i)^{\frac{1}{i}}, n \geq i \geq 1$ to be concave on R^i_+. This is certainly true if $p \in AFP(n,n)$. Thus, we get from Corollary(2.2) that $Cap(q_i) \geq g(i)Cap(q_{i-1})$. Finally

$$(\partial x_1)...(\partial x_n)p(0) = Cap(q_1) \geq Cap(p) \prod_{2\leq i\leq n} g(i) = \frac{n!}{n^n}Cap(p).$$

Remark 2.3: Using the third item in Lemma (2.1) and the same method, we can prove that if $p \in SLCP(n,n)$ then $(\partial x_1)...(\partial x_n)p(0) \geq Cap(p)\prod_{2\leq i\leq n} L(\min(deg_p(i),i))$. If, say $deg_p(i) \leq 2$, this gives a bound
$(\partial x_1)...(\partial x_n)p(0) \geq Cap(p)\frac{1}{2}(1+\sqrt{2})^{-n+2}$. ∎

3 Comments, Open problems

1. The inequality (11) is a far going generalization of the famous Van der Waerden conjecture on permanents of doubly-stochastic matrices([16], [4], [3], [2]). See more on this combinatorial connection in [9], [11], [5].
   The "convex relaxation" approach to Newton-Alexandrov(like) inequalities in Theorem(1.12) was introduced by the author in [10] for determinantal polynomials $\det(\sum_{1\leq i\leq m} x_i A_i)$, where $A_1,...,A_m$ are $n\times n$ hermitian PSD matrices.

2. In the case of **H-Stable** polynomials, Corollary (1.14) can be made much more precise:

$$a_{r_1,...,r_m} > 0 \Longleftrightarrow \sum_{j \in S} r_j \leq Rank_p(S) : S \subset \{1, ..., m\} \tag{12}$$

   The characterization (12) is a far going generalization of Hall-Rado theorems.
   The paper [9] provides algorithmic applications of these results: strongly polynomial deterministic algorithms for the membership problem as for the support as well Newton polytope of **H-Stable** polynomials $p \in Hom_+(m, n)$, given as oracles.
   We don't know whether (12) works for $p \in AFP(m, n)$ and $p \in SLCP(m, n)$.

3. What are the exact constant in Theorem(1.12)? To be precise, let us define

$$NW(n) =: - \inf_{p \in HSP(n,n)} \frac{MC(LC_p)}{n}.$$

   What is the exact value of $NW(n)$ or its exact assymptotics? Is the polynomial in Example(1.15) extremal?

4. Can recently refuted Okounkov's conjecture [17], in the representation theory, on log-concavity of multiplicities be fixed/generalized in the way similar to Theorem(1.12)?

5. Stable multivariate polynomials form a backbone of linear multivariate control. If $p \in HSP_+(n, n)$ then $Cap(p) = \inf_{Re(z_i)>0} \frac{|p(z_1,...,z_n)|}{\prod_{1 \leq i \leq n} Re(z_i)}$. In other words, the capacity can be viewed as a measure of stability. What is a meaning of capacity if terms of control/dynamics?

6. Can our results be generalized to the fractional derivatives?

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