Lower bounds for approximate factorizations via semidefinite programming (extended abstract)*

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The problem of approximately factoring a real or complex multivariate polynomial f seeks minimal perturbations Δf to the coefficients of the input polynomial f so that the deformed polynomial $f + \Delta f$ has the desired factorization properties. Efficient algorithms exist that compute the nearest real or complex polynomial that has non-trivial factors (see [3, 6] and the literature cited there). Here we consider the solution of the arising optimization problems using polynomial optimization (POP) via semidefinite programming (SDP). We restrict to real coefficients in the input and output polynomials.

1. OPTIMIZING FACTOR COEFFICIENTS

In [4] we formulate the problem of computing for an absolutely irreducible input polynomial $f \in \mathbb{R}[x, y]$ the nearest polynomial with a factor $g = g_{0,0} + g_{1,0}x + g_{0,1}y + \cdots + g_{k,0}x^k \in \mathbb{R}[x, y]$ of given total degree k as a multivariate rational function optimization problem

$$\min_{g,\deg(g)\le k} u_f(g_{0,0}, \ldots, g_{k,0})/v_f(g_{0,0}, \ldots, g_{k,0}), \qquad (1)$$

where u, v are real polynomials in the coefficients of the factor g. Since the polynomial v is a positive polynomial, the optimization problem (1) can be written as a constrained POP [9]

$$\max_{r,g} r,\ u_f(g_{0,0}, \ldots, g_{k,0}) - rv_f(g_{0,0}, \ldots, g_{k,0}) \ge 0, \qquad (2)$$

and solved by sums-of-squares (SOS) relaxation. Note that the optimum $r^{[\mathrm{OPT}]} = \|\Delta f\|_2^2$.

Example 1.1. Consider the polynomials

$$f_1 = (x + y + 1)(x - 2y + 1) + 0.1\,x,$$
$$f_2 = \left(x^2+xy+2\,y-1\right)\left(x^3+y^2x-y+7\right) +0.2\,x \text{ [5, Example 3]},$$
$$f_3 = \left(x^2+3\,y^2+4\,x+1\right)(x-2\,y-1) +0.1\,x^3+0.2\,y^2+0.3\,xy.$$

We compute the optimal solution of the constrained POP (2) with the MATLAB package SOSTOOLS [10].

Ex. 1.1.1. Polynomial f_1: For $k = 1$, u, v are real polynomials in three variables of degree 6, and $\|\Delta f_1^{[\mathrm{SOS}]}\|_2 \approx 2.61661\cdot 10^{-2} \approx \|\Delta f_1^{[\mathrm{OPT}]}\|_2$, the latter because the solution agrees with the approximate factorization (upper bound) computed by our SVD+Gauss-Newton algorithm [6].

Ex. 1.1.2. Polynomial f_2: For $k = 1$, u, v are real polynomials in three variables of degree 30, and $\|\Delta f_2^{[\mathrm{SOS}]}\|_2 \approx 0.60422$. It is obvious that f_2 is closer to a polynomial with degree 2 and 3 factors, we tried to solve the optimization problem (2) for $k = 2$ or $k = 3$. However, the expressions of u, v become very complicated and we can not solve the problem yet.

Ex. 1.1.3. Polynomial f_3: For $k = 1$, u, v are real polynomials in three variables of degree 12, and $\|\Delta f_3^{[\mathrm{SOS}]}\|_2 \approx 2.0165 \cdot 10^{-1} \approx \|\Delta f_3^{[\mathrm{OPT}]}\|_2$, which again agrees with the SVD+Gauss-Newton factorization.

2. OPTIMIZING INPUT COEFFICIENT DEFORMATIONS

In [3, 5] we employ the approach in [2, 11, 12] in the hybrid symbolic-numeric setting. Our algorithms can use the following fact, here stated for 2 variables.

Fact: *Let $f \in \mathbb{C}[x, y]$ be a polynomial of degrees $\deg_x(f) = d_x \ge 2$ and $\deg_y(f) = d_y \ge 2$ in the variables x and y, respectively. Suppose $\mathrm{GCD}(f, \partial_x f) = 1$ and suppose that f has r irreducible factors over $\mathbb{C}$. Furthermore consider the coefficient matrix $R(f)$ of the homogeneous linear system $\partial_y(g/f) = \partial_x(h/f)$ in the coefficients of $g, h \in \mathbb{C}[x, y]$ with $\deg(g), \deg(h) \le \deg(f)$ and $\deg_x(g) \le d_x - 2$, $\deg_y(h) \le d_y - 1$. Then the nullspace of $R(f)$ (the space of the combined coefficient vectors $\mathbf{x}$ of g, h) has dimension $r - 1$. In particular, f is irreducible over $\mathbb{C}$ if and only if $R(f)$ has a trivial nullspace, i.e., $R(f)$ is of full rank.*

The problem of approximate factorization can then be formulated as a structured deformation of a matrix to achieve rank deficiency at least $r - 1$:

$$\left.\begin{array}{ll}\min_{Q,\Delta f} & \|\Delta f\|_2^2 \\ \text{s. t.} & R(f + \Delta f)Q = 0, \text{where } Q^H Q = I_{r-1}.\end{array}\right\} \qquad (3)$$

Note that like in the GCD case [7], the optimal solution to (3) may not yield a factorizable polynomial since the optimal solution $f + \Delta f$ may have a lower degree or a factor of multiplicity ≥ 2.

Nonetheless, (3) is a constrained polynomial optimization problem and we can deploy codes based on Lagrangian mul-

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tipliers (STLS) and semidefinite programming (SDP) for its solution. In contrast to our earlier SVD+Gauss-Newton and STLS approaches [6], SDP-based solutions are certified to be the global optima (within the floating point error). If we permit relaxation [8], the computed values $\|\Delta f\|$ yield lower bounds for the irreducibility radii [5].

We have tested the SDP approach on three small examples with $r = 2$, using the Matlab SparsePOP code [13]. Here Q is a single vector $\mathbf{x}$ of length 1. We have observed that the order 2 Lasserre relaxation produces the actual global solutions for our examples. It may be possible to prove this fact. Those global optima agree with the local optimal solutions computed by our SVD+Gauss-Newton and STLS approachs [6]. Unfortunately, the arising SDPs are quite large and the SDP approach cannot solve the larger benchmarks in [6].

The optimal objective value of order 1 Lasserre relaxation for (3) is always zero, hence this relaxation produces no lower bound. One can formulate for a given regularization parameter ρ the randomized problem

$$\left.\begin{array}{ll} \min_{\mathbf{x},\Delta f} & \|\Delta f\|_2^2 + \rho\|\mathbf{x}\|_2^2 \\ \text{s. t.} & R(f+\Delta f)\mathbf{x} = 0 \text{ and} \\ & \mathbf{v}^T\mathbf{x} = 1 \text{ where } \mathbf{v} \text{ is a fixed random vector.} \end{array}\right\} \quad (4)$$

The statement of a lower bound is weaker (only with high probability and accounting the regularization error), but the order 1 Lasserre solution reveals some information. Similar to [7], one can also select a column $b(f)$ in $R(f)$ and instead use the constraint $A(f + \Delta f)\mathbf{y} = b(f + \Delta f)$, where $A(f)$ is $R(f)$ with column $b(f)$ removed.

Example 2.1 Set $\rho = 10^{-6}$. We use the polynomials f_1, f_2 and f_3 in Example 1.1.

Ex. 2.1.1. Polynomial f_1: We compute the lower bound by solving the order 1 and order 2 Lasserre relaxations for the problem (4): $\|\Delta f_1^{[1]}\|_2 \approx 2.63075 \cdot 10^{-3}$, $\|\Delta f_1^{[2]}\|_2 \approx 9.07773 \cdot 10^{-3}$, while the SVD lower bound [5] is $\approx 1.03581 \cdot 10^{-2}$. The solution produced by the order 2 Lasserre for (3) is optimal (see Example 1.1.1). The corresponding SDPs have the following sizes: Order 1 Lasserre relaxation for (4): 44 constraints; 3 semidefinite blocks of sizes 4, 4, 5 respectively; and 7 free variables. Order 2 Lasserre for (3): 838 constraints, 3 semidefinite blocks of sizes 26, 39, 40 respectively; and 326 free variables.

Ex. 2.1.2. Polynomial f_2: The lower bounds computed by the order 1 Lasserre relaxation for (4) and SVD [5] are, respectively $\|\Delta f_2^{[1]}\|_2 \approx 1.54592 \cdot 10^{-3}$, $\|\Delta f_2^{[\mathrm{SVD}]}\|_2 \approx 1.77862 \cdot 10^{-3}$. The corresponding SDP has the following sizes: 896 constraints, 3 semidefinite blocks of sizes 37, 40, 40 respectively; and 36 free variables. Unfortunately, the order 2 Lasserre relaxation for this example is too large to be currently solved.

Ex. 2.1.3. Polynomial f_3: The lower bounds computed by the order 1 Lasserre relaxation for (4) and SVD [5] are, respectively $\|\Delta f_3^{[1]}\|_2 \approx 2.09142 \cdot 10^{-3}$, $\|\Delta f_3^{[\mathrm{SVD}]}\|_2 \approx 2.89648 \cdot 10^{-2}$. The corresponding SDP has the following sizes: 244 constraints, 3 semidefinite blocks of sizes 17,18,20; and 16 free variables.

We succeeded in solving the order 2 Lasserre relaxation for (3) for the optimal solution (see Example 1.1.3): The resulting SDP has 9939 constaints, 3 semidefinite blocks of sizes 121, 150, 162, and 5832 free variables. It took 90 minutes of computing time to solve the associated SDP to 8 digits of accuracy on NCSU's distributed high performance IBM cluster "Henry2."

3. DISCUSSION

We have tested SDP-based constrained polynomial optimization packages on the optimization problems arising in our approximate factorization algorithms. Both our algorithms in [4] and in [6] lead to very large semidefinite programs. We plan to investigate SDP relaxations of (3) and (4) that give better lower bounds than the order 1 Lasserre SDP relaxation and are smaller in size than the order 2 Lasserre SDP relaxation applied to these formulations. Solution of these SDP relaxations using parallel interior point software such as CSDP [1] and SDPARA [14] on a high performance cluster is also planned.

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