Two Row Mixed Integer Cuts Via Lifting*

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Abstract

Recently, Andersen et al. [1], Borozan and Cornuéjols [7] and Cornuéjols and Margot [10] characterized extreme inequalities of a system of two rows with two free integer variables and nonnegative continuous variables. These inequalities are either split cuts or intersection cuts (Balas [3]) derived using maximal lattice-free convex sets. In order to use these inequalities to obtain cuts from two rows of a general simplex tableau, one approach is to extend the system to include all possible nonnegative integer variables (giving the two-row mixed integer infinite-group problem), and to develop lifting functions giving the coefficients of the integer variables in the corresponding inequalities. In this paper, we study the characteristics of these lifting functions.

We begin by observing that functions giving valid coefficients for the nonnegative integer variables can be constructed by lifting a subset of the integer variables and then applying the fill-in procedure presented in Johnson [25]. We present conditions for these 'general fill-in functions' to be extreme for the two-row mixed integer infinite-group problem. We then show that there exists a unique 'trivial' lifting function that yields extreme inequalities when starting from a maximal lattice-free triangle with multiple integer points in the relative interior of one of its sides, or a maximal lattice-free triangle with integral vertices and one integer point in the relative interior of each side. In all other cases (maximal lattice-free triangle with one integer point in the relative interior of each side and non-integral vertices, and maximal lattice-free quadrilaterals), non-unique lifting functions may yield distinct extreme inequalities. For the case of a triangle with one integer point in the relative interior of each side and non-integral vertices, we present sufficient conditions to yield an extreme inequality for the two-row mixed integer infinite-group problem.

1 Introduction

The Gomory mixed integer cuts (GMIC) and valid inequalities based on single row mixed integer group relaxations have been studied both theoretically and computationally, whereas valid inequalities from two and multiple rows have so far almost exclusively been studied theoretically. Our goal is to present valid inequalities generated from two-row mixed integer group problems that have the potential to be useful computationally. They can be used directly to generate valid inequalities from any two rows of an optimal simplex tableau, are strong in a well-defined sense and have similar properties to the most effective single-row inequality, the GMIC.

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Below we briefly discuss earlier work and the motivation for the approach that is taken in this paper. The GMIC was proposed by Gomory [17], and has been shown to be one of the most effective cuts used in general purpose mixed integer programming solvers, see Balas et al. [4] and Bixby and Rothberg [6]. Group cutting planes based on relaxations of a single row of a mixed integer program, of which GMIC is a special case, were presented by Gomory [18], Gomory and Johnson [20, 21] in the 70's, and more recently in Gomory, Johnson, and Evans [23], Gomory and Johnson [22], Aráoz et al. [2], Miller, Li and Richard [28], Richard, Li, and Miller [30] and Dash and Günlük [12]. This has led to computational work to test whether the other inequalities based on single row group relaxation are effective computationally; see Cornúejols, Li and Vandenbussche [9], Fischetti and Saturni [16] and Dash and Günlük [11]. In general the results have been disappointing and the GMIC seems to be the most effective single row mixed integer group inequality. One possible explanation for this is the fact that the GMIC has the strongest coefficients for the continuous variables among all single row group inequalities.

In Johnson [25] multiple row group inequalities were studied and in Gomory and Johnson [22] the potential advantages of valid inequalities based on multiple constraints were discussed. In particular, one weakness of the single row inequalities is that the continuous variables are modeled by aggregating them into two continuous variables, based on the signs of the coefficients. Group cuts based on multiple rows overcome this limitation and can more accurately represent the structure of the columns corresponding to continuous variables. Some extreme inequalities for two-row mixed integer group problems are presented in Dey and Richard [14, 13].

A slightly different viewpoint has been taken recently by Andersen et al. [1], Borozan and Cornuéjols [7] and Cornuéjols and Margot [10]. They have analyzed a system of two rows with two free integer variables and nonnegative continuous variables. They show that extreme inequalities of the system

$$f + \sum_{w \in \mathbb{Q}^2} w y(w) \in \mathbb{Z}^2, \tag{1}$$
$$y(w) \geq 0 \ \forall w \in \mathbb{Q}^2, \quad y \text{ has finite support}, \quad f \in \mathbb{Q}^2 \setminus \mathbb{Z}^2$$

are either split cuts or intersection cuts (Balas [3]) that can be derived using maximal lattice-free convex sets.

Our approach builds on this work. Given that the GMIC is the most effective single row group inequality and has the strongest coefficients on the continuous variables, we attempt to keep similar properties when generating inequalities from two rows. Thus we view the construction of the GMIC in the following way:

1. Starting from a simplex tableau of a MIP, create a single-row mixed-integer group relaxation.
2. Fix the nonnegative integer variables of mixed-integer group relaxation to zero and generate an extreme (facet-defining) inequality with respect to the continuous variables.
3. Lift the nonnegative integer variables into this cut to obtain an inequality that is extreme for the one-row mixed integer infinite-group problem (see Nemhauser and Wolsey [29] for an overview on lifting).

We apply the same approach to the two row case. The recent results of Andersen et al. [1], Borozan and Cornuéjols [7] and Cornuéjols and Margot [10] tells us how to approach step 2. Our contribution is to accomplish the two-row counterpart of step 3, i.e., to lift integer variables into the extreme inequalities for (1) in order to obtain new extreme inequalities for the two-row mixed integer infinite-group problem. The new inequalities derived in this way may thus be considered as the two-row counterparts to the GMIC; they are both extreme inequalities for the mixed integer infinite-group problem and have the strongest possible coefficients for the continuous variables. A related approach has been discussed by Gomory [19]. An extended abstract of some of the results in this paper is presented in Dey and Wolsey [15].

The rest of the paper is organized as follows. In Section 2, we present some preliminaries about the mixed integer infinite-group problem and the continuous infinite-group problem, and classify *maximal lattice-free convex sets* in $\mathbb{R}^2$. In particular, we show that convex maximal lattice-free sets with non empty interiors in $\mathbb{R}^2$ are splits, triangles with multiple integer points in the relative interior of one side, triangles with integral vertices and a single integer point in the relative interior of each side, triangles with non-integral vertices and a single integer point in the relative interior of each side, and quadrilaterals. In Section 3, we illustrate the characterization and analysis of lifting functions for the case when the lattice-free convex set is the split cylinder and derive the well known split cut. In Section 4, we show that when the inequality for (1) is related to a maximal lattice-free triangle with either multiple integer points in the relative interior of one side, or integral vertices and one integer point in the relative interior of each side, then there exists a unique lifting function such that the resultant inequality is extreme for the two-row mixed-integer infinite-group problem. In Section 5, we present a modified version of the *fill-in procedure* of Gomory and Johnson [20] and Johnson [25], which is closely related to the lifting of integer variables in valid inequalities for the mixed integer infinite-group problem. The tools developed here are a generalization of the tools used in the previous sections and allow the analysis of the more complex cases. In Section 6, using these tools we show that when we start with an inequality for (1) which is related to a maximal lattice-free triangle with one integer point in the relative interior of each side and non-integral vertices or a maximal lattice-free quadrilateral, there does not exist a unique lifting function. For the case of lattice-free triangles with one integer point in the relative interior of each side and non-integral vertices, we present sufficient conditions for the lifting functions to generate an extreme inequality for the two-row mixed-integer infinite-group problem. In Section 7, we illustrate examples of these new inequalities. We conclude in Section 8.

2 Preliminaries

We begin this section with a concise description of the mixed integer infinite-group problem. Then we focus on the continuous version of it. Finally we end this section with a classification of maximal lattice-free convex sets in $\mathbb{R}^2$.

2.1 Mixed integer infinite-group problem

Observe that the integer variables in (1) have no sign restrictions. This corresponds to the so-called group relaxation that was first defined and studied by Gomory [18], Gomory and Johnson [20, 21, 23, 22] and Johnson [25]. We present notation and a brief overview of the mixed integer infinite-group problem and establish its relationship to (1).

Let I^m represent the infinite-group of real m-dimensional vectors where addition is taken modulo 1 componentwise, i.e., $I^m = \{(u_1, u_2, ...u_m) \mid 0 \leq u_i < 1, i \in \{0, 1, ..., m\}\}$. Let S^m represent the set of real m-dimensional vectors $w = (w_1, w_2, ..., w_m)$ that satisfy $max_{1\leq i\leq m}|w_i| = 1$. For an element $u \in \mathbb{R}^m$, we use the symbol $\mathbb{P}(u)$ to denote the element in I^m whose i^{th} entry is $u_i(mod\ 1)$. We use the symbol $\bar{0}$ to represent the zero vector in $\mathbb{R}^m$ and I^m and the symbol $\bar{1}$ to represent the vector $(1, ..., 1)$. We use the symbols $+$ and $-$ to represent addition and substraction in both I^m and $\mathbb{R}^m$.

The mixed integer infinite-group problem is defined next.

Definition 1 ([21], [25]) *Let U be a subgroup of I^m and W be any subset of S^m. Then the mixed integer infinite-group problem, denoted $MI(U, W, r)$, is defined as the set of pairs of functions $x : U \to \mathbb{Z}_+$ and $y : W \to \mathbb{R}_+$ that satisfy*

1. $\sum_{u\in U} ux(u) + \mathbb{P}(\sum_{w\in W} wy(w)) = r$, $r \in I^m$,
2. *x and y have finite supports.* □

The key observation connecting $MI(I^2, S^2, r)$ to (1) is the following: *If all the $x(u)$'s are fixed to zero in $MI(I^2, S^2, r)$, the problem would reduce to that presented in (1) (with columns suitably scaled) where $r = \mathbb{P}(-f)$.*[1]

[1]Note here that columns corresponding to the continuous variables are assumed to be rational in (1). However, we will assume that $W = S^2$ which allows the use of results from Johnson [25]. This is a only a minor technical assumption as we will show that results obtained using only rational columns for (1) apply to the case when columns are irrationals.

Throughout this paper, we will use the symbol r to represent the right-hand-side of group problem and f as the constant in (1), with $r = \mathbb{P}(-f)$.

Next we present the definition of valid inequalities for the infinite-group problem.

Definition 2 ([21], [25]) *A valid inequality for $MI(U, W, r)$ is defined as a pair of functions, $\phi : U \to \mathbb{R}_+$ and $\mu_\phi : W \to \mathbb{R}_+$, such that $\sum_{u \in U} \phi(u)x(u) + \sum_{w \in W} \mu_\phi(w)y(w) \geq 1$, $\forall (x, y) \in MI(U, W, r)$, where $\phi(\bar{0}) = 0$.* □

Since valid inequalities for the group problem are functions defined over I^m and S^m, we will use the terms valid inequality and valid function interchangeably.

See Gomory and Johnson [22] for a presentation of how these inequalities can be used to generate valid cutting planes for two rows of a simplex tableau. Gomory and Johnson [21] and Johnson [25] present a hierarchy of valid inequalities which include valid, subadditive, minimal and extreme inequalities. We present next the concept of a minimal inequality, which is essentially an inequality that is not dominated by any other inequality.

Definition 3 ([21], [25]) *A valid function (ϕ, π) is minimal for $MI(U, W, r)$ if there do not exist valid functions (ϕ^*, π^*) for $MI(U, W, r)$ different from (ϕ, π) such that $\phi^*(u) \leq \phi(u)$ $\forall u \in U$ and $\pi^*(w) \leq \pi(w)$ $\forall w \in W$.* □

Next we define the notion of extreme inequalities.

Definition 4 ([21], [25]) *A valid function (ϕ, π) is extreme for $MI(U, W, r)$ if there do not exist valid functions (ϕ_1, π_1) and (ϕ_2, π_2) for $MI(U, W, r)$ such that $(\phi_1, \pi_1) \neq (\phi_2, \pi_2)$ and $(\phi, \pi) = \frac{1}{2}(\phi_1, \pi_1) + \frac{1}{2}(\phi_2, \pi_2)$.* □

Gomory and Johnson [20, 21] and Johnson [25] prove the following result.

Theorem 5 ([21], [25]) *If (ϕ, π) is extreme for $MI(U, W, r)$, then it is minimal for $MI(U, W, r)$.* □

2.2 Two row continuous infinite-group problem

As discussed in the previous section, (1) is essentially equivalent to the continuous infinite-group problem[2] $MI(\emptyset, S^2, r)$. Our first objective is to obtain inequalities with the strongest possible coefficients for the continuous problem. We lift in the integer variables to obtain extreme inequalities for $MI(I^2, S^2, r)$.

For notational convenience, we assume that the columns of the continuous variables in the group problem $MI(I^2, S^2, r)$ are from $\mathbb{R}^2$ (i.e., not just the scaled vectors) and π, the valid function corresponding to continuous variables, is defined over $\mathbb{R}^2$. (The relationship between the functions $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ and $\mu_\phi : S^2 \to \mathbb{R}_+$ in Definition 2 is straightforward. In what follows, π will be positively homogenous, and we can construct μ_ϕ in a well-defined fashion by restricting the domain of π to S^2. Conversely, given μ_ϕ, π is the gauge function which is the homogenous extension of μ_ϕ.)

We begin with the definition of a maximal lattice-free set, that is the key component in the description of minimal inequalities for $MI(\emptyset, S^2, r)$.

Definition 6 ([27]) *A set S is called a maximal lattice-free convex set in $\mathbb{R}^2$ if it is convex,*

1. *interior$(S) \cap \mathbb{Z}^2 = \emptyset$, and*
2. *There exists no convex set S' satisfying (1), such that $S \subsetneq S'$.* □

We state the following theorem, modified from Borozan and Cornuéjols [7]; see also Theorem 1 in Andersen et al. [1].

[2]Some authors have used the term continuous group problem to imply the infinite-group problem, as the underlying group is a 'continuous' set. However, we use the term to imply the problem whose variables are all non-negative continuous (except for the free integer variables of the group problem).

Theorem 7 ([7]) *An inequality of the form* $\sum_{w \in \mathbb{Q}^2} \tilde{\pi}(w)y(w) \geq 1$ *is minimal for (1) if the closure of*

$$P(\tilde{\pi}) = \{w \in \mathbb{Q}^2 | \tilde{\pi}(w - f) \leq 1\} \tag{2}$$

in $\mathbb{R}^2$ *is a maximal lattice-free convex set. Moreover, given a maximal lattice-free convex set* P *such that* $f \in$ *interior*(P)*, the function* $\tilde{\pi} : \mathbb{Q}^2 \to \mathbb{R}_+$ *defined as*

$$\tilde{\pi}(w) = \begin{cases} 0 & \textit{if } w \in \textit{recession cone of } P \\ \lambda & \textit{if } f + \frac{w}{\lambda} \in \textit{Boundary}(P) \end{cases} \tag{3}$$

is a minimal valid inequality for (1). □

It is possible to analyze the case when $f \in \text{Boundary}(P(\pi))$. However, in this paper we focus on the case when $f \in \text{interior}(P(\pi))$.

It can be verified that if a function $\tilde{\pi} : \mathbb{Q}^2 \to \mathbb{R}_+$ corresponding to a maximal lattice-free set P is minimal (extreme resp.) for (1) and $f \in \text{interior}(P)$, then $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ defined as

$$\pi(w) = \begin{cases} 0 & \text{if } w \in \text{recession cone of } P \\ \lambda & \text{if } f + \frac{w}{\lambda} \in \text{Boundary}(P) \end{cases} \tag{4}$$

is minimal (extreme resp.) for $MI(\emptyset, S^2, r)$. This is just a technical verification and we relegate the proof to Appendix 1. (Note that since f is rational in (1), we assume that r (i.e., $\mathbb{P}(-f)$) is rational in the rest of the paper.)

For any minimal valid function π, we denote the corresponding lattice-free maximal set by $P(\pi)$.

In order to build some intuition concerning the difference between extreme inequalities for the mixed integer and the continuous group problems, we present an example from the one-row case. Figure 1 shows two extreme inequalities for $MI(I^1, S^1, 0.5)$. The pair of functions (ϕ_1, π_1), plotted in bold, was shown to be extreme for $MI(I^1, S^1, 0.5)$ in Gomory and Johnson [22]. The functions $(\phi_{GMIC}, \pi_{GMIC})$ plotted in dashed lines is the GMIC which is also extreme from $MI(I^1, S^1, 0.5)$. Therefore, from the perspective of the mixed integer problem, both inequalities are strong. However, if we just compare the functions π_1 and π_{GMIC}, we observe that π_{GMIC} dominates π_1. Therefore, while π_{GMIC} is extreme for $MI(\emptyset, S^1, 0.5)$, π_1 is not even minimal for $MI(\emptyset, S^1, 0.5)$.

In this paper, we take functions $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ that are the two-row versions of the functions π_{GMIC} (whose characterization was partially given in Theorem 7) and determine functions $\phi : I^2 \to \mathbb{R}_+$ such that (ϕ, π) is extreme for $MI(I^2, S^2, r)$.

2.3 Classification of maximal lattice-free sets in $\mathbb{R}^2$

Theorem 7 shows the relationship between minimal inequalities for $MI(\emptyset, S^2, r)$ and maximal lattice-free sets. In this section, we present a classification of maximal lattice-free convex sets in $\mathbb{R}^2$ that is suitable for the study in this paper.

In the rest of the paper, we use the term 'interior of a side' to imply the relative interior of a line segment in $\mathbb{R}^2$.

We begin with a result from Lovász [27].

Theorem 8 ([27]) *A maximal lattice-free convex set in the plane is one of the following:*

1. *A line with irrational slope, i.e.,* $a_1x_1 + a_2x_2 = b$*, where* $\frac{a_1}{a_2}$ *is irrational and* $b \notin \mathbb{Z}a_1 + \mathbb{Z}a_2$*,*
2. *A split, i.e., the set* $\{(x_1, x_2) \in \mathbb{R}^2 \mid b \leq a_1x_1 + a_2x_2 \leq b + 1\}$ *where* $a_1, a_2, b \in \mathbb{Z}$ *and* a_1*,* a_2 *are coprime.*
3. *A triangle with at least one integer point in the interior of each of its edges.*

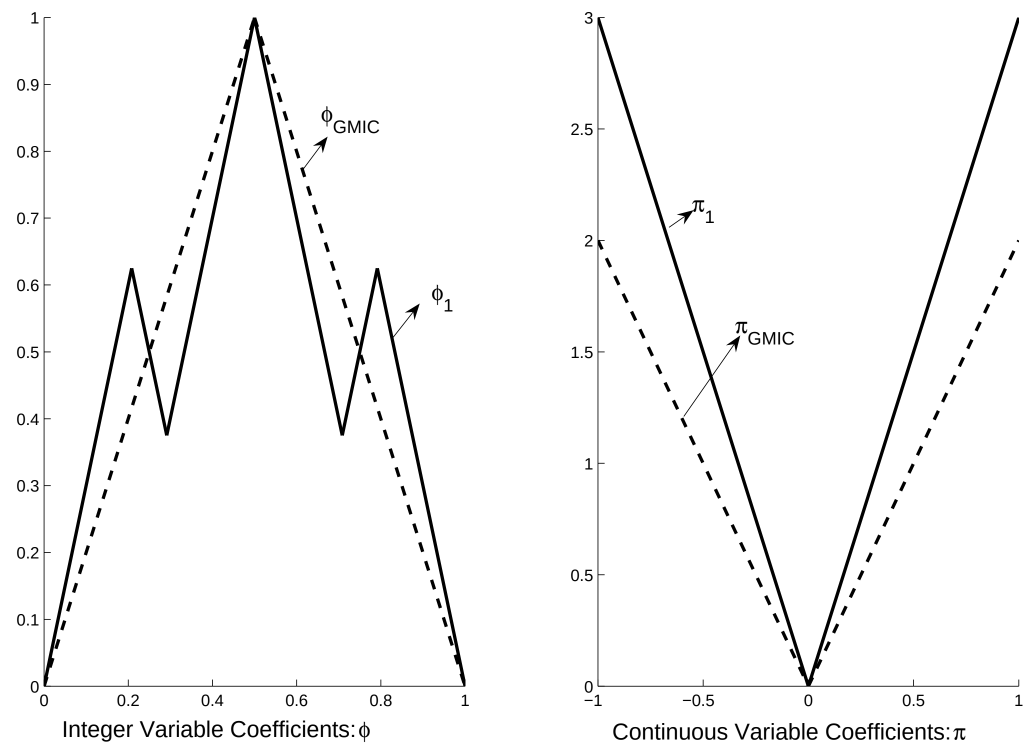

Figure 1: Extreme functions for $MI(I^1, S^1, 0.5)$

4. *A quadrilateral containing exactly four integer points with exactly one of them in the interior of each of its edges.*

Note that except for the first case, all the other sets have non-empty interiors. Here we focus our attention on the last three classes of maximal lattice-free sets. We classify the maximal lattice-free triangles in $\mathbb{R}^2$ more precisely in Proposition 13. The key result used in the proof of Proposition 13 is the following theorem from Andersen et al. [1].

Theorem 9 ([1]) *Let $Q \in \mathbb{R}^2$ be a convex polygon with integer vertices that has no integer points in its interior. Then*

1. *Q has at most four vertices.*
2. *If Q has four vertices, then at least two of it four facets are parallel.*
3. *If Q is not a triangle with integer points in the interior of all three facets, then there exists parallel lines $a_1w_1 + a_2w_2 = c$ and $a_1w_1 + a_2w_2 = c + 1$ such that $Q \subseteq \{b \leq a_1w_1 + a_2w_2 \leq b + 1\}$.*

We next present a lemma allowing us to put the maximal lattice-free convex set in 'standard' form.

Lemma 10 (Standardization) *Let P be a maximal lattice-free convex set (with non-empty interior) with v an integer point in the interior of one of its sides. Then there exists a unimodular matrix M such that the set $\{x \in \mathbb{R}^2 \,|\, x = M(u - v), u \in P\}$ is a maximal lattice-free convex set with the points $(0, 0)$, $(1, 0)$, and $(1, 1)$ in the interior of its sides.*

Proof. Since v is an integer point in the interior of one of the side of P, we have that $P - \{v\}$ is a maximal lattice-free convex set with $\bar{0} := (0,0)$ in the interior of one of its sides. By Theorem 8, P is either a split or a triangle or a quadrilateral. In each case it is possible to select two integer points $s, t \in \mathbb{Z}^2$ in the interior of the sides of P such that s, t, v are affinely independent and the interior of the line segments $[\bar{0}, s-v]$, $[\bar{0}, t-v]$, and $[s-v, t-v]$ contain no integer points. Therefore, by part 3 of Theorem 9, we can assume without loss of generality that $s - v = (a_1, a_2)$ and $t - v = (p, q)$ with $a_2 p - a_1 q = 1$. Setting $M = \begin{bmatrix} -q & p \\ a_2 & -a_1 \end{bmatrix}$ completes the proof. $\square$

The next lemma analyzes the standard triangles with each side containing one integer point in its interior. See Cornúejols and Margot [10] for an alternative proof.

Lemma 11 *Let P be a maximal lattice-free triangle with the points $(0,0)$, $(1,0)$, and $(0,1)$ being the only integer points lying in the interior of different sides. Let s_1 be the side of P passing thorough $(1,0)$ and let s_2 be the side of P passing through the point $(0,1)$. Then one of the following holds:*

1. *s_1 and s_2 intersect at $(1,1)$. The vertices of P are $(-1,1)$, $(1,1)$, and $(1,-1)$.*
2. *s_1 and s_2 intersect at a point outside the unit square. Let $-m_1$, m_2 and $-m_3$ be the slopes of s_1, s_2 and s_3 respectively. Either $1 < m_1 < \infty$, $0 < m_2 < \infty$ and $0 < m_3 < 1$ or $-\infty < m_1 < 0$, $-1 < m_2 < 0$ and $1 < m_3 < \infty$. All the vertices of P are non-integral.*

Proof. Let s_3 be the side that passes through the point $(0,0)$.

1. Consider the case in which s_1 and s_2 intersect at $(1,1)$. Therefore s_1 is vertical and s_2 has a slope of 0. If the slope of s_3 is lesser that -1, the point $(1,-1)$ will belong to the interior of s_1. If the slope of s_3 is greater that -1, the point $(-1,1)$ will belong to the interior of s_2. Therefore the slope of s_3 is -1, and the vertices of P are $(-1,1)$, $(1,1)$, and $(1,-1)$.

2. We first prove that s_1 and s_2 do not intersect in $S \setminus \{(1,1)\}$ where S is the unit square. Assume by contradiction that s_1 and s_2 intersect in $S \setminus \{(1,1)\}$. Therefore the slope of s_2 is less than (or equal to) 0 and s_1 is vertical or the slope of s_1 is negative. Moreover if s_1 is vertical, then the slope of s_2 is negative. Similarly if slope of s_2 is 0, then s_1 cannot be vertical. WLOG assume that s_1 is not vertical (the other case can be proven in the same way). Since P is lattice-free and contains only one integer point in the interior of each of its sides, the slope of s_3 is greater than -1 (Otherwise since the slope of s_1 is negative, the point $(1,-1)$ will belong to P). Thus the slope of s_3 is less than -1. This however implies that the point $(-1,1)$ belongs in the interior of P (if s_2 has a negative slope) or in the interior of s_2 (if the slope of s_2 is 0), a contradiction.

   We next prove that the vertices of P are all non-integral. Since s_1 and s_2 intersect outside the unit square, there are three possible cases:

   (a) The slope of s_1 is negative (and not vertical). The slope of s_2 is positive.
   (b) The slope of s_1 is positive (and not vertical). The slope of s_2 is negative.
   (c) The slope of s_1 is non-negative. The slope of s_2 is non-negative: As s_1 and s_2 do not intersect at $(1,1)$ in this case, this would imply that $(1,1)$ belongs to the interior of P or interior of either s_1 or s_2. This is a contradiction. Therefore this case is not possible.

   We assume WLOG that the slope of s_1 is negative (and not vertical) and the slope of s_2 is positive. (The other case can be proven in the same way).

   Vertex between s_1 and s_2: Note that s_2 cannot be vertical, since, otherwise s_2 and s_3 would meet at $(0,0)$, which would make $(0,0)$ a non interior point for s_3. Since s_2 has a positive slope and is not vertical and s_1 has a negative slope (and not vertical) they cannot meet at an integral vertex.

Vertex between s_2 and s_3: The slope of s_3 cannot be zero, since otherwise s_1 and s_3 would intersect at $(1,0)$, which would make $(1,0)$ a non interior point for s_1. This implies that slope of s_3 is negative. Since s_2 has a positive slope and s_3 has a negative slope they cannot meet at an integral vertex.

Vertex between s_3 and s_1: The slope of s_1 must be lesser than -1, since otherwise s_1 and s_2 would intersect at $(0,1)$, which would make $(0,1)$ a non interior point for s_2. If the slope of s_3 is lesser than (or equal) to -1, then $(1,-1)$ will belong either to the interior of P or interior of s_3 which is not possible. Therefore, slope of s_3 is greater than -1. Let m_1 be the negative of the slope of s_1 and m_3 is the negative of the slope of s_3. Then $m_1 > 1$ and $0 < m_3 < 1$ and the point of intersection of s_1 and s_3 is $\left(\frac{m_1}{m_1-m_3}, \frac{-m_1 m_3}{m_1-m_3}\right)$. Suppose

$$\frac{m_1}{m_1 - m_3} = k, \tag{5}$$

where $k \in \mathbb{Z}$. Then we have that $\frac{-m_1 m_3}{m_1-m_3} = -m_3 k \in \mathbb{Z}$. This implies that

$$m_3 = \frac{p}{k}, \tag{6}$$

where $p \in \mathbb{Z}$ and $1 \leq p \leq k-1$ since $0 < m_3 < 1$. However, substituting this in (5), we obtain that $m_1 = \frac{p}{k-1} \leq 1$, a contradiction. □

If M is an unimodular matrix, a vector $u \in \mathbb{Z}^2$ iff $Mu \in \mathbb{Z}^2$. Using this property and Lemmas 10 and 11 we can verify the following proposition.

Proposition 12 *Let P be a maximal lattice-free triangle with one integer point in the interior of each side. Then either all the vertices of P are integral, or none of them are integral.*

We conclude this section by consolidating all the results of this section.

Proposition 13 (Classification) *Let P be a maximal lattice-free convex set with a non-empty interior in $\mathbb{R}^2$. Then P is any one of the following:*

1. *The set $\{(x_1, x_2) \in \mathbb{R}^2 \,|\, b \leq a_1 x_1 + a_2 x_2 \leq b+1\}$ where $a_1, a_2, b \in \mathbb{Z}$ and a_1, a_2 are coprime.*
2. *A maximal lattice-free triangle in $\mathbb{R}^2$. In this case exactly one of the following is true:*
   - *(a) One side of P contains more than one integer point in its interior.*
   - *(b) All the vertices are integral and each side contains one integer point in its interior.*
   - *(c) The vertices are non-integral and each side contains one integer point in its interior.*
3. *A lattice-free quadrilateral and each of its sides contains exactly one integer point in its interior.* □

3 The trivial fill-in function

We begin this section by introducing a candidate for the lifting function, that we call the *trivial fill-in function*. It will be shown that this function is, in fact, the unique lifting function in many cases, i.e., the trivial fill-in function defined over I^2 together with the function $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ (obtained starting with a maximal lattice-free set $P(\pi)$ and applying (4)) leads to extreme inequalities for the mixed integer group problem. The objective of this section is to present the method for proving that the resulting inequality is extreme and illustrating this procedure on the split cut.

We begin with a lemma that motivates the definition of the trivial fill-in function.

Lemma 14 *Let $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ be a valid function for $MI(\emptyset, S^2, r)$. Consider the function $\phi : I^2 \to \mathbb{R}_+$ defined as $\phi(u) = \pi(\hat{u})$ where $\hat{u} \in \mathbb{R}^2$ is any point such that $\mathbb{P}(\hat{u}) = u$. Then (ϕ, π) is a valid inequality for $MI(I^2, S^2, r)$.*

Proof: Consider any $(\bar{x}, \bar{y}) \in MI(I^2, S^2, r)$, i.e., $(\bar{x}, \bar{y})$ satisfying $\sum_{u \in I^2} u\bar{x}(u) + \mathbb{P}(\sum_{w \in \mathbb{R}^2} w\bar{y}(w)) = r$. Now consider the point $\tilde{y}$ defined as follows:

1. First set $\tilde{y}(w) = \bar{y}(w)\ \forall w \in \mathbb{R}^2$.

2. For every $u \in I^2$, such that $\bar{x}(u) > 0$ (note there is a finite number of such $u \in I^2$), update $\tilde{y}(w) := \tilde{y}(w) + \bar{x}(u)$, where $w = \hat{u}$.

Observe that $\mathbb{P}(\sum_{w \in \mathbb{R}^2} w\tilde{y}(w)) = \sum_{u \in I^2} u\bar{x}(u) + \mathbb{P}(\sum_{w \in \mathbb{R}^2} w\bar{y}(w)) = r$. Moreover the support of $\tilde{y}$ is finite as the support of $\bar{x}$ and $\bar{y}$ were finite. Therefore, we have that $\tilde{y} \in MI(\emptyset, S^2, r)$. Also observe that $\sum_{\bar{x}(u)>0} \phi(u)\bar{x}(u) + \sum_{\bar{y}(w)>0} \pi(w)\bar{y}(w) = \sum_{w \in \mathbb{R}^2} \pi(w)\tilde{y}(w)$. By validity of π for $MI(\emptyset, S^2, r)$, we obtain that $\sum_{w \in \mathbb{R}^2} \pi(w)\tilde{y}(w) \geq 1$. Thus we have that $\sum_{\bar{x}(u)>0} \phi(u)\bar{x}(u) + \sum_{\bar{y}(w)>0} \pi(w)\bar{y}(w) \geq 1$. $\square$

Therefore, we observe that if we set the value of $\phi(u)$ to be that of $\pi(\hat{u})$ for any $\hat{u}$ such that $\mathbb{P}(\hat{u}) = u$, then $\phi(u)$ is a valid coefficient for $x(u)$. Since we want to obtain the best possible coefficient for the integer variables, we choose $\hat{u}$ so as to obtain the smallest possible coefficient for $\phi(u)$.

Definition 15 (Trivial Fill-in Function) *Let π be a valid inequality corresponding to the maximal lattice-free convex set $P(\pi)$. The trivial fill-in function, denoted $\phi^{\bar{0}} : I^2 \to \mathbb{R}_+$, is defined to be: $\phi^{\bar{0}}(u) = inf\{\pi(w) \,|\, \mathbb{P}(w) = u, w \in \mathbb{R}^2\}$.*

The reason for the notation and the nomenclature of the trivial fill-in function will become apparent when a generalization of this function is analyzed in Section 5. One interpretation of the trivial fill-in function is that its computation is equivalent to applying the procedure for strengthening coefficients of integer variables presented in Balas and Jeroslow [5]. The focus next is to present the procedure to prove that the trivial fill-in function provides the strongest possible coefficient for integer variables in certain cases.

Theorem 5 shows that an extreme inequality for the two-row infinite-group problem must be minimal. Therefore to prove that $(\phi^{\bar{0}}, \pi)$ is extreme for $MI(I^2, S^2, r)$, we must first show that $(\phi^{\bar{0}}, \pi)$ is minimal for $MI(I^2, S^2, r)$. To do this we use the following characterization by Johnson [25].

Theorem 16 (Theorem 6.1, [25]) *The pair of functions $\phi : I^2 \to \mathbb{R}_+$ and $\pi : W \to \mathbb{R}_+$ is a minimal valid inequality for the convex hull of $MI(I^2, S^2, r)$ if and only if*

1. *ϕ is subadditive, i.e., $\phi(u^1) + \phi(u^2) \geq \phi(u^1 + u^2)\ \forall u^1, u^2 \in I^2$*

2. *$\phi(u) + \phi(r - u) = \phi(r) = 1$ for all $u \in I^2$,*

3. *$\pi(w) = lim_{h \to 0^+} \frac{\phi(\mathbb{P}(hw))}{h}\ \forall w \in S^2$,*

where $r \neq \bar{0}$. $\square$

The next proposition allows us to simplify calculations by showing that it suffices to only deal with standard maximal lattice-free convex sets as described in Lemma 10. Let $P(\pi)$ be a maximal lattice-free convex set with $f \in$ interior$(P(\pi))$ and let M be a two-by-two unimodular matrix. Let $P^M(\pi)$ be the maximal lattice-free set defined as $P^M(\pi) = \{x \,|\, x = M(u - v), u \in P(\pi)\}$ where $v \in \mathbb{Z}^2$. Let π^M be the function corresponding to the maximal lattice-free convex set $P^M(\pi)$ with $M(f - v) \in P^M(\pi)$. By definition, if $\pi^M(w) = \lambda$, then $M(f - v) + \frac{w}{\lambda} \in$ Boundary$P^M(\pi)$. Therefore, $\pi(M^{-1}w) = \lambda = \pi^M(w)$ since $f + \frac{M^{-1}w}{\lambda} \in$ Boundary$P(\pi)$. Let $\phi : I^2 \to \mathbb{R}_+$ be a valid function corresponding to the integer coefficients in $MI(I^2, S^2, r)$. Similar to the construction of π^M, define $\phi^M : I^2 \to \mathbb{R}_+$ as $\phi^M(u) = \phi(\mathbb{P}(M^{-1}u))$.

Proposition 17 *Let $P(\pi)$ be a maximal lattice-free convex set with a point $f \in$ interior$(P(\pi))$. Let M be a two-by-two unimodular matrix. Let $P^M(\pi)$ be the maximal lattice-free set defined above. Define the functions $\phi^M : I^2 \to \mathbb{R}_+$ and $\pi^M : \mathbb{R}^2 \to \mathbb{R}_+$ as $\phi^M(u) = \phi(\mathbb{P}(M^{-1}u))$ and $\pi^M(w) = \pi(M^{-1}w)$. Then (ϕ, π) is a minimal (extreme resp.) inequality for $MI(I^2, S^2, r)$ iff (ϕ^M, π^M) is a minimal (extreme rep.) inequality for $MI(I^2, S^2, \mathbb{P}(Mr))$.*

Proof: First note that since M^{-1} is also unimodular it is enough to verify that if (ϕ, π) is a minimal (extreme resp.) inequality for $MI(I^2, S^2, r)$, then (ϕ^M, π^M) is a minimal (extreme resp.) inequality for $MI(I^2, S^2, \mathbb{P}(Mr))$. We first verify that if (ϕ, π) is a minimal inequality for $MI(I^2, S^2, r)$ then (ϕ^M, π^M) is a minimal inequality for $MI(I^2, S^2, \mathbb{P}(Mr))$. Using Theorem 16 we need to verify the following conditions:

1. ϕ^M is subadditive: $\phi^M(u) + \phi^M(v) = \phi(\mathbb{P}(M^{-1}u)) + \phi(\mathbb{P}(M^{-1}v)) \geq \phi(\mathbb{P}(M^{-1}u) + \mathbb{P}(M^{-1}v)) = \phi(\mathbb{P}(M^{-1}(u+v))) = \phi^M(u+v)$. The inequality follows from the subadditivity of ϕ, since (ϕ, π) is minimal.

2. $\phi^M(u) + \phi^M(\mathbb{P}(Mr) - u) = 1$: $\phi^M(u) + \phi^M(\mathbb{P}(Mr) - u) = \phi(\mathbb{P}(M^{-1}u)) + \phi(\mathbb{P}(M^{-1}(\mathbb{P}(Mr) - u))) = \phi(\mathbb{P}(M^{-1}u)) + \phi(r - \mathbb{P}(M^{-1}u)) = 1$.

3. $\phi^M(\mathbb{P}(Mr)) = 1$: $\phi^M(\mathbb{P}(Mr)) = \phi(\mathbb{P}(M^{-1}\mathbb{P}(Mr))) = \phi(r) = 1$. The last equality follows from the minimality of (ϕ, π).

4. $\lim_{h\to 0^+} \frac{\phi^M(\mathbb{P}(wh))}{h} = \pi^M(w)$: $\lim_{h\to 0^+} \frac{\phi^M(\mathbb{P}(wh))}{h} = \lim_{h\to 0^+} \frac{\phi(\mathbb{P}(M^{-1}\mathbb{P}(wh)))}{h} = \lim_{h\to 0^+} \frac{\phi(\mathbb{P}(M^{-1}wh))}{h} = \pi(M^{-1}w) = \pi^M(w)$. The next to last equality follows from the minimality of (ϕ, π).

Next assume by contradiction that (ϕ, π) is an extreme valid inequality for $MI(I^2, S^2, r)$, but (ϕ^M, π^M) is not an extreme valid inequality for the problem $MI(I^2, S^2, \mathbb{P}(Mr))$. Observe that if (ϕ, π) is an extreme inequality for $MI(I^2, S^2, r)$, then it must be minimal for $MI(I^2, S^2, r)$. Therefore, (ϕ^M, π^M) is a minimal inequality for $MI(I^2, S^2, \mathbb{P}(Mr))$. If (ϕ^M, π^M) is not extreme, then $(\phi^M, \pi^M) = \frac{1}{2}(\phi_1, \pi_1) + \frac{1}{2}(\phi_2, \pi_2)$ where $(\phi_1, \pi_1) \neq (\phi_2, \pi_2)$ are two minimal inequalities for $MI(I^2, S^2, \mathbb{P}(Mr))$. However, this implies that $(\phi, \pi) = \frac{1}{2}(\phi_1^{M^{-1}}, \pi_1^{M^{-1}}) + \frac{1}{2}(\phi_2^{M^{-1}}, \pi_2^{M^{-1}})$, a contradiction to the extremality of (ϕ, π). $\square$

Proposition 17 and Lemma 10 show that it is enough to analyze 'standard' maximal lattice-free convex sets that have the integer points $(0,0)$, $(1,0)$ and $(0,1)$ in the interior of the boundary of the set.

The next two propositions show that some of the conditions of Theorem 16 are always satisfied by $\phi^{\bar{0}}$.

Proposition 18 $\phi^{\bar{0}}(u^1) + \phi^{\bar{0}}(u^2) \geq \phi^{\bar{0}}(u^1 + u^2)\ \forall u^1, u^2 \in I^2$.

Proof: For any $u^1, u^2 \in I^2$ and any $\epsilon > 0$, by the definition of $\phi^{\bar{0}}$ there exists $\bar{u}^i \in \mathbb{R}^2$ such that $\phi^{\bar{0}}(u^i) > \pi(\bar{u}^i) - \frac{\epsilon}{2}$. Therefore, $\phi^{\bar{0}}(u^1) + \phi^{\bar{0}}(u^2) > \pi(\bar{u}^1) + \pi(\bar{u}^2) - \epsilon \geq \pi(\bar{u}^1 + \bar{u}^2) - \epsilon \geq \phi^{\bar{0}}(\bar{u}^1 + \bar{u}^2) - \epsilon$. Since ϵ can be made as small as possible, this completes the proof. $\square$

The next proposition shows that $\phi^{\bar{0}}(r) = 1$.

Proposition 19 *If there exists $\tilde{y} \in MI(\emptyset, S^2, r)$ such that $\sum \pi(w)\tilde{y}(w) = 1$, then $\phi^{\bar{0}}(r) = 1$.*

Proof:

1. $\phi^{\bar{0}}(r) \geq 1$: Assume by contradiction that $\phi^{\bar{0}}(r) < 1$. Then there exists $k_1, k_2 \in \mathbb{Z}$ such that $\pi(r_1 + k_1, r_2 + k_1) = \phi^{\bar{0}}(r) < 1$. Let
$$\tilde{y}(w) = \begin{cases} 1 & \text{if } w = (r_1 + k_1, r_2 + k_2) \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$
As $\tilde{y}$ is a valid solution to $MI(\emptyset, S^2, r)$, and since we have $\sum \pi(w)y(w) \geq 1$ for all feasible y, this is a contradiction.

2. $\phi^{\bar{0}}(r) \leq 1$: The set of solutions to $MI(\emptyset, S^2, r)$ are of the form:
$$\{y | \sum wy(w) = (r_1 + k_1, r_2 + k_2)\}, \tag{8}$$
where $k_1, k_2 \in \mathbb{Z}$. By assumption, there exists a solution $\tilde{y}$ such that $\sum \pi(w)\tilde{y}(w) = 1$ and $\sum w\tilde{y}(w) = (r_1 + \tilde{k}_1, r_2 + \tilde{k}_2)$, where $\tilde{k}_1, \tilde{k}_2 \in \mathbb{Z}$. Then, by subadditivity of π, we have $1 = \sum \pi(w)\tilde{y}(w) \geq \pi(\sum w\tilde{y}(w)) = \pi(r_1 + \tilde{k}_1, r_2 + \tilde{k}_2) \geq \phi^{\bar{0}}(r)$.

$\square$

Since π will be always assumed to be extreme, we will assume that $\phi^{\bar{0}}(r) = 1$.

Suppose now that we obtain a trivial fill-in function that can be verified to be minimal. Proposition 20 next shows that if we start from an extreme inequality π for $MI(\emptyset, S^2, r)$ and if (ϕ, π) is the unique minimal function for $MI(I^2, S^2, r)$, then (ϕ, π) is extreme for $MI(I^2, S^2, r)$. This result will allow us to verify that the trivial-fill-in function is extreme in certain cases.

Proposition 20 *Let π be an extreme inequality for $MI(\emptyset, S^2, r)$. If $\phi : I^2 \to \mathbb{R}_+$ is the unique function such that (ϕ, π) is minimal for $MI(I^2, S^2, r)$, then (ϕ, π) is extreme for $MI(I^2, S^2, r)$.*

Proof: Assume by contradiction that (ϕ, π) is not extreme. Then there exists two valid functions (ϕ_1, π_1) and (ϕ_2, π_2) such that $(\phi_1, \pi_1) \neq (\phi_2, \pi_2)$ and $(\phi, \pi) = \frac{1}{2}(\phi_1, \pi_1) + \frac{1}{2}(\phi_2, \pi_2)$. It can be easily verified that (ϕ_i, π_i) must be minimal. (Otherwise, there exists $(\phi', \pi') < (\phi_1, \pi_1)$ which is valid for $MI(I^2, S^2, r)$. However, this shows that there exists $(\phi'', \pi'') < (\phi, \pi)$ which is valid; a contradiction to the minimality of (ϕ, π)).

Now note that $\pi_1 = \pi_2 = \pi$ since π is an extreme inequality for $MI(\emptyset, S^2, r)$. However since $\phi : I^2 \to \mathbb{R}_+$ is the unique function such that (ϕ, π) is minimal, it implies that $\phi_1 = \phi_2 = \phi$, which is the required contradiction. $\square$

Now we have all the tools to outline the steps used to prove that $(\phi^{\bar{0}}, \pi)$ is the unique extreme function for $MI(I^2, S^2, r)$.

1. Assume that the set $P(\pi)$ is standard (i.e., assume that using Lemma 10 a suitable unimodular matrix M and integer point $\bar{v}$ is constructed so that $P^M(\pi)$ is standard, and it is enough to prove that (ϕ^M, π^M) is extreme by the result of Proposition 17).

2. Show that

   (a) $\phi^{\bar{0}}(u) + \phi^{\bar{0}}(r - u) = 1 \ \forall u \in I^2$. We will define a set $D \subset \mathbb{R}^2$ such that if $\hat{u} \in D$, then $\phi^{\bar{0}}(\mathbb{P}(\hat{u})) + \phi^{\bar{0}}(r - \mathbb{P}(\hat{u})) = 1$. Therefore proving $\phi^{\bar{0}}(u) + \phi^{\bar{0}}(r - u) = 1 \ \forall u \in I^2$ will amount to proving that $\mathbb{P}(D) = I^2$.

   (b) $\pi(w) = lim_{h \to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(hw))}{h} \ \forall w \in S^2$.

   These steps will ensure that $(\phi^{\bar{0}}, \pi)$ is minimal for $MI(I^2, S^2, r)$.

3. Finally show that $\phi^{\bar{0}}$ is the unique function such that $(\phi^{\bar{0}}, \pi)$ is minimal for $MI(I^2, S^2, r)$. If π is extreme for $MI(\emptyset, S^2, r)$, this will imply that $(\phi^{\bar{0}}, \pi)$ is extreme for $MI(\emptyset, S^2, r)$ by the result of Proposition 20.

We next illustrate these steps in the case of the split cuts.

Example 21 (Split Cut) *Let the maximal lattice-free set $P(\pi)$ be the split, i.e., the set $\{(x_1, x_2) \in \mathbb{R}^2 \,|\, b \leq a_1x_1 + a_2x_2 \leq b + 1\}$ where $a_1, a_2, b \in \mathbb{Z}$ and a_1, a_2 are coprime. Then $(\phi^{\bar{0}}, \pi)$ is the unique extreme function for $MI(I^2, W, r)$.*

Proof: As a first step, it is enough to analyze a standard maximal lattice-free convex set with integer points $(0, 0)$, $(1, 0)$ and $(0, 1)$ in the interior of the boundary of the set. With out loss of generality we analyze the set $\{(w_1, w_2) \in \mathbb{R}^2 \,|\, 0 \leq w_1 \leq 1\}$. (The other splits which have the integer points $(0, 0)$, $(1, 0)$ and $(0, 1)$ in the interior of the boundary can be transformed to this set with the use of a suitable unimodular matrix). Let $f := (f_1, f_2)$ be in the interior of $P(\pi)$, where $r = \mathbb{P}(-f)$.

Using $P(\pi)$ and (4) we obtain that,

$$\pi(w_1, w_2) = \begin{cases} \frac{w_1}{1-f_1} & \text{if } w_1 \geq 0 \\ \frac{-w_1}{f_1} & \text{if } w_1 \leq 0. \end{cases} \tag{9}$$

Next we find a set $D_{\text{split}} \subset \mathbb{R}^2$ such that for all $\hat{u} \in D_{\text{split}}$, $\phi^{\bar{0}}(\mathbb{P}(\hat{u})) + \phi^{\bar{0}}(r - \mathbb{P}(\hat{u})) = 1$. Using π it can be verified that D_{split} is the set $\{(w_1, w_2) | 0 \leq w_2 \leq 1, -f_1 \leq w_1 \leq 1 - f_1\}$. Now since $\mathbb{P}(D_{split}) = I^2$ we obtain that $\phi^{\bar{0}}(u) + \phi^{\bar{0}}(r - u) = 1 \ \forall u \in I^2$. It can be verified in this case that $\pi(w) = lim_{h \to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(hw))}{h} \ \forall w \in S^2$.

Finally, it remains to prove that $(\phi^{\bar{0}}, \pi)$ is the unique minimal function: We prove this by showing that for any valid function (ϕ, π) of $MI(I^2, S^2, r)$, $\phi(u) \geq \phi^{\bar{0}}(u)$ for all $u \in I^2$. Consider the case when $u := (u_1, u_2) \in I^2$, and $u_1 \leq r_1$. Then construct the point,

$$\begin{aligned} \bar{x}(v) &= \begin{cases} 1 & \text{if } v = u \\ 0 & \text{otherwise} \end{cases}, \\ \bar{y}(w) &= \begin{cases} 1 & \text{if } w = (1 - f_1 - u_1, -u_2 - f_2) \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

It can be verified that $(\bar{x}, \bar{y}) \in MI(I^2, S^2, r)$. Since (ϕ, π) is a valid function, we obtain that $\sum_{v \in I^2} \phi(v)\bar{x}(\bar{v}) + \sum_{w \in \mathbb{R}^2} \pi(w)\bar{y}(w) \geq 1$ or $\phi(u) \geq 1 - \pi(1 - f_1 - u_1, -u_2 - f_2) = \frac{u_1}{r_1} = \phi^{\bar{0}}(u)$. There is a similar proof for the case $u_1 > r_1$. Thus, $(\phi^{\bar{0}}, \pi)$ is the unique minimal function. Finally, since π is extreme for $MI(\emptyset, S^2, r)$ (Cornuéjols and Margot [10]), we have by the use of Proposition 20 that $(\phi^{\bar{0}}, \pi)$ is extreme for $MI(\emptyset, S^2, r)$. $\square$

Since the 'continuous part' of the GMIC inequality is the *only* extreme inequality for $MI(\emptyset, S^1, r)$, an observation to be deduced from Example 21 is that for the one-row group problem, the trivial-fill-in function is sufficient to obtain extreme inequality for $MI(I^1, S^1, r)$. In the next section we show that this behavior is observed for some cases in the two-row group problem as well.

4 Unique lifting functions

In the example in the previous section, it was shown that when $P(\pi)$ is a split, the trivial fill-in function along with π, i.e., $(\phi^{\bar{0}}, \pi)$ is extreme for $MI(I^2, S^2, r)$. The key step in the proof was the determination of the set $D \subset \mathbb{R}^2$ such that $\phi^{\bar{0}}(\mathbb{P}(\hat{u})) + \phi^{\bar{0}}(\mathbb{P}(r - \hat{u})) = 1$ where $\hat{u} \in D$. In the next definition we present the corresponding set $D(\pi) \subset \mathbb{R}^2$ for the case of bounded convex maximal lattice-free sets which have similar properties to those of D_{split}.

Definition 22 *Let $P(\pi)$ be a lattice-free triangle or quadrilateral. Let d^1 , d^2, ..., d^k with $k \in \{3, 4\}$, be vectors such that $d^i + f$ are the vertices of $P(\pi)$. Let s_i be the line segment between vertices $d^i + f$ and $d^{i+1} + f$ (where $d^4 := d^1$, $d^5 := d^1$ when $P(\pi)$ is triangle, quadrilateral respectively). Let p^i be the set of integer points in the interior of s_i. Let the cone formed by the extreme rays d^i, d^{i+1} be denoted by C^i. For an integer point $X^{ij} \in p^i$, let $\delta^{ij} d^i + (1 - \delta^{ij}) d^{i+1} + f = X^{ij}$ where $0 < \delta^{ij} < 1$. Define the sets $D_{ij}(\pi), D(\pi) \subset \mathbb{R}^2$ as $D_{ij}(\pi) = \{\rho d^i + \gamma d^{i+1} | 0 \leq \rho \leq \delta^{ij}, 0 \leq \gamma \leq 1 - \delta^{ij}\}$ and $D(\pi) = \cup_{i,j} D_{ij}(\pi)$.* $\square$

The next proposition establishes some of the crucial properties of $D(\pi)$.

Proposition 23 *Let $P(\pi)$ be a bounded maximal lattice-free convex set. For any $v \in D(\pi)$ the following are true:*

1. *There exists a point $(\bar{x}, \bar{y}) \in MI(I^2, S^2, r)$ with $\bar{x}(\mathbb{P}(v)) > 0$ which satisfies the inequality $(\phi^{\bar{0}}, \pi)$ at equality.*
2. *$\phi^{\bar{0}}(\mathbb{P}(v)) = \pi(v)$.*
3. *$\phi^{\bar{0}}(\mathbb{P}(v)) + \phi^{\bar{0}}(\mathbb{P}(r - v)) = 1$.*
4. *If $(\bar{\phi}, \pi)$ is any valid inequality for $MI(I^2, S^2, r)$, then $\bar{\phi}(\mathbb{P}(v)) \geq \phi^{\bar{0}}(\mathbb{P}(v))$.*

Proof.

1. Since $v \in D_{ij}(\pi)$, let $v = \rho d^i + \gamma d^{i+1}, \rho \leq \delta^{ij}$, $\gamma \leq 1 - \delta^{ij}$. Consider the point $v' = (\delta^{ij} - \rho)d^i + (1 - \delta^{ij} - \gamma)d^{i+1}$. Since $\rho \leq \delta^{ij}, \gamma \leq 1 - \delta^{ij}$, $v' \in C^i$. Now consider the solution:

$$\bar{x}(u) = \begin{cases} 1 & \text{if } u = \mathbb{P}(v) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

$$\bar{y}(w) = \begin{cases} 1 & \text{if } w = v' \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

   Then,

$$\begin{aligned} \sum_{u \in I^2} u\bar{x}(u) + \sum_{w \in \mathbb{R}^2} w\bar{y}(w) + f &= \mathbb{P}(v) + v' + f \\ &\equiv (v + v' + f)(mod \bar{1}) \\ &= \rho d^i + \gamma d^{i+1} + (\delta^{ij} - \rho)d^i \\ &\quad +(1 - \delta^{ij} - \gamma)d^{i+1} + f \\ &= X^{ij} \in \mathbb{Z}^2 \end{aligned}$$

   Also

$$\begin{aligned} \sum_{u \in I^2} \phi^{\bar{0}}(u)\bar{x}(u) + \sum_{w \in \mathbb{R}^2} \pi(w)\bar{y}(w) &\leq \pi(v) + \pi(v') \\ &= (\rho + \gamma) \\ &\quad +(\delta^{ij} - \rho + 1 - \delta^{ij} - \gamma) \\ &= 1 \end{aligned} \tag{12}$$

   Finally, (12) holds at equality because of the validity of $(\phi^{\bar{0}}, \pi)$ (from Lemma 14).

2. Follows from (12).

3. Consider the point v' constructed in proof of part 1. Since $\mathbb{P}(v) + v' + f \in \mathbb{Z}$, we have that $\mathbb{P}(v) \equiv (-f - v')(mod \bar{1})$ or $\mathbb{P}(r - v) = \mathbb{P}(v')$ since $r = \mathbb{P}(-f)$. Now the result follows from (12).

4. Since $(\bar{x}, \bar{y})$ ((10) and (11)) is a valid solution of $MI(I^2, W, r)$, we obtain that $\sum_{u \in I^2} \bar{\phi}(u)\bar{x}(u) + \sum_{w \in \mathbb{R}^2} \pi(w)\bar{y}(w) \geq 1$ or $\bar{\phi}(\mathbb{P}(v)) \geq 1 - \pi(v') = \phi^{\bar{0}}(\mathbb{P}(v))$. □

Next we show that in the case when $P(\pi)$ is a bounded maximal lattice-free convex set we can verify that $lim_{h \to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(wh))}{h} = \pi(w) \; \forall \; w \in \mathbb{R}^2$.

Corollary 24 *Let $P(\pi)$ be a bounded maximal lattice-free convex set. Then $lim_{h \to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(wh))}{h} = \pi(w) \; \forall \; w \in \mathbb{R}^2$.*

Proof: Let $w \in \mathbb{R}^2$ belong to the cone formed by d^i and d^{i+1}, i.e., $w = \alpha d^i + \beta d^{i+1}$, where $\alpha \geq 0$, and $\beta \geq 0$. We know that $\exists \; \delta^{ij}$ such that $0 < \delta^{ij} < 1$ and $\delta^{ij} d^i + (1 - \delta^{ij})d^{i+1} + f = X^{ij} \in p^i$ and $D_{ij}(\pi) = \{w \in C^i | w = \rho d^i + \gamma d^{i+1}, \rho \leq \delta^{ij}, \gamma \leq 1 - \delta^{ij}\}$. Therefore for sufficiently small positive h we have that $wh \in D_{ij}(\pi)$ since $h\alpha \leq \delta^{ij}$ and $h\beta \leq 1 - \delta^{ij}$. Therefore using Proposition 23, we obtain that $lim_{h \to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(wh))}{h} = lim_{h \to 0^+} \frac{\pi(wh)}{h} = \pi(w)$. □

Thus when $P(\pi)$ is a bounded maximal lattice-free convex set, a number of the steps required to prove that $(\phi^{\bar{0}}, \pi)$ is an extreme function for $MI(I^2, S^2, r)$ have been verified to be true. A key difference between D_{split} and $D(\pi)$ is that while $\mathbb{P}(D_{\text{split}}) = I^2$, there is no guarantee that $\mathbb{P}(D(\pi)) = I^2$. This difference leads to a richer class of extreme inequalities as we shall see in later sections. We now consolidate the results of Theorem 16, Propositions 18, 19, 23 and Corollary 24 in the following result.

Theorem 25 *If $P(\pi)$ is a bounded maximal lattice-free convex set in $\mathbb{R}^2$ such that π is extreme for $MI(\emptyset, S^2, r)$ and $I^2 = \mathbb{P}(D(\pi))$, then $(\phi^{\bar{0}}, \pi)$ is an extreme function for $MI(I^2, S^2, r)$. Moreover, there exists no function $\phi : I^2 \to \mathbb{R}_+$ such that $\phi \neq \phi^{\bar{0}}$ and (ϕ, π) is an extreme function for $MI(I^2, S^2, r)$.*

Proof: By Propositions 18, we obtain that $\phi^{\bar{0}}$ is subadditive. By Proposition 19, we obtain that $\phi^{\bar{0}}(r) = 1$. If $I^2 = \mathbb{P}(D(\pi))$, then by Proposition 23, we obtain that $\phi^{\bar{0}}(u) + \phi^{\bar{0}}(r - u) = 1 \ \forall u \in I^2$. By Corollary 24, $lim_{h \to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(wh))}{h} = \pi(w) \ \forall \ w \in \mathbb{R}^2$. Therefore, $(\phi^{\bar{0}}, \pi)$ is minimal for $MI(I^2, S^2, r)$. Finally, by (4) of Proposition 23 we obtain that for any valid inequality (ϕ, π) of $MI(I^2, S^2, r)$, $\phi(u) \geq \phi^{\bar{0}}(u) \ \forall u \in \mathbb{P}(D(\pi)) = I^2$. Thus $(\phi^{\bar{0}}, \pi)$ is the unique minimal inequality for $MI(I^2, S^2, r)$. Now since π is extreme for $MI(\emptyset, S^2, r)$ the result follows from Proposition 20. $\square$

In the rest of the section, we verify that $I^2 = \mathbb{P}(D(\pi))$ for some classes of $P(\pi)$ which proves that the fill-in function is the unique lifting function generating extreme inequalities for the infinite-group problem.

4.1 $P(\pi)$ is a triangle with multiple integer points in the interior of one side

In this section, we consider a set $P(\pi)$ which is a triangle with multiple integer points in the interior of one side. We begin with a variant of Lemma 10.

Proposition 26 *Let P be a triangle with multiple integer points in the interior of one side. Then there exists an unimodular matrix M such that the set $\{x \in \mathbb{R}^2 \,|\, x = M(u - v), u \in P\}$ is a maximal lattice-free convex set with the points $(0,0)$ and $(1,0)$ on one side and $(0,1)$ and $(1,1)$ in the interior of other two sides.*

Proof: Let $s, t \in \mathbb{Z}^2$ be two adjacent points in the interior of one side of the triangle and let $g, h \in \mathbb{Z}^2$ be two points in the interior of the other sides of the triangle. Again we have that the interior of the line segments $[s,t]$, $[t,g]$, $[g,h]$ and $[h,s]$ are empty. Therefore invoking Theorem 9 and using a proof similar to that of Lemma 10 we can obtain the required result. $\square$

Notation: (Refer to Figure 2.) Any point w will be represented as $w := (w_1, w_2)$. For example, $a^1 := (a^1_1, a^1_2)$. We denote the length of a line segment pq by $|pq|$. Let $P(\pi)$ be a maximal lattice-free triangle with the points $(0,0)$ and $(1,0)$ being adjacent integer points in the interior of one side and $(0,1)$ and $(1,1)$ in the interior of the other two sides. We use the following notation for points in this section:

1. The points a^1, a^2 and a^3 represent the vertices of the lattice-free triangle $P(\pi)$.
2. $b^1 := (1,1)$ is the integer point in the interior of the side a^1a^2.
3. $b^2 := (0,1)$ is the integer point in the interior of the side a^2a^3.
4. $b^3 := (0,0)$ and $b^4 := (1,0)$ are adjacent integer points in the interior of the side a^3a^1.
5. The union of quadrilaterals $fc^1b^1e^1$, $fc^2b^2e^2$, $fc^3b^3e^3$, and $fc^4b^4e^4$ represents a subset of the set $D(\pi) + \{f\}$. (In particular, c^1 lies on fa^1, e^1 lies on fa^2 and $f + (c^1 - f) + (e^1 - f) = b^1$. c^2 lies on fa^2, e^2 lies on fa^3 and $f + (c^2 - f) + (e^2 - f) = b^2$. c^3 lies on fa^3, e^3 lies on fa^1 and $f + (c^3 - f) + (e^3 - f) = b^3$. c^4 lies on fa^3, e^4 lies on fa^1 and $f + (c^4 - f) + (e^4 - f) = b^4$).
6. Let g be the point where b^3e^3 and b^4c^4 intersect.
7. We will assume that $f_1 \geq a^2_1$: therefore let i be the point of intersection between b^2e^2 extended and b^3e^3, and let j be the point of intersection between b^2e^2 extended and a^3a^1.

Proposition 27 *Let $P(\pi)$ be a maximal lattice-free triangle with the points $(0,0)$ and $(1,0)$ on one side and $(0,1)$ and $(1,1)$ in the interior of the other two sides. If we assume that $f_1 \geq a^2_1$, then*

1. *e^1 and c^2 are the same point.*

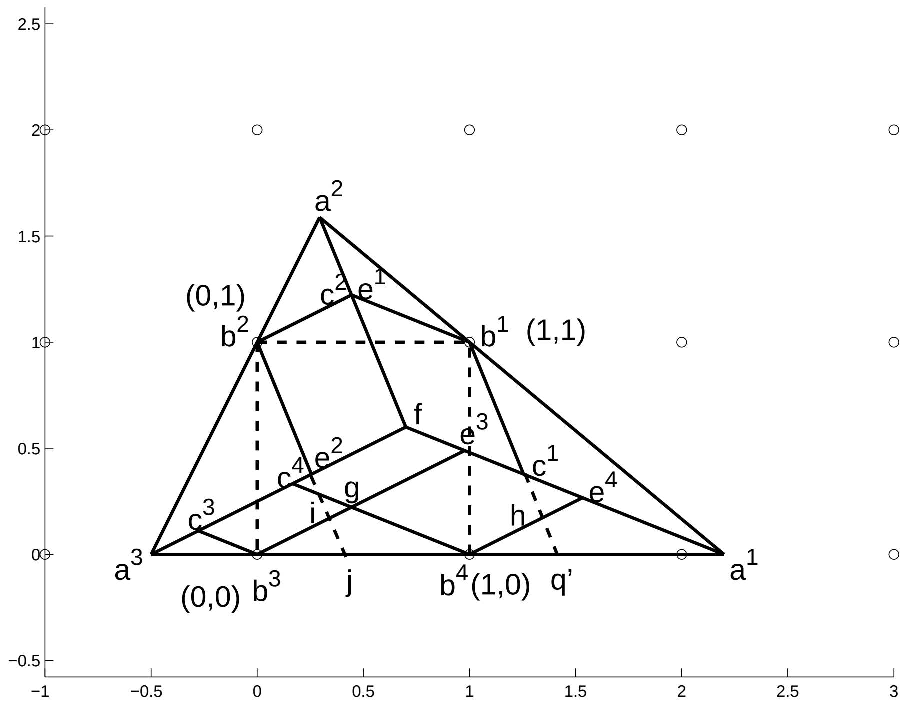

Figure 2: A maximal lattice-free triangle with the points $(0,0)$ and $(1,0)$ on one side and $(0,1)$ and $(1,1)$ in the interior of other two sides

2. *Triangle b^3gb^4 is symmetric to triangle $b^2c^2b^1$.*

3. *There exists a point h such that b^1c^1 extended to b^1h intersects b^4e^4.*

4. *Triangle b^1hb^4 is symmetric to b^2ib^3.*

Proof.

1. Since b^2c^2 is parallel to a^3f and b^1e^1 is parallel to a^1f, we have that $\frac{|a^2c^2|}{|a^2f|} = \frac{|a^2b^2|}{|a^2a^3|}$ and $\frac{|a^2e^1|}{|a^2f|} = \frac{|a^2b^1|}{|a^2a^1|}$. Now since a^3a^1 is parallel to b^2b^1 we obtain $\frac{|a^2b^2|}{|a^2a^3|} = \frac{|a^2b^1|}{|a^2a^1|}$ which implies that $|a^2c^2| = |a^2e^1|$.

2. This follows from the fact that $|b^2b^1| = |b^3b^4|$ and the angles gb^3b^4 and gb^4b^3 are equal to $c^2b^2b^1$ and $c^2b^1b^2$ (since b^3b^4 is parallel to b^2b^1, b^3e^3 is parallel to b^2c^2 and b^4c^4 is parallel to b^1c^2).

3. Note that b^2b^3 is parallel to b^1b^4. Since b^3 is in the interior of a^3a^1 we must have $\frac{|a^1b^1|}{|a^1a^2|} > \frac{|a^1b^4|}{|a^1a^3|}$. Since $\frac{|a^1e^4|}{|a^1f|} = \frac{|a^1b^4|}{|a^1a^3|}$ and $\frac{|a^1b^1|}{|a^1a^2|} = \frac{|a^1c^1|}{|a^1f|}$, we obtain that

$$|a^1e^4| < |a^1c^1|. \tag{13}$$

Let q' be the point of intersection of extension of b^1c^1 to a^3a^1. Therefore b^1b^2jq' forms a parallelogram and $|jq'| = |b^2b^1| = |b^3b^4| = 1$. Note now that j lies to the right of b^3 and therefore

$$|a^1q'| < |a^1b^4| \tag{14}$$

Using (13) and (14) we obtain the desired result.

4. We know that $|b^2b^3| = |b^1b^4| = 1$. Moreover, angles b^3b^2i and b^2ib^3 are equal to angles b^4b^1h and hb^4b^1 respectively, since b^2b^3 is parallel to b^4b^1, b^2i is parallel to b^1h, and b^3i is parallel to b^4h. $\square$

Proposition 27 gives us all the tools that are needed to prove the following result.

Theorem 28 *If $P(\pi)$ is a maximal lattice-free triangle with multiple integer points in the interior of one side, then $(\phi^{\bar{0}}, \pi)$ is an extreme inequality for $MI(I^2, S^2, r)$. Moreover, there exists no function $\phi : I^2 \rightarrow \mathbb{R}_+$ such that $\phi \neq \phi^{\bar{0}}$ and (ϕ, π) is an extreme function for $MI(I^2, S^2, r)$.*

Proof. By Propositions 26 and 17 it is enough to prove this result for the maximal lattice-free triangle $P(\pi)$ with the points $(0, 0)$ and $(1, 0)$ on one side and $(0, 1)$ and $(1, 1)$ in the interior of other two sides. By Theorem 5.4 from Cornuéjols and Margot [10], we obtain that π is extreme for $MI(\emptyset, S^2, r)$. Therefore, by Theorem 25 it is enough to show that $\mathbb{P}(D(\pi)) = I^2$. This is equivalent to showing that $\mathbb{P}(D(\pi) + \{f\}) = I^2$

Refer to Figure 2. We present the proof in the case when $f_1 \geq a_1^2$. A similar proof can be presented for the case when $f_1 \leq a_1^2$. Note now that the union of the parallelograms $fc^1b^1e^1$, $fc^2b^2e^2$, $fc^3b^3e^3$ and $fc^4b^4e^4$ is a subset of $D(\pi) + \{f\}$. Now using Proposition 27, we obtain that triangles b^3gb^4 and $b^2c^2b^1$ are symmetric. Since, b^1, b^2, b^3 and b^4 are integer points, and b^1b^2 is parallel to b^3b^4 the fractional parts of points in the triangles b^3gb^4 and $b^2c^2b^1$ are exactly the same. Similarly, b^1hb^4 is symmetric to b^2ib^3 and a similar result regarding fractional parts may be obtained. As the triangles $b^2c^2b^1$ and b^1hb^4 belong to $D(\pi) + \{f\}$, all the fractional parts within the quadrilateral $b^1b^2b^3b^4$ belong to $D(\pi) + \{f\}$, completing the proof. $\square$

The class of inequalities presented in this section share two important properties with the GMIC: The function π is extreme for $MI(\emptyset, S^2, r)$ and the *trivial fill-in function* together with π produces extreme inequalities for the two-row mixed integer infinite-group problem, i.e., $(\phi^{\bar{0}}, \pi)$ is extreme for $MI(I^2, S^2, r)$. Therefore, in this context, these new inequalities are the 'closest' two-row counter parts of the Gomory mixed integer cut.

It is interesting to examine if any of the known classes of inequalities for two row relaxations of MIPs are related to the lifted intersection cuts. In the next subsection, we show that a subclass of the sequential-merge inequalities (Dey and Richard [13]) are related to maximal lattice-free triangles with multiple integer points in the interior of one side. Thus Theorem 28 provides an alternative proof for the extremity of this subclass of the sequential-merge inequalities. (Note that not all sequential-merge inequalities have minimal coefficients for the continuous variables, and therefore not all sequential-merge inequalities are related to the inequalities presented in this paper. Also it is easily verified that not all lifted inequalities starting from triangles with multiple integer points in the interior of one side are sequential-merge inequalities.) The original derivation of sequential-merge inequalities is very different from the lifting approach used in this paper. Therefore, while on the one hand, the original derivation provides an algebraic framework for deriving these inequalities by the application of a sequence of GMICs to two rows of a simplex tableau, on the other hand the relationship of these inequalities to maximal lattice-free triangles shows that the coefficients for the continuous variables cannot be improved.

4.1.1 Sequential-Merge inequalities

The following result is modified from Dey and Richard [13].

Theorem 29 ([13]) *Let $\xi_r(u)$ represent the coefficient of an integer variable in the Gomory mixed integer cut, where u is the fractional part of the coefficient of the integer variable in the tableau row, and r is the fractional part of the right-hand-side of the tableau row (the function plotted in dashed lines in Figure 1 is $\xi_{0.5}$), i.e.,*

$$\xi_r(u) = \begin{cases} \frac{u}{r} & \text{if } u \leq r \\ \frac{1-u}{1-r} & \text{otherwise .} \end{cases} \tag{15}$$

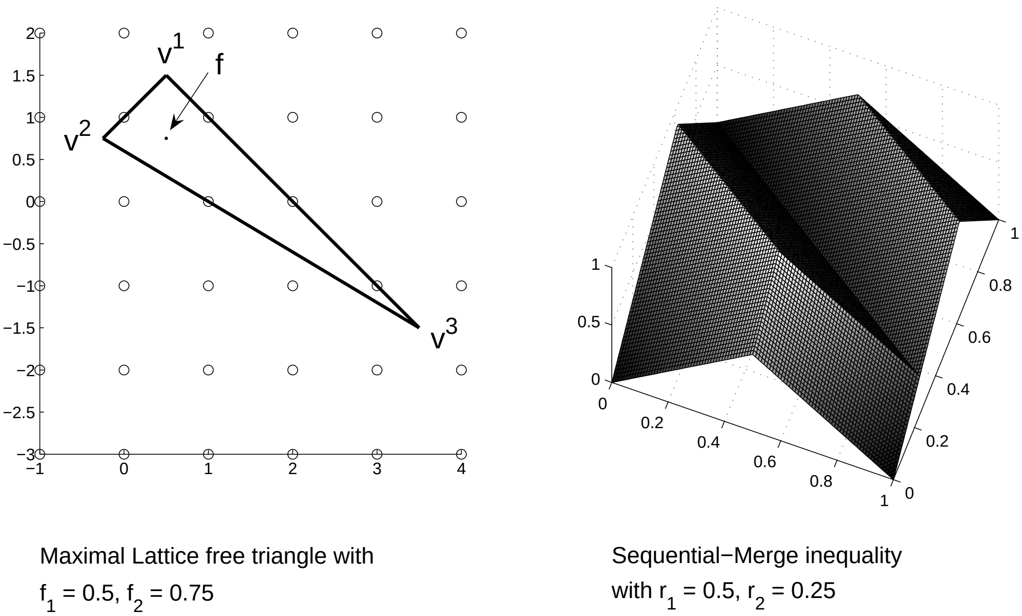

Figure 3: A subset of sequential-merge inequalities is a special case of a trivial fill-in function starting from a lattice-free triangle with more than one integer point in the interior of one side.

For $0 < r_1, r_2 < 1$, the function (ϕ^{sm}, π^{sm}) is an extreme inequality for $MI(I^2, S^2, (r_1, r_2))$ where

$$\phi^{sm}(x_1, x_2) = \frac{r_2\xi_{r_2}(x_2) + r_1\xi_{r_1}(\mathbb{P}(x_1 + x_2 - r_2\xi_{r_2}(x_2)))}{r_1 + r_2}, \tag{16}$$

and

$$\pi^{sm}(w_1, w_2) = \begin{cases} \frac{\frac{w_1+w_2}{r_1+r_2}}{} & \text{if } w_1 \geq 0, w_2 \geq 0 \\ \frac{1}{r_1+r_2}(w_2 - \frac{r_1 w_1}{1-r_1}) & \text{if } w_1 \leq 0, w_2 \geq 0 \\ \frac{w_1+w_2}{r_1+r_2} & \text{if } w_2 \leq 0, w_1 + \frac{w_2}{1-r_2} \geq 0 \\ \frac{1}{r_1+r_2}(\frac{-r_2 w_2}{1-r_2} - \frac{r_1}{1-r_1}(w_1 + \frac{w_2}{1-r_2})) & \text{if } w_2 \leq 0, w_1 + \frac{w_2}{1-r_2} \leq 0 \end{cases}$$

□

The intuitive explanation for (16) is to first generate a GMIC from the second row of the tableau, then add this GMIC to the first row of the tableau and finally obtain a GMIC for this combined row. The important observation is that there is an algorithmic relationship between this inequality and GMICs.

Now let $f := (f_1, f_2)$ where $f_1 = 1 - r_1$ and $f_2 = 1 - r_2$. We can construct the set $P(\pi^{sm})$, as $P(\pi^{sm}) = \{w \in \mathbb{R}^2 \mid \pi^{sm}(w - f) \leq 1\}$. We obtain the triangle with vertices:

1. $v^1 : (f_1, 2 - f_1)$,

2. $v^2 : (\frac{-f_1(1-f_2)}{1-f_1}, f_2)$,

3. $v^3 : (\frac{1-f_1}{1-f_2} + 1 + f_1, -\frac{f_2(1-f_1)}{1-f_2})$.

First note that $P(\pi^{sm})$ contains no integer point in the interior, since otherwise π^{sm} will not be a valid inequality. Now we can verify that the integer points in the interior of its sides are:

1. v^1v^2: $(0,1)$. $(\frac{1-f_2}{2-f_1-f_2}v^1 + \frac{1-f_1}{2-f_1-f_2}v^2 = (0,1))$.

2. v^2v^3: $(1,0)$. $(\frac{1-f_1}{2-f_1-f_2}v^2 + \frac{1-f_2}{2-f_1-f_2}v^3 = (1,0))$.

3. v^1v^3: There are $\bar{k}$ points where $\bar{k} = \left\lfloor \frac{2-f_2-f_1f_2}{1-f_2} \right\rfloor$. It can be verified that $\bar{k} \geq 2$. The integer points are of the form $\frac{(2-f_1-f_2)-(k-f_1)(1-f_2)}{2-f_1-f_2}v^1 + \frac{(k-f_1)(1-f_2)}{2-f_1-f_2}v^3$ where $1 \leq k \leq \bar{k}$. The first two points corresponding to $k = 1$ and $k = 2$ are:
   - $(\frac{1-f_1f_2}{2-f_1-f_2}v^1 + \frac{1-f_1-f_2+f_1f_2}{2-f_1-f_2}v^3 = (1,1))$.
   - $(\frac{f_2-f_1f_2}{2-f_1-f_2}v^1 + \frac{(2-f_2)(1-f_2)}{2-f_1-f_2}v^3 = (2,0))$.

It can also be verified that the function ϕ^{sm} is the trivial fill-in function. Figure 3 shows a maximal lattice-free triangle with 3 integer points in the interior of one of its side. This generates the sequential-merge cut with $r_1 = 0.5$ and $r_2 = 0.25$.

4.2 $P(\pi)$ is a triangle with a single integer point in the interior of each side and integral vertices

In Lemma 11 it was shown that the standard triangle with only the points $(0,0)$, $(1,0)$, and $(1,1)$ in the interior of its sides and integral vertices is the triangle whose vertices are $(-1,1)$, $(1,1)$, and $(1,-1)$. We verify that when starting with such a $P(\pi)$ (or a set that is obtain by application of unimodular matrix to this set), the inequality $(\phi^{\bar{0}}, \pi)$ is extreme for $MI(I^2, S^2, r)$.

Notation: (Refer to Figure 4.) We use the following notation for points in this section:

1. $a^1 := (1,-1)$
2. $a^2 := (1,1)$
3. $a^3 := (-1,1)$. The points a^1, a^2 and a^3 represent the vertices of the lattice-free triangle $P(\pi)$.
4. $b^1 := (1,0)$ is the integer point in the interior of the side a^1a^2.
5. $b^2 := (0,1)$ is the integer point in the interior of the side a^2a^3.
6. $b^3 := (0,0)$ is the integer point in the interior of the side a^3a^1.
7. The union of quadrilaterals $fc^1b^1e^1$, $fc^2b^2e^2$, and $fc^3b^3e^3$ represents $D(\pi) + \{f\}$. (In particular, c^1 lies on fa^1, e^1 lies on fa^2 and $f + (c^1 - f) + (e^1 - f) = b^1$. c^2 lies on fa^2, e^2 lies on fa^3 and $f + (c^2 - f) + (e^2 - f) = b^2$. c^3 lies on fa^3, e^3 lies on fa^1 and $f + (c^3 - f) + (e^3 - f) = b^3$.)

Theorem 30 *If $P(\pi)$ is a maximal lattice-free triangle with integral vertices and one integer point in the interior of each side, then $(\phi^{\bar{0}}, \pi)$ is extreme for $MI(I^2, S^2, r)$. Moreover, there exists no function $\phi : I^2 \to \mathbb{R}_+$ such that $\phi \neq \phi^{\bar{0}}$ and (ϕ, π) is an extreme function for $MI(I^2, S^2, r)$.*

Proof. By Propositions 26 and 17 it is enough to prove this result for the maximal lattice-free triangle $P(\pi)$ with the points $(1,0)$, $(0,1)$ and $(0,0)$ in the interior of sides. By Theorem 5.4 from Cornuéjols and Margot [10], we obtain that π is extreme for $MI(\emptyset, S^2, r)$. Therefore, by Theorem 25 it is enough to show that $\mathbb{P}(D(\pi)) = I^2$. This is equivalent to showing that $\mathbb{P}(D(\pi) + \{f\}) = I^2$.

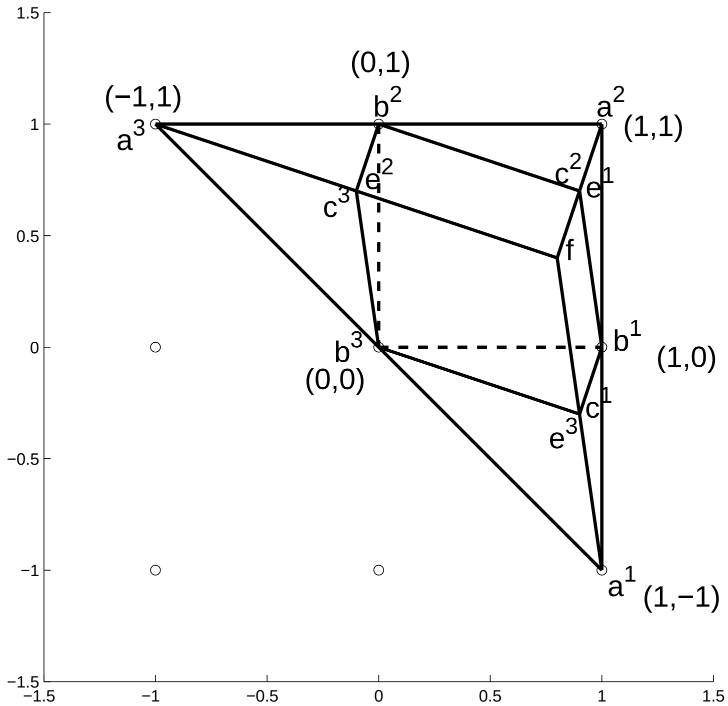

Figure 4: The Triangle with integral vertices and the points $(1,0)$, $(0,1)$, $(0,0)$ in the interior of its sides.

Refer to Figure 4. We first claim that the point e^1 and c^2 are the same point. This is because $\frac{|a^2c^2|}{|a^2f|} = \frac{|a^2b^2|}{|a^2a^3|} = \frac{|a^2b^1|}{|a^2a^1|} = \frac{|a^2e^1|}{|a^2f|}$. Similarly it can be verified that e^2 and c^3 are the same point and e^3 and c^1 are the same point.

Now it can be verified that the triangle $b^3b^1e^3$ is symmetric to triangle $b^2a^2c^2$. This is because $|b^2a^2| = |b^3b^1| = 1$ and angles $b^1b^3e^3$ and $b^3b^1e^3$ are equal to angles $a^2b^2c^2$ and $b^2a^2c^2$ respectively. Since b^3, b^1, a^2 and b^2 are integral and b^3b^1 is parallel to b^2a^2, the fractional part of points in triangle $b^3b^1e^3$ is exactly the same as those in triangle $b^2a^2c^2$. Similarly, the fractional parts of the points in triangles $b^3c^3b^2$ and $b^1e^1a^2$ are exactly the same. Note that triangles $b^3b^1e^3$ and $b^3c^3b^2$ belong to $D(\pi) + \{f\}$. Since, the quadrilateral $b^2b^3b^1c^2$ belongs to $D(\pi) + \{f\}$, we obtain that all the fractional parts in the unit square $b^2b^3b^1a^2$ belong to $D(\pi) + \{f\}$. □

Figure 5 shows an example of a lattice-free triangle with one integer point in the interior of each side and integral vertices. It also shows the corresponding trivial fill-in function which is extreme for $MI(I^2, S^2, r)$. This family of inequalities is related to an example first presented in Cook et al. [8] and generalized in Li and Richard [26]. In particular, while an infinite number of GMICs are not enough to solve this example, one cut from this family added to the problem furnishes the convex hull of integer feasible solutions.

5 Coefficients for integer variables in general: the fill-in procedure

In the previous section, it was shown that for some classes of lattice-free convex sets, the corresponding trivial fill-in functions provide unique extreme inequalities for the two-row mixed integer infinite-group problem. The

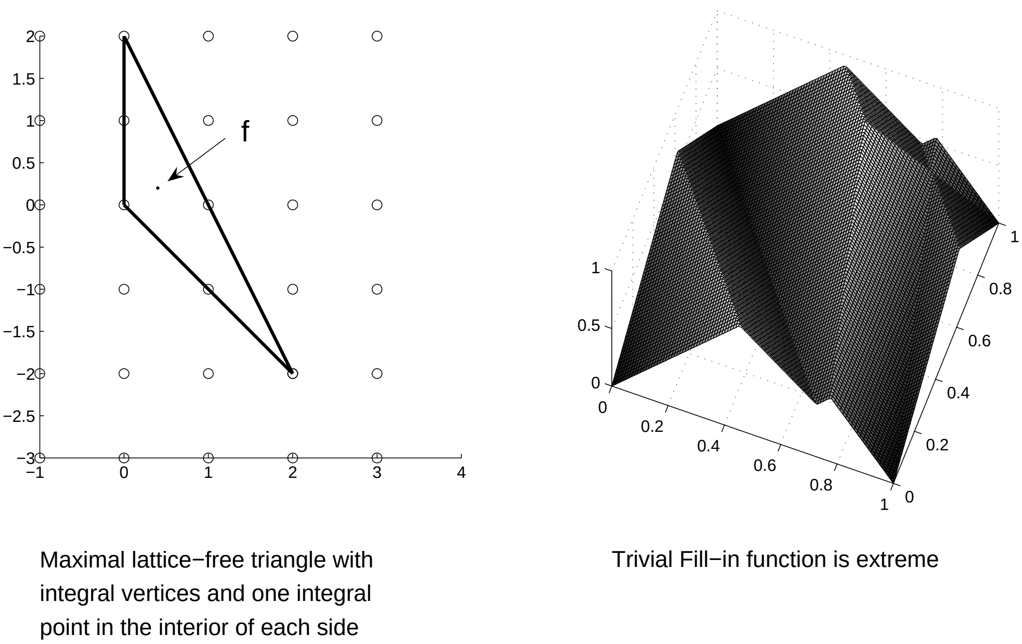

Figure 5: Lattice-free triangle with interval vertices and single integer point in the interior of each side

chief ingredient of the proof was to show that $\mathbb{P}(D(\pi)) = I^2$ and then to apply Theorem 25. The natural question then is what happens in the other cases when $\mathbb{P}(D(\pi)) \subsetneq I^2$ (We will prove that $\mathbb{P}(D(\pi)) \subsetneq I^2$ for all other cases of maximal lattice-free triangles and quadrilaterals). It will be shown in this section that in this case $(\phi^{\bar{0}}, \pi)$ is not minimal for $MI(I^2, S^2, r)$ (and therefore not extreme). To generate extreme inequalities in this case, we will then present a generalization of a procedure developed by Gomory and Johnson [21] and Johnson [25] called the fill-in procedure. We will end this section with an analysis of conditions under which the fill-in procedure produces extreme inequalities.

5.1 Trivial fill-in function is not minimal if $\mathbb{P}(D(\pi)) \subsetneq I^2$

To analyze the strength of the trivial fill-in procedure in the case when $\mathbb{P}(D(\pi)) \subsetneq I^2$, we first present a proposition characterizing the set $\mathbb{R}^2 \setminus D(\pi)$.

Proposition 31 *If $u \in C^i \setminus D(\pi)$, then there does not exist any $v \in C^i$, $n \in \mathbb{Z}_+$ such that $n \geq 1$ and $nu + v + f \in p^i$.*

Proof. Assume by contradiction that there exists a $v \in C^i$, $n \in \mathbb{Z}_+$ such that $n \geq 1$ and $nu + v + f \in p^i$. Let $\delta^{ij} d^i + (1 - \delta^{ij}) d^j + f = X^{ij}$. Since $u \in C^i \setminus D(\pi)$, by the definition of $D(\pi)$ $u = \alpha d^i + \beta d^{i+1}$ where either $\alpha > \delta^{ij}$ or $\beta > (1 - \delta^{ij})$. Now $v = (\delta^{ij} - n\alpha) d^i + (1 - \delta^{ij} - n\beta) d^{i+1}$ which implies that $v \notin C^i$, a contradiction. $\square$

Next we present a property of the function π corresponding to a maximal lattice-free bounded set $P(\pi)$ that will be useful in proving that the function $\phi^{\bar{0}}$ is not minimal when $\mathbb{P}(D(\pi)) \subsetneq I^2$.

Proposition 32 *Let $P(\pi)$ be a bounded maximal lattice-free convex set in $\mathbb{R}^2$ (i.e., either a triangle or a quadrilateral). Let $w^1, w^2 \in \mathbb{R}^2$. Suppose that for some i, $w^1 \in C^i$, and either $w^2 \notin C^i$ or $w^1 + w^2 \notin C^i$, then $\pi(w^1) + \pi(w^2) > \pi(w^1 + w^2)$.*

Proof: First note that it is enough to prove that if $w^2 \notin C^i$ then $\pi(w^1) + \pi(w^2) > \pi(w^1 + w^2)$. This is because $w^1 \in C^i$ and $w^1 + w^2 \notin C^i$ imply that $w^2 \notin C^i$.

In the first step, we show that if w^1 and w^2 are in adjacent cones then $\pi(w^1) + \pi(w^2) > \pi(w^1 + w^2)$. WLOG assume that $w^1 \in C^1$ and $w^2 \in C^2$. Note that w^1 and w^2 do not belong to the edge $C^1 \cap C^2$ since then they would belong to the same cone. Assume by contradiction that $\pi(w^1) + \pi(w^2) = \pi(w^1 + w^2)$. There are two cases:

1. The angle spanned by the two cones is less than 180^o (This case occurs only for quadrilaterals). Let $\pi(x) = \alpha^i x_1^i + \beta^i x_2^i$ when $(x_1^i, x_2^i) \in C^i$. Note that by construction of π, it can be verified that $(\alpha^1, \beta^1) \neq (\alpha^2, \beta^2)$. Let $c := (c_1, c_2)$ be the direction of $C^1 \cap C^2$. It can be verified that the unique solution to the equation, $(\alpha^1 - \alpha^2)x + (\beta^1 - \beta^2)y = 0$ is $x = c_1$ and $y = c_2$. Since the angle spanned by the two adjacent cones is less than 180^os, $w^1 + w^2$ belongs to one of these two cones. WLOG assume that $w^1 + w^2 \in C^2$. Therefore, we have $(\alpha^1 w_1^1 + \beta^1 w_2^1) + (\alpha^2 w_1^2 + \beta^2 w_2^2) = \alpha_1^2(w_1^1 + w_1^2) + \beta_1^2(w_2^1 + w_2^2)$, or $\alpha^1 w_1^1 + \beta^1 w_2^1 = \alpha^2 w_1^1 + \beta^2 w_2^1$ which is a contradiction since $w^1 \notin C^1 \cap C^2$.

2. The angle spanned by the two cones is greater than 180^o (This case can occur both for triangles and quadrilaterals). Note first that w^1 and w^2 do not belong to the edge $C^1 \cap C^2$. Also if both w^1 and w^2 belong to the other extreme rays of C^1 and C^2 respectively, then w^1 and w^2 either belong to the same cone when $P(\pi)$ is a triangle, or they belong to two adjacent cones which span an angle less than 180^o. Therefore, the result will follow using case 1 above. Hence, we assume WLOG that w^1 belongs to the interior of C^1. Then it can be verified that there exists a direction d such that $d \in C^1$, $-d \in C^2$, $w^1 - \epsilon d \in C^1$, $w^2 + \epsilon d \in C^2$ where $\epsilon > 0$. If $w^2 \in C^2 \cap C^3$, then let $-d$ be the extreme ray shared by C^2 and C^3. Since $w^1, \epsilon d, w^1 - \epsilon d \in C^1$, we obtain that $\pi(w^1 - \epsilon d) + \pi(\epsilon d) = \pi(w^1)$ or $\pi(w^1 - \epsilon d) < \pi(w^1)$. Similarly, $\pi(w^2 + \epsilon d) + \pi(-\epsilon d) = \pi(w^2)$ and $\pi(w^2 + \epsilon d) < \pi(w^2)$. Therefore, we obtain that $\pi(w^1 - \epsilon d) + \pi(w^2 + \epsilon d) < \pi(w^1) + \pi(w^2) = \pi(w^1 + w^2)$ a contradiction to subadditivity of π.

The above (case 2) proves the result when $P(\pi)$ is a triangle. When $P(\pi)$ is a quadrilateral, we need to verify that $\pi(w^1) + \pi(w^2) > \pi(w^1 + w^2)$, when $w^1 \in C^1$ and $w^2 \in C^3$, i.e., when w^1 and w^2 belong to non-adjacent cones. First note that if w^1 and w^2 belong to the boundary of C^1 and C^3, then either both of them belong to the same cone or they belong to adjacent cones. Therefore, we may assume that w^1 and w^2 are in the interior of C^1 and C^3 respectively. Using this, it can again be verified that there exists a direction d such that $d \in C^1$, $-d \in C^3$, $w^1 - \epsilon d \in C^1$, $w^2 + \epsilon d \in C^2$ where $\epsilon > 0$. The rest of the proof is similar to part (2) above. $\square$

We finally present the proof of the fact that $(\phi^{\bar{0}}, \pi)$ is not minimal when $\mathbb{P}(D(\pi)) \subsetneq I^2$.

Proposition 33 *Let $P(\pi)$ be a lattice-free bounded convex set. Suppose $u^* \notin D(\pi)$ and $\phi^{\bar{0}}(\mathbb{P}(u^*)) = \pi(u^*)$.*

1. *Then the following system has no solution*

$$x(\mathbb{P}(u^*))\mathbb{P}(u^*) + \sum_{w \in \mathbb{R}^2} w y(w) + f \quad \in \quad \mathbb{Z}^2 \tag{17}$$

$$\phi^{\bar{0}}(\mathbb{P}(u^*))x(\mathbb{P}(u^*)) + \sum_{w \in \mathbb{R}^2} \pi(w) y(w) \quad = \quad 1 \tag{18}$$

$$x(\mathbb{P}(u^*)) \in \mathbb{Z}, x(\mathbb{P}(u^*)) \geq 1, y \quad \geq \quad 0 \tag{19}$$

2. *$\phi^{\bar{0}}(\mathbb{P}(u^*)) + \phi^{\bar{0}}(\mathbb{P}(r - u^*)) > 1$.*

Proof.

1. If $u^* \notin P(\pi)$ the result is obvious since $\phi^{\bar{0}}(\mathbb{P}(u^*)) > 1$. Consider the case when $u^* \in P(\pi) \setminus D(\pi)$. WLOG assume that $u^* \in C^1$. Assume by contradiction that there exists $(\bar{x}, \bar{y})$ that satisfies (17), (18), and (19). Therefore,

$$\bar{x}(\mathbb{P}(u^*))\mathbb{P}(u^*) + \sum_{w\in\mathbb{R}^2} w\bar{y}(w) + f = X \in \mathbb{Z}^2 \tag{20}$$

   Since $X \in \mathbb{Z}^2$ and $P(\pi)$ is lattice-free, $\pi(X - f) \geq 1$. Now

$$\begin{aligned} 1 &= \phi^{\bar{0}}(\mathbb{P}(u^*))\bar{x}(\mathbb{P}(u^*)) + \sum_{w\in\mathbb{R}^2} \pi(w)\bar{y}(w) \\ &= \pi(u^*)\bar{x}(\mathbb{P}(u^*)) + \sum_{w\in\mathbb{R}^2} \pi(w)\bar{y}(w) \\ &\geq \pi(u^*)\bar{x}(\mathbb{P}(u^*)) + \pi(\sum_{w\in\mathbb{R}^2} w\bar{y}(w)) \\ &\geq \pi(X - f) \end{aligned} \tag{21}$$

   Therefore, (21) is satisfied at equality and $\pi(X - f) = 1$ or $X \in P(\pi)$. Moreover by Proposition 32, $\pi(u) + \pi(v) = \pi(u+v)$ iff $u, v, u+v \in C^i$. Since $u^* \in C^1$, $\bar{x}(\mathbb{P}(u^*)) \geq 1$, and (21) is satisfied at equality, we obtain that $X - f \in C^1$ or $X - f \in p^1 - f$. We also obtain that $\sum_{w\in\mathbb{R}^2} w\bar{y}(w) \in C^1$. However, as $u^* \in C^1 \setminus D(\pi)$ we obtain using Proposition 31 that there does not exists a vector $v \in C^1$ such that $v + nu^* \in p^1 - f$ where $n \in \mathbb{Z}_+$ and $n \geq 1$, which is the required contradiction to (20).

2. This follows from the proof of part (1) since

$$\phi^{\bar{0}}(\mathbb{P}(r - u^*)) = \min\{\sum_{w\in\mathbb{R}^2} \pi(w)y(w) \,|\, \mathbb{P}(u^*) + \sum_{w\in\mathbb{R}^2} wy(w) + f \in \mathbb{Z}^2\}.$$

□

If there exists a point $u \in I^2$ such that $u \notin \mathbb{P}(D(\pi))$ then $\phi^{\bar{0}}(u) + \phi^{\bar{0}}(r - u) > 1$ implying that the function is not minimal by the use of Theorem 16. Therefore there must exist some other valid function (ϕ, π) such that $\phi(u) = \phi^{\bar{0}}(u)\ \forall u \in \mathbb{P}(D(\pi))$ and either $\phi(u) < \phi^{\bar{0}}(u)$ or $\phi(r - u) < \phi^{\bar{0}}(r - u)$ (or both) $\forall u \in I^2 \setminus \mathbb{P}(D(\pi))$.

5.2 General fill-in function: definition and validity

In this section we present a general version of the fill-in procedure developed by Gomory and Johnson [21] and Johnson [25] that will be used to generate valid inequalities for $MI(I^2, S^2, r)$ starting from inequalities of $MI(\emptyset, S^2, r)$. It will follow from the definition of these functions that all extreme inequalities must be general fill-in functions.

Definition 34 (Fill-in Function) *Let $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ be a valid and minimal function corresponding to a bounded maximal lattice-free set $P(\pi)$. Let G be any subgroup of I^2. Let (V, π) be a valid subadditive function for $MI(G, S^2, r)$. The function $\phi^{G,V} : I^2 \to \mathbb{R}_+$ is defined as follows:*

$$\phi^{G,V}(u) = inf_{v\in G, w\in\mathbb{R}^2} \{V(v) + \pi(w) \mid \mathbb{P}(w) = u - v\}. \tag{22}$$

The fill-in procedure may be interpreted as a two-step lifting scheme. In the first step we obtain the inequality (V, π) by lifting integer variables corresponding to columns in the set G. The function V may depend on the order of lifting of these variables, i.e., for a given subgroup G there may exist two different functions V_1 and V_2 such that both functions eventually yield strong cutting planes for $MI(I^2, S^2, r)$. Once the integer variables corresponding to columns in the set G are lifted, the lifting in the second step (of the rest of the integer variables) is completely defined by the choice of G and V.

It can be verified that the construction of $\phi^{G,V}$ is equivalent to the original fill-in procedure of Johnson [25] when we start with a subgroup G of finite order. The advantage of allowing general subgroups G in the fill-in procedure is twofold. First *given π, since G is allowed to be any subgroup of I^2, trivially, every extreme inequality (ϕ, π) for the $MI(I^2, S^2, r)$ is a fill-in function (by selecting G to be I^2).* Second, this definition allows for construction of fill-in functions starting from infinite subgroups G of I^2; such as $\{(x,x)|x \in I^1\}$. However, when using the fill-in procedure, we mainly consider subgroups G that are finite.

When G is the trivial subgroup, i.e., $G = \{\bar{0}\}$ and $V(\bar{0}) = 0$, we obtain the trivial fill-in function: $\phi^{\bar{0},V(\bar{0})=0}$ which we have represented for simplicity as $\phi^{\bar{0}}$. It follows from Johnson [25] that when G is finite, $(\phi^{G,V}, \pi)$ is a valid inequality for $MI(I^2, S^2, r)$. We next verify that this more general version of the fill-in procedure also generates valid functions for $MI(I^2, S^2, r)$.

We use a result from Johnson [25] (Lemmas 4.4, 4.5 and Corollary 5.4) that gives the following sufficient conditions for valid subadditive functions for $MI(I^2, W, r)$.

Proposition 35 ([25]) *For a pair of functions (ϕ, π) to be valid and subadditive for $MI(I^2, S^2, r)$, the following conditions are sufficient:*

1. *ϕ is subadditive, i.e., $\phi(u) + \phi(v) \geq \phi(u+v)$ $\forall u, v \in I^2$,*
2. *π is convex,*
3. *$\pi(w) \geq lim_{h\to 0^+} \frac{\phi(\mathbb{P}(hw))}{h}$ $\forall w \in S^2$,*
4. *$\phi(r) \geq 1$.*

We next show that $\phi^{G,V}$ is a valid function for $MI(I^2, S^2, r)$. Before we proceed, we make an observation used in the proof of the next proposition. Let G_1 be a subgroup of G_2 and let $V_2 : G_2 \to \mathbb{R}_+$. If $V_1 : G_1 \to \mathbb{R}^2$ is defined as $V_1(u) = V_2(u)$, then $\phi^{G_1,V_1}(v) \geq \phi^{G_2,V_2}(v)$ $\forall v \in I^2$. Therefore, $\phi^{G,V}(v) \leq \phi^{\bar{0}}(v)$ $\forall v \in I^2$ for any subgroup G of I^2.

Proposition 36 *Let $P(\pi)$ be a maximal lattice-free convex set. Then*

1. *$\phi^{G,V}$ is subadditive.*
2. *$\pi(w) = lim_{h\to 0^+} \frac{\phi^{G,V}(\mathbb{P}(hw))}{h}$ $\forall w \in S^2$*
3. *$\phi^{G,V}(r) = 1$.*

Proof. By the use of Proposition 35, we verify the following conditions

1. $\phi^{G,V}$ is subadditive: We want to show that for any $u^1, u^2 \in I^2$, $\phi^{G,V}(u^1) + \phi^{G,V}(u^2) \geq \phi^{G,V}(u^1 + u^2)$. By definition of $\phi^{G,V}$, for any $\epsilon_i > 0$, $\exists$ $v^i \in G, w^i \in \mathbb{R}^2$ such that $\phi^{G,V}(u^1) \geq V(v^1) + \pi(w^1) - \epsilon_1$ and $\phi^{G,V}(u^2) \geq V(v^2) + \pi(w^2) - \epsilon_2$, with $u^i = \mathbb{P}(v^i + w^i)$ for $i = 1, 2$. Then
$$\begin{aligned} \phi^{G,V}(u^1) + \phi^{G,V}(u^2) &\geq V(v^1) + \pi(w^1) + V(v^2) + \pi(w^2) - \epsilon_1 - \epsilon_2 \\ &\geq V(v^1 + v^2) + \pi(w^1 + w^2) - \epsilon_1 - \epsilon_2 \\ &\geq \phi^{G,V}(u^1 + u^2) - (\epsilon_1 + \epsilon_2) \end{aligned} \tag{23}$$
Since $\epsilon_1 + \epsilon_2$ can be made arbitrarily small (by suitably selecting v^i, w^i), it follows that $\phi^{G,V}(u^1) + \phi^{G,V}(u^2) \geq \phi^{G,V}(u^1 + u^2)$.

2. $\pi(w) \geq lim_{h\to 0^+} \frac{\phi^{G,V}(\mathbb{P}(hw))}{h}$ $\forall w \in S^2$: Observe that since $\phi^{\bar{0}}(u) \geq \phi^{G,V}(u)$ $\forall u \in I^2$, $\pi(w) = lim_{h\to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(hw))}{h} \geq lim_{h\to 0^+} \frac{\phi^{G,V}(\mathbb{P}(hw))}{h}$ $\forall w \in S^2$, where the first equality follows from Corollary 24. Note also that for any $w \in \mathbb{R}^2$, for sufficiently small $h > 0$, $hw \in D(\pi)$ (see proof of Corollary 24). Therefore, by part 4 of Proposition 23 we obtain that the last inequality in an equality, i.e., $\pi(w) = lim_{h\to 0^+} \frac{\phi^{\bar{0}}(\mathbb{P}(hw))}{h} = lim_{h\to 0^+} \frac{\phi^{G,V}(\mathbb{P}(hw))}{h}$ $\forall w \in S^2$.

3. $\phi^{G,V}(r) = 1$: From Proposition 19, we obtain that $\phi^{G,V}(r) \leq \phi^{\bar{0}}(r) = 1$. We next show that $\phi^{G,V}(r) \geq 1$. By definition of $\phi^{G,V}$, for any $\epsilon > 0$, $\exists\ \bar{v} \in G$, $\bar{w} \in \mathbb{R}^2$ with $\bar{v} + \mathbb{P}(\bar{w}) = r$ such that $\phi^{G,V}(r) \geq V(\bar{v}) + \pi(\bar{w}) - \epsilon$. Since (V, π) is a valid function for $MI(G, S^2, r)$, we have that $V(\bar{v}) + \pi(\bar{w}) \geq 1$. Therefore, for any $\epsilon > 0$, $\phi^{G,V}(r) \geq 1 - \epsilon$. □

Since π is convex (Borozan and Cornuéjols [7]), we obtain by the use of Proposition 35 that $(\phi^{G,V}, \pi)$ is a valid function for $MI(I^2, S^2, r)$.

Before we present more results analyzing the strength of the fill-in function, we show that the function $\phi^{G,V}$ can be evaluated in finite time for each $u \in I^2$, when G is a finite group. This follows from the next proposition.

Proposition 37 *Let $P(\pi)$ be a maximal lattice-free bounded convex set. If G is a finite subgroup of I^2, then there exist nonnegative integers N_1 and N_2 such that*

$$\phi^{G,V}(u) = \phi(v) + \pi(w_1 + k_1, w_2 + k_2) \tag{24}$$

for some $v \in G$, $(w_1, w_2) \in [0,1) \times [0,1)$, and integers k_1 and k_2 with $|k_1| \leq N_1$, $|k_2| \leq N_2$ $\forall u \in I^2$.

Proof. By the definition of the fill-in function, $\phi^{G,V}(u) = \inf_{v \in G, w \in \mathbb{R}^2}\{\phi(v) + \pi(w) \,|\, \mathbb{P}(w) = u - v\}$. Since G is finite, we may write $\phi^{G,V}(u) = \min_{v \in G}(\phi(v) + \inf_{w \in \mathbb{R}^2}\{\pi(w) \,|\, \mathbb{P}(w) = u - v\})$. Therefore to prove this result, it is sufficient to show that there exist nonnegative integers N_1 and N_2 such that $\forall (w_1, w_2) \in [0,1) \times [0,1)$,

$$\inf_{n_1, n_2 \in \mathbb{Z}}\{\pi(w_1 + n_1, w_2 + n_2)\} = \pi(w_1 + k_1, w_2 + k_2) \tag{25}$$

where k_1 and k_2 are integers satisfying $|k_1| \leq N_1$, $|k_2| \leq N_2$.

Since $P(\pi)$ is bounded, $\pi(w_1, w_2) > 0$ $\forall (w_1, w_2) \in \mathbb{R}^2 \setminus \{\bar{0}\}$. Let $d := (d_1, d_2)$ be the unit vector in the direction of minimum slope. By assumption this minimum slope is positive. Since the value of π is bounded over the set $[0,1) \times [0,1)$, let $k = \sup\{\pi(w_1, w_2) \mid (w_1, w_2) \in [0,1) \times [0,1)\}$. Let l be the real such that $\pi(ld) = k$. Set $N_1 = N_2 = \lceil l \rceil$. Now for any (w_1, w_2) and n_1, $n_2 \in \mathbb{Z}$ such that either $|n_1| > N_1$ or $|n_2| > N_2$ (or both), we have $\pi(w_1 + n_1, w_2 + n_2) \geq \pi(\|(w_1 + n_1, w_2 + n_2)\| d) \geq \pi(ld) = k \geq \pi(w_1, w_2)$. □

5.3 Strength of fill-in functions

We next study conditions under which $(\phi^{G,V}, \pi)$ is a minimal function for $MI(I^2, S^2, r)$. Notice that Proposition 36 establishes all but one of the conditions needed to prove that $(\phi^{G,V}, \pi)$ is a minimal function for $MI(I^2, S^2, r)$. We record this result next.

Corollary 38 *Let $P(\pi)$ be a maximal lattice-free bounded convex set. Then the valid function $(\phi^{G,V}, \pi)$ is minimal for $MI(I^2, S^2, r)$ iff $\phi^{G,V}(u) + \phi^{G,V}(r - u) = 1$ $\forall u \in I^2$.*

We next present conditions for the function $(\phi^{G,V}, \pi)$ to be an extreme function for $MI(I^2, S^2, r)$. For the case of the trivial fill-in function, we showed in Proposition 20 that if $\phi^{\bar{0}}$ is the unique function such that $(\phi^{\bar{0}}, \pi)$ is minimal for $MI(I^2, S^2, r)$, and π is extreme for $MI(\emptyset, S^2, r)$, then $(\phi^{\bar{0}}, \pi)$ is extreme for $MI(I^2, S^2, r)$. We now develop similar conditions for the function $(\phi^{G,V}, \pi)$ to be an extreme function for $MI(I^2, S^2, r)$.

If the function $(\phi^{G,V}, \pi)$ is minimal, then we next show that it must be the unique minimal function under certain conditions. This result is a consequence of the following result from Johnson [25] that states that minimal inequalities must be subadditive.

Theorem 39 ([25]) *If (ϕ, π) is a minimal inequality for $MI(U, W, r)$ for some subgroup U of I^2, then*

1. *$\phi(u) + \phi(v) \geq \phi(u + v)$ $\forall u, v \in U$.*
2. *$\phi(u) + \sum_{w \in W} \pi(w) y(w) \geq \phi(v)$ whenever $u + \mathbb{P}(\sum_{w \in W} w y(w)) = v$.*
3. *$\sum_{w \in W} \pi(w) y(w) \geq \pi(w')$ whenever $\sum_{w \in W} w y(w) = w'$.* □

The next result is a modified version of the uniqueness result for the case of a general fill-in function.

Lemma 40 *Let $(\phi^{G,V}, \pi)$ be minimal for $MI(I^2, S^2, r)$. If (ϕ', π) is a valid minimal function for $MI(I^2, S^2, r)$ such that $\phi'(u) = V(u)\ \forall u \in G$, then $\phi'(v) = \phi^{G,V}(v)\ \forall v \in I^2$.*

Proof. Assume by contradiction that there exists a valid minimal function (ϕ', π) with $\phi' \neq \phi^{G,V}$ and $\phi'(u) = V(u)\ \forall u \in G$. Since $(\phi^{G,V}, \pi)$ is minimal, there exists a point $u^* \in I^2$ such that $\phi'(u^*) > \phi^{G,V}(u^*)$. Let $\phi'(u^*) - \phi^{G,V}(u^*) = \epsilon$. Now by definition of $\phi^{G,V}$, $\exists\ u \in G$ and $w \in \mathbb{R}^2$ such that $\phi^{G,V}(u^*) \geq V(u) + \pi(w) - \frac{\epsilon}{2}$ where $u^* - u = \mathbb{P}(w)$. Therefore we obtain that $\phi'(u^*) = \phi^{G,V}(u^*) + \epsilon \geq V(u) + \pi(w) + \frac{\epsilon}{2}$ or $\phi'(u^*) > V(u) + \pi(w) = \phi'(u) + \pi(w)$. Since ϕ' is minimal, this contradicts Theorem 39. Therefore, $\phi'(u) = \phi^{G,V}(u)\ \forall u \in I^2$. □

Now we have all the tools to derive the main result of this section.

Theorem 41 *Let (V, π) be minimal for $MI(G, S^2, r)$. $(\phi^{G,V}, \pi)$ is an extreme valid inequality for $MI(I^2, S^2, r)$ iff (V, π) is extreme for $MI(G, S^2, r)$ and $(\phi^{G,V}, \pi)$ is minimal for $MI(I^2, S^2, r)$.*

Proof: $\Leftarrow$ Assume first that (V, π) is extreme for $MI(G, S^2, r)$ and $(\phi^{G,V}, \pi)$ is minimal for $MI(I^2, S^2, r)$. Suppose that $(\phi^{G,V}, \pi)$ is not extreme. So there exist valid functions (ϕ_1, π_1) and (ϕ_2, π_2) for $MI(I^2, S^2, r)$ such that $\phi^{G,V} = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$, $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ and $(\phi_1, \pi_1) \neq (\phi_2, \pi_2)$. Since $(\phi^{G,V}, \pi)$ is minimal, it can be shown that (ϕ_1, π_1) and (ϕ_2, π_2) are also minimal (See proof of Theorem 3.4 in Johnson [25]). Moreover, since (V, π) is extreme, $\phi_1(u) = \phi_2(u) = V(u)\ \forall u \in G$ and $\pi_1(w) = \pi_2(w)\ \forall w \in \mathbb{R}^2$. However by Lemma 40, this implies that $\phi_1(u) = \phi_2(u) = \phi^{G,V}(u)\ \forall u \in I^2$ as $\phi^{G,V}$ is the unique minimal function, which is the required contradiction.

$\Rightarrow$ If $(\phi^{G,V}, \pi)$ is not minimal for $MI(I^2, S^2, r)$, then clearly $(\phi^{G,V}, \pi)$ is not extreme for $MI(I^2, S^2, r)$.

Finally assume that (V, π) is not extreme for $MI(G, S^2, r)$, i.e., $V = \frac{1}{2}V_1 + \frac{1}{2}V_2$ and $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ where (V_i, π_i) is valid minimal inequality for $MI(G, S^2, r)$ for $i = 1, 2$ and $(V_1, \pi_1) \neq (V_2, \pi_2)$. (Since (V, π) is minimal, (V_i, π_i) must be minimal). We have to show that $(\phi^{G,V}, \pi)$ is not an extreme valid inequality for $MI(I^2, S^2, r)$.

We first show that $\phi^{G,V} \geq \frac{1}{2}\phi^{G,V_1} + \frac{1}{2}\phi^{G,V_2}$. For any $u \in I^2$ and any $\epsilon > 0$, by definition of $\phi^{G,V}$ $\exists\ \bar{u} \in G$, $\bar{w} \in \mathbb{R}^2$ such that $\phi^{G,V}(u) \geq V(\bar{u}) + \pi(\bar{w}) - \epsilon$. Therefore, $\phi^{G,V}(u) \geq V(\bar{u}) + \pi(\bar{w}) - \epsilon = \frac{1}{2}V_1(\bar{u}) + \frac{1}{2}V_2(\bar{u}) + \frac{1}{2}\pi_1(\bar{w}) + \frac{1}{2}\pi_2(\bar{w}) - \epsilon \geq \frac{1}{2}\phi^{G,V_1}(u) + \frac{1}{2}\phi^{G,V_2}(u) - \epsilon$. Since ϵ can be made as small as possible, we obtain that $\phi^{G,V} \geq \frac{1}{2}\phi^{G,V_1} + \frac{1}{2}\phi^{G,V_2}$. Therefore, we have that $(\phi^{G,V}, \pi) \geq \frac{1}{2}(\phi^{G,V_1}, \pi_1) + \frac{1}{2}(\phi^{G,V_2}, \pi_2)$. Clearly if $\pi_1 \neq \pi_2$, we have that $(\phi^{G,V}, \pi)$ is not extreme since $(\phi^{G,V}, \pi) \geq \frac{1}{2}(\phi^{G,V_1}, \pi_1) + \frac{1}{2}(\phi^{G,V_2}, \pi_2)$. If $\pi = \pi_1 = \pi_2$, and $V_1 \neq V_2$, we need to show that $\phi^{G,V_1} \neq \phi^{G,V_2}$ to complete the proof. Assume by contradiction that $\phi^{G,V_1} = \phi^{G,V_2}$. By definition, $\phi^{G,V_1}(u) \leq V_1(u)$ and $\phi^{G,V_1}(u) = \phi^{G,V_2}(u) \leq V_2(u)\ \forall u \in G$. Since $V_1 \neq V_2$, (ϕ^{G,V_1}, π) strictly dominates (V_1, π_1) or (V_2, π_2) for $MI(G, S^2, r)$. This contradicts the minimality of (V_i, π_i). □

Theorem 41 shows that if (V, π) is extreme and $(\phi^{G,V}, \pi)$ is minimal, then $(\phi^{G,V}, \pi)$ is extreme for $MI(I^2, S^2, r)$. This statement may be interpreted as infinite dimensional version of results for lifting in finite dimensions, i.e., we are performing strong lifting on a facet-defining inequality for a low dimensional polytope to form a facet-defining inequality of a higher dimensional polytope. On the other hand, in Proposition 46 in the next section, we present a result that does not assume that (V, π) is extreme. The next section illustrates some examples with distinct functions ϕ_1 and ϕ_2 forming extreme inequalities with the same function π. The key idea in constructing such functions is to start from different subgroups G and corresponding functions V. In this way it will be possible to construct different functions $\phi^{G,V}$ such that the different functions are extreme for the two-row mixed integer infinite-group problem.

6 Non-Unique lifting functions

In this section, we analyze the inequalities obtained by starting from the other two classes of maximal lattice-free convex sets, namely triangles with one integer point in the interior of each side with non-integral vertices and quadrilaterals. For each set we use the results of Section 5.1 to prove that the trivial fill-in function is not minimal. In the case of triangles with one integer point in the interior of each side and non-integral vertices we also present some sufficient conditions for the general fill-in function to be extreme.

We begin this section with a tool for the analysis of the area of $D(\pi)$. This result will be used to show that $\mathbb{P}(D(\pi)) \subsetneq I^2$.

Proposition 42 *Let $P(\pi)$ be a maximal lattice-free bounded convex set. For any $f := (f_1, f_2) \in P(\pi)$ let $A(f) = Area(D(\pi))$. If there exists only one integer point in the interior of each side of $P(\pi)$, then A is an affine function of f, i.e., $A(f) = \alpha_0 + \alpha_1 f_1 + \alpha_2 f_2$ for some $\alpha_0, \alpha_1, \alpha_2$.*

Proof: Since there exists only one integer point in the interior of each side of $P(\pi)$, we obtain that $\text{Area}(D(\pi)) = \sum_{ij} \text{Area}(D_{ij}(\pi))$. Therefore, to prove the result it is sufficient to show that $\text{Area}(D_{ij})$ is an affine function with respect to the position of f. For simplicity denote the points $f + d^i$, $f + d^{i+1}$, $f + \delta^{ij} d^i$, $f + (1 - \delta^{ij})d^{i+1}$ and X^j by a, b, c, d and e respectively. Hence the quadrilateral $fced$ represents D_{ij}.

Therefore $Area(fced) = Area(abf) - Area(aec) - Area(ebd)$. Since ec is parallel to bf and ed is parallel to af, we have that $\text{Area}(aec) = \lambda^2 \text{Area}(bfa)$ and $\text{Area}(ebd) = (1-\lambda)^2 \text{Area}(bfa)$ where $\lambda = \frac{|ae|}{|ab|}$ is independent of f. We now obtain that $Area(fced) = Area(abf) - Area(aec) - Area(ebd) = \mu Area(abf)$, where $\mu = 2(\lambda - \lambda^2)$ is independent of the position of f. Thus $Area(D_{ij}(\pi)) = \mu Area(abf)$. Now since f always lies on one side of the line segment ab, the area of the triangle abf varies affinely with the position of f. Therefore, we obtain the required result. □

6.1 $P(\pi)$ is a triangle with a single integer point in the interior of each side and non-integral vertices

In this section, we first show that unlike the previous cases, if $P(\pi)$ is a triangle with single integer point in the interior of each side and non-integral vertices, then $(\phi^{\bar{0}}, \pi)$ is not minimal. We then present some sufficient conditions for the generation of an extreme inequality using the fill-in procedure.

To prove that $(\phi^{\bar{0}}, \pi)$ is not minimal, we show that $\mathbb{P}(D(\pi))$ is a proper subset of I^2. This is achieved by verifying that the area of $D(\pi)$ is less than 1 in this case.

Notation: (Refer to Figure 6.) Let $P(\pi)$ be a maximal lattice-free triangle with the points $(1, 0)$, $(0, 1)$ and $(0, 0)$ in the interior of its sides. We use the following notation for points in this section:

1. The points a^1, a^2 and a^3 represent the vertices of the lattice-free triangle $P(\pi)$.
2. $b^1 := (1, 0)$ is the integer point in the interior of the side $a^1 a^2$.
3. $b^2 := (0, 1)$ is the integer point in the interior of the side $a^2 a^3$.
4. $b^3 := (0, 0)$ is the integer point in the interior of the side $a^3 a^1$.
5. The union of quadrilaterals $fc^1b^1e^1$, $fc^2b^2e^2$, $fc^3b^3e^3$, and $fc^4b^4e^4$ represents $D(\pi)+\{f\}$. (In particular, c^1 lies on fa^1, e^1 lies on fa^2 and $f + (c^1 - f) + (e^1 - f) = b^1$. c^2 lies on fa^2, e^2 lies on fa^3 and $f + (c^2 - f) + (e^2 - f) = b^2$. c^3 lies on fa^3, e^3 lies on fa^1 and $f + (c^3 - f) + (e^3 - f) = b^3$).

Proposition 43 *If $P(\pi)$ is a lattice-free triangle with a single integer point in the interior of each side and non-integral vertices, then $(\phi^{\bar{0}}, \pi)$ is not minimal for $MI(I^2, S^2, r)$.* □

Proof. Note first that translation and a linear transformation by a unimodular matrix do not change the area of a set. Therefore, it is enough to analyze the standard triangles, i.e., we analyze triangles with $(0, 0)$, $(1, 0)$, and $(0, 1)$ in the interior of its sides are. Let s_1, s_2 and s_3 be the sides of $P(\pi)$ passing through $(1, 0)$, $(0, 1)$, and $(0, 0)$ respectively. Henceforth we assume WLOG that slope of s_1 is negative and the slope of s_2 is positive (and s_1 is not vertical). (See part 2 of Lemma 11).

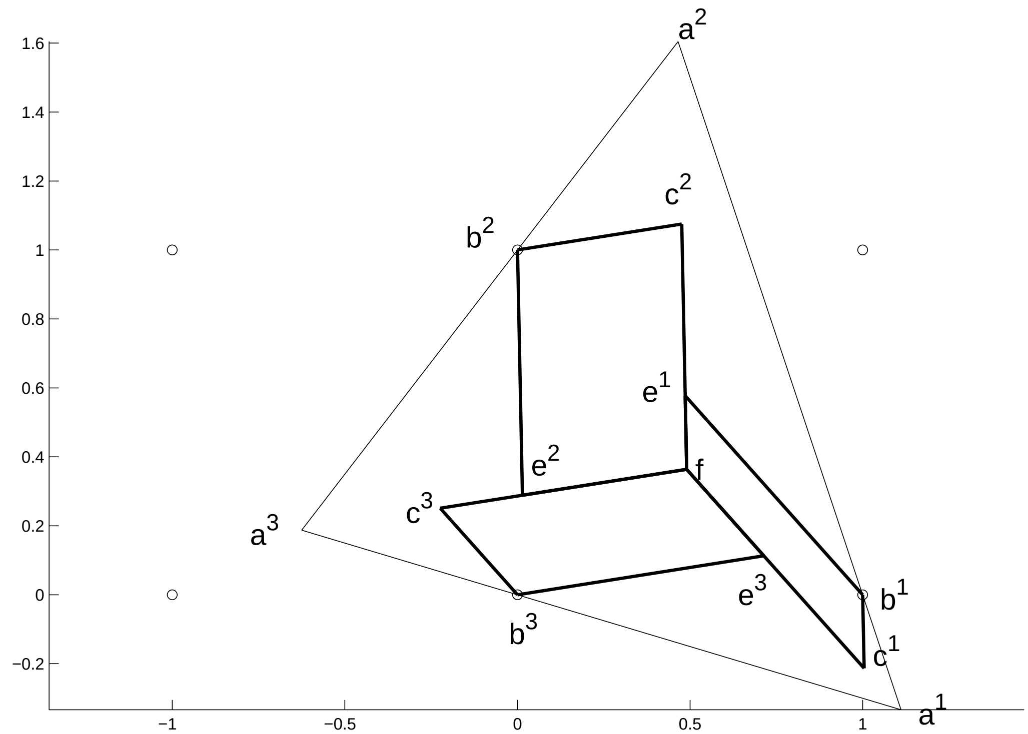

Figure 6: An example of triangle with one integer point in the interior of each side

Let m_1 be the negative of the slope of s_1, m_2 be the slope of s_2 and m_3 be the negative of the slope of s_3. We may assume WLOG that the slope of s_1 is negative and the slope of s_2 is positive (and s_1 is not vertical).

We know that f is in the strict interior of the triangle $a^1a^2a^3$. As shown by Proposition 42, the area is an affine function of the position of f. Therefore the area of $D(\pi)$ is maximized when f is the same point as either a^1, a^2 or a^3. We consider each of these cases next. The three cases are shown in Figure 7.

1. f is same as a^1. The area of $D(\pi)$ is the area of the parallelogram $a^1e^2b^2c^2$. The equation of the line passing through c^2a^1 is $m_1x + y = m_1$. The coordinates of c^2 are $\left(\frac{m_1-1}{m_1-m_3}, \frac{m_1(1-m_3)}{m_1-m_3}\right)$. Using this information, we can compute the area of $a^1e^2b^2c^2$ to be $\frac{m_1-1}{m_1-m_3}$. As $m_1 > 1$ and $0 < m_3 < 1$, we obtain that $\text{Area}(a^1e^2b^2c^2) < 1$.

2. f is same as a^2. The area of $D(\pi)$ is the area of the parallelogram $a^2e^3b^3c^3$. The equation of the line passing through a^2c^3 is $-m_2x + y = 1$. The coordinates of e^3 are $\left(\frac{m_1}{m_1+m_2}, \frac{m_1m_2}{m_1+m_2}\right)$. Using this information, we can compute the area of $a^2e^3b^3c^3$ to be $\frac{m_1}{m_1+m_2}$. As $m_1 > 0$ and $m_2 > 0$, we obtain that $\text{Area}(a^2e^3b^3c^3) < 1$.

3. f is same as a^3. The area of $D(\pi)$ is the area of the parallelogram $a^3e^1b^1c^1$. The equation of the line passing through a^3c^1 is $m_3x + y = 0$. The coordinates of e^1 are $\left(\frac{m_3-1}{m_2+m_3}, \frac{m_3(m_2+1)}{m_2+m_3}\right)$. Using this information, we can compute the area of $a^3e^1b^1c^1$ to be $\frac{(1+m_2)m_3}{m_2+m_3}$. As $m_2 > 0$ and $m_3 < 1$, we obtain that $\text{Area}(a^3e^1b^1c^1) < 1$.

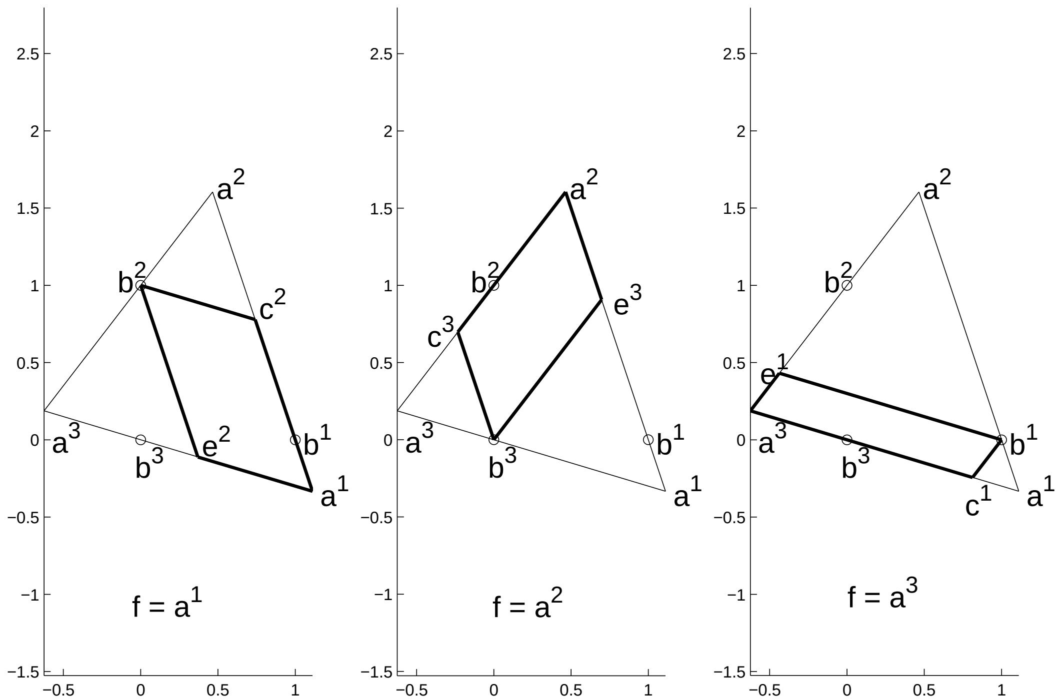

Figure 7: f is the same as vertex of the triangle $P(\pi)$

Thus $Area(D(\pi)) < 1$. This implies that $\mathbb{P}(D(\pi))$ is a proper subset of I^2. Therefore, we obtain using Proposition 33 that $(\phi^{\bar{0}}, \pi)$ is not minimal. $\square$

The next example shows that not only is the function $(\phi^{\bar{0}}, \pi)$ not minimal, but one may have $\phi^{\bar{0}}(u) > 1$ for some values of u.

Example 44 *Let $P(\pi)$ be the triangle with vertices $(0.25, 1.25)$, $(-0.75, 0.25)$, and $(1.25, -5/12)$ and let $f = (0.5, 0.5)$. Then it can be verified that $P(\pi)$ is a lattice-free triangle with only one integer point in the interior of each of its sides and non-integral vertices. $\phi^{\bar{0}}(0.1, 0.2) = 1.1$ and $\phi^{\bar{0}}$ is not minimal. There are two distinct functions ϕ_1 and ϕ_2 such that both (ϕ_1, π) and (ϕ_2, π) are extreme. (The proof of the extremality of these functions is similar to the proof of Theorem 7.1 in Dey and Richard [14]). See Figure 8.* $\square$

We showed in Section 4.1.1 that some subfamilies of the sequential-merge inequalities were lifted extreme inequalities for the two-row mixed integer infinite-group problem when starting with maximal lattice-free triangles with multiple integer point in the interior of one side. In the next example we show that the family of mixed MIR inequalities of Günlük and Pochet [24] can be derived from triangles with one integer point in the interior of each side and non-integral vertices.

Example 45 (Mixing) *The mixing set with two rows is defined as follows:*

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} s_1 + \begin{pmatrix} -1 \\ 0 \end{pmatrix} s_2 + \begin{pmatrix} 0 \\ -1 \end{pmatrix} s_3 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \quad s_1, s_2, s_3 \geq 0, \quad x_1, x_2 \in \mathbb{Z}, \tag{26}$$

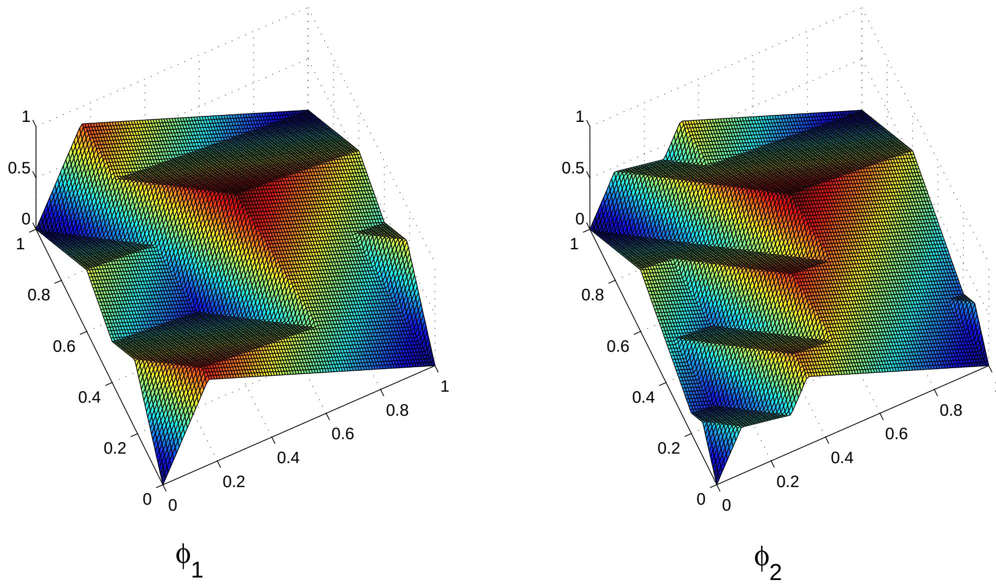

Figure 8: There exist distinct functions ϕ_1 and ϕ_2 such that (ϕ_1, π) and (ϕ_2, π) are extreme.

where $1 > r_2 > r_1 \geq 0$. *It has been shown that the mixing inequality,* $s_1 \geq r_1(1 + x_1) + (r_2 - r_1)(1 + x_2)$ *is facet-defining for (26). If we substitute for* x_1 *and* x_2*, we obtain the inequality:*

$$\frac{f_2}{-f_1^2 - f_2^2 + f_1 f_2 + f_1} s_1 + \frac{-f_1 + 1}{-f_1^2 - f_2^2 + f_1 f_2 + f_1} s_2 + \frac{f_1 - f_2}{-f_1^2 - f_2^2 + f_1 f_2 + f_1} s_3 \geq 1 \tag{27}$$

where $f_1 = 1 - r_1$ *and* $f_2 = 1 - r_2$. *It can be verified that this inequality can be derived using (4) from the lattice-free triangle whose vertices are*

1. $v^1 := (c, f_2 - f_1 + c)$.
2. $v^2 := (\frac{cf_2 - f_1}{c + f_2 - f_1 - 1}, f_2)$.
3. $v^3 := (f_1, \frac{(c + f_2 - f_1)(f_1 - 1)}{(c-1)})$.

where $c = \frac{f_1 - f_1^2 + 2f_1 f_2 - f_2^2}{f_2}$. *The point* (f_1, f_2) *belongs to the interior of this triangle. It is also verifiable that the only integer points within each edge of the triangle are:* $(0, 0)$, $(1, 0)$ *and* $(1, 1)$.

In [24], the mixing inequalities were used for generating cutting planes for general simplex tableau by using a procedure equivalent to the trivial fill-in procedure. Proposition 43 indicates that the trivial fill-in procedure does not generate the best possible coefficients for all non-basic integer variables, and that these coefficients can be improved by use of a general fill-in procedure. □

In the next section we will present some conditions for generating extreme inequalities for $MI(I^2, S^2, r)$.

6.1.1 Some conditions for extremality of $\phi^{G,V}$

In Section 5, we presented the fill-in procedure for generating a valid function for $MI(I^2, S^2, r)$ starting from a valid function (V, π) for $MI(G, S^2, r)$. We begin this section by presenting a specific method of selecting G and V. This method is based on first lifting one integer variable corresponding to an element in G, and then obtaining the other coefficients by sequence-independent lifting.

For some $u \in I^2$, let

$$V(u) = \max_{n\in\mathbb{Z},n\geq 1}\left\{\frac{1-\pi(w)}{n} \,|\, \mathbb{P}(w) = r - nu\right\}. \tag{28}$$

Then $V(u)$ is the smallest value such that for all $x(u) \geq 1$ and integer, and $y : \mathbb{R}^2 \to \mathbb{R}$ satisfying $(ux(u) + \sum_{w\in\mathbb{R}^2} wy(y))(mod\bar{1}) \equiv r$, we have that $V(u)x(u) + \sum_{w\in\mathbb{R}^2} \pi(w)y(w) \geq 1$. $V(u)$ is therefore the *exact lifting coefficient* for the integer variable $x(u)$ and we call this step, "lifting the point u". Let G be the subgroup of I^2 generated by u. Once we obtain the lifting coefficient for $x(u)$, we propose to obtain $V(v)$, the coefficients for other $x(v)$s, where $v \in G$, $v \neq u$ in the following fashion,

$$V(v) = \min_{n\in\mathbb{Z}_+}\{nV(u) + \pi(w) \,|\, \mathbb{P}(w) = v - nu\}. \tag{29}$$

It is easily verified that V is subadditive. The validity of this function (similar to the fill-in function) follows thus: Let $(x, y) \in MI(G, S^2, r)$, then $\sum_{v\in G} V(v)x(v) + \sum_{w\in\mathbb{R}^2} \pi(w)y(w) \geq V(\sum_{v\in G} vx(v)) + \pi(\sum_{w\in\mathbb{R}^2} wy(w)) \geq \bar{n}V(u) + \pi(r - \bar{n}u) \geq 1$. (The next to last inequality follows from the definition of $V(v)$, $v \neq u$ and the last inequality follows from the definition of $V(u)$).

The exact lifting followed by the sequence-independent lifting described above does not guarantee an inequality (V, π) that is minimal for $MI(G, I^2, r)$. We next develop a slight variant of Theorem 41 for the case when G and V are chosen based on (28) and (29). The result of Proposition 46 does not assume that (V, π) is minimal for $MI(G, S^2, r)$.

Proposition 46 *Let u be the generator of the cyclic subgroup G. Define $V : G \to \mathbb{R}_+$ by first lifting u and then sequence-independent lifting the elements $G \setminus \{u\}$, as in (28) and (29). If π is extreme for $MI(\emptyset, S^2, r)$ and $(\phi^{G,V}, \pi)$ is minimal, then $(\phi^{G,V}, \pi)$ is extreme for $MI(I^2, S^2, r)$.*

Proof: Assume by contradiction that $(\phi^{G,V}, \pi)$ is not extreme for $MI(I^2, S^2, r)$. Then $(\phi^{G,V}, \pi) = \frac{1}{2}(\phi_1, \pi_1) + \frac{1}{2}(\phi_2, \pi_2)$ where (ϕ_i, π_i) are valid minimal functions and $(\phi_1, \pi_1) \neq (\phi_2, \pi_2)$. (Note that since $(\phi^{G,V}, \pi)$ is minimal, (ϕ_i, π_i) must be minimal).

First observe that $\pi_1 = \pi_2$ since π is extreme for $MI(\emptyset, S^2, r)$.

Next we claim that $\phi_1(u) = \phi_2(u) = V(u)$. Assume by contradiction that $\phi_1(u) \neq \phi_2(u)$. WLOG let $\phi_1(u) < V(u)$. By definition of $V(u)$, there exists $\bar{n} \in \mathbb{Z}_+, \bar{w} \in \mathbb{R}^2$ such that $\bar{n}V(u)+\pi(\bar{w}) = 1$ and $\bar{w}+\bar{n}u \equiv r$. This implies that $\exists(\bar{x}, \bar{y}) \in MI(I^2, S^2, r)$ where $\bar{x}(u) = \bar{n}$, $\bar{x}(v) = 0\ \forall v \in I^2, v \neq u$ and $y(\bar{w}) = 1$. Therefore, we obtain that $\phi_1(u)\bar{n} + \pi(\bar{w}) < 1$, giving us the required contradiction.

Finally, we claim that $\phi_1 = \phi_2 = \phi^{G,V}$. Note first that combining the definition of fill-in function with (29), we obtain that $\phi^{G,V}(v) = \inf_{n\in\mathbb{Z}_+}\{nV(u)+\pi(w) \,|\, nu+\mathbb{P}(w) = v\}\ \forall v \in I^2$. Suppose that $\phi_1(v) = \phi^{G,V}(v)+\delta$, where $\delta > 0$ (Since $\phi^{G,V}$ is minimal and $\phi_1 \neq \phi^{G,V}$, we must obtain this condition for some $v \in I^2$ and some $\delta > 0$). By definition of $\phi^{G,V}$, there exists n^v, w^v such that $\phi^{G,V}(v) \geq n^vV(u) + \pi(w^v) - \frac{\delta}{2}$. Since $\phi_1(u) = V(u)$, we obtain that $\phi_1(v) \geq \delta + n^v\phi_1(u) + \pi(w^v) - \frac{\delta}{2}$. Therefore, we obtain that $\phi_1(v) > n^v\phi_1(u) + \pi(w^v)$. This contradicts Theorem 39 as ϕ_1 is minimal. □

When G is a cyclic subgroup generated by u and V is defined as in (28) and (29), we denote $\phi^{G,V}$ by ϕ^u. (Note this nomenclature is consistent with the notation for the trivial fill-in function). As noted in the proof of Proposition 46,

$$\phi^u(v) = \inf_{n\in\mathbb{Z}_+}\{nV(u) + \pi(w) \,|\, \mathbb{P}(w) = v - nu\} \quad \forall v \in I^2 \setminus \{u\} \tag{30}$$

The main result of this section is Theorem 52 in which we present sufficient condition for $(\phi^{\bar{v}_0}, \pi)$ to be extreme for $MI(I^2, S^2, r)$ for a specific point $\bar{v}_0 \in I^2$. We begin with some definitions.

Definition 47 *(Refer to Figure 9.) Let $P(\pi)$ be a maximal lattice-free triangle with $(0,0)$, $(1,0)$, and $(0,1)$ in the interior of its sides. We use the following notation for the rest of this section:*

1. *The points a^1, a^2 and a^3 represent the vertices of the lattice-free triangle $P(\pi)$.*
2. *$b^1 := (1,0)$ is the integer point in the interior of the side a^1a^2.*
3. *$b^2 := (0,1)$ is the integer point in the interior of the side a^2a^3.*
4. *$b^3 := (0,0)$ is the integer point in the interior of the side a^3a^1.*

Let

$$\begin{aligned} D_{12}(\pi) &= \{\eta d^1 + \gamma d^2 \mid 0 \le \eta \le \delta^{12}, 0 \le \gamma \le (1-\delta^{12})\} \\ &\qquad (\textit{Quadrilateral } fc^1b^1e^1 - \{f\}) \\ D_{23}(\pi) &= \{\eta d^2 + \gamma d^3 \mid 0 \le \eta \le \delta^{23}, 0 \le \gamma \le (1-\delta^{23})\} \\ &\qquad (\textit{Quadrilateral } fc^2b^2e^2 - \{f\}) \\ D_{31}(\pi) &= \{\eta d^3 + \gamma d^1 \mid 0 \le \eta \le \delta^{31}, 0 \le \gamma \le (1-\delta^{31})\} \\ &\qquad (\textit{Quadrilateral } fc^3b^3e^3 - \{f\}) \end{aligned}$$

To describe $D(\pi) + \{f\}$, we need the following points,

1. *c^1: $f + \delta^{12}d^1$.*
2. *e^1: $f + (1-\delta^{12})d^2$.*
3. *c^2: $f + \delta^{23}d^2$.*
4. *e^2: $f + (1-\delta^{23})d^3$.*
5. *c^3: $f + \delta^{31}d^3$.*
6. *e^3: $f + (1-\delta^{31})d^1$.*

The set $D(\pi) + \{f\}$ is represented by the quadrilaterals: $fc^1b^1e^1$, $fc^2b^2e^2$, and $fc^3b^3e^3$. We use some other points in $\mathbb{R}^2$ which are described next:

1. *g: $f + (1-\delta^{12})d^2 + (\delta^{12} - 1 + \delta^{31})d^1$. (Note: $g = b^1 - e^3 + f$.)*
2. *i: $f + (1-\delta^{12})d^2 + (\delta^{12} - 1 + \delta^{31})d^1 - (\delta^{31} - 1 + \delta^{23})d^3$. (Note: $i = g - c^3 + e^2$.)*
3. *j: $f + \delta^{23}d^2 + (\delta^{12} - 1 + \delta^{31})d^1 - (\delta^{31} - 1 + \delta^{23})d^3$. (Note: $j = i - e^1 + c^2$.)*
4. *m: Mid point of i and j.*
5. *k: $f + \delta^{23}d^2 - (\delta^{31} - 1 + \delta^{23})d^3$. (Note: $k = j - g + e^1$.)*
6. *l: midpoint of e^1 and c^2.*
7. *u_0: $f + \left(\frac{1-\delta^{12}+\delta^{23}}{2}\right) d^2 + (\delta^{12} + \delta^{31} - 1)d^1$. (Note: $u_0 = g - i + m$.)*
8. *v_0: $f + \left(\frac{1-\delta^{12}+\delta^{23}}{2}\right) d^2 + (1 - \delta^{31} - \delta^{23})d^3$. (Note: $v_0 = k - m + i$.)*

It can be verified that $(u_0 - f) + (v_0 - f) + f = (1,1)$.

Proposition 48 *Let δ^{23}, δ^{31}, and δ^{12} be as defined above. Then $1-\delta^{23} < \delta^{31}$, $1-\delta^{31} < \delta^{12}$, and $1-\delta^{12} < \delta^{23}$.*

Proof. Refer to Figure 9. To prove this claim we need to show $|fc^3| > |fe^2|$, $|fc^1| > |fe^3|$, and $|fc^2| > |fe^1|$.

1. $|fc^3| > |fe^2|$: Since c^3b^3 is parallel to fa^1, we obtain that $\frac{|a^3c^3|}{|a^3f|} = \frac{|a^3b^3|}{|a^3a^1|}$. Similarly, $\frac{|a^3e^2|}{|a^3f|} = \frac{|a^3b^2|}{|a^3a^2|}$. Now observe that $\frac{|a^3b^2|}{|a^3a^2|} > \frac{|a^3b^3|}{|a^3a^1|}$ since b^2b^3 is vertical while a^2a^1 has a slope less that 0 (Proposition 11). Therefore, $|a^3e^2| > |a^3c^3|$.

2. $|fc^1| > |fe^3|$: Proof similar to previous case.

3. $|fc^2| > |fe^1|$: We have that $\frac{|a^2b^2|}{|a^2a^3|} = \frac{|a^2c^2|}{|a^2f|}$ and $\frac{|a^2b^1|}{|a^2a^1|} = \frac{|a^2e^1|}{|a^2f|}$. Now since slope of a^3a^1 is strictly less than -1 (Lemma 11) and the slope of b^2b^1 is -1, therefore $\frac{|a^2b^1|}{|a^2a^1|} > \frac{|a^2b^2|}{|a^2a^3|}$ which proves the result. $\square$

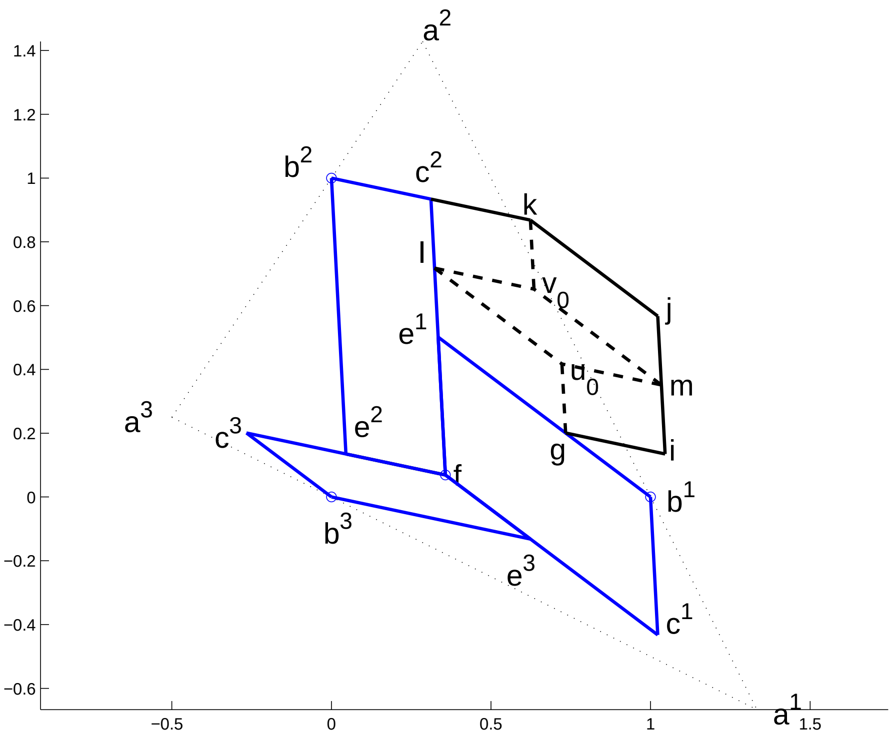

Figure 9: $T(\pi)$

Definition 49 *Let $T(\pi) \subset \mathbb{R}^2$ be the set $(D(\pi) + \{f\}) \cup$ the hexagon c^2e^1gijk.*

For any point $p \in \mathbb{R}^2$, we denote the point $p - f$ as $\bar{p}$. The next lemma records a crucial result for the proof of Theorem 52. The proof is given in Appendix 2.

Lemma 50 $\mathbb{P}(T(\pi)) = I^2$. $\square$

We use a variant of a theorem from Gomory and Johnson in proving the next result.

Theorem 51 ([20]) *If $\phi : I^2 \to \mathbb{R}_+$ is a valid function for $MI(I^2, \emptyset, r)$ and if $\phi(u) + \phi(r-u) \le 1$ $\forall u \in I^2$, then ϕ is subadditive.*

Throughout the proof of the following theorem, we use the fact that $(1,0) = f + \delta^{12}d^1 + (1-\delta^{12})d^2$, $(0,1) = f + \delta^{23}d^2 + (1-\delta^{23})d^3$, and $(0,0) = f + \delta^{31}d^3 + (1-\delta^{31})d^1$.

Theorem 52 *If $V(\mathbb{P}(\bar{v}_0)) = 1 - \pi(\bar{u}_0)$, then $(\phi^{\mathbb{P}(\bar{v}_0)}, \pi)$ is an extreme valid inequality for $MI(I^2, S^2, r)$.*

Proof. Since π is extreme for $MI(\emptyset, S^2, r)$, using Proposition 46, we need to show that $(\phi^{\mathbb{P}(\bar{v}_0)}, \pi)$ is minimal for $MI(I^2, S^2, r)$ to prove this result.

Using (30) we know that

$$\phi^{\mathbb{P}(\bar{v}_0)}(u) = \inf_n \{nV(\mathbb{P}(\bar{v}_0)) + \pi(w) \,|\, u = \mathbb{P}(\bar{v}_0 + w)\} \quad \forall u \in I^2. \tag{31}$$

To prove that $\phi^{\mathbb{P}(\bar{v}_0)}$ is minimal, we need to verify the conditions of Theorem 16. However, using directly the definition of $(\phi^{\mathbb{P}(\bar{v}_0)}, \pi)$ is not convenient; as for any $u \in I^2$, we cannot be sure of the value of $\bar{n} \in \mathbb{Z}_+$, $\bar{w} \in \mathbb{R}^2$, such that $\bar{n}\mathbb{P}(\bar{v}_0) + \mathbb{P}(\bar{w}) = u$ and $\phi^{\mathbb{P}(\bar{v}_0)}(u) = \bar{n}V(\mathbb{P}(\bar{v}_0)) + \pi(\bar{w})$.

Therefore, instead of working with $\phi^{\mathbb{P}(\bar{v}_0)}$, we prove this result by creating an upper bound $\tilde{\phi}$ on $\phi^{\mathbb{P}(\bar{v}_0)}$. We then show that $\tilde{\phi}$ satisfies the symmetry conditions, i.e., $\tilde{\phi}(u) + \tilde{\phi}(r-u) \le 1$ $\forall u \in I^2$. Since $\phi^{\mathbb{P}(\bar{v}_0)}$ is a valid function by Proposition 36 and $\tilde{\phi}$ is an upper bound, $\tilde{\phi}$ is a valid function. Now using Theorem 51 we will obtain that $\tilde{\phi}$ must be subadditive. It will also be verified that $\tilde{\phi}$ satisfies $\lim_{h\to 0^+} \frac{\tilde{\phi}(\mathbb{P}(wh))}{h} = \pi(w)$ $\forall w \in S^2$ and $\tilde{\phi}(r) = 1$. This will show that $(\tilde{\phi}, \pi)$ is minimal for $MI(I^2, S^2, r)$. This will imply that $\tilde{\phi}$ is the same function as $\phi^{\mathbb{P}(\bar{v}_0)}$, thus completing the proof.

The proof has two main steps. Step one involves creating the function $\tilde{\phi} : I^2 \to \mathbb{R}_+$ and showing that this function is an upper bound on the function $\phi^{\mathbb{P}(\bar{v}_0)}$. Step two involves proving that $\tilde{\phi}(u) + \tilde{\phi}(r-u) \le 1$ $\forall u \in I^2$, $\lim_{h\to 0^+} \frac{\tilde{\phi}(\mathbb{P}(wh))}{h} = \pi(w)$ $\forall w \in S^2$, and $\tilde{\phi}(r) = 1$.

<u>Step 1</u>: To define the function $\tilde{\phi}$, we first define a function $\phi_1 : T(\pi) \to \mathbb{R}_+$. By Lemma 50, we know that $\mathbb{P}(T(\pi)) = I^2$. This allows us to define $\tilde{\phi} : I^2 \to \mathbb{R}_+$ as:

$$\tilde{\phi}(u) = \min\{\phi_1(w) | \mathbb{P}(\bar{w}) = u\}. \tag{32}$$

We next present the function ϕ_1. Refer to figure 9. We use the symbols Q^{11}, and Q^{21} to represent the quadrilaterals $e^1 g u_0 l$ and $gimu_0$ respectively.

$$\phi_1(u) = \begin{cases} \pi(u-f) & \text{if } u \in (D(\pi) + \{f\}) \cup Q^{11} \\ \pi(u-(1,0)-f) & \text{if } u \in Q^{21} \\ V(\mathbb{P}(\bar{v}_0)) + \pi(u-v_0) & \text{otherwise.} \end{cases} \tag{33}$$

<u>Claim:</u> ϕ_1 is well-defined, i.e., we check if $\phi_1(u)$ has the same value if u belongs to different categories in (33). Since by assumption $V(\mathbb{P}(\bar{v}_0)) = 1 - \pi(\bar{u}_0)$ and $\pi(\bar{u}_0) = \pi\left(\left(\frac{1-\delta^{12}+\delta^{23}}{2}\right)d^2 + (\delta^{12}+\delta^{31}-1)d^1\right) = \frac{\delta^{12}+\delta^{23}-1}{2} + \delta^{31}$, we obtain $V(\mathbb{P}(\bar{v}_0)) = \frac{3}{2} - \frac{\delta^{12}}{2} - \frac{\delta^{23}}{2} - \delta^{31}$.

1. u belongs to the line segment $c^2 l$: It is easily verified that the function is linear (both the first and third case) over this interval. Therefore it is enough to check the value of the function ϕ_1 at $u = c^2$ and $u = l$.
   - $u = c^2$: From the first case in (33), $\phi_1(c^2) = \pi(\bar{c^2}) = \pi(\delta^{23}d^2) = \delta^{23}$. From the third case in (33), $\phi_1(c^2) = V(\mathbb{P}(\bar{v}_0)) + \pi(c^2 - v_0)$, or

$$\begin{aligned} \phi_1(c^2) &= \frac{3}{2} - \frac{\delta^{12}}{2} - \frac{\delta^{23}}{2} - \delta^{31} \\ &\quad + \pi\left(\left(\frac{-1+\delta^{12}+\delta^{23}}{2}\right)d^2 + (\delta^{31}+\delta^{23}-1)d^3\right) \\ &= \delta^{23}. \end{aligned}$$

- $u=l$: From the first case in (33), $\phi_1(l)=\pi(\bar{l})=\pi(\frac{(\delta^{23}-\delta^{12}+1)}{2}d^2)=\frac{\delta^{23}-\delta^{12}+1}{2}$. From the third case in (33), $\phi_1(l)=V(\mathbb{P}(\bar{v_0}))+\pi(l-v_0)$, or

$$\begin{aligned}\phi_1(l) &= \frac{3}{2}-\frac{\delta^{12}}{2}-\frac{\delta^{23}}{2}-\delta^{31}+\pi((\delta^{31}+\delta^{23}-1)d^3)\\ &= \frac{\delta^{23}-\delta^{12}+1}{2}.\end{aligned}$$

2. u belongs to the line segment lu_0: It is easily verified that the function is linear (both the first and third case) over this interval. Therefore it is enough to check the value of the function ϕ_1 at $u=l$ and $u=u_0$.

   - $u=l$: Verified.
   - $u=u_0$: From the first case in (33), $\phi_1(u_0)=\pi(\bar{u_0})=\frac{\delta^{12}+\delta^{23}-1}{2}+\delta^{31}$. From the third case in (33), $\phi_1(u_0)=V(\mathbb{P}(\bar{v_0}))+\pi(u_0-v_0)$.

$$\begin{aligned}\phi_1(u_0) &= \frac{3}{2}-\frac{\delta^{12}}{2}-\frac{\delta^{23}}{2}-\delta^{31}\\ &\quad+\pi((\delta^{12}+\delta^{31}-1)d^1+(\delta^{23}+\delta^{31}-1)d^3)\\ &= \frac{\delta^{12}+\delta^{23}-1}{2}+\delta^{31}.\end{aligned}$$

3. u belongs to the line segment u_0g: It is easily verified that the function is linear (both the first and second case) over this interval.

   - $u=u_0$: From the first case in (33), $\phi_1(u_0)=\pi(\bar{u_0})=\frac{\delta^{12}+\delta^{23}-1}{2}+\delta^{31}$. From the second case in (33), $\phi_1(u_0)=\pi(u_0-(1,0)-f)$. It is easily verified that $(1,0)=(1,0)-(0,0)=(\delta^{12}+\delta^{31}-1)d^1+(1-\delta^{12})d^2-\delta^{31}d^3$. Therefore, $\phi_1(u_0)=\pi(u_0-(1,0)-f)=\pi((\frac{\delta^{12}+\delta^{23}-1}{2})d^2+\delta^{31}d^3)=\frac{\delta^{12}+\delta^{23}-1}{2}+\delta^{31}$.
   - $u=g$: From the first case in (33), $\phi_1(g)=\pi(\bar{g})=(\delta^{31}+\delta^{12}-1)d^1+(1-\delta^{12})d^2=\delta^{31}$. From the second case in (33), $\phi_1(g)=\pi(g-(1,0)-f)=\pi(\delta^{31}d^3)=\delta^{31}$.

4. u belongs to the line segment u_0m: It is easily verified that the function is linear (both the second and third case) over this interval. Therefore, it is enough to check the value of the function ϕ_1 at $u=u_0$ and $u=m$.

   - $u=u_0$: Verified.
   - $u=m$: From the second case in (33), we obtain that $\phi_1(m)=\pi(m-f-(1,0))$. Therefore $\phi_1(m)=\pi(\left(\frac{1-\delta^{12}+\delta^{23}}{2}\right)d^2+(\delta^{12}+\delta^{31}-1)d^1+(1-\delta^{31}-\delta^{23})d^3)-(\delta^{12}+\delta^{31}-1)d^1-(1-\delta^{12})d^2+\delta^{31}d^3=\pi(\frac{-1+\delta^{12}+\delta^{23}}{2}d^2+(1-\delta^{23})d^3)=\frac{1+\delta^{12}-\delta^{23}}{2}$. From the third case in (33), $\phi_1(m)=V(\mathbb{P}(\bar{v_0}))+\pi(m-v_0)$,

$$\begin{aligned}\phi_1(m) &= \frac{3}{2}-\frac{\delta^{12}}{2}-\frac{\delta^{23}}{2}-\delta^{31}+\pi((\delta^{12}+\delta^{31}-1)d^1)\\ &= \frac{1+\delta^{12}-\delta^{23}}{2}.\end{aligned}$$

Finally, we verify that $\tilde{\phi}$ is an upper bound on ϕ^{v_0}: This follows from the definition of $\phi^{\mathbb{P}(\bar{v}_0)}(u)=\inf_{n\in\mathbb{Z}_+}\{nV(\mathbb{P}(\bar{v}_0))+\pi(w)\,|\,\mathbb{P}(w)+n\mathbb{P}(\bar{v}_0)=u\}$. Now this claim easily follows from (33) and (32).

Step 2:

- $\lim_{h\to 0^+} \frac{\tilde{\phi}(\mathbb{P}(wh))}{h} = \pi(w)\ \forall w \in S^2$: This follows from the fact that $\tilde{\phi}(u) = \phi^{\bar{0}}(u)\ \forall u \in D(\pi)$ and Corollary 24.

- $\tilde{\phi}(r) = 1$: We know that $\tilde{\phi}$ is an upper bound to $\phi^{\mathbb{P}(\bar{v}_0)}$. Therefore, $\tilde{\phi}(r) \geq 1$. Moreover we have that $\tilde{\phi}(r) \leq \phi_1(\bar{b_1}) = 1$.

- Finally we show that $\tilde{\phi}(u) + \tilde{\phi}(r-u) \leq 1\ \forall u \in I^2$: For $u \in I^2$, we call $r - u \in I^2$ the complementary point. By the definition of ϕ_1 and $\tilde{\phi}$, it is easily verified that $\tilde{\phi}(u) + \tilde{\phi}(r-u) \leq 1\ \forall u \in D(\pi)$. We now present some key complementary points:

  1. Complement of $\mathbb{P}(\bar{e^1})$ is $\mathbb{P}(\bar{j})$: $\bar{e}^1 + \bar{j} + f = e^1 + j - f = e^1 + (i - e^1 + c^2) - f = i + c^2 - f = (g - c^3 + e^2) + c^2 - f = b^1 - e^3 + f - c^3 + e^2 + c^2 - f = b^1 + (-e^3 - c^3 - f) + (e^2 + c^2 + f) = (1, 1)$. Therefore, $\mathbb{P}(\bar{e^1}) + \mathbb{P}(\bar{j}) = -f = r$.
  2. Complement of $\mathbb{P}(\bar{g})$ is $\mathbb{P}(\bar{k})$: $\bar{g} + \bar{k} + f = \bar{e}^1 + \bar{j} + f = (1, 1)$.
  3. Complement of $\mathbb{P}(\bar{u_0})$ is $\mathbb{P}(\bar{v}_0)$: $\bar{u}_0 + \bar{v}_0 + f = \bar{g} + \bar{k} + f = (1, 1)$.
  4. Complement of $\mathbb{P}(\bar{l})$ is $\mathbb{P}(\bar{m})$: $\bar{l} + \bar{m} + f = \bar{e}^1 + \bar{j} + f = (1, 1)$.
  5. Complement of $\mathbb{P}(\bar{i})$ is $\mathbb{P}(\bar{c^2})$: $\bar{i} + \bar{c}^2 + f = \bar{l} + \bar{m} + f = (1, 1)$.

  Note that the function ϕ_1 is linear in each of the following quadrilaterals: kv_0mj, kv_0lc^2, lv_0mu_0, le^1gu_0, and u_0gim. Therefore to prove that $\tilde{\phi}(u) + \tilde{\phi}(r-u) \leq 1\ \forall u \in I^2 \setminus \mathbb{P}(D(\pi))$, it is enough to check the following five cases:

  1. $\tilde{\phi}(\mathbb{P}(\bar{e^1})) + \tilde{\phi}(\mathbb{P}(\bar{j})) \leq 1$: $\tilde{\phi}(\mathbb{P}(\bar{e^1})) \leq \phi_1(e^1) = 1 - \delta^{12}$. $\tilde{\phi}(\mathbb{P}(\bar{j})) \leq \phi_1(j) = V(\mathbb{P}(\bar{v}_0)) + \pi(j - v_0) = \frac{3}{2} - \frac{\delta^{12}}{2} - \frac{\delta^{23}}{2} - \delta^{31} + \pi((\delta^{12} + \delta^{31} - 1)d^1 + (\frac{\delta^{23}+\delta^{12}-1}{2})d^2) = \delta^{12}$. Therefore, $\tilde{\phi}(\mathbb{P}(\bar{e^1})) + \tilde{\phi}(\mathbb{P}(\bar{j})) \leq 1$.
  2. $\tilde{\phi}(\mathbb{P}(\bar{g})) + \tilde{\phi}(\mathbb{P}(\bar{k})) \leq 1$: $\tilde{\phi}(\mathbb{P}(\bar{g})) \leq \phi_1(g) = \pi(\bar{g}) = \delta^{31}$. $\tilde{\phi}(\mathbb{P}(\bar{k})) \leq \phi_1(k) = V(\mathbb{P}(\bar{v}_0)) + \pi(\frac{\delta^{23}+\delta^{12}-1}{2}d^2) = 1 - \delta^{31}$.
  3. $\tilde{\phi}(\mathbb{P}(\bar{u_0})) + \tilde{\phi}(\mathbb{P}(\bar{v}_0)) \leq 1$: $\tilde{\phi}(\mathbb{P}(\bar{u_0})) + \tilde{\phi}(\mathbb{P}(\bar{v}_0)) \leq \tilde{\phi}(\mathbb{P}(\bar{v}_0)) + \phi_1(u_0) \leq 1$.
  4. $\tilde{\phi}(\mathbb{P}(\bar{l})) + \tilde{\phi}(\mathbb{P}(\bar{m})) \leq 1$: $\tilde{\phi}(\mathbb{P}(\bar{l})) \leq \phi_1(l) = \frac{\delta^{23}+1-\delta^{12}}{2}$. $\tilde{\phi}(\mathbb{P}(\bar{m})) \leq \phi_1(m) = \frac{1+\delta^{12}-\delta^{23}}{2}$.
  5. $\tilde{\phi}(\mathbb{P}(\bar{i})) + \tilde{\phi}(\mathbb{P}(\bar{c^2})) \leq 1$: $\tilde{\phi}(\mathbb{P}(\bar{c^2})) \leq \phi_1(c^2) = \delta^{23}$. $\tilde{\phi}(\mathbb{P}(\bar{i})) \leq \phi_1(i) = \pi(i - (1, 0) - f) = \pi((1 - \delta^{23})d^3) = 1 - \delta^{23}$. □

As an example, note that the function ϕ_1 illustrated in Figure 8 is the function $\phi^{\mathbb{P}(\bar{v}_0)}$.

6.2 $P(\pi)$ is a quadrilateral

In this section, we consider a set $P(\pi)$ which is a quadrilateral. We will prove that the trivial fill-in procedure does not generate extreme inequalities for $MI(I^2, S^2, r)$ in this case. We begin with a variant of Lemma 10.

Proposition 53 *Let $P(\pi)$ be a maximal lattice-free quadrilateral. Then there exists a unimodular matrix M and $v \in \mathbb{Z}^2$ such that the set $\{x \in \mathbb{R}^2 \,|\, x = M(u - v), u \in P\}$ is a maximal lattice-free quadrilateral with the points $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ in the interior of its four sides.*

Proof: Same as Proof of Proposition 26. □

Notation: (Refer to Figure 10.) Let $P(\pi)$ be a maximal lattice-free quadrilateral with the points $(1, 0)$, $(1, 1)$, $(0, 1)$, and $(0, 0)$ in the interior of its sides. We use the following notation for points in this section:

1. The points a^1, a^2, a^3, and a^4 represent the vertices of the lattice-free quadrilateral $P(\pi)$.

2. $b^1 := (1, 1)$ is the integer point in the interior of the side a^1a^2.

3. $b^2 := (0,1)$ is the integer point in the interior of the side a^2a^3.

4. $b^3 := (0,0)$ is the integer point in the interior of the side a^3a^4.

5. $b^4 := (1,0)$ is the integer point in the interior of the side a^4a^1.

The following result is from Cornuéjols and Margot [10].

Theorem 54 ([10]) *π is extreme for $MI(\emptyset, S^2, r)$ iff there does not exist an $h \in \mathbb{R}_+$ such that*

$$\frac{|b^i - a^i|}{|b^i - a^{i+1}|} = \begin{cases} h & \text{if } i = 1,3 \\ \frac{1}{h} & \text{if } i = 2,4. \end{cases} \tag{34}$$

□

Therefore the interesting case for analysis is the case when there exists no such h.

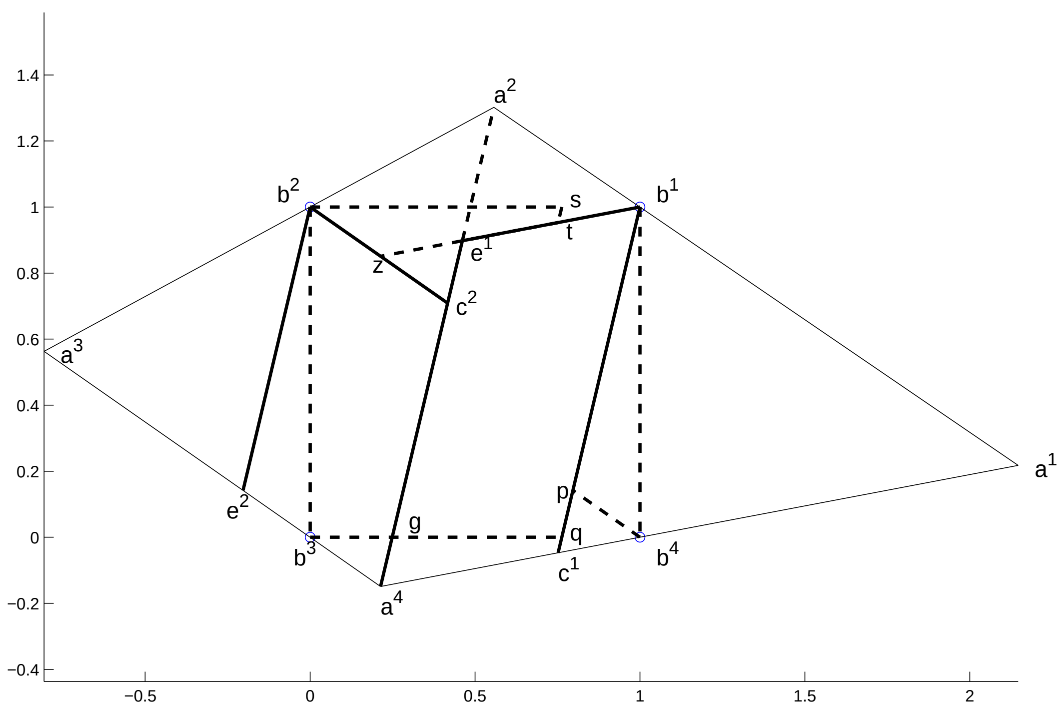

Figure 10: The maximum area of $D(\pi)$ when $P(\pi)$ is a maximal lattice-free quadrilateral

Proposition 55 *Let $P(\pi)$ be maximal lattice-free quadrilateral. If there exists no $h \in \mathbb{R}_+$ satisfying (34), then $Area(D(\pi)) < 1$.*

Proof. Using Propositions 53 and 17 it is enough to analyze the case when $P(\pi)$ is a quadrilateral with the points $(1,0)$, $(1,1)$, $(0,1)$, $(0,0)$ in the interior of its sides. Let s_1, s_2, s_3, s_4 be the sides of the quadrilateral which the points $(1,1)$, $(0,1)$, $(0,0)$, and $(1,0)$ respectively. Let $-m_1$, m_2, $-m_3$, and m_4 be the slopes of the sides s_1, s_2, s_3, s_4 respectively. Let a^1, a^2, a^3, and a^4 be the vertex between s_4 and s_1, s_1 and s_2, s_2 and s_3, and s_3 and s_4 respectively.

First note that it can be verified that $0 < m_1, m_2, m_3, m_4 < \infty$. (If $m_1 = 0$, then b^2 becomes a vertex of $P(\pi)$, a contradiction. The other conditions can be proven similarly). Refer to Figure 10. Note that $a^1_2 = a^3_2$ and $a^2_1 = a^4_1$ iff $m_1 m_3 = m_2 m_4$ since the vertices of $P(\pi)$ are $a^1 := \left(\frac{m_4+1+m_1}{m_4+m_1}, \frac{m_4}{m_4+m_1}\right)$, $a^2 := \left(\frac{m_1}{m_2+m_1}, \frac{m_1}{m_2+m_1}\right)$, $a^3 := \left(\frac{-1}{m_2+m_3}, \frac{m_3}{m_2+m_3}\right)$, and $a^4 := \left(\frac{m_4}{m_4+m_3}, \frac{-m_4 m_3}{m_4+m_3}\right)$.

It can also be verified that the condition $m_1 m_3 = m_2 m_4$ holds iff there exists $h \in \mathbb{R}_+$ satisfying (34). Therefore, we have that $a^1_2 \neq a^3_2$ and $a^2_1 \neq a^4_1$.

Note that since there is a unique integer point in the interior of each side, using Proposition 42, we know that $D(\pi)$ has a maximum area when f is one of the vertices of $P(\pi)$. We show next that if f is the same as a^4 then $\text{Area}(D(\pi)) < 1$. The case when f is the same as a^1, a^2 or a^3 can be proven similarly.

When f is the same as the point a^4, $D(\pi) + \{f\}$ is represented by the two parallelograms $e^2b^2c^2a^4$ and $e^1b^1c^1a^4$. (Here we have assumed that the slope of a^2a^4 is positive. If the slope of a^2a^4 is negative a similar proof can be given. Note that a^2a^4 is vertical iff $m_1m_3 = m_2m_4$). Let pb^4 be parallel to e^2b^3. Therefore $p_2 > 0$ as the slope of s_3 is negative. Let b^3b^4 intersect a^2a^4 at g and b^1c^1 at q. $q_1 < 1$ since a^2a^4 has a positive slope.

Let $s = q + (0,1)$, $z = a^4 + (0,1)$, and $t = c^1 + (0,1)$.

Since the slope of a^2a^4 is positive, z lies to the left of the line a^2a^4. Moreover since b^2b^3 is parallel and equal in length to za^4, we have that $b^2b^3a^4z$ is a parallelogram. Therefore z lies in the interior of the line segment b^2c^2. Similarly, e^1 and t lie in the interior of the line segment b^1z. Also it can be proven that c^2 lies in the interior of a^4e^1. Therefore, $\text{Area}(ze^1c^2) > 0$.

Next observe that $\text{Area}(b^2stz) = \text{Area}(b^3qc^1a^4)$ since $b^2stz = b^3qc^1a^4 + (0,1)$.

The line segment b^2e^2 is parallel to b^1p, as both are parallel to a^2a^4. Also pb^4 is parallel to e^2b^3 by construction. Moreover, $|b^2b^3| = |b^1b^4| = 1$. Therefore, the triangle $b^2e^2b^3$ is symmetric to b^1b^4p implying that $\text{Area}(e^2b^2b^3) = \text{Area}(pb^1b^4)$.

Finally observe that $\text{Area}(D(\pi)) = \text{Area}(e^2b^2c^2a^4) + \text{Area}(e^1b^1c^1a^4) = \text{Area}(b^2b^3gc^2) + \text{Area}(b^3ga^4) + \text{Area}(b^2b^3e^2) + \text{Area}(b^1qge^1) + \text{Area}(qc^1a^4g) = \text{Area}(b^2b^3gc^2) + \text{Area}(b^2b^3e^2) + \text{Area}(b^1qge^1) + \text{Area}(b^3qc^1a^4)$ $= \text{Area}(b^2b^3gc^2) + \text{Area}(b^1b^4p) + \text{Area}(b^1qge^1) + \text{Area}(b^2stz) = \text{Area}(b^1b^2b^3b^4) - \text{Area}(stb^1) - \text{Area}(ze^1c^2) - \text{Area}(pqb^4)$. Now since we have that $\text{Area}(stb^1) > 0$, $\text{Area}(ze^1c^2) > 0$, and $\text{Area}(pqb^4) > 0$, it follows that $\text{Area}(D(\pi)) < 1$. □

The following corollary is easily verified.

Corollary 56 *Let π be an extreme inequality corresponding to a maximal lattice-free quadrilateral $P(\pi)$. Then $(\phi^{\bar{0}}, \pi)$ is not extreme for $MI(I^2, S^2, r)$.* □

7 Example of cutting planes from two-row of a simplex tableau

The main aim of this paper has been to study the characteristics of lifting functions for integer variables derived from two rows of a general simplex tableau. Here we sketch a procedure to generate cutting planes for two rows of a simplex tableau using the results presented earlier.

Assume that we have two rows of a simplex tableau with n_1 nonbasic integer variables, n_2 nonbasic continuous variables and x_{B_1}, x_{B_2} are basic variables that are required to be integer.

$$
\begin{aligned}
x_{B_1} + \sum_{i=1}^{n_1} a^i_1 x_i + \sum_{j=1}^{n_2} b^j_1 y_j &= r_1 \\
x_{B_2} + \sum_{i=1}^{n_1} a^i_2 x_i + \sum_{j=1}^{n_2} b^j_2 y_j &= r_2 \\
x_B \in \mathbb{Z}^2, \quad x \in \mathbb{Z}^{n_1}_+, \quad y \in \mathbb{R}^{n_2}_+ & \qquad (35)
\end{aligned}
$$

Apply the following steps:

1. Fix the non-basic integer variables to zero.
2. Select three (four for a quadrilateral inequality) continuous variables y_{j_1}, y_{j_2}, and y_{j_3} such that the positive combination of b^{j_1}, b^{j_1}, and b^{j_3} spans $\mathbb{R}^2$.
3. Find a maximal lattice-free triangle $P(\pi)$ such that the inequality π is extreme for the problem,
$$x_B + b^{j_1}y_{j_1} + b^{j_2}y_{j_2} + b^{j_3}y_{j_3} = r, \quad x_B \in \mathbb{Z}^2, y_{j_1}, y_{j_2}, y_{j_3} \geq 0.$$
4. Lift the other continuous variables, i.e., use the function π to generate the coefficients for the other continuous variables.
5. Lift the non-basic integer variables into this cut.
   - If $P(\pi)$ is a triangle with multiple integer points in the interior of one side or a triangle with integral vertices and one integer point in the interior of each side, then use the trivial fill-in function to lift all the integer variables.
   - If $P(\pi)$ is a triangle with a single integer point in the interior of each side and non-integral vertices (or a quadrilateral), then do the following: Select an integer variable x_i corresponding to column a^i where $u^i = \mathbb{P}(a^i)$. Find the value of the term $\phi^{\bar{0}}(u^i) + \phi^{\bar{0}}(r - u^i)$. If $\phi^{\bar{0}}(u^i) + \phi^{\bar{0}}(r - u^i) = 1$, then $u^i \in \mathbb{P}(D(\pi))$. (This can be verified based on Propositions 23 and 33). $\phi^{\bar{0}}(u^i)$ is the best possible coefficient. Denote the set of variables such that $\phi^{\bar{0}}(u^i) + \phi^{\bar{0}}(r - u^i) = 1$ by N_T. For the other variables with $\phi^{\bar{0}}(u^i) + \phi^{\bar{0}}(r - u^i) > 1$, try to improve upon the coefficient obtained using the trivial fill-in function. Let N_L be the list of such variables. Lift a subset N_I of these variables using traditional lifting giving an inequality of the form
$$\sum_{i\in N_T} \phi^{\bar{0}}(u^i)x_i + \sum_{i\in N_I} V_i x_i + \sum_{j=1}^{n_2} \pi(b^j)y_j \geq 1. \tag{36}$$
     (V_is are the coefficients obtained using lifting.) Finally, apply either the general fill-in function (by computing a valid function $V : G \to \mathbb{R}_+$, with G the subgroup generated by the elements in N_I), or a function similar to (29) to obtain the coefficients for the integer variables in the set $N_L \setminus N_I$, giving the inequality
$$\sum_{i\in N_T} \phi^{\bar{0}}(u^i)x_i + \sum_{i\in N_I} V_i x_i + \sum_{i\in N_L\setminus N_I} \phi(u^i)x_i + \sum_{j=1}^{n_2} \pi(b^j)y_j \geq 1, \tag{37}$$
     where ϕ is computed as
$$\phi(v) = \min_{n_i\in\mathbb{Z}_+}\{\sum_{i\in N_I} n_i V_i + \pi(w) \,|\, \mathbb{P}(w) = v - \sum_{i\in I} n_i u^i\}. \tag{38}$$
     It is easily verified that this function is valid. However, this function may not be minimal in all cases.

   As shown in Proposition 37, the value of the trivial fill-in function can be found by evaluating the value of the function π at a finite number of points; this number is bounded by the square of the inverse of the smallest gradient of π (or equivalently the length of the longest extreme ray of $P(\pi)$ i.e., $\max\{|a^i f| \,|\, a^i$ is a vertex of $P(\pi)\}$) times $\sup\{\pi(w) \,|\, w \in [0,1) \times [1,0)\}$. On the other hand, since the fill-in function is calculated via a minimization problem and the cut obtained has the $\geq$ symbol, it is not necessary to solve the fill-in process to optimality to obtain a valid cutting plane (unlike the traditional lifting process).

We next present an example illustrating some of the steps outlined above.

Example 57 *Consider the following instance:*

$$\begin{aligned}
&\begin{pmatrix} 1 \\ 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} y_2 + \begin{pmatrix} -1 \\ -2 \end{pmatrix} y_3 + \begin{pmatrix} -3 \\ -7 \end{pmatrix} y_4 + \begin{pmatrix} -4/5 \\ 6/5 \end{pmatrix} x_1 \\
&+ \begin{pmatrix} 19/10 \\ 23/10 \end{pmatrix} x_2 + \begin{pmatrix} 3/10 \\ -7/5 \end{pmatrix} x_3 + \begin{pmatrix} -2/3 \\ 11/6 \end{pmatrix} x_4 + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_{B_1} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_{B_2} \\
&x_B \in \mathbb{Z}^2, \quad x \in \mathbb{Z}^4_+, \quad y \in \mathbb{R}^4_+
\end{aligned} \tag{39}$$

- *Choose three continuous variables: y_1, y_2, y_3. A maximal lattice-free triangle generating a facet for*

$$\begin{aligned}
&\begin{pmatrix} 1 \\ 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} y_2 + \begin{pmatrix} -1 \\ -2 \end{pmatrix} y_3 + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_{B_1} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_{B_2} \\
&x_B \in \mathbb{Z}^2, \quad y_1, y_2, y_3 \in \mathbb{R}_+,
\end{aligned} \tag{40}$$

  is given by the vertices: $(1.5, 0.5)$, $(0.5, 1.5)$, and $(-0.5, -1.5)$. This triangle $P(\pi)$ is illustrated in Figure 11. Given $P(\pi)$, calculate $\pi(-3, -7) = 4$. Note now that $P(\pi)$ has multiple integer points in the interior of one side. Therefore, it is enough to use the trivial fill-in procedure to lift the integer variables. We now illustrate the computation of $\phi^{\bar{0}}(\mathbb{P}(-4/5, 6/5))$. Since, in the case of this $P(\pi)$, there exists, $w \in D(\pi)$ such that $\mathbb{P}(w) = \mathbb{P}(-4/5, 6/5)$, and $D(\pi) \subset P(\pi) \subset \{(w_1, w_2) \,|\, -0.5 \leq w_1 \leq 1.5, -1.5 \leq w_2 \leq 1.5\}$ we do the following:

$$\phi^{\bar{0}}(0.2, 0.2) = min \begin{cases} \pi(0.2, -0.8) = 1 \\ \pi(0.2, 0.2) = 0.4 \\ \pi(0.2, 1.2) = 1.4 \\ \pi(1.2, -0.8) = 2 \\ \pi(1.2, 0.2) = 1.4 \\ \pi(1.2, 1.2) = 2.4 \end{cases} \tag{41}$$

  By computing the trivial fill-in function for the other integer variables, we obtain the inequality $y_1 + y_2 + y_3 + 4y_4 + 0.4x_1 + 0.6x_2 + 0.7x_3 + 0.5x_4 \geq 1$.

- *Now choose the following three continuous variables: y_1, y_2, y_4. A maximal lattice-free triangle generating a facet for*

$$\begin{aligned}
&\begin{pmatrix} 1 \\ 0 \end{pmatrix} y_1 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} y_2 + \begin{pmatrix} -3 \\ -7 \end{pmatrix} y_4 + \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_{B_1} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} x_{B_2} \\
&x_B \in \mathbb{Z}^2, \quad y_1, y_2, y_4 \in \mathbb{R}_+,
\end{aligned} \tag{42}$$

  is given by the vertices: $(7/10)*(1,0)+(0.5,0.5)$, $(7/4)*(0,1)+(0.5,0.5)$, and $(7/26)*(-3,-7)+(0.5,0.5)$. This triangle is illustrated in Figure 12.

  Given $P(\pi)$, one can check that $\pi(-1, -2) = 7/15$. Note now that as $P(\pi)$ has one integer point in the interior of each side and the vertices of $P(\pi)$ are non-integral, we need to check whether the trivial fill-in function is sufficient to obtain strong coefficients. To compute the trivial-fill-in function in this case note that $l = \sqrt{(3^2 + 7^2)}.max\{\pi(w) \,|\, w \in [0,1] \times [0,1]\} \approx 13.9$. (Notation of Proposition 37). Therefore, $\phi^{\bar{0}}(u_1, u_2) = Min_{0 \leq |n_1|, |n_2| \leq 14} \pi(u_1 + n_1, u_2 + n_2)$.

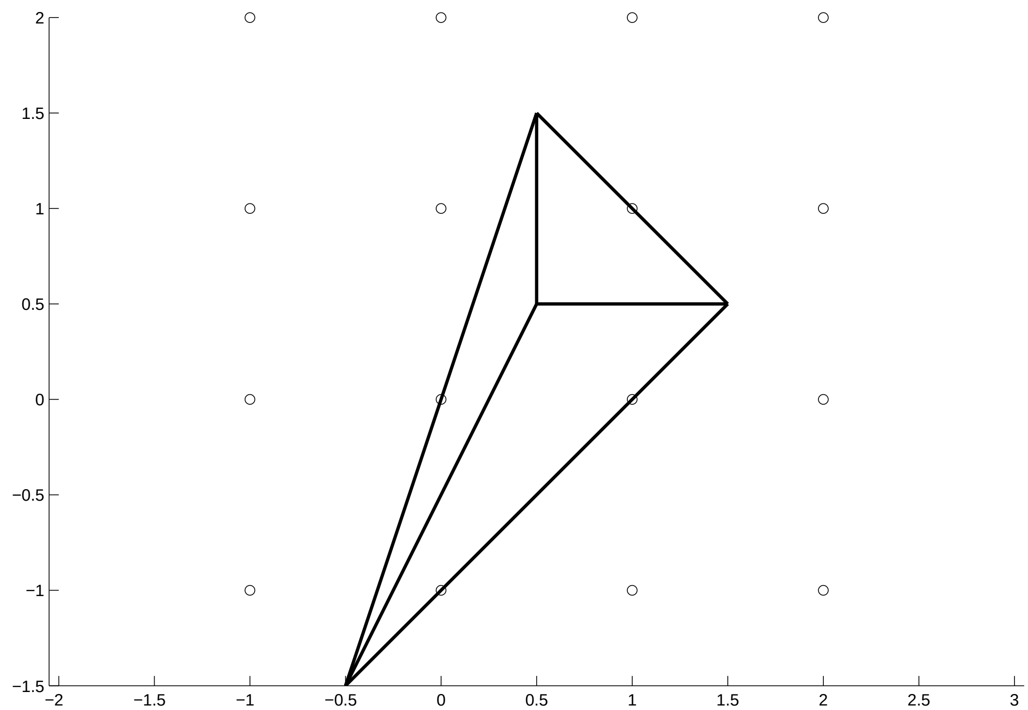

Figure 11: $P(\pi)$ to generate facet for (40)

1. *x_1: $u^1 = \mathbb{P}((-4/5, 6/5)) = (0.2, 0.2)$. Now $\phi^{\bar{0}}(0.2, 0.2) + \phi^{\bar{0}}(0.3, 0.3) = 1$ implying that $\phi^{\bar{0}}(0.2, 0.2) = 2/5$ is the cut coefficient.*
2. *x_2: $u^2 = \mathbb{P}(19/10, 23/10) = (0.9, 0.3)$. Now $\phi^{\bar{0}}(0.9, 0.3) + \phi^{\bar{0}}(0.6, 0.2) = 1$ implying that $\phi^{\bar{0}}(0.9, 0.3) = 3/7$ is the cut coefficient.*
3. *x_3: $u^3 = \mathbb{P}((3/10, -7/5)) = (0.3, 0.6)$. Now $\phi^{\bar{0}}(0.3, 0.6) + \phi^{\bar{0}}(0.2, 0.9) > 1$. Therefore the value of coefficient $\phi^{\bar{0}}(0.3, 0.6) = 27/35$ can be possibly improved.*
4. *x_4: $u^4 = \mathbb{P}((-2/3, 11/6)) = (1/3, 5/6)$. Now $\phi^{\bar{0}}(1/3, 5/6) + \phi^{\bar{0}}(1/6, 2/3) > 1$. Therefore the value of coefficient $\phi^{\bar{0}}(1/3, 5/6) = 2/3$ can be possibly improved.*

Select x_4 for exact lifting, i.e., solve the problem:

$$max_{n\in\mathbb{Z}, n\geq 1}\left\{\frac{1-\pi(w)}{n}|w \equiv (0.5, 0.5) - n(1/3, 5/6)\right\}$$
$$= 8/21 < 2/3 = \phi^{\bar{0}}(1/3, 1/6).$$

Now the generalized fill-in function coefficient for u^3 is given by $\phi^{u^4}(u^3) = inf_{n\in\mathbb{Z}_+}\{8/21n + \pi(w) \,|\, \mathbb{P}(w) = u^2 - nu^4\} = 0.6 < \phi^{\bar{0}}(0.3, 0.6)$. Thus, the coefficients for both x_3 and x_4 have been decreased and the resulting inequality is: $10/7y_1 + 4/7y_2 + 7/15y_3 + 26/7y_4 + 0.4x_1 + 3/7x_2 + 0.6x_3 + 8/21x_4 \geq 1$. □

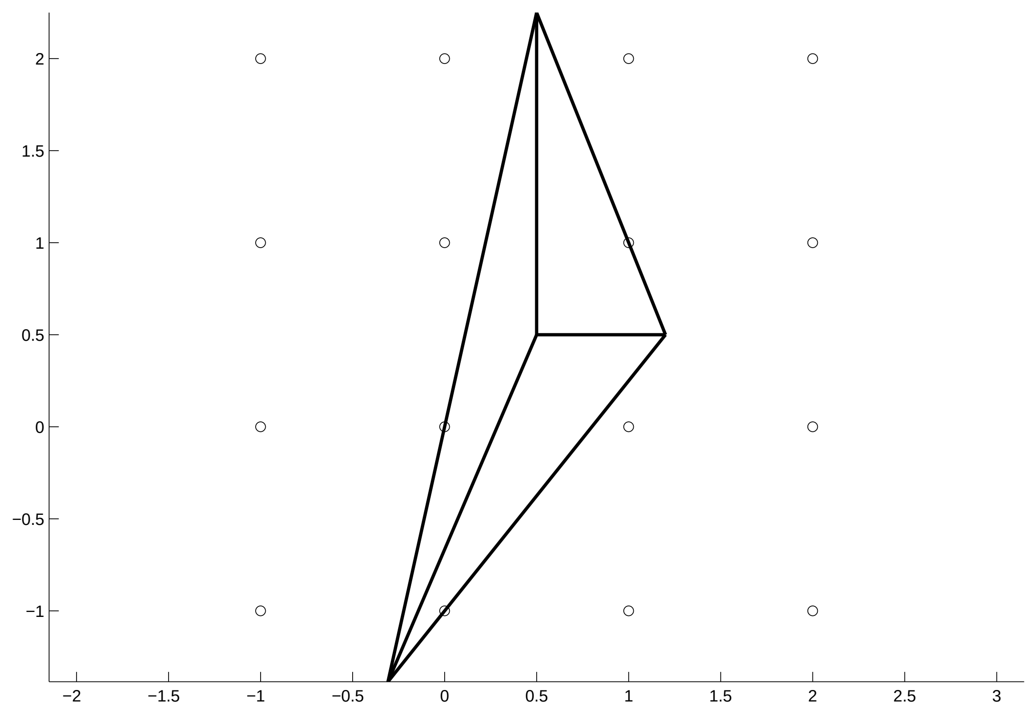

Figure 12: $P(\pi)$ to generate facet for (42)

8 Conclusion

In this paper, we have presented new cutting planes that can be generated from two rows of a simplex tableau. The approach used to derive these inequalities was to lift nonnegative integer variables into extreme inequalities for a system of two rows with two free integer variables and nonnegative continuous variables.

We have presented general conditions for fill-in functions to be extreme for the two-row mixed integer infinite-group problem and have proved that a unique lifting function exists in the case when the original inequality for the continuous variables corresponds to either a maximal lattice-free triangle with multiple integer points in the interior of one of its sides or a maximal lattice-free triangle with integral vertices and one integer point in the interior of each side. The resulting lifted inequality is extreme for $MI(I^2, S^2, r)$. This class of inequalities may be considered as the closest two-row counter parts of the Gomory mixed integer cut as the function π is extreme for $MI(\emptyset, S^2, r)$ and the trivial fill-in function is extreme for $MI(I^2, S^2, r)$.

In all other cases the lifting functions may not be unique. In Theorem 52, we showed that under suitable conditions, starting with a specific cyclic subgroup of I^2 and using the fill-in procedure leads to extreme inequalities for $MI(I^2, S^2, r)$ when the inequality π corresponds to a lattice-free triangle with non-integral vertices and one integer point in the interior of each side.

There is the possibility that these new extreme inequalities for $MI(I^2, S^2, r)$ may be useful computationally, since the coefficients for the continuous variables in these inequalities are not dominated by those of any other inequality for the two-row infinite-group problem and some of these inequalities cannot be obtained finitely using only inequalities generated from a single row.

Future research directions include analysis of maximal lattice-free convex sets in higher dimensions, obtaining closed-form expressions for the trivial fill-in functions (we showed that some subclasses of the trivial fill-in functions are sequential-merge inequalities or mixing inequalities for which closed form expressions are known) and extensive computational experiments.

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Appendix 1

In this section we show that if a function $\tilde{\pi} : \mathbb{Q}^2 \to \mathbb{R}_+$ as defined in (3) is minimal (extreme resp.) for (1) and $f \in \text{interior}(P(\pi))$, then π as defined in (4) is minimal (extreme resp.) for $MI(\emptyset, S^2, r)$.

The proof of the following proposition is exactly the same as the proof of Theorem 7 from Borozan and Cornuéjols [7] and related to the proof of Theorem 39 from Johnson [25].

Proposition 58 *If $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is minimal inequality for $MI(\emptyset, S^2, r)$ then*

1. *π is positively homogenous.*
2. *π is subadditive.*

Since the function $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is positively homogenous and subadditive, it is convex, see Rockafeller [31]. Moreover, if $\pi(w)$ is finite for every $w \in \mathbb{R}^2$, then it is continuous.

Proposition 59 ([31]) *If $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is a finite, subadditive, and positively homogenous function, then π is a continuous function.*

Proposition 60 *If $P(\pi)$ is a maximal lattice-free set with $f \in$ interior$(P(\pi))$ and $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ is defined as (4), then π is minimal.*

Proof: Note that π is a continuous function by construction. Assume by contradiction that π is not minimal. Therefore, there exists a valid minimal function $\pi' : \mathbb{R}^2 \to \mathbb{R}_+$ such that $\pi > \pi'$. Since π is a finite function, this implies that π' is finite. Using Proposition 58, π' is positively homogenous and subadditive. Thus using Proposition 59, π' is continuous. However, by Theorem 7, $\pi(u) = \pi'(u)\ \forall u \in \mathbb{Q}^2$. Since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, continuity of π and π' implies that $\pi = \pi'$. $\square$

Proposition 61 *Let $\tilde{\pi} : \mathbb{Q}^2 \to \mathbb{R}_+$ be an inequality for (1) corresponding to maximal lattice-free convex set $P(\pi)$ with $f \in$ interior$(P(\pi))$. Let $\pi : \mathbb{R}^2 \to \mathbb{R}_+$ be as defined in (4). If $\tilde{\pi}$ is extreme for (1), then π is extreme for $MI(\emptyset, S^2, r)$.*

Proof: Observe that $\pi(u) = \tilde{\pi}(u)\ \forall u \in \mathbb{Q}^2$. Assume by contradiction that there exist two valid functions $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}_+$ such that $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ and $\pi_1 \neq \pi_2$. By Proposition 60, π is a minimal inequality. This implies that π_1 and π_2 are minimal. Thus, using Proposition 58, π_1 and π_2 are positively homogenous and subadditive. Moreover, since π is finite by construction, π_1 and π_2 and finite. Thus, using Proposition 59, π_1 and π_2 are continuous.

Since $\tilde{\pi}$ is extreme, we have $\tilde{\pi}(u) = \pi_i(u)\ \forall u \in \mathbb{Q}^2$. Since $f \in$ interior$(P(\pi))$, π is a continuous function. However, since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$, this implies that $\pi_1 = \pi_2$, a contradiction. $\square$

Appendix 2

To prove that $\mathbb{P}(T(\pi)) = I^2$, we first present a preliminary result that identifies relations between some points on the boundary of $T(\pi)$.

Proposition 62 *For $T(\pi)$ (see Definition 49),*

1. $c^3 = g - (1, 0)$.
2. $e^2 = i - (1, 0)$.
3. $e^3 = k - (0, 1)$.
4. $c^1 = j - (0, 1)$.

Proof.

1. $g - c^3 = f + (1-\delta^{12})d^2 + (\delta^{12} - 1 + \delta^{31})d^1 - f - \delta^{31}d^3 = f + (1-\delta^{12})d^2 + (\delta^{12} - 1 + \delta^{31})d^1 + (1-\delta^{31})d^1 = f + (1-\delta^{12})d^2 + \delta^{12}d^1 = (1, 0)$.
2. $i - e^2 = f + (1-\delta^{12})d^2 + (\delta^{12} - 1 + \delta^{31})d^1 - (\delta^{31} - 1 + \delta^{23})d^3 - f - (1-\delta^{23})d^3 = (1-\delta^{12})d^2 + (\delta^{12} - 1 + \delta^{31})d^1 - \delta^{31}d^3 = (1-\delta^{12})d^2 + (\delta^{12} - 1 + \delta^{31})d^1 + f + (1-\delta^{31})d^1 = f + (1-\delta^{12})d^2 + \delta^{12}d^1 = (1, 0)$.
3. $k - e^3 = f + \delta^{23}d^2 - (\delta^{31} - 1 + \delta^{23})d^3 - f - (1-\delta^{31})d^1 = \delta^{23}d^2 - (\delta^{31} - 1 + \delta^{23})d^3 - (1-\delta^{31})d^1 = \delta^{23}d^2 - (\delta^{31} - 1 - \delta^{23})d^3 + f + \delta^{31}d^3 = f + \delta^{23}d^2 + (1-\delta^{23})d^3 = (0, 1)$.
4. $j - c^1 = f + \delta^{23}d^2 + (\delta^{12} - 1 + \delta^{31})d^1 - (\delta^{31} - 1 + \delta^{23})d^3 - f - \delta^{12}d^1 = \delta^{23}d^2 + (-1 + \delta^{31})d^1 - (\delta^{31} - 1 - \delta^{23})d^3 = \delta^{23}d^2 + (-1 + \delta^{31})d^1 + (1-\delta^{23})d^3 + f + (1-\delta^{31})d^1 = (0, 1)$. $\square$

Proposition 63 $\mathbb{P}(T(\pi)) = I^2$.

Proof. Refer to Figure 13. Let b^4 be the point $(1, 1)$. We perform the following operations on $T(\pi)$:

1. Let Δ_1 be the triangle $b^1c^1e^3$. Construct $T_1 = (T(\pi) \setminus \Delta_1) \cup (\Delta_1 + \{(0, 1)\})$. (It can be verified that this operation is general and can always be applied. We obtain the set illustrated in the second frame in Figure 13 since $kj = e^3c^1 + \{(0, 1)\}$. Note that since the operation creating T_1 involves relative motion of a subset of $T(\pi)$ by integral amount, $\mathbb{P}(T(\pi)) = \mathbb{P}(T_1)$.

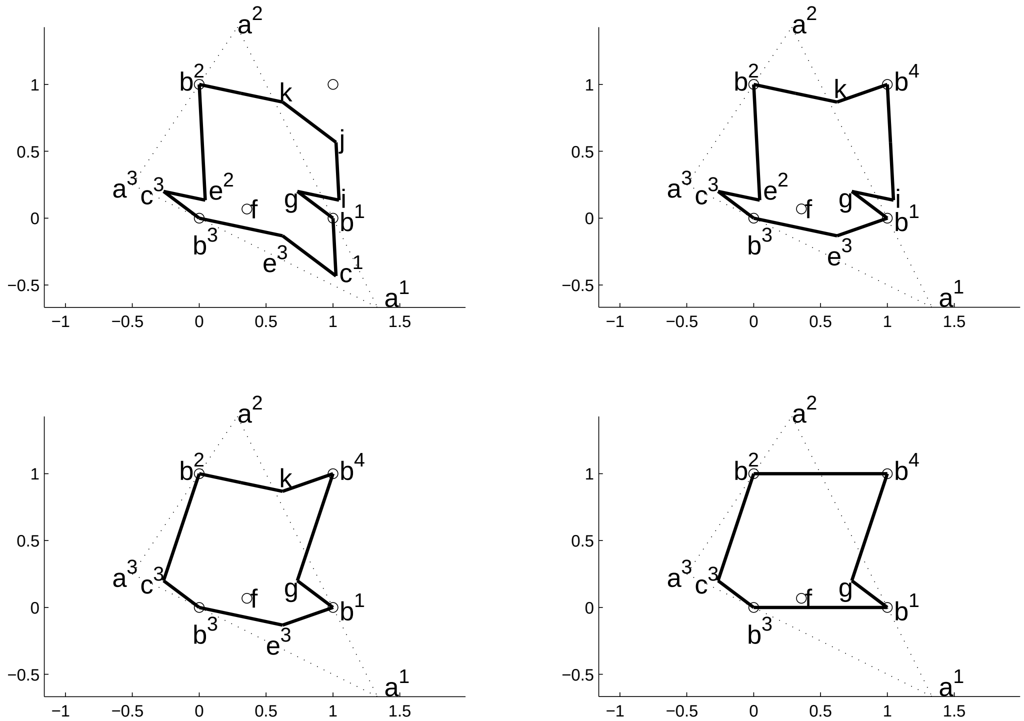

Figure 13: Some lattice-preserving operations

2. Let Δ_2 be the triangle gib^4. Construct $T_2 = (T_1 \setminus \Delta_2) \cup (\Delta_2 - \{(1,0)\})$. We obtain the set illustrated in the third frame in Figure 13 since $gi = c^3e^2 + \{(1,0)\}$ and $ib^4 = e^2b^2 + \{(1,0)\}$. Again $\mathbb{P}(T_2) = \mathbb{P}(T_1)$.

3. If $k_2 \leq 1$, then let Δ_3 be the triangle $b^3e^3b^1$ and construct $T_3 = (T_2 \setminus \Delta_3) \cup (\Delta_3 + \{(0,1)\})$. If $k_2 \geq 1$, then let Δ_3 be the triangle b^2kb^4 and construct $T_3 = (T_2 \setminus \Delta_3) \cup (\Delta_3 - \{(0,1)\})$. We obtain the set illustrated in the fourth frame in Figure 13 since $b^2k = b^3e^3 + \{(0,1)\}$ and $kb^4 = e^3b^1 + \{(0,1)\}$. Again $\mathbb{P}(T_3) = \mathbb{P}(T_2)$.

4. If $c_1^3 \leq 0$, then let Δ_4 be the triangle $b^3c^3b^2$. Construct $T_4 = (T_3 \setminus \Delta_4) \cup (\Delta_4 + \{(1,0)\})$. If $c_1^3 \geq 0$, then let Δ_4 be the triangle b^1gb^4. Construct $T_4 = (T_3 \setminus \Delta_4) \cup (\Delta_4 - \{(1,0)\})$. Since $b^3c^3 = b^1g - \{(1,0)\}$ and $c^3b^2 = gb^4 - \{(1,0)\}$ we obtain that $I^2 = T_4$. Therefore, $I^2 = \mathbb{P}(T_4) = \mathbb{P}(T_3) = \mathbb{P}(T(\pi))$. $\square$