Basis partition of the space of linear programs through a differential equation *

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Abstract: The space of linear programs (LP) can be partitioned into a finite number of sets, each corresponding to a basis. This partition is thus called *the basis partition.* The closed-form solution on the space of LP can be determined with the basis partition if we can characterize the basis partition. A differential equation on the Grassmann manifold which represents the space of LP provides a powerful tool for characterizing the basis partition. In paper [3], the author presented some basic concepts and properties of this differential equation. This paper continues the research of [3] and presents three useful properties.

Keywords: Linear programming, Space of linear programs, Basis partition, Grassmannian/Grassmann manifold, Projection matrix, Differential equation.

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1 Introduction

We refer to the collection of all linear programming (LP) instances as *the space of linear programs*, denoted by SLP. For each LP instance we define an *optimal basis* as the index set of constraints which are active at every optimal solution. Since an LP instance has a strictly complementary solution, the optimal basis can be defined uniquely. For each index set B, we denote by $SLP(B)$ the set of LP instances which have the common optimal basis B. The family $\{SLP(B) : \text{ all index sets } B\}$ is a partition of SLP. We refer to this partition as *the basis partition*.

The notion of *basis* comes along with the notion of *complementarity*. Thus, the basis partition can be defined on any space of optimization problems and complementarity problems which have a finite number of coefficients. In principle, the ideas and results of the basis partition on the space of LP can be extended to the space of linear complementarity problems, the space of quadratic programs, the space of semidefinite programs, and other more general conic programs.

If we know the optimal basis of an LP instance, we can find the solution of the LP instance by solving a system of linear equations, i.e., we can find the solution in terms of coefficients of the LP instance. This means, we can find a closed-form solution on the SLP if the basis partition is known explicitly.

Many problems involve an infinite number of optimization instances, just to name a few, parametric optimization, bilevel optimization, stochastic programming, and constrained dynamical systems, e.g. dynamical systems with state variables and control variables where control variables are determined by an optimization problem whose coefficients depend on state variables. Closed-form solutions on the space of optimization instances are useful, sometimes even necessary, for solving such problems. A way, perhaps the only way, to generate closed-form solutions on the space of optimization instances is to explicitly determine the basis partition of the space.

However, to know the basis partition explicitly is a very difficult issue, far more difficult than to solve an individual LP instance. We have now very powerful tools for solving individual LP instances, such as the simplex method and the interior point method. However, we have known very little about the basis partition, because no effective tools have been developed to find its structures.

In our previous paper [3], a novel tool, a differential equation on the space of projection matrices (namely, the Grassmann manifold), is presented to characterize the basis partition of the space of LP. We have presented basic concepts and some properties of the basis partition and the differential equation in [3]. This paper continues this research and presents three main results:

(i) We establish a one-to-one correspondence between a path and an equilibrium-eigenvector pair.

(ii) We find an LP representation for a path in terms of the corresponding equilibrium and eigenvector.

(iii) We characterize sources and sinks of attraction regions and their boundaries (viewed as stable/unstable manifolds) through a simple calculation of dimensions of these sources and sinks.

The rest of this paper is organized as follows. Section 2 collates concepts and notions which display the field we are studying. Section 3 presents some properties of the solution of the differential equation. Sections 4, 5 and 6 present the three main results of this paper.

Throughout this paper we use the following notations. For any vectors $x, s \in R^n$ and scalar $\alpha \in R$, we denote $x \circ s = (x_1 s_1, \ldots, x_n s_n)^T$, $x^\alpha = (x_1^\alpha, \ldots, x_n^\alpha)^T$, and $\lfloor x \rceil = \mathrm{diag}\,(\mathrm{x})$. We use the symbol $\mathbf{1}$ for the vector of all ones regardless of its dimension. For any map $f : \mathcal{V} \to \mathcal{W}$, we denote by $Df(p) : T_p\mathcal{V} \to T_{f(p)}\mathcal{W}$ the Fréchet derivative of f at $p \in \mathcal{V}$.

2 Preliminaries

This section collates concepts and notions which display the field we are studying.

Consider the linear program:

$$\begin{aligned} \min \quad & c^T x \\ s.t. \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{2.1}$$

and its dual

$$\max \quad b^T y$$

$$\text{s.t.} \quad A^T y + s = c \qquad (2.2)$$
$$s \geq 0$$

where $A \in R^{m \times n}$ is of full row rank, and b, c, x and s are vectors of appropriate dimensions. We say that this linear program is of dimension (n, m).

Definition 2.1 *We refer to a set of coefficients* (A, b, c) *as a* **strictly feasible instance** *(in short,* **instance***) of linear programming if the primal and dual problems, (2.1) and (2.2), have strictly feasible solutions, i.e. feasible solutions with* $x > 0$ *and* $s > 0$*. We denote by* $SLP(n, m)$ *the set of all strictly feasible instances of dimension* (n, m)*. We call* $SLP(n, m)$ **the space of linear programs***.*

Definition 2.2 *An index set* $B \subset \{1, \ldots, n\}$ *is said to be the* **optimal basis** *of* (A, b, c) *if for each* $i \in B$ *the dual constraint* $a_i^T y \leq c_i$ *is satisfied at equality for every dual optimal solution* y *and for each* $i \notin B$ *the primal constraint* $x_i \geq 0$ *is satisfied at equality for every primal optimal solution* x*.*

Note that an instance need not be nondegenerate. A basis can be any subset, even an empty set or a full set.

It is known that for any feasible instance (A, b, c), there exists a **unique** optimal basis (B, N). Furthermore, there exists an optimal solution (x, y) such that

$$Ax = b, \quad A_B^T y = c_B, \quad A_N^T y < c_N, \quad x_B > 0, \quad x_N = 0.$$

Such a solution is called a *strictly complementary optimal solution.*

Since each instance (A, b, c) possesses a unique basis B, we can partition $SLP(n, m)$ into $\{SLP(B) : B \subset \{1, \ldots, n\}, |B| = m\}$, where $SLP(B)$ is the set of all (A, b, c) whose basis is B. This partition is referred to as **the basis partition** of $SLP(n, m)$.

A novel tool we use to characterize the basis partition is a differential equation which is defined on **the space of projection matrices**

$$G(n, m) := \{M \in S^n : MM = M, \ \text{rank}(M) = m\}.$$

where S^n is the set of all symmetric $n \times n$-matrices. The space of projection matrices is also known as **the Grassmann manifold**. The differential equation we use to characterize the basis partition is, cf. [2] and [3],

$$M' = h(M),$$

where the derivative $'$ is taken with respect to $t \in (-\infty, +\infty)$ and

$$h(M) := M\lfloor M\mathbf{1}\rceil + \lfloor M\mathbf{1}\rceil M - 2M\lfloor M\mathbf{1}\rceil M.$$

We denote by $M(t)$ or $M(t, M_0)$ the solution of $M' = h(M)$ with $M(0) = M_0$. For clarity of notation, sometimes we use the map $\phi : R \times G(n, m) \to G(n, m)$ which is defined by $\phi(t, M_0) = M(t, M_0)$. The map ϕ is called the *flow* of $M' = h(M)$ in the literature of dynamical systems. Furthermore, for any $t \in R$, we define $\phi_t : G(n, m) \to G(n, m)$ by $\phi_t(M) = \phi(t, M)$. It is known that $\phi_t \in C^\infty$ since $h \in C^\infty$.

A close relationship between $SLP(n, m)$ and $G(n, m)$ and the basis partitions on them was shown in [3]. Here we briefly summarize this relationship.

Throughout the paper, we define the map $\Upsilon : R^{m\times n} \to G(n, m)$ by

$$\Upsilon(A) = A^T(AA^T)^{-1}A, \tag{2.3}$$

and the map $\Gamma : SLP(n, m) \to G(n, m)$ by

$$\Gamma(A, b, c) = \Upsilon(A\lfloor x\rceil), \tag{2.4}$$

where (x, s, y) is the analytic center of (A, b, c), i.e., the unique solution of the system

$$Ax = b, \quad A^T y + s = c, \quad x \circ s = \mathbf{1}, \quad x > 0, s > 0.$$

Conversely, for any $M \in G(n, m)$ we can construct an instance (A, b, c) such that $\Gamma(A, b, c) = M$, see Lemma 2.7 in [3]. Thus, the map Γ is surjective but not injective.

For any strictly feasible instance $(A, b, c) \in SLP(n, m)$, the map from (A, b, c) to the analytic center (x, s, y) and the map Υ are both smooth. Therefore, the map Γ is smooth.

For any strictly feasible instance, the perturbed KKT system:

$$\begin{array}{rcl} x \circ s & = & e^{-t}\mathbf{1} \\ Ax & = & b \\ A^T y + s & = & c \\ x > 0, & & s > 0 \end{array} \tag{2.5}$$

has unique solution for any $t \in R$. We refer to this solution $(x(t), s(t))$, $t \in R$, as the **central path** of the LP instance. The limit of the central path $(\bar{x}, \bar{s}) = \lim_{t\to+\infty}(x(t), s(t))$ is a pair strictly complementary optimal solution of the primal and dual problems.

Any instance $(A, b, c) \in SLP(n, m)$ defines a path $M(t)$ in two ways: (i) $M(t)$ is the solution of $M' = h(M)$ with $M(0) = \Gamma(A, b, c)$, and (ii) $M(t) = \Upsilon(A\lfloor x(t)\rceil)$ where $x(t)$ is the central path of (A, b, c). By Theorem 2.9 in [3], these two ways define the same path. Furthermore, Theorem 3.3 in [3] shows that the limit point $\bar{M} = \lim_{t\to+\infty} M(t)$ is an *equilibrium*, i.e. $h(\bar{M}) = 0$, which exhibits

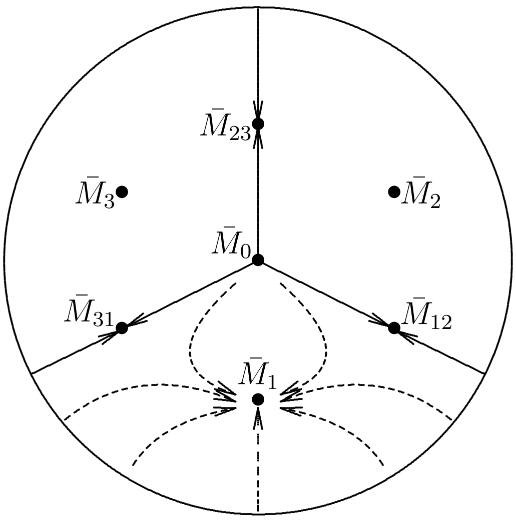
$$\bar{M}\mathbf{1} = \begin{pmatrix} \mathbf{1}_B \\ 0_N \end{pmatrix}$$

where the index set B is precisely the optimal basis of the instance (A, b, c).

On the space $G(n, m)$, we can define a basis partition $\{G(B) : B \subset \{1, \ldots, n\}\}$, where $G(B)$ is the set of all $M \in G(n, m)$ such that the limit point $\bar{M} = \lim_{t\to+\infty} \phi_t(M)$ satisfies $\bar{M}\mathbf{1} = \begin{pmatrix} \mathbf{1}_B \\ 0_N \end{pmatrix}$.

It is remarkable that the basis partition of $G(n, m)$ is completely and solely defined by the dynamical system $M' = h(M)$, while, in the meantime, the partitions of $SLP(n, m)$ and $G(n, m)$ are related by $\Gamma(SLP(B)) = G(B)$. By virtue of this relationship, we can study the basis partition of $SLP(n, m)$ via the basis partition of $G(n, m)$. This approach will prove to be an essential advance in the study of the basis partition, due to the well-structured space $G(n, m)$ and the dynamical system $M' = h(M)$ on it.

The following picture shows the basis partition of $G(3, 1)$:



where equilibrium points are

$$\bar{M}_0 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\bar{M}_1 = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \ \bar{M}_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix}, \ \bar{M}_3 = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$$

$$\bar{M}_{12} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \bar{M}_{23} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \ \bar{M}_{31} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

and all points on the circle in the form of

$$\bar{M} = uu^T \qquad \text{for } u \in R^3 \text{ with } uu^T = 1 \text{ and } u^T\mathbf{1} = 0.$$

The basis partition of $G(3,1)$ consists of 8 sets $\{G(B) : B \subset \{1,2,3\}\}$ which are described as follows:

- For the basis $B = \{i\}$, $i = 1,2,3$, $G(\{i\})$ is an attraction region which is the open set containing the stable equilibrium $\bar{M}_i$;
- For the basis $B = \{i,j\}$, $G(\{i,j\})$ is the line segment containing the unstable equilibrium $\bar{M}_{ij}$;
- For the basis $B = \{1,2,3\}$, $G(\{1,2,3\})$ is the singleton $\{\bar{M}_0\}$;
- For the basis $B = \emptyset$, $G(\emptyset)$ is the circle. (Note that for any point $\bar{M}$ on the circle, $\bar{M}\mathbf{1} = 0$, thus $B = \emptyset$.)

An annotation is necessary: The opposite points on the circle are identical. Thus, the bottom boundary of $G(\{1\})$, namely the bottom one third of the circle, is also the boundary on the top of $G(\{2\})$ and $G(\{3\})$, i.e. this boundary is the boundary between $G(\{1\})$ and $G(\{2\})$ and between $G(\{1\})$ and $G(\{3\})$.

3 Some properties of paths $M(t)$

Lemma 3.1 *Let $M(t)$ be a path satisfying $M' = h(M)$. For any $t_0, t_1, t_2, t_3 \in R$, if $\Gamma(A,b,c) = M(t_0)$, then $\Gamma(e^{t_1}A, e^{t_2}b, e^{t_3}c) = M(t_0 - t_1 + t_2 + t_3)$.*

Proof. Let $M(t \mid t_0)$ denote the solution of $M'(t \mid t_0) = h(M(t \mid t_0))$ with $M(0 \mid t_0) = M(t_0)$. Then it is known that $M(t \mid t_0) = M(t + t_0)$ for any t and t_0.

For any $(A, b, c) = \Gamma^{-1}(M(0 \mid t_0))$, let $(x(t), s(t), y(t))$ be the central path of (A, b, c), which satisfies

$$\begin{aligned} Ax(t) &= b \\ A^T y(t) + s(t) &= c \\ x(t)s(t) &= e^{-t}\mathbf{1}. \end{aligned} \tag{3.1}$$

By Theorem 2.9 in [3], we have

$$M(t \mid t_0) \quad = \quad \lfloor x(t) \rceil A^T (A \lfloor x(t) \rceil^2 A^T)^{-1} A \lfloor x(t) \rceil.$$

Let $(\tilde{x}(t), \tilde{s}(t), \tilde{y}(t))$ be the central path of $(e^{t_1}A, e^{t_2}b, e^{t_3}c)$. Then it satisfies

$$\begin{aligned} e^{t_1} A\tilde{x}(t) &= e^{t_2} b \\ e^{t_1} A^T \tilde{y}(t) + \tilde{s}(t) &= e^{t_3} c \\ \tilde{x}(t)\tilde{s}(t) &= e^{-t}\mathbf{1}. \end{aligned} \tag{3.2}$$

By the definition of Γ, we have

$$\begin{aligned} \Gamma(e^{t_1}A, e^{t_2}b, e^{t_3}c) &= \lfloor \tilde{x}(0) \rceil (e^{t_1}A)^T ((e^{t_1}A) \lfloor \tilde{x}(0) \rceil^2 (e^{t_1}A)^T)^{-1} (e^{t_1}A) \lfloor \tilde{x}(0) \rceil \\ &= \lfloor \tilde{x}(0) \rceil A^T (A \lfloor \tilde{x}(0) \rceil^2 A^T)^{-1} A \lfloor \tilde{x}(0) \rceil \end{aligned}$$

Define $\hat{x}(t) = e^{t_1 - t_2}\tilde{x}(t)$, $\hat{y}(t) = e^{t_1 - t_3}\tilde{y}(t)$ and $\hat{s}(t) = e^{-t_3}\tilde{s}(t)$. It follows from (3.2) that

$$\begin{aligned} A\hat{x}(t) &= b \\ A^T \hat{y}(t) + \hat{s}(t) &= c \\ \hat{x}(t)\hat{s}(t) &= e^{-(t - t_1 + t_2 + t_3)}\mathbf{1}. \end{aligned} \tag{3.3}$$

Comparing (3.1) and (3.3), we observe that $\hat{x}(t) = x(t - t_1 + t_2 + t_3)$. Thus,

$$\begin{aligned} \Gamma(e^{t_1}A, e^{t_2}b, e^{t_3}c) &= \lfloor \tilde{x}(0) \rceil A^T (A \lfloor \tilde{x}(0) \rceil^2 A^T)^{-1} A \lfloor \tilde{x}(0) \rceil \\ &= \lfloor \hat{x}(0) \rceil A^T (A \lfloor \hat{x}(0) \rceil^2 A^T)^{-1} A \lfloor \hat{x}(0) \rceil \\ &= \lfloor x(-t_1 + t_2 + t_3) \rceil A^T (A \lfloor x(-t_1 + t_2 + t_3) \rceil^2 A^T)^{-1} A \lfloor x(-t_1 + t_2 + t_3) \rceil \\ &= M(-t_1 + t_2 + t_3 \mid t_0) \\ &= M(t_0 - t_1 + t_2 + t_3). \end{aligned}$$

This proves the lemma. □

Theorem 3.2 *For any $t_0, t_1, t_2, t_3 \in R$, the map*

$$\varphi : \Gamma^{-1}(M(t_0)) \to \Gamma^{-1}(M(t_0 - t_1 + t_2 + t_3))$$

defined by

$$\varphi(A, b, c) = (e^{t_1}A, e^{t_2}b, e^{t_3}c)$$

is bijective.

Proof. By Lemma 3.1, for any $(A, b, c) \in \Gamma^{-1}(M(t_0))$,

$$(e^{t_1}A, e^{t_2}b, e^{t_3}c) \in \Gamma^{-1}(M(t_0 - t_1 + t_2 + t_3)).$$

Thus the map φ is well defined.

Also by Lemma 3.1, we can define a map

$$\psi : \Gamma^{-1}(M(t_0 - t_1 + t_2 + t_3)) \to \Gamma^{-1}(M(t_0))$$

by $\psi(A, b, c) = (e^{-t_1}A, e^{-t_2}b, e^{-t_3}c)$.

For any $(A, b, c) \in \Gamma^{-1}(M(t_0))$, we have

$$\begin{aligned}\psi \circ \varphi(A, b, c) &= \psi(\varphi(A, b, c)) \\ &= \psi(e^{t_1}A, e^{t_2}b, e^{t_3}c) \\ &= (A, b, c)\end{aligned}$$

This shows that the map $\psi \circ \varphi : \Gamma^{-1}(M(t_0)) \to \Gamma^{-1}(M(t_0))$ is the identity map.

Similarly, we can show that

$$\varphi \circ \psi : \Gamma^{-1}(M(t_0 - t_1 + t_2 + t_3)) \to \Gamma^{-1}(M(t_0 - t_1 + t_2 + t_3))$$

is the identity map.

Therefore, $\varphi : \Gamma^{-1}(M(t_0)) \to \Gamma^{-1}(M(t_0 - t_1 + t_2 + t_3))$ is an bijection. □

Corollary 3.3

$$\begin{aligned}\Gamma^{-1}(M(t)) &= \{e^t(A, b, c) \mid (A, b, c) \in \Gamma^{-1}(M(0))\} \\ &= \{(e^{-t}A, b, c) \mid (A, b, c) \in \Gamma^{-1}(M(0))\} \\ &= \{(A, e^t b, c) \mid (A, b, c) \in \Gamma^{-1}(M(0))\} \\ &= \{(A, b, e^t c) \mid (A, b, c) \in \Gamma^{-1}(M(0))\}\end{aligned}$$

Lemma 3.4 *If $A \in R^{m\times n}$ is of full row rank and A^TA is a projection matrix, then*

$$AA^T = I.$$

Proof. Let $M = A^TA$ be a projection matrix. Then we have $MM = M$. This implies $A^TAA^TA = A^TA$. Multiplying A on left and A^T on right, we obtain $(AA^T)^3 = (AA^T)^2$. Since A has full row rank, it follows that $AA^T = I$. □

Lemma 3.5 *Let $\bar{M}, M_k \in G(n,m)$ with $M_k \to \bar{M}$ as $k \to \infty$.*

(i) For any $\bar{A} \in R^{m\times n}$ with $\bar{M} = \bar{A}^T\bar{A}$, there exist $A_k \in R^{m\times n}$ such that $M_k = A_k^TA_k$ and $A_k \to \bar{A}$ as $k \to \infty$.

(ii) For any $\bar{A} \in R^{m\times n}$ with $\bar{M} = \bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A}$, there exist $A_k \in R^{m\times n}$ such that $M_k = A_k^T(A_kA_k^T)^{-1}A_k$ and $A_k \to \bar{A}$ as $k \to \infty$.

Proof. (i) Because $M_k\bar{A}^T \to \bar{M}\bar{A}^T$ as $k \to \infty$ and $\bar{M}\bar{A}^T = \bar{A}^T$ by Lemma 3.4, $M_k\bar{A}^T$ has full column rank for large k (assumed for all k for simplicity). Define

$$A_k = (\bar{A}M_k\bar{A}^T)^{-1/2}\bar{A}M_k. \tag{3.4}$$

As $k \to \infty$, $M_k \to \bar{M}$, thus

$$A_k \to (\bar{A}\bar{M}\bar{A}^T)^{-1/2}\bar{A}\bar{M} = \bar{A}.$$

It remains to show $A_k^TA_k = M_k$. From the definition (3.4), we have

$$\begin{aligned} A_kA_k^T &= (\bar{A}M_k\bar{A}^T)^{-1/2}\bar{A}M_kM_k\bar{A}^T(\bar{A}M_k\bar{A}^T)^{-1/2} \\ &= (\bar{A}M_k\bar{A}^T)^{-1/2}\bar{A}M_k\bar{A}^T(\bar{A}M_k\bar{A}^T)^{-1/2} \\ &= I. \end{aligned} \tag{3.5}$$

For any $A_k^Tu \in Rang(A_k^T)$ with $u \in R^m$, let $z = \bar{A}^T(\bar{A}M_k\bar{A}^T)^{-1/2}u \in R^n$. Then $A_k^Tu = M_kz \in Rang(M_k)$. Thus, $Rang(A_k^T) \subseteq Rang(M_k)$. Since $M_k\bar{A}^T$ has full column rank, so has A_k^T. This implies that the subspaces $Rang(A_k^T)$ and $Rang(M_k)$ have the same dimension m. Therefore,

$$Rang(A_k^T) = Rang(M_k). \tag{3.6}$$

Since M_k is a projection matrix, (3.6) implies

$$M_k = A_k^T (A_k A_k^T)^{-1} A_k.$$

Then, by (3.5), we have

$$M_k = A_k^T A_k.$$

(ii) Now suppose that $\bar{M} = \bar{A}^T (\bar{A}\bar{A}^T)^{-1} \bar{A}$. Define $\hat{A} = (\bar{A}\bar{A}^T)^{-1/2} \bar{A}$. Then $\hat{A}^T \hat{A} = \bar{M}$. As shown in part (i), there exist $\hat{A}_k$ such that $\hat{A}_k^T \hat{A}_k = M_k$ and $\hat{A}_k \to \hat{A}$ as $k \to \infty$. Define $A_k = (\bar{A}\bar{A}^T)^{1/2} \hat{A}_k$. Then we have

$$A_k \to (\bar{A}\bar{A}^T)^{1/2} \hat{A} = \bar{A},$$

and

$$A_k^T (A_k A_k^T)^{-1} A_k = \hat{A}_k^T (\hat{A}_k \hat{A}_k^T)^{-1} \hat{A}_k = \hat{A}_k^T \hat{A}_k = M_k.$$

□

Lemma 3.6 *Let $A \in R^{m \times n}$ and $W \in R^{(n-m) \times n}$ be of full row rank and complementary to each other. i.e. $AW^T = 0$. Let $c, d \in R^n$ be any vectors. The path $M(t)$ is associated with (A, Ac, d) if and only if the path $I - M(t)$ is associated with (W, Wd, c). In particular,*

$$\Gamma(W, Wd, c) = I - \Gamma(A, Ac, d).$$

Remark: Rigorously speaking, $(A, Ac, d) \in SLP(n, m)$ and $(W, Wd, c) \in SLP(n, n-m)$. Thus, the two Γ in the above equation are defined on different spaces.

Proof. Let $\tilde{A} \in R^{m \times n}$ and $\tilde{W} \in R^{(n-m) \times n}$ be of full row rank and satisfy $\tilde{A}\tilde{W}^T = 0$. We have

$$\begin{aligned} I &= \begin{pmatrix} \tilde{A}^T & \tilde{W}^T \end{pmatrix} \left(\begin{pmatrix} \tilde{A} \\ \tilde{W} \end{pmatrix} \begin{pmatrix} \tilde{A}^T & \tilde{W}^T \end{pmatrix} \right)^{-1} \begin{pmatrix} \tilde{A} \\ \tilde{W} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{A}^T & \tilde{W}^T \end{pmatrix} \begin{pmatrix} \tilde{A}\tilde{A}^T & 0 \\ 0 & \tilde{W}\tilde{W}^T \end{pmatrix}^{-1} \begin{pmatrix} \tilde{A} \\ \tilde{W} \end{pmatrix} \\ &= \tilde{A}^T (\tilde{A}\tilde{A}^T)^{-1} \tilde{A} + \tilde{W}^T (\tilde{W}\tilde{W}^T)^{-1} \tilde{W}. \end{aligned} \tag{3.7}$$

Next, we show that

$$\begin{aligned} Ax &= Ac \\ A^T y + s &= d \\ x \circ s &= e^{-t} \mathbf{1} \\ x, s &> 0, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} W^T u + x &= c \\ Ws &= Wd \\ x \circ s &= e^{-t}\mathbf{1} \\ x, s &> 0, \end{aligned} \tag{3.9}$$

have the same solution (x, s), (with appropriate y and u).

Let (x, s, y) be the solution of (3.8). Let

$$\begin{pmatrix} v \\ u \end{pmatrix} = (A^T, W^T)^{-1}(c - x).$$

That is,

$$c - x = A^T v + W^T u.$$

The first equation in (3.8) leads to

$$0 = A(c - x) = AA^T v + AW^T u = AA^T v,$$

which yields $v = 0$. It follows that

$$c - x = W^T u.$$

Thus, (x, u) satisfies the first equation in (3.9). Multiplying the second equation in (3.8) with W, we obtain the second equation in (3.9). Therefore, (x, s, u) is a solution of (3.9). Analogously, we can show that any solution (x, s) of (3.9) is a solution of (3.8), (y and u are determined accordingly).

Now, the path $M(t)$ which is associated with (A, Ac, d), namely with the solution of (3.8), is defined by

$$M(t) = \lfloor x \rceil A^T (A \lfloor x \rceil^2 A^T)^{-1} A \lfloor x \rceil,$$

and the path $\hat{M}(t)$ which is associated with (W, Wd, c), namely with the solution of (3.9), is defined by

$$\hat{M}(t) = \lfloor s \rceil W^T (W \lfloor s \rceil^2 W^T)^{-1} W \lfloor s \rceil.$$

Let $\tilde{A} = A\lfloor x \rceil$ and $\tilde{W} = W\lfloor s \rceil$. Using $\lfloor x \rceil \lfloor s \rceil = \lfloor x \circ s \rceil = e^{-t} I$, we have $\tilde{A}\tilde{W}^T = e^{-t} A W^T = 0$. Thus, by (3.7), $\hat{M}(t) = I - M(t)$. □

4 Limiting equilibrium and direction of a path

An attraction region consists of all paths which converge to a stable equilibrium. The boundary of attraction regions consists of paths which converge to unstable equilibria. Typically, at an equilibrium $\bar{M}$, $Dh(\bar{M})$ has positive and negative eigenvalues. One can imagine that a path $M(t)$ will converge to an equilibrium in an eigenvector direction for a negative eigenvalue as $t \to +\infty$ and converge to an equilibrium in an eigenvector direction for a positive eigenvalue as $t \to -\infty$. In order to characterize the attraction regions and their boundaries, we will be interested in the *set* of all paths which converge to an equilibrium (either $t \to +\infty$ or $t \to -\infty$). This motivates us to investigate the following sets:

$$\mathcal{W}^s(\bar{M}) := \{M \in G(n,m) \mid \lim_{t \to +\infty} \phi_t(M) = \bar{M}\} \tag{4.1}$$

and

$$\mathcal{W}^u(\bar{M}) := \{M \in G(n,m) \mid \lim_{t \to -\infty} \phi_t(M) = \bar{M}\}. \tag{4.2}$$

In the literature of dynamical systems, $\mathcal{W}^s(\bar{M})$ is called a *stable manifold* and $\mathcal{W}^u(\bar{M})$ an *unstable manifold*, see e.g. (1.3.6) and (1.3.7) in [1]. Here, the definitions of (4.1) and (4.2) are simpler than that in [1], because we have shown that every path $M(t)$ in $G(n,m)$ converges to two equilibria as $t \to +\infty$ and $t \to -\infty$, respectively. Thus, $\mathcal{W}^s(\bar{M})$ and $\mathcal{W}^u(\bar{M})$ are well defined and are the same as defined by (1.3.7) in [1].

The basic properties of the stable/unstable manifolds are as follows, see e.g. Theorem 3.2.1 in [1]: Let $E^-(\bar{M})/E^+(\bar{M})$ be the generalized eigenspace which is the span of all eigenvectors for negative/positive real-part eigenvalues of $Dh(\bar{M})$. The stable/unstable manifold $\mathcal{W}^s(\bar{M})/\mathcal{W}^u(\bar{M})$ has the same dimension as $E^-(\bar{M})/E^+(\bar{M})$ and is tangent at $\bar{M}$ to $E^-(\bar{M})/E^+(\bar{M})$, i.e.

$$T_{\bar{M}}\mathcal{W}^s(\bar{M}) = E^-(\bar{M}), \qquad T_{\bar{M}}\mathcal{W}^u(\bar{M}) = E^+(\bar{M}). \tag{4.3}$$

Note that $Dh(\bar{M})$ has only one negative eigenvalue -1 and one positive eigenvalue 1. Thus, we can simply write

$$E^+(\bar{M}) = \{U \in T_{\bar{M}}(G(n,m)) \mid Dh(\bar{M})U = U\}, \tag{4.4}$$
$$E^-(\bar{M}) = \{U \in T_{\bar{M}}(G(n,m)) \mid Dh(\bar{M})U = -U\}. \tag{4.5}$$

Equilibria are clustered into a number of connected sets. Each such connected set is called an *equilibrium cluster* and is a submanifold. For any basis B and a rank m_B, there is a unique equilibrium cluster as follows:

$$G_c(B,m_B) := \{M = \begin{pmatrix} M_B & 0 \\ 0 & M_N \end{pmatrix} \mid M_B\mathbf{1}_B = \mathbf{1}_B,\ M_N\mathbf{1}_N = 0,$$
$$M_B \in G(n_B,m_B),\ M_N \in G(n_N,m_N)\}$$

where $N = \{1,\ldots,n\}\setminus B$ and $n_B = |B|, n_N = n-n_B, m_N = m-m_B$, assuming $n_B \geq m_B \geq 0$ and $n_N \geq m_N \geq 0$. A remarkable fact shown by Lemma 4.6 in [3] is that any tangent to $G_c(B,m_B)$ at M is an eigenvector of $Dh(M)$ for the eigenvalue $\lambda = 0$. As in the literature of dynamical systems, we will also call $G_c(B,m_B)$ a *center manifold*. However, we should notice that $G_c(B,m_B)$ is a very special center manifold because every point in $G_c(B,m_B)$ is an equilibrium.

For an equilibrium cluster $G_c(B,m_B)$, we define

$$\begin{aligned} \mathcal{W}^s(B,m_B) &:= \{M \in G(n,m) : \lim_{t\to+\infty} \phi_t(M) \in G_c(B,m_B)\} = \cup_{\bar{M}\in G_c(B,m_B)}\mathcal{W}^s(\bar{M}) \\ \mathcal{W}^u(B,m_B) &:= \{M \in G(n,m) : \lim_{t\to-\infty} \phi_t(M) \in G_c(B,m_B)\} = \cup_{\bar{M}\in G_c(B,m_B)}\mathcal{W}^u(\bar{M}). \end{aligned}$$

We call both of $\mathcal{W}^s(\bar{M})$ and $\mathcal{W}^s(B,m_B)$ the stable manifold. Their distinction is obvious and thus no confusion will be caused.

Every stable equilibrium $M(B)$ is itself an equilibrium cluster (a singleton). For $G_c(B,m_B) = \{M(B)\}$, all paths in the stable manifold $\mathcal{W}^s(B,m_B)$ converge to the stable equilibrium $M(B)$. Thus, this stable manifold is the attraction region $G(B)$. Stable manifolds associated with unstable equilibrium clusters are boundaries of attraction regions.

The property (4.3) (a manifold is tangent at an equilibrium to an eigenspace) holds true under very mild conditions which are satisfied by most dynamical systems. However, the property (4.3) does not mean that each path is tangent at an equilibrium to an eigenvector. The latter property is significantly stronger than (4.3) and is very useful. It implies that each path is uniquely determined by an equilibrium and an eigenvector. That means, all information about a path is completely carried by an equilibrium and an eigenvector. Noteworthily, equilibria and eigenvectors have very simple structures. Unfortunately, for most dynamical systems, when a path converges to an equilibrium, it need not converge in a constant direction, e.g. it may converge to an equilibrium in circles. Remarkably, every path of the dynamical system $M' = h(M)$ does converge to an equilibrium in a constant direction. This is the main result of this section.

Before the main result, we shall first present some lemmas.

Lemma 4.1 *The map $\Phi : \mathcal{W}^s(B, m_B) \to G_c(B, m_B)$, defined by $\Phi(M) = \lim_{t\to+\infty} \phi_t(M)$, is continuous.*

Proof. For any $\tilde{M}^0 \in \mathcal{W}^s(B, m_B)$, there exists a $\tilde{A} \in R^{m\times n}$ satisfying

$$\tilde{A}^T(\tilde{A}\tilde{A}^T)^{-1}\tilde{A} = \tilde{M}^0.$$

Because the limit point $\Phi(\tilde{M}^0)$ is in $G_c(B, m_B)$, by Corollary 3.5 in [3], $rank(\tilde{M}^0_{BB}) = m_B$ and $rank(\tilde{M}^0_{NN}) = m - m_B$, where $N = \{1, \dots, n\} \setminus B$. Let J with $|J| = m_B$ be an index set such that $\tilde{A}_{JB}$ consists of a maximum set of linearly independent rows of $\tilde{A}_B$ and let $K = \{1, \dots, m\} \setminus J$. We write

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{JB} & \tilde{A}_{JN} \\ \tilde{A}_{KB} & \tilde{A}_{KN} \end{pmatrix}.$$

We can assume $\tilde{A}_{KB} = 0$, because otherwise we can find a nonsingular matrix $\tilde{Q}$ such that $\hat{A} = \tilde{Q}\tilde{A}$ has the submatrix $\hat{A}_{KB} = 0$ and still satisfies $\hat{A}^T(\hat{A}\hat{A}^T)^{-1}\hat{A} = \tilde{M}^0$. Since $\tilde{A}$ is of full row rank and $\tilde{A}_{KB} = 0$, the rows of $\tilde{A}_{KN}$ are linearly independent.

By Lemma 2.7 in [3] we can construct an instance $(\tilde{A}, \tilde{b}, \tilde{c}) = (\tilde{A}, \tilde{A}\mathbf{1}, \mathbf{1})$ which satisfies $\Gamma(\tilde{A}, \tilde{b}, \tilde{c}) = \tilde{M}_0$. Suppose the central path $(\tilde{x}(t), \tilde{s}(t))$ of $(\tilde{A}, \tilde{b}, \tilde{c})$ converges to the strictly complementary solution $(\tilde{x}^*, \tilde{s}^*)$. By Theorem 3.3, (3.9) and (3.10) in [3], the limit point $\Phi(\tilde{M}_0)$ is determined by

$$\Phi(\tilde{M}_0) = \begin{pmatrix} \lfloor \tilde{x}^*_B \rceil \tilde{A}^T_{JB}(\tilde{A}_{JB}\lfloor \tilde{x}^*_B \rceil^2 \tilde{A}^T_{JB})^{-1}\tilde{A}_{JB}\lfloor \tilde{x}^*_B \rceil & 0 \\ 0 & \lfloor \tilde{s}^*_N \rceil^{-1}\tilde{A}^T_{KN}(\tilde{A}_{KN}\lfloor \tilde{s}^*_N \rceil^{-2}\tilde{A}^T_{KN})^{-1}\tilde{A}_{KN}\lfloor \tilde{s}^*_N \rceil^{-1} \end{pmatrix}.$$

Note that $\tilde{x}^*_B$ and $\tilde{s}^*_N$ are analytic centers of the optimal faces of $(\tilde{A}, \tilde{b}, \tilde{c})$ uniquely determined by (B, N), more precisely,

$$\tilde{x}^*_B = \text{argmax}\{\sum_{i\in B} \ln x_i \mid \tilde{A}_B x_B = \tilde{b}, x_B > 0\} \tag{4.6}$$

$$\tilde{y}^* = \text{argmax}\{\sum_{i\in N} \ln(\tilde{c}_i - \tilde{A}^T_i y) \mid \tilde{A}^T_B y = \tilde{c}_B, \tilde{A}^T_N y < \tilde{c}_N\}, \tag{4.7}$$

and $\tilde{s}^*_N = \tilde{c}_N - \tilde{A}^T_N \tilde{y}^*$.

For any $M^0 \in \mathcal{W}^s(B, m_B)$ near $\tilde{M}^0$, by Lemma 3.5, we can construct a matrix

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}_{JB} & \mathcal{A}_{JN} \\ \mathcal{A}_{KB} & \mathcal{A}_{KN} \end{pmatrix}$$

which is close to $\tilde{A}$ and satisfies $\mathcal{A}^T(\mathcal{A}\mathcal{A}^T)^{-1}\mathcal{A} = M^0$. Since $\mathcal{A}$ is close to $\tilde{A}$, $\mathcal{A}_{JB}$ remains full row rank (as $\tilde{A}_{JB}$) and $\mathcal{A}_{KB} \approx 0$. Since $|J| = m_B$, $\mathcal{A}_{JB}$ consists of a maximum set of linearly independent rows of $\mathcal{A}_B$. Thus, there exists a matrix Q such that $\mathcal{A}_{KB} = Q\mathcal{A}_{JB}$. Denote $D = \begin{pmatrix} I & 0 \\ -Q & I \end{pmatrix}$ and $A = D\mathcal{A}$. Then

$$A = \begin{pmatrix} \mathcal{A}_{JB} & \mathcal{A}_{JN} \\ 0 & \mathcal{A}_{KN} - Q\mathcal{A}_{JN} \end{pmatrix}$$

satisfies $A^T(AA^T)^{-1}A = M^0$ and $A_{KB} = 0$. Since $\mathcal{A}_{KB} \approx 0$ implies $Q \approx 0$, A is close to $\mathcal{A}$ and thus close to $\tilde{A}$.

Let $(A, b, c) = (A, A\mathbf{1}, \mathbf{1})$. Then $\Gamma(A, b, c) = M^0$. Let (x^*, s^*) be the strictly complementary solution of (A, b, c). By Theorem 3.3, (3.9) and (3.10) in [3], the limit matrix $\Phi(M^0)$ is determined by

$$\Phi(M_0) = \begin{pmatrix} \lfloor x_B^* \rceil A_{JB}^T (A_{JB} \lfloor x_B^* \rceil^2 A_{JB}^T)^{-1} A_{JB} \lfloor x_B^* \rceil & 0 \\ 0 & \lfloor s_N^* \rceil^{-1} A_{KN}^T (A_{KN} \lfloor s_N^* \rceil^{-2} A_{KN}^T)^{-1} A_{KN} \lfloor s_N^* \rceil^{-1} \end{pmatrix}.$$

Because both M^0 and $\tilde{M}^0$ belonging to $\mathcal{W}^s(B, m_B)$, the optimal solutions (x^*, s^*) and $(\tilde{x}^*, \tilde{s}^*)$ have the same basis (B, N). Thus the (x_B^*, s_N^*) is the determined by (4.6) and (4.7) with $(\tilde{A}, \tilde{b}, \tilde{c})$ being replaced by (A, b, c). With (A, b, c) close to $(\tilde{A}, \tilde{b}, \tilde{c})$, (x_B^*, s_N^*) is close to $(\tilde{x}_B^*, \tilde{s}_N^*)$.

As $M_0 \to \tilde{M}_0$, we have $(A, b, c) \to (\tilde{A}, \tilde{b}, \tilde{c})$ and $(x_B^*, s_N^*) \to (\tilde{x}_B^*, \tilde{s}_N^*)$, thus $\Phi(M_0) \to \Phi(\tilde{M}_0)$. This shows the continuity of Φ. □

Grönwall inequality has many variants. The following is one of them. These variants have different forms, but they can be proved in similar ways. For completeness, we give the proof below.

Lemma 4.2 *(Grönwall inequality). Let $\varphi(t) \geq 0$ be continuous for $t \geq 0$ and $\int_0^\infty \varphi(t)dt < \infty$. Suppose that, for $t \geq 0$, $u(t) \geq 0$ is continuous and satisfies the inequality*

$$u(t) \leq K + \int_0^t \varphi(s) u^{1+\alpha}(s) ds$$

for some constants $K > 0$ and $\alpha \geq 0$.

(i) If $\alpha > 0$ and $\alpha K^\alpha \int_0^\infty \varphi(s)ds < 1$, then

$$u(t) \leq \frac{K}{[1 - \alpha K^\alpha \int_0^t \varphi(s)ds]^{1/\alpha}}, \qquad \forall\, t \in [0, \infty).$$

(ii) If $\alpha = 0$, then

$$u(t) \leq K \exp(\int_0^t \varphi(s)ds), \qquad \forall\, t \in [0, \infty).$$

Proof. Let

$$w(t) = K + \int_0^t \varphi(s) u^{1+\alpha}(s) ds.$$

Then $u(t) \leq w(t)$ and

$$w'(t) = \varphi(t) u^{1+\alpha}(t) \leq \varphi(t) w^{1+\alpha}(t).$$

That is,

$$\frac{w'(t)}{w^{1+\alpha}(t)} \leq \varphi(t). \tag{4.8}$$

If $\alpha > 0$, integrating on both sides of (4.8) yields

$$-\frac{1}{\alpha}(w^{-\alpha}(t) - K^{-\alpha}) \leq \int_0^t \varphi(s)ds.$$

It follows

$$w(t) \leq \frac{K}{[1 - \alpha K^\alpha \int_0^t \varphi(s)ds]^{1/\alpha}}.$$

on $t \geq 0$. This shows part (i) of the lemma because $u(t) \leq w(t)$.

If $\alpha = 0$, integrating on both sides of (4.8) yields

$$\ln w(t) - \ln K \leq \int_0^t \varphi(s)ds.$$

It follows that

$$w(t) \leq K \exp(\int_0^t \varphi(s)ds).$$

This shows part (ii) of the lemma. □

Now we begin to show that, for the dynamical system $M' = h(M)$, any *equilibrium-eigenvector pair*, i.e. a pair consisting of an equilibrium $\bar{M}$ and an eigenvector $\bar{U}$ of $Dh(\bar{M})$, determines a unique path. It is well known that each regular (nonequilibrium) point, as an initial point, determines a unique path. However, an equilibrium point, as an initial point, does not determine a path, because the vector field at an equilibrium is zero. In order to determine a path starting from an equilibrium, we need, in addition, a direction. Now the question is whether there exists a unique path for each equilibrium-direction pair. For most dynamical systems, the answer is negative. Fortunately, for the dynamical system $M' = h(M)$, due to its simple spectral structure, we will be able to show the existence and uniqueness of a path starting from an equilibrium along an eigenvector direction. First, we show this property for the simplest case as in the following lemma.

Lemma 4.3 *Let $\mathcal{N}_0$ be an open neighborhood of 0 in R^n. Suppose that $g : \mathcal{N}_0 \to R^n$ is continuously differentiable with Lipschitz continuous Dg at 0 and satisfies $g(0) = 0$ and $Dg(0) = 0$. Let $x(t, x^0)$ be the solution of*

$$x' = \lambda x + g(x), \quad x(0, x^0) = x^0, \tag{4.9}$$

where $\lambda \neq 0$ is a constant. Then the following statements hold.

(i) There exists a $\delta > 0$ such that for any $x^0 \in R^n$ with $\|x^0\| \leq \delta$, the solution $x(t, x^0)$ of (4.9) satisfies

$$x(t, x^0) = e^{\lambda t} v + O(e^{2\lambda t} \|x^0\|^2), \tag{4.10}$$

for $\lambda t \to -\infty$ (i.e., $t \to -\infty$ if $\lambda > 0$ and $t \to \infty$ if $\lambda < 0$), where

$$v = x^0 + \int_0^\infty e^{-\lambda s} g(x(s, x^0)) ds. \tag{4.11}$$

Furthermore,

$$\int_0^\infty e^{-\lambda s} g(x(s, x^0)) ds = O(\|x^0\|^2). \tag{4.12}$$

(ii) Let Ψ be a map defined by $\Psi(x^0) = x^0 + \int_0^\infty e^{-\lambda s} g(x(s, x^0)) ds$. There exist open neighborhoods of 0, $\mathcal{N}$ and $\mathcal{M}$ in R^n, such that the map $\Psi : \mathcal{N} \to \mathcal{M}$ is invertible. The map Ψ and its inverse Ψ^{-1} are continuously differentiable in $\mathcal{N}$ and $\mathcal{M}$, respectively.

Proof. We first prove for the case $\lambda < 0$.

Proof of (i). $x(t, x^0)$ is a solution of (4.9) if and only if it satisfies

$$x(t, x^0) = e^{\lambda t} x^0 + \int_0^t e^{\lambda(t-s)} g(x(s, x^0)) ds. \tag{4.13}$$

By the assumption of g, there exist positive numbers σ and δ_0 such that

$$\|g(x)\| \leq \sigma \|x\|^2, \quad \forall \, \|x\| \leq \delta_0.$$

Because the Jacobian of the right hand side of (4.9) at $x = 0$ is λI, $x = 0$ is a stable point of (4.9). Thus, there is $\delta_1 > 0$ such that for any $\|x^0\| \leq \delta_1$ the solution $x(t, x^0)$ satisfies

$$\|x(t, x^0)\| \leq \delta_1 \quad \forall \, t \geq 0.$$

Thus, for $\|x^0\| \le \min\{\delta_0, \delta_1\}$, we have

$$\begin{aligned}
\frac{\|x(t,x^0)\|}{e^{\lambda t}} &\le \|x^0\| + \int_0^t e^{-\lambda s}\|g(x(s,x^0))\|ds \\
&\le \|x^0\| + \int_0^t e^{-\lambda s}\sigma\|x(s,x^0)\|^2 ds \\
&= \|x^0\| + \int_0^t e^{\lambda s}\sigma\left(\frac{\|x(s,x^0)\|}{e^{\lambda s}}\right)^2 ds
\end{aligned}$$

By Grönwall inequality ($\alpha = 1$),

$$\begin{aligned}
\frac{\|x(t,x^0)\|}{e^{\lambda t}} &\le \frac{\|x^0\|}{1 - \|x^0\|\sigma\int_0^t e^{\lambda s}ds} \\
&\le \frac{\|x^0\|}{1 - \|x^0\|\sigma|\lambda|^{-1}}
\end{aligned}$$

Let $\delta > 0$ satisfy $\delta\sigma|\lambda|^{-1} < 1$, and denote $\beta = (1 - \delta\sigma|\lambda|^{-1})^{-1}$. Then for any $\|x^0\| \le \delta$, we have

$$\frac{\|x(t,x^0)\|}{e^{\lambda t}} \le \beta\|x^0\|, \quad \forall\, t \ge 0. \tag{4.14}$$

This leads to

$$\begin{aligned}
\int_t^\infty e^{-\lambda s}\|g(x(s,x^0))\|ds &\le \int_t^\infty e^{\lambda s}\sigma\left(\frac{\|x(s,x^0)\|}{e^{\lambda s}}\right)^2 ds \\
&\le \int_t^\infty e^{\lambda s}\sigma\beta^2\|x^0\|^2 ds \\
&= e^{\lambda t}\sigma\beta^2|\lambda|^{-1}\|x^0\|^2, \quad \forall\, t \ge 0.
\end{aligned} \tag{4.15}$$

Now, it follows from (4.13) that

$$\begin{aligned}
x(t,x^0) &= e^{\lambda t}v - e^{\lambda t}\int_t^\infty e^{-\lambda s}g(x(s,x^0))ds \\
&= e^{\lambda t}v + O(e^{2\lambda t}\|x^0\|^2).
\end{aligned}$$

This shows (4.10).

Taking $t = 0$ in (4.15), we have $\int_0^\infty e^{-\lambda s}g(x(s,x^0))ds \le \sigma\beta^2|\lambda|^{-1}\|x^0\|^2$. This shows (4.12).

Proof of (ii). For any fixed t, regard $x(t,\cdot) : x^0 \to x(t,x^0)$ as a map, where $x(t,x^0)$ is determined by (4.13). We first estimate the Jacobian of the map $x(t,\cdot)$.

$$Dx(t,x^0) = e^{\lambda t}I + \int_0^t e^{\lambda(t-s)}Dg(x(s,x^0))Dx(s,x^0)ds.$$

By conditions on g, there exist $\delta_0 > 0$ and $\sigma_1 > 0$ such that $\|Dg(x)\| \leq \sigma_1 \|x\|$ for all $x \in R^n$ with $\|x\| \leq \delta_0$. Then, together with (4.14), we have

$$\begin{aligned} \frac{\|Dx(t,x^0)\|}{e^{\lambda t}} &\leq 1 + \int_0^t e^{-\lambda s} \|Dg(x(s,x^0))\| \cdot \|Dx(s,x^0)\| ds \\ &\leq 1 + \int_0^t \sigma_1 e^{-\lambda s} \|x(s,x^0)\| \cdot \|Dx(s,x^0)\| ds \\ &\leq 1 + \int_0^t \sigma_1 \beta e^{\lambda s} \|x^0\| \cdot \frac{\|Dx(s,x^0)\|}{e^{\lambda s}} ds \end{aligned}$$

By Grönwall inequality ($\alpha = 0$),

$$\begin{aligned} \frac{\|Dx(t,x^0)\|}{e^{\lambda t}} &\leq \exp\left(\int_0^t \sigma_1 \beta e^{\lambda s} \|x^0\| ds\right) \\ &\leq \exp(\sigma_1 \beta \delta_0 / |\lambda|), \quad \forall\, t \geq 0, \|x^0\| \leq \delta_0. \end{aligned} \tag{4.16}$$

The Jacobian of Ψ is

$$D\Psi(x^0) = I + \int_0^{+\infty} e^{-\lambda s} Dg(x(s,x^0)) Dx(s,x^0) ds.$$

Using $\|Dg(x)\| \leq \sigma_1 \|x\|$, (4.14) and (4.16), we have

$$\begin{aligned} \left\| \int_0^{+\infty} e^{-\lambda s} Dg(x(s,x^0)) Dx(s,x^0) ds \right\| &\leq \int_0^{+\infty} \sigma_1 \beta \exp(\sigma_1 \beta \delta_0 / |\lambda|) \|x^0\| e^{\lambda s} ds \\ &= (\sigma_1 \beta / |\lambda|) \exp(\sigma_1 \beta \delta_0 / |\lambda|) \|x^0\|, \end{aligned}$$

Therefore, for $\|x^0\| < |\lambda| (\sigma_1 \beta)^{-1} \exp(-\sigma_1 \beta \delta_0 / |\lambda|)$, the Jacobian $D\Psi(x^0)$ is bounded and non-singular.

By the inverse function theorem, there exist an open neighborhood $\mathcal{N} = \{x^0 \in R^n \mid \|x^0\| < \delta\}$ for some $\delta > 0$, such that Ψ is an one-to-one map from $\mathcal{N}$ to $\mathcal{M} = \Psi(\mathcal{N})$, and $\mathcal{M}$ is an open neighborhood of $\Psi(0) = 0$ in R^n. Furthermore, the map Ψ and its inverse Ψ^{-1} are continuously differentiable in $\mathcal{N}$ and $\mathcal{M}$, respectively. (ii) is proved.

For $\bar{\lambda} > 0$ and $\bar{t} \to -\infty$, we need only a transformation $t = -\bar{t}$. Denote $x(t) = \bar{x}(\bar{t})$ and $\lambda = -\bar{\lambda}$. Then $x'(t) = -\bar{x}'(\bar{t})$. Thus

$$\bar{x}'(\bar{t}) = \bar{\lambda} \bar{x}(\bar{t}) + g(\bar{x}(\bar{t}))$$

is equivalent to

$$x'(t) = -\bar{\lambda} \bar{x}(\bar{t}) - g(\bar{x}(\bar{t})) = \lambda x(t) - g(x(t)).$$

Thus, the above proof applies. □

For almost all of equilibria of $M' = h(M)$, the Jacobian $Dh(M)$ has more than one eigenvalues, thus Lemma 4.3 cannot be used directly. Our strategy is applying Lemma 4.3 to certain stable/unstable submanifolds each of which is associated with only one eigenvalue. This can be achieved because the spectral structure of $Dh(M)$ is extremely simple. There are only three eigenvalues: 0, 1 and -1.

Theorem 4.4 *(i) For any equilibrium point $\bar{M}$ and any eigenvector $\bar{U}$ of $Dh(\bar{M})$ associated with a nonzero eigenvalue λ ($\lambda = 1$ or -1), there exists a unique $M^0 \in G(n, m)$ such that the solution $M(t)$ of $M' = h(M)$ with $M(0) = M^0$ satisfies*

$$M(t) = \bar{M} + e^{\lambda t}\bar{U} + O(e^{2\lambda t}), \tag{4.17}$$

for $\lambda t \to -\infty$.

(ii) For any $M^0 \in \mathcal{W}^s(\bar{M})$, there exists a unique $\bar{U} \in E^-(\bar{M})$ such that the solution $M(t)$ of $M' = h(M)$ with $M(0) = M^0$ satisfies (4.17) for $\lambda = -1$. Furthermore, the map from $M^0 \in \mathcal{W}^s(\bar{M})$ to $\bar{U} \in E^-(\bar{M})$ is continuous. The same result holds for $\mathcal{W}^u(\bar{M})$, $\lambda = 1$ and $\bar{U} \in E^+(\bar{M})$.

Proof. In the following proof, we will only consider $\lambda = -1$. For $\lambda = 1$, the proof is precisely the same, with only the stable manifold changed to the unstable manifold.

Let $\mathcal{W}^s(\bar{M})$ be of dimension k and let $\varphi : \mathcal{U} \to \mathcal{V}$ be a diffeomorphism with $\varphi(\bar{M}) = 0$, where $\mathcal{U} \subset \mathcal{W}^s(\bar{M})$ is an open neighborhood of $\bar{M}$ and $\mathcal{V} \subset R^k$ is an open neighborhood of 0.

From $M' = h(M)$ we can derive an ODE on R^k in terms of $x = \varphi(M)$:

$$\begin{aligned} x' &= D\varphi(M)M' \\ &= D\varphi(M)h(M) \\ &= D\varphi(\varphi^{-1}(x))h(\varphi^{-1}(x)) =: f(x). \end{aligned}$$

At $x = 0$, we have

$$f(0) = D\varphi(\varphi^{-1}(0))h(\varphi^{-1}(0)) = D\varphi(\bar{M})h(\bar{M}) = 0,$$

The Jacobian of f is

$$Df(x) = D^2\varphi(\varphi^{-1}(x))D\varphi^{-1}(x)h(\varphi^{-1}(x)) + D\varphi(\varphi^{-1}(x))Dh(\varphi^{-1}(x))D\varphi^{-1}(x).$$

Because $\bar{M} = \varphi^{-1}(0)$ and $h(\bar{M}) = 0$,

$$Df(0) = D\varphi(\bar{M})Dh(\bar{M})D\varphi^{-1}(0). \tag{4.18}$$

For any $v \in R^k$ and small $t \in R$, $\varphi^{-1}(tv)$ is a curve on $\mathcal{W}^s(\bar{M})$ passing through $\varphi^{-1}(0) = \bar{M}$. Thus, the vector $\frac{d}{dt}\varphi^{-1}(tv)|_{t=0}$ is a tangent of $\mathcal{W}^s(\bar{M})$ at $\bar{M}$. Since $\frac{d}{dt}\varphi^{-1}(tv)|_{t=0} = D\varphi^{-1}(0)v$ and $D\varphi^{-1}(x) = [D\varphi(\varphi^{-1}(x))]^{-1} = [D\varphi(M)]^{-1}$, we see that $[D\varphi(M)]^{-1}v \in T_{\bar{M}}(\mathcal{W}^s(\bar{M}))$. Conversely, for any $\bar{U} \in T_{\bar{M}}(\mathcal{W}^s(\bar{M}))$, there exists a curve $M(t)$ with $M(0) = \bar{M}$ and $M'(0) = \bar{U}$. Let $\xi(t) = \varphi(M(t))$. Then $\xi(t)$ is a curve in $\mathcal{V}$ passing through $x = 0$. Let $v = \xi'(0)$. Then $v \in R^k$ and $v = D\varphi(M(0))M'(0) = D\varphi(\bar{M})\bar{U}$. Thus, $\bar{U} = [D\varphi(\bar{M})]^{-1}v$. This shows

$$T_{\bar{M}}(\mathcal{W}^s(\bar{M})) = \{[D\varphi(\bar{M})]^{-1}v \mid v \in R^k\}. \tag{4.19}$$

This, combined with (4.3), implies

$$E^-(\bar{M}) = \{[D\varphi(\bar{M})]^{-1}v \mid v \in R^k\}. \tag{4.20}$$

This shows that $[D\varphi(\bar{M})]^{-1}v$ is an eigenvector of $Dh(\bar{M})$ for $\lambda = -1$, i.e.,

$$Dh(\bar{M})[D\varphi(\bar{M})]^{-1}v = \lambda[D\varphi(\bar{M})]^{-1}v, \quad \forall\, v \in R^k.$$

This yields

$$Dh(\bar{M})[D\varphi(\bar{M})]^{-1} = \lambda[D\varphi(\bar{M})]^{-1}.$$

It follows from (4.18) that

$$\begin{aligned} Df(0) &= D\varphi(\bar{M})Dh(\bar{M})[D\varphi(\bar{M})]^{-1} \\ &= \lambda D\varphi(\bar{M})[D\varphi(\bar{M})]^{-1} \\ &= \lambda I. \end{aligned}$$

Now we denote $g(x) = f(x) - \lambda x$ and write $x' = f(x)$ as

$$x' = \lambda x + g(x).$$

Then g is analytic on the open set $\mathcal{V}$ of R^k and satisfies $g(0) = 0$ and $Dg(0) = Df(0) - \lambda I = 0$. By Lemma 4.3 (ii), for any $v \in R^k$ with small $\|v\|$, there exists a unique x^0 such that the solution $x(t, x^0)$ of $x' = \lambda x + g(x)$ with $x(0, x^0) = x^0$ satisfies

$$x(t, x^0) = e^{\lambda t}v + O(e^{2\lambda t}), \quad \forall\, \lambda t \le 0. \tag{4.21}$$

For any eigenvector $\bar{U}$ of $Dh(\bar{M})$ for $\lambda = -1$, by (4.20) there is a $v \in R^k$ such that $\bar{U} = [D\varphi(\bar{M})]^{-1}v$. Let the initial point x^0 and the solution $x(t, x^0)$ be determined by v satisfying (4.21), let $M^0 = \varphi^{-1}(x^0)$ and $M(t, M^0) = \varphi^{-1}(x(t, x^0))$. Then, $M(t, M^0)$ is the solution of $M' = h(M)$ with $M(0, M^0) = M^0$, satisfying

$$\begin{aligned} M(t, M^0) &= \varphi^{-1}(x(t, x^0)) \\ &= \varphi^{-1}(0) + D\varphi^{-1}(0)x(t, x^0) + O(\|x(t, x^0)\|^2) \\ &= \bar{M} + [D\varphi(\bar{M})]^{-1}x(t, x^0) + O(\|x(t, x^0)\|^2) \\ &= \bar{M} + e^{\lambda t}[D\varphi(\bar{M})]^{-1}v + O(e^{2\lambda t}) \\ &= \bar{M} + e^{\lambda t}\bar{U} + O(e^{2\lambda t}). \end{aligned}$$

The uniqueness of the initial point M^0 which satisfies the above can be shown as follows. Let M^0 and $\tilde{M}^0$ be initial points which correspond to the same $\bar{M}$ and $\bar{U}$. Let $x^0 = \varphi(M^0)$ and $\tilde{x}^0 = \varphi(\tilde{M}^0)$. Then, the solutions $x(t, x^0)$ and $x(t, \tilde{x}^0)$ converge to the same point $0 = \varphi(\bar{M})$ in the same direction $v = [D\varphi^{-1}(0)]^{-1}\bar{U}$. By Lemma 4.3 (ii), we have $x^0 = \tilde{x}^0$. This implies $M^0 = \varphi^{-1}(x^0) = \varphi^{-1}(\tilde{x}^0) = \tilde{M}^0$. Therefore, the initial point M^0 is uniquely determined by $\bar{M}$ and $\bar{U}$. This proves (i).

For any $M^0 \in \mathcal{W}^s(\bar{M})$, let $x^0 = \varphi(M^0)$. Define $v = x^0 + \int_0^\infty e^{-\lambda s} g(x(s, x^0))ds$ and $\bar{U} = D\varphi^{-1}(0)v$. Then the solution $M(t, M^0)$ of $M' = h(M)$ with $M(0, M^0) = M^0$ satisfies

$$M(t, M^0) = \bar{M} + e^{\lambda t}\bar{U} + O(e^{2\lambda t}).$$

By Lemma 4.3, the map from x^0 to v is continuous, thus the map from M^0 to $\bar{U}$ is continuous. This proves (ii). □

Corollary 4.5 *There are one-to-one correspondences between*

(i) a point $M \in G(n, m)$,

(ii) a path $M(t)$,

(iii) an equilibrium-eigenvector pair $(\bar{M}, \bar{U})$ for $\lambda = -1$, and

(iv) an equilibrium-eigenvector pair $(\bar{M}, \bar{U})$ for $\lambda = 1$.

A stable manifold $\mathcal{W}^s(B, m_B)$ consists of paths which converge to equilibria in $G_c(B, m_B)$ in directions of eigenvectors. Thus, the stable manifold $\mathcal{W}^s(B, m_B)$ is isomorphic to the set of equilibrium-eigenvector pairs

$$\Sigma^-(B, m_B) := \{(M, U) \mid M \in G_c(B, m_B),\ U \in E^-(M)\}. \tag{4.22}$$

We refer to $\Sigma^-(B, m_B)$ as the *sink* of the stable manifold $\mathcal{W}^s(B, m_B)$. Similarly, an unstable manifold $\mathcal{W}^u(B, m_B)$ is isomorphic to the *source*

$$\Sigma^+(B, m_B) := \{(M, U) \mid M \in G_c(B, m_B),\ U \in E^+(M)\}. \tag{4.23}$$

The analysis of the sinks/sources is easier than the analysis of the stable/unstable manifolds, because equilibria and eigenvectors have simple structures. We will investigate $\Sigma^-(B, m_B)$ and $\Sigma^+(B, m_B)$ in details in Section 6.

5 LP representation of equilibrium-eigenvector pairs

We can map an LP instance (A, b, c) to a projection matrix $M = \Gamma(A, b, c)$, then define a path $M(t)$ by $M' = h(M)$ with $M(0) = \Gamma(A, b, c)$. This path $M(t)$ converges to an equilibrium $\bar{M}$ in a direction $\bar{U}$ in the sense of (4.17). We will say that (A, b, c) is mapped to $(\bar{M}, \bar{U})$ and denote this map by

$$(\bar{M}, \bar{U}) = \Lambda(A, b, c).$$

Conversely, given an equilibrium-eigenvector pair $(\bar{M}, \bar{U})$, in this section we will construct an instance (A, b, c) which is mapped to $(\bar{M}, \bar{U})$. We will call such an instance (A, b, c) an *LP representation* of $(\bar{M}, \bar{U})$.

The motivations of LP representations are of two-fold. First, the characterization of the partition on the space of (A, b, c) is our ultimate goal. Thus, we need an LP representation of $(\bar{M}, \bar{U})$ to carry the partition of the space of $(\bar{M}, \bar{U})$ onto the partition of the space of (A, b, c). Second, the central path $x(t)$ and the optimal solution x^* of an LP instance (A, b, c) are helpful in constructing the corresponding path $M(t)$ and the limiting equilibrium $\bar{M}$. For example, if we want to find the sink of a given source, we can construct an LP representation (A, b, c) of the given source (M_-, U_-) and find the optimal solution x^* of (A, b, c), then construct the sink point M_+ by Theorem 3.3 in [3].

One will see many applications of the LP representation to the characterization of attraction regions and their boundaries in our coming papers.

In this section, we will show how to construct an LP instance for a path in terms of the corresponding equilibrium-eigenvector pair. Finding an LP representation of $(\bar{M}, \bar{U})$ turns out to be nontrivial due to the fact that the pair $(\bar{M}, \bar{U})$ is a limiting feature of a path.

Theorem 5.1 *For any $\bar{M} \in G(n,m)$ with $\bar{M}\mathbf{1} = 0$ and any eigenvector $\bar{U} = h_{\bar{M}}(d)$, with $\bar{M}d = d$, of $Dh(\bar{M})$ for $\lambda = 1$, let $\bar{A} \in R^{m\times n}$ satisfy $\bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A} = \bar{M}$. Then*

$$\Lambda(\bar{A}, \bar{A}d, \mathbf{1}) = (\bar{M}, \bar{U}).$$

Proof. First, we show that the instance $(A, b, c) = (\bar{A}, \bar{A}d, \mathbf{1})$ is strictly feasible. Note that $\bar{M}\mathbf{1} = 0$ implies $\bar{A}\mathbf{1} = 0$. Thus, for any $\alpha \in R$, $x = d + \alpha\mathbf{1}$ satisfies $\bar{A}x = \bar{A}d$. Choosing $\alpha > 0$ sufficiently large, we can have $x = d + \alpha\mathbf{1} > 0$. Now, $y = 0$ and $s = \mathbf{1} > 0$ satisfies $\bar{A}^T y + s = \mathbf{1}$. Thus, $(\bar{A}, \bar{A}d, \mathbf{1})$ has strictly feasible solutions.

For given $\bar{M}$ and $\bar{U}$, by Theorem 4.4 (i), there exists a unique (initial) point M^0 such that the unique solution $M(t)$ of $M' = h(M)$ with $M(0) = M^0$ satisfies (4.17). In order to show $\Lambda(\bar{A}, \bar{A}d, \mathbf{1}) = (\bar{M}, \bar{U})$, we need only to show that the path $M(t)$ is defined by $(\bar{A}, \bar{A}d, \mathbf{1})$, namely, $\Gamma(\bar{A}, \bar{A}d, \mathbf{1}) = M^0$.

First, we assume that $\bar{A}^T\bar{A} = \bar{M}$.

Because $M(-k) \to \bar{M}$ as $k \to \infty$, by Lemma 3.5, there exist $A_k \in R^{m\times n}$ satisfying $A_k^T A_k = M(-k)$ and $A_k \to \bar{A}$ as $k \to \infty$. Then, it follows from (4.17) that

$$\begin{aligned} A_k^T A_k \mathbf{1} &= M(-k)\mathbf{1} \\ &= e^{-k}\bar{U}\mathbf{1} + O(e^{-2k}) \\ &= e^{-k}d + O(e^{-2k}). \end{aligned}$$

Since $A_k^T A_k$ is a projection matrix, by Lemma 3.4, we have $A_k A_k^T = I$. Thus,

$$A_k\mathbf{1} = e^{-k}A_k d + O(e^{-2k}).$$

Let $f_k = e^k A_k \mathbf{1}$. Since $A_k \to \bar{A}$, we have $f_k \to \bar{A}d$ as $k \to \infty$.

Because $A_k^T A_k = M(-k)$, by Lemma 2.7 in [3], we have $\Gamma(A_k, A_k\mathbf{1}, \mathbf{1}) = M(-k)$, namely, $\Gamma(A_k, e^{-k}f_k, \mathbf{1}) = M(-k)$. By Lemma 3.1, $\Gamma(A_k, f_k, \mathbf{1}) = M(0) = M^0$.

Since $\Gamma : SLP(n,m) \to G(n,m)$ is continuous, we have

$$M^0 = \lim_{k\to\infty} \Gamma(A_k, f_k, \mathbf{1}) = \Gamma(\bar{A}, \bar{A}d, \mathbf{1}).$$

Now, consider any $\bar{A}$ satisfying $\bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A} = \bar{M}$. Let $Q = (\bar{A}\bar{A}^T)^{-1/2}$ and $\tilde{A} = Q\bar{A}$. Then $\tilde{A}^T\tilde{A} = \bar{M}$. As we showed above, $\Lambda(\tilde{A}, \tilde{A}d, \mathbf{1}) = (\bar{M}, \bar{U})$.

Because for any invertible matrix $Q \in R^{m\times m}$, (QA, Qb, c) and (A, b, c) define the same path $M(t)$, we have

$$\Lambda(\bar{A}, \bar{A}d, \mathbf{1}) = \Lambda(Q\bar{A}, Q\bar{A}d, \mathbf{1}) = \Lambda(\tilde{A}, \tilde{A}d, \mathbf{1}) = (\bar{M}, \bar{U}).$$

□

Theorem 5.2 *For any $\bar{M} \in G(n, m)$ with $\bar{M}\mathbf{1} = \mathbf{1}$ and any eigenvector $\bar{U} = h_{\bar{M}}(d)$, with $\bar{M}d = 0$, of $Dh(\bar{M})$ for $\lambda = 1$, let $\bar{A} \in R^{m\times n}$ satisfy $\bar{A}^T(\bar{A}\bar{A}^T)^{-1}\bar{A} = \bar{M}$. Then*

$$\Lambda(\bar{A}, \bar{A}\mathbf{1}, -d) = (\bar{M}, \bar{U}).$$

Proof. Let $\hat{M} = I - \bar{M}$. Then $\hat{M}\mathbf{1} = 0$ and $\hat{M}d = d$. Choose a full row rank $W \in R^{(n-m)\times n}$ with $\bar{A}W^T = 0$. Then, we have $\hat{M} = W^T(WW^T)^{-1}W$. Let $\hat{U} = h_{\hat{M}}(-d)$. Then, by Theorem 5.1, the path $\hat{M}(t)$ defined by $(W, -Wd, \mathbf{1})$ satisfies

$$\hat{M}(t) = \hat{M} + e^t\hat{U} + O(e^{2t}) = \hat{M} + e^t h_{\hat{M}}(-d) + O(e^{2t}) \qquad \text{for } t \to -\infty.$$

Let $M(t) = I - \hat{M}(t)$. Since $\bar{A}W^T = 0$, by Lemma 3.6, the path $M(t)$ is associated with $(\bar{A}, \bar{A}\mathbf{1}, -d)$.

Since $h_{I-M}(d) = h_M(d)$, we have

$$M(t) = \bar{M} - e^t h_{\bar{M}}(-d) + O(e^{2t}) = \bar{M} + e^t h_{\bar{M}}(d) + O(e^{2t}) = \bar{M} + e^t\bar{U} + O(e^{2t}).$$

□

In general, we have

Theorem 5.3 *Let $\bar{M} = \begin{pmatrix} \bar{M}_B & 0 \\ 0 & \bar{M}_N \end{pmatrix}$ with $\bar{M}_B\mathbf{1}_B = \mathbf{1}_B$ and $\bar{M}_N\mathbf{1}_N = 0$ and $\bar{U} = h_{\bar{M}}(d) = \begin{pmatrix} h_{\bar{M}_B}(d_B) & 0 \\ 0 & h_{\bar{M}_N}(d_N) \end{pmatrix}$ with $\bar{M}_Bd_B = 0$ and $\bar{M}_Nd_N = d_N$. Here $\bar{U}$ is an eigenvector of $Dh(\bar{M})$ for $\lambda = 1$. Let $A_{JB} \in R^{m_B\times n_B}$ and $A_{KN} \in R^{m_N\times n_N}$ satisfy $A_{JB}^T(A_{JB}A_{JB}^T)^{-1}A_{JB} = \bar{M}_B$ and $A_{KN}^T(A_{KN}A_{KN}^T)^{-1}A_{KN} = \bar{M}_N$. Then we can construct an instance (A, b, c),*

$$A = \begin{pmatrix} A_{JB} & 0 \\ 0 & A_{KN} \end{pmatrix}, \quad b = \begin{pmatrix} A_{JB}\mathbf{1}_B \\ A_{KN}d_N \end{pmatrix}, \quad c = \begin{pmatrix} -d_B \\ \mathbf{1}_N \end{pmatrix},$$

such that

$$\Lambda(A, b, c) = (\bar{M}, \bar{U}).$$

Proof. Applying Theorem 5.1 to $(\bar{M}_B, h_{\bar{M}_B}(d_B))$ and Theorem 5.2 to $(\bar{M}_N, h_{\bar{M}_N}(d_N))$, we know that the instances $(A_{JB}, A_{JB}\mathbf{1}_B, -d_B)$ and $(A_{KN}, A_{KN}d_N, \mathbf{1}_N)$ define paths $M_B(t)$ and $M_N(t)$ satisfying

$$M_B(t) = \bar{M}_B + e^t h_{\bar{M}_B}(d_B) + O(e^{2t}), \quad M_N(t) = \bar{M}_N + e^t h_{\bar{M}_N}(d_N) + O(e^{2t}), \quad t \to -\infty. \tag{5.1}$$

Let $x_B(t)$ and $x_N(t)$ be the central paths of these two instances, i.e. they satisfy the system (2.5). Then it is easy to see that $x(t) = (x_B(t), x_N(t))$ is the central path of the instance (A, b, c) defined in the theorem. Therefore, the path defined by (A, b, c) can be written as

$$\begin{aligned} M(t) &= \lfloor x(t) \rceil A^T (A \lfloor x(t) \rceil^2 A^T)^{-1} A \lfloor x(t) \rceil \\ &= \begin{pmatrix} M_B(t) & 0 \\ 0 & M_N(t) \end{pmatrix}, \end{aligned}$$

where $M_B(t) = \Upsilon(A_{JB} \lfloor x_B(t) \rceil)$ and $M_N(t) = \Upsilon(A_{KN} \lfloor x_N(t) \rceil)$.

Since $x_B(t)$ and $x_N(t)$ are the central paths of $(A_{JB}, A_{JB}\mathbf{1}_B, -d_B)$ and $(A_{KN}, A_{KN}d_N, \mathbf{1}_N)$, the paths $M_B(t)$ and $M_N(t)$ defined by them satisfy (5.1). Hence,

$$M(t) = \bar{M} + e^t h_{\bar{M}}(d) + O(e^{2t}), \quad t \to -\infty.$$

□

There is another type of eigenvectors for $\lambda = 1$ which we consider in the following theorem.

Theorem 5.4 *Let* $\bar{M} = \begin{pmatrix} \bar{M}_B & 0 \\ 0 & \bar{M}_N \end{pmatrix}$ *with* $\bar{M}_B \mathbf{1}_B = \mathbf{1}_B$ *and* $\bar{M}_N \mathbf{1}_N = 0$ *and* $\bar{U} = \begin{pmatrix} 0 & U_0 \\ U_0^T & 0 \end{pmatrix}$ *with* $\bar{M}_B U_0 = 0$ *and* $U_0 \bar{M}_N = U_0$*. Here* $\bar{U}$ *is an eigenvector of* $Dh(\bar{M})$ *for* $\lambda = 1$*. Let* $A_{JB} \in R^{m_B \times n_B}$ *and* $A_{KN} \in R^{m_N \times n_N}$ *satisfy* $A_{JB}^T (A_{JB} A_{JB}^T)^{-1} A_{JB} = \bar{M}_B$ *and* $A_{KN}^T (A_{KN} A_{KN}^T)^{-1} A_{KN} = \bar{M}_N$*. Then we can construct an instance* (A, b, c)*,*

$$A = \begin{pmatrix} A_{JB} & 0 \\ A_{KN} U_0^T & A_{KN} \end{pmatrix}, \quad b = \begin{pmatrix} A_{JB} \mathbf{1} \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ \mathbf{1}_N \end{pmatrix}, \tag{5.2}$$

such that

$$\Lambda(A, b, c) = (\bar{M}, \bar{U}).$$

Proof. As in the proof of Theorem 5.1, we need only to consider A_{JB} and A_{KN} satisfying $A_{JB}^T A_{JB} = \bar{M}_B$ and $A_{KN}^T A_{KN} = \bar{M}_N$.

Note that $A_{JB} U_0 = 0$, $U_0 \mathbf{1}_N = U_0 \bar{M}_N \mathbf{1}_N = 0$ and $U_0^T \mathbf{1}_B = U_0^T \bar{M}_B \mathbf{1}_B = 0$. One can verify that

$$x(t) = \begin{pmatrix} \mathbf{1}_B \\ e^{-t} \mathbf{1}_N \end{pmatrix}, \quad s(t) = \begin{pmatrix} e^{-t} \mathbf{1}_B \\ \mathbf{1}_N \end{pmatrix}, \quad y(t) = \begin{pmatrix} -e^{-t} A_{JB} \mathbf{1}_B \\ 0 \end{pmatrix}$$

are the central path of (A,b,c) defined by (5.2), i.e. satisfying (2.5). Using this central path, we can calculate the path $M(t)$ as follows:

$$A\lfloor x(t)\rceil = \begin{pmatrix} A_{JB} & 0 \\ A_{KN}U_0^T & e^{-t}A_{KN} \end{pmatrix},$$

$$\begin{aligned} A\lfloor x(t)\rceil^2 A^T &= \begin{pmatrix} I_J & 0 \\ 0 & e^{-2t}I_K + A_{KN}U_0^TU_0A_{KN}^T \end{pmatrix} \\ &= \begin{pmatrix} I_J & 0 \\ 0 & e^{-t}I_K \end{pmatrix}\begin{pmatrix} I_J & 0 \\ 0 & I_K + e^{2t}A_{KN}U_0^TU_0A_{KN}^T \end{pmatrix}\begin{pmatrix} I_J & 0 \\ 0 & e^{-t}I_K \end{pmatrix}, \end{aligned}$$

thus

$$(A\lfloor x(t)\rceil^2 A^T)^{-1} = \begin{pmatrix} I_J & 0 \\ 0 & e^{t}I_K \end{pmatrix}\begin{pmatrix} I_J & 0 \\ 0 & I_K + O(e^{2t}) \end{pmatrix}\begin{pmatrix} I_J & 0 \\ 0 & e^{t}I_K \end{pmatrix}.$$

$$\begin{aligned} M(t) &= \lfloor x(t)\rceil A^T \begin{pmatrix} I_J & 0 \\ 0 & e^{t}I_K \end{pmatrix}\begin{pmatrix} I_J & 0 \\ 0 & I_K + O(e^{2t}) \end{pmatrix}\begin{pmatrix} I_J & 0 \\ 0 & e^{t}I_K \end{pmatrix} A\lfloor x(t)\rceil \\ &= \begin{pmatrix} A_{JB}^T & e^tU_0A_{KN}^T \\ 0 & A_{KN}^T \end{pmatrix}\begin{pmatrix} A_{JB} & 0 \\ e^tA_{KN}U_0^T & A_{KN} \end{pmatrix} + O(e^{2t}) \\ &= \begin{pmatrix} \bar{M}_B & e^tU_0 \\ e^tU_0^T & \bar{M}_N \end{pmatrix} + O(e^{2t}) \\ &= \bar{M} + e^t\bar{U} + O(e^{2t}). \end{aligned}$$

□

The above two theorems show the LP representations for two types of eigenvectors. For general eigenvector $\bar{U} = \begin{pmatrix} h_{\bar{M}_B}(d_B) & U_0 \\ U_0^T & h_{\bar{M}_N}(d_N) \end{pmatrix}$, based on our numerical experiments, we conjecture that

$$A = \begin{pmatrix} A_{JB} & 0 \\ A_{KN}U_0^T & A_{KN} \end{pmatrix}, \quad b = \begin{pmatrix} A_{JB}\mathbf{1}_B \\ A_{KN}d_N \end{pmatrix}, \quad c = \begin{pmatrix} -d_B \\ \mathbf{1}_N \end{pmatrix},$$

is an LP representation. However, we have not been able to give a rigorous proof yet.

Now, we turn to eigenvectors with $\lambda = -1$.

Theorem 5.5 *Let*

$$\bar{M} = \begin{pmatrix} \bar{M}_B & 0 \\ 0 & \bar{M}_N \end{pmatrix} \in G(n,m), \quad \bar{U} = \begin{pmatrix} -h_{\bar{M}_B}(U_0\mathbf{1}_N) & U_0 \\ U_0^T & -h_{\bar{M}_N}(U_0^T\mathbf{1}_B) \end{pmatrix} \in T_{\bar{M}}G(n,m)$$

where $\bar{M}_B \in G(n_B, m_B)$, $\bar{M}_N \in G(n_N, m_N)$ *and* $U_0 \in R^{n_B \times n_N}$ *satisfy*

$$\bar{M}_B \mathbf{1} = \mathbf{1}, \quad \bar{M}_N \mathbf{1}_N = 0, \quad \bar{M}_B U_0 = U_0, \quad U_0 \bar{M}_N = 0.$$

Here $\bar{U}$ *is a eigenvector of* $Dh(\bar{M})$ *for* $\lambda = -1$. *Let* $A_{JB} \in R^{m_B \times n_B}$ *and* $A_{KN} \in R^{m_N \times n_N}$ *satisfy* $A_{JB}^T (A_{JB} A_{JB}^T)^{-1} A_{JB} = \bar{M}_B$ *and* $A_{KN}^T (A_{KN} A_{KN}^T)^{-1} A_{KN} = \bar{M}_N$. *Then we can construct an instance* $(\bar{A}, \bar{b}, \bar{c})$,

$$\bar{A} = \begin{pmatrix} A_{JB} & A_{JB} U_0 \\ 0 & A_{KN} \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} A_{JB} \mathbf{1}_B \\ 0 \end{pmatrix} \quad \text{and} \quad \bar{c} = \begin{pmatrix} 0_B \\ \mathbf{1}_N \end{pmatrix},$$

such that

$$\Lambda(\bar{A}, \bar{b}, \bar{c}) = (\bar{M}, \bar{U}).$$

Proof. We need only consider matrix A_{JB} with $A_{JB}^T A_{JB} = \bar{M}_B$, i.e. $A_{JB} A_{JB}^T = I$, because for any nonsingular Q matrices A_{JB} and $Q A_{JB}$ define the same $\bar{M}_B$. Similarly, we assume $A_{KN}^T A_{KN} = \bar{M}_N$.

For any small $\epsilon > 0$, define

$$A^\epsilon = \begin{pmatrix} A_{JB} - \epsilon C_B & A_{JB} U_0 \\ 0 & A_{KN} - \epsilon C_N \end{pmatrix}, \quad b^\epsilon = \begin{pmatrix} A_{JB} \mathbf{1}_B + \epsilon A_{JB} U_0 \mathbf{1}_N \\ 0 \end{pmatrix}, \quad c^\epsilon = \begin{pmatrix} \epsilon \mathbf{1}_B \\ \mathbf{1}_N \end{pmatrix},$$

where

$$C_B = A_{JB} \lfloor U_0 \mathbf{1}_N \rceil (I - \bar{M}_B), \quad C_N = A_{KN} \lfloor U_0^T \mathbf{1}_B \rceil (I - \bar{M}_N).$$

Denote

$$\bar{x}^\epsilon = \begin{pmatrix} \mathbf{1}_B + \epsilon U_0 \mathbf{1}_N \\ 0 \end{pmatrix}, \quad \bar{y}^\epsilon = \begin{pmatrix} \epsilon A_{JB} \mathbf{1}_B \\ 0 \end{pmatrix}, \quad \bar{s}^\epsilon = \begin{pmatrix} 0 \\ \mathbf{1}_N - \epsilon U_0^T \mathbf{1}_B \end{pmatrix}.$$

From $(I - \bar{M}_B) \mathbf{1}_B = 0$ and $(I - \bar{M}_B) U_0 = 0$ it follows that $C_B \mathbf{1}_B = 0$ and $C_B U_0 = 0$. This leads to

$$A^\epsilon \bar{x}^\epsilon = b^\epsilon.$$

Using $A_{JB}^T A_{JB} \mathbf{1}_B = \bar{M}_B \mathbf{1}_B = \mathbf{1}_B$ and $C_B^T A_{JB} \mathbf{1}_B = (I - \bar{M}_B) \lfloor U_0 \mathbf{1}_N \rceil \mathbf{1}_B = (I - \bar{M}_B) U_0 \mathbf{1}_N = 0$, we have

$$\begin{aligned} A^{\epsilon T} \bar{y}^\epsilon &= \epsilon \begin{pmatrix} A_{JB}^T - \epsilon C_B^T \\ U_0^T A_{JB}^T \end{pmatrix} A_{JB} \mathbf{1}_B \\ &= \epsilon \begin{pmatrix} \mathbf{1}_B \\ U_0^T \mathbf{1}_B \end{pmatrix}. \end{aligned}$$

This shows that

$$A^{\epsilon T}\bar{y}^\epsilon + \bar{s}^\epsilon = c^\epsilon.$$

For sufficiently small $\epsilon > 0$, both $\bar{x}^\epsilon$ and $\bar{s}^\epsilon$ are positive. Thus $(\bar{x}^\epsilon, \bar{s}^\epsilon, \bar{y}^\epsilon)$ is a strictly complementary optimal solution to $(A^\epsilon, b^\epsilon, c^\epsilon)$ and has the optimal basis (B, N).

For any $\epsilon > 0$, denote by $(x^\epsilon(\cdot), s^\epsilon(\cdot))$ the central path of $(A^\epsilon, b^\epsilon, c^\epsilon)$. Define $t_\epsilon = -\ln\epsilon$, i.e. $\epsilon = e^{-t_\epsilon}$. Using $C_B\mathbf{1}_B = 0$, $A_{KN}\mathbf{1}_N = 0$, $A_{KN}U_0^T = 0$ and $C_N\mathbf{1}_N = A_{KN}\lfloor U_0^T\mathbf{1}_B\rceil\mathbf{1}_N = A_{KN}U_0^T\mathbf{1}_B = 0$, one can verify that

$$A^\epsilon\begin{pmatrix}\mathbf{1}_B\\ e^{-t_\epsilon}\mathbf{1}_N\end{pmatrix} = b^\epsilon, \quad A^{\epsilon T}0 + \begin{pmatrix}e^{-t_\epsilon}\mathbf{1}_B\\ \mathbf{1}_N\end{pmatrix} = c^\epsilon, \quad \begin{pmatrix}\mathbf{1}_B\\ e^{-t_\epsilon}\mathbf{1}_N\end{pmatrix}\circ\begin{pmatrix}e^{-t_\epsilon}\mathbf{1}_B\\ \mathbf{1}_N\end{pmatrix} = e^{-t_\epsilon}\mathbf{1}.$$

This shows that $(\begin{pmatrix}\mathbf{1}_B\\ e^{-t_\epsilon}\mathbf{1}_N\end{pmatrix}, \begin{pmatrix}e^{-t_\epsilon}\mathbf{1}_B\\ \mathbf{1}_N\end{pmatrix})$ is a point on the central path of $(A^\epsilon, b^\epsilon, c^\epsilon)$ at $t = t_\epsilon$. This means

$$x^\epsilon(t_\epsilon) = \begin{pmatrix}\mathbf{1}_B\\ e^{-t_\epsilon}\mathbf{1}_N\end{pmatrix}, \quad s^\epsilon(t_\epsilon) = \begin{pmatrix}e^{-t_\epsilon}\mathbf{1}_B\\ \mathbf{1}_N\end{pmatrix}.$$

(Note that $(x^\epsilon(t), s^\epsilon(t)) = (\begin{pmatrix}\mathbf{1}_B\\ e^{-t}\mathbf{1}_N\end{pmatrix}, \begin{pmatrix}e^{-t}\mathbf{1}_B\\ \mathbf{1}_N\end{pmatrix})$ need not be true for $t \neq t_\epsilon$.)

For each $\epsilon > 0$, let the path $M^\epsilon(t)$ be defined by the instance $(A^\epsilon, b^\epsilon, c^\epsilon)$. By Theorem 4.4 (ii), there exists an equilibrium $\bar{M}^\epsilon = \lim_{t\to+\infty} M^\epsilon(t) \in G(n,m)$ and an eigenvector $U^\epsilon \in T_{\bar{M}^\epsilon}G(n,m)$ such that

$$M^\epsilon(t) = \bar{M}^\epsilon + e^{-t}U^\epsilon + O(e^{-2t}), \quad \forall\, t > 0. \tag{5.3}$$

Since the path $M^\epsilon(t)$ is defined by $(A^\epsilon, b^\epsilon, c^\epsilon)$, we have

$$\begin{aligned} M^\epsilon(t_\epsilon) &= \lfloor x^\epsilon(t_\epsilon)\rceil A^{\epsilon T}(A^\epsilon\lfloor x^\epsilon(t_\epsilon)\rceil^2 A^{\epsilon T})^{-1}A^\epsilon\lfloor x^\epsilon(t_\epsilon)\rceil \\ A^\epsilon\lfloor x^\epsilon(t_\epsilon)\rceil &= \begin{pmatrix} A_{JB} - \epsilon C_B & \epsilon A_{JB}U_0 \\ 0 & \epsilon(A_{KN} - \epsilon C_N)\end{pmatrix}.\end{aligned}$$

Because $C_B A_{JB}^T = 0$, $C_N A_{KN}^T = 0$ and $U_0 A_{KN}^T = 0$,

$$\begin{aligned} A^\epsilon\lfloor x^\epsilon(t_\epsilon)\rceil^2 A^{\epsilon T} &= \begin{pmatrix} I_J + O(\epsilon^2) & O(\epsilon^3) \\ O(\epsilon^3) & \epsilon^2(I_K + O(\epsilon^2))\end{pmatrix} \\ &= \begin{pmatrix} I_J & 0 \\ 0 & \epsilon I_K\end{pmatrix}(I + O(\epsilon^2))\begin{pmatrix} I_J & 0 \\ 0 & \epsilon I_K\end{pmatrix}\end{aligned}$$

Thus,

$$(A^\epsilon\lfloor x^\epsilon(t_\epsilon)\rceil^2 A^{\epsilon T})^{-1} = \begin{pmatrix} I_J & 0 \\ 0 & \epsilon^{-1} I_K\end{pmatrix}(I - O(\epsilon^2))\begin{pmatrix} I_J & 0 \\ 0 & \epsilon^{-1} I_K\end{pmatrix}$$

Now, we have

$$\begin{aligned}
M^\epsilon(t_\epsilon) &= \begin{pmatrix} A_{JB}^T - \epsilon C_B^T & 0 \\ \epsilon U_0^T A_{JB}^T & A_{KN}^T - \epsilon C_N^T \end{pmatrix} \begin{pmatrix} A_{JB} - \epsilon C_B & \epsilon A_{JB} U_0 \\ 0 & A_{KN} - \epsilon C_N \end{pmatrix} + O(\epsilon^2) \\
&= \begin{pmatrix} \bar{M}_B - \epsilon h_{\bar{M}_B}(U_0 \mathbf{1}_N) & \epsilon U_0 \\ \epsilon U_0^T & \bar{M}_N - \epsilon h_{\bar{M}_N}(U_0^T \mathbf{1}_B) \end{pmatrix} + O(\epsilon^2) \\
&= \bar{M} + \epsilon \bar{U} + O(\epsilon^2). \qquad (5.4)
\end{aligned}$$

In the above, we use

$$\begin{aligned}
A_{JB}^T C_B + C_B^T A_{JB} &= h_{\bar{M}_B}(U_0 \mathbf{1}_N) \\
A_{JB}^T A_{JB} U_0 &= \bar{M}_B U_0 = U_0 \\
A_{KN}^T C_N + C_N^T A_{KN} &= h_{\bar{M}_N}(U_0^T \mathbf{1}_B).
\end{aligned}$$

(Note that (5.4) is only shown to be true at $t = t_\epsilon$, but not for all t.)

By Theorem 3.3, (3.9) and (3.10) in [3] (note that A^ϵ satisfies the assumption $A_{KB}^\epsilon = 0$), $\bar{M}^\epsilon$, the limit of $M^\epsilon(t)$, is determined by A^ϵ and $(\bar{x}^\epsilon, \bar{s}^\epsilon)$. More precisely, we have

$$\bar{M}^\epsilon = \begin{pmatrix} \bar{M}_B^\epsilon & 0 \\ 0 & \bar{M}_N^\epsilon \end{pmatrix},$$

where

$$\begin{aligned}
\bar{M}_B^\epsilon &= \lfloor \bar{x}_B^\epsilon \rceil {A_{JB}^\epsilon}^T (A_{JB}^\epsilon \lfloor \bar{x}_B^\epsilon \rceil^2 {A_{JB}^\epsilon}^T)^{-1} A_{JB}^\epsilon \lfloor \bar{x}_B^\epsilon \rceil \\
\bar{M}_N^\epsilon &= \lfloor \bar{s}_N^\epsilon \rceil^{-1} {A_{KN}^\epsilon}^T (A_{KN}^\epsilon \lfloor \bar{s}_N^\epsilon \rceil^{-2} {A_{KN}^\epsilon}^T)^{-1} A_{KN}^\epsilon \lfloor \bar{s}_N^\epsilon \rceil^{-1}.
\end{aligned}$$

For $\bar{M}_B^\epsilon$,

$$\begin{aligned}
A_{JB}^\epsilon \lfloor \bar{x}_B^\epsilon \rceil &= (A_{JB} - \epsilon A_{JB} \lfloor U_0 \mathbf{1}_N \rceil (I - \bar{M}_B))(I + \epsilon \lfloor U_0 \mathbf{1}_N \rceil) \\
&= A_{JB} + \epsilon A_{JB} \lfloor U_0 \mathbf{1}_N \rceil \bar{M}_B + O(\epsilon^2),
\end{aligned}$$

$$\begin{aligned}
(A_{JB}^\epsilon \lfloor \bar{x}_B^\epsilon \rceil^2 {A_{JB}^\epsilon}^T)^{-1} &= (I + 2\epsilon A_{JB} \lfloor U_0 \mathbf{1}_N \rceil A_{JB}^T + O(\epsilon^2))^{-1} \\
&= I - 2\epsilon A_{JB} \lfloor U_0 \mathbf{1}_N \rceil A_{JB}^T + O(\epsilon^2),
\end{aligned}$$

$$\begin{aligned}
\bar{M}_B^\epsilon &= (A_{JB} + \epsilon A_{JB} \lfloor U_0 \mathbf{1}_N \rceil \bar{M}_B)^T (I - 2\epsilon A_{JB} \lfloor U_0 \mathbf{1}_N \rceil A_{JB}^T)(A_{JB} + \epsilon A_{JB} \lfloor U_0 \mathbf{1}_N \rceil \bar{M}_B) + O(\epsilon^2) \\
&= \bar{M}_B + O(\epsilon^2).
\end{aligned}$$

For $\bar{M}_N^\epsilon$,

$$\begin{aligned}
A_{KN}^\epsilon \lfloor \bar{s}_N^\epsilon \rceil^{-1} &= (A_{KN} - \epsilon C_N)(I_N - \epsilon \lfloor U_0^T \mathbf{1}_B \rceil)^{-1} \\
&= (A_{KN} - \epsilon C_N)(I_N + \epsilon \lfloor U_0^T \mathbf{1}_B \rceil) + O(\epsilon^2) \\
&= A_{KN} + \epsilon A_{KN} \lfloor U_0^T \mathbf{1}_B \rceil \bar{M}_N + O(\epsilon^2).
\end{aligned}$$

Now, similar to the above, we have

$$\bar{M}_N^\epsilon = \bar{M}_N + O(\epsilon^2).$$

Hence,

$$\bar{M}^\epsilon = \bar{M} + O(\epsilon^2). \tag{5.5}$$

By (5.3),

$$U^\epsilon = \frac{M^\epsilon(t) - \bar{M}^\epsilon}{e^{-t}} + O(e^{-t}).$$

Using (5.4) and (5.5) for $\epsilon = e^{-t_\epsilon}$, we have

$$U^\epsilon = \bar{U} + O(\epsilon). \tag{5.6}$$

Because the mapping from (A, b, c) to $M^0 = \Gamma(A, b, c)$ and the mapping from M^0 to $(\bar{M}, \bar{U})$ are continuous, the mapping Λ from (A, b, c) to $\Lambda(A, b, c) = (\bar{M}, \bar{U})$ is continuous. Using (5.5) and (5.6), we have

$$\Lambda(\bar{A}, \bar{b}, \bar{c}) = \lim_{\epsilon \to 0} \Lambda(A^\epsilon, b^\epsilon, c^\epsilon) = \lim_{\epsilon \to 0} (\bar{M}^\epsilon, U^\epsilon) = (\bar{M}, \bar{U}).$$

That is, the instance $(\bar{A}, \bar{b}, \bar{c})$ defines the path $M(t)$ converging to $\bar{M}$ in direction $\bar{U}$. □

6 Sources, sinks and their dimensions

An effective method for characterizing sinks and sources of stable/unstable manifolds is to compute their dimensions.

We denote $\bar{n} = m(n - m)$ which is the dimension of $G(n, m)$. Note that the dimension of an attraction region is the same as the dimension of $G(n, m)$.

We are particularly interested in stable and unstable manifolds of dimension $\bar{n}$ and of dimension $\bar{n} - 1$, because these manifolds comprise the attraction regions and the major boundaries of the attraction regions.

Theorem 6.1

$$dim(\Sigma^+(B, m_B)) = \bar{n} - (n_N - m_N)m_B \tag{6.1}$$

$$dim(\Sigma^-(B, m_B)) = \bar{n} - m_N(n_B - m_B) - (n_B - m_B) - m_N. \tag{6.2}$$

Proof. For simplicity, we omit (B, m_B), writing Σ^+, etc, instead of $\Sigma^+(B, m_B)$, etc.

Since $\Sigma^+ = \cup_{M \in G_c} E^+(M)$ is the bundle of subspaces $E^+(M)$ with the base G_c and $E^+(M)$ have the same dimension, written as $\dim(E^+)$, for all $M \in G_c$, we have

$$\dim(\Sigma^+) \;=\; \dim(G_c) + \dim(E^+).$$

Similarly,

$$\dim(\Sigma^-) \;=\; \dim(G_c) + \dim(E^-).$$

By Lemmas 4.3 and 4.5 in [3],

$$\begin{aligned}\dim(E^+) &= n_B - m_B + m_N + (n_B - m_B)m_N \\ \dim(E^-) &= m_B(n_N - m_N)\end{aligned}$$

By Lemma 4.6 in [3],

$$\dim(G_c) \;=\; (m_B - 1)(n_B - m_B) + (n_N - m_N - 1)m_N.$$

Elemental calculations amount to (6.1) and (6.2). □

Now we can use formulas (6.1) and (6.2) to fully describe the $\bar{n}$- and $(\bar{n} - 1)$-dimensional sources and sinks. We can also describe other dimensional sources and sinks in the similar way, but the description will be more involved.

• **$\bar{n}$-dimensional sources.**

By (6.1), $\Sigma^+(B, m_B)$ is an $\bar{n}$-dimensional source iff

$$(n_N - m_N)m_B = 0$$

iff

$$n_N - m_N = 0 \quad \text{or} \quad m_B = 0.$$

If $n_B > 0$, then $M_B\mathbf{1}_B = \mathbf{1}_B$ implies that M_B has at least one nonzero eigenvalue, and thus $m_B = rank(M_B) > 0$. Therefore, $m_B = 0$ implies $n_B = 0$ (i.e. $B = \emptyset$) and $(n_N, m_N) = (n, m)$. If $n_N > 0$, then $M_N\mathbf{1}_N = 0$ implies that M_N has at least one eigenvalue of zero, and thus M_N is not of full rank, i.e. $n_N > m_N$. Therefore, $n_N - m_N = 0$ implies $n_N = 0$ and $(n_B, m_B) = (n, m)$ (i.e. $B = \{1, \ldots, n\}$). So we have two types of $\bar{n}$-dimensional source:

Source I: $n_B = m_B = 0$ and $(n_N, m_N) = (n, m)$. This implies $B = \emptyset$, i.e. $M = M_N$. Thus,

$$G_c(B, m_B) = \{M \in G(n, m) \mid M\mathbf{1} = 0\}$$

$$E^+(M) = \{h_M(d) \mid d \in R^n,\ Md = d\}.$$

Source II: $n_N = m_N = 0$ and $(n_B, m_B) = (n, m)$. This implies $B = \{1, \ldots, n\}$, i.e. $M = M_B$. Thus,

$$G_c(B, m_B) = \{M \in G(n, m) \mid M\mathbf{1} = \mathbf{1}\},$$

$$E^+(M) = \{h_M(d) \mid d \in R^n,\ Md = 0\}.$$

Remark: Since these are the only $\bar{n}$ sources, the two unstable manifolds from these two sources comprise almost the entire $G(n, m)$, except a $(\bar{n} - 1)$-dimensional set.

Partitions of source I and source II can substantially characterize the partition of $G(n, m)$. Moreover, the two sources are related through the exchange of M and $I - M$, thus, by virtue of Lemma 3.6, the analysis of one source will suffice.

• $\bar{n}$-dimensional sinks.

By (6.2), $\Sigma^-(B, m_B)$ is an $\bar{n}$-dimensional sink iff

$$m_N(n_B - m_B) + (n_B - m_B) + m_N = 0$$

iff

$$n_B = m_B \quad \text{and} \quad m_N = 0.$$

This implies that $n_B = m_B = m$ and $n_N = n - m$. Thus, every $\bar{n}$-dimensional sink can contain only a single point which is the stable point M associated with a basis B, i.e.

$$M = diag(M_{ii}),\ M_{ii} = 1\,\forall\, i \in B,\ M_{ii} = 0\,\forall\, i \notin B.$$

We denote by $M(B)$ this stable point. Thus

$$G_c(B, m_B) = \{M(B)\}$$

Since $M(B) = \begin{pmatrix} I_B & 0 \\ 0 & 0 \end{pmatrix}$, $M_B = I_B$ and $M_N = 0$. Thus, for any $U_0 \in R^{m \times (n-m)}$, $M_B U_0 = U_0$, $U_0 M_N = 0$ and $h_{M_B}(U_0 \mathbf{1}_N) = 0$. Hence, any eigenvector of $Dh(M(B))$ for $\lambda = -1$ is in the form of $U = \begin{pmatrix} 0 & U_0 \\ U_0^T & 0 \end{pmatrix}$. For any $U = \begin{pmatrix} U_{BB} & U_{BN} \\ U_{BN}^T & U_{NN} \end{pmatrix} \in T_{M(B)}(G(n, m))$,

$$U = M(B)U + UM(B) = \begin{pmatrix} 2U_{BB} & U_{BN} \\ U_{BN}^T & 0 \end{pmatrix}$$

implies $U_{BB} = 0$ and $U_{NN} = 0$. Thus any $U \in T_{M(B)}(G(n,m))$ is an eigenvector of $Dh(M(B))$ for $\lambda = -1$. This shows

$$\begin{aligned} E^-(M(B)) &= \{U = \begin{pmatrix} 0 & U_0 \\ U_0^T & 0 \end{pmatrix} \; : \; \forall\, U_0 \in R^{m\times(n-m)}\} \\ &= T_{M(B)}(G(n,m)), \end{aligned}$$

$$\Sigma^-(B,m) = \{(M(B),U) : \forall\, U \in T_{M(B)}(G(n,m))\}.$$

This $\Sigma^-(B,m)$ is the sink of the attraction region $G(M(B))$ regarded as a stable manifold.

• $(\bar{n}-1)$**-dimensional sinks.** By condition (6.2), a sink is of dimension $\bar{n}-1$ iff

$$m_N(n_B - m_B) + m_N + (n_B - m_B) = 1.$$

Thus, there are two types of $(\bar{n}-1)$-dimensional sink:

Sink I: $n_B = m_B$ and $m_N = 1$. These imply that $n_B = m_B = m-1$ and $n_N = n-m+1$. Thus the associated equilibrium cluster is

$$G_c(B,m_B) \;\; = \;\; \left\{ M = \begin{pmatrix} I_{m-1} & 0 \\ 0 & uu^T \end{pmatrix} : u \in R^{n-m+1}, u^Tu = 1, u^T\mathbf{1} = 0 \right\}$$

and the eigenspace is

$$E^-(M) \;\; = \;\; \{\begin{pmatrix} 0 & U_0 \\ U_0^T & -h_{uu^T}(U_0^T\mathbf{1}_B) \end{pmatrix} : U_0u = 0\}.$$

Sink II: $n_B = m_B+1$ and $m_N = 0$. These imply that $n_B = m+1$, $m_B = m$ and $n_N = n-m-1$. Thus the associated equilibrium cluster is

$$G_c(B,m_B) \;\; = \;\; \left\{ M = \begin{pmatrix} I_{m+1} - vv^T & 0 \\ 0 & 0 \end{pmatrix} : v \in R^{m+1}, v^Tv = 1, v^T\mathbf{1} = 0 \right\}$$

and the eigenspace is

$$E^-(M) = \{\begin{pmatrix} -h_{vv^T}(U_0\mathbf{1}_N) & U_0 \\ U_0^T & 0 \end{pmatrix} : v^TU_0 = 0\}.$$

These sinks are the sinks of $(\bar{n}-1)$-dimensional boundaries, thus are particularly important.

• $(\bar{n}-1)$**-dimensional sources.**

By condition (6.1), a source is of dimension $\bar{n}-1$ iff

$$(n_N - m_N)m_B = 1.$$

This implies $m_B = 1$ and $n_N - m_N = 1$. Thus, we have $n_B = n - m$, $m_B = 1$, $n_N = m$ and $m_N = m - 1$. There is only one equilibrium which satisfies these conditions, that is, $G_c(B, m_B) = \{M\}$ where

$$M = \begin{pmatrix} \frac{1}{n-m}\mathbf{1}_B\mathbf{1}_B^T & 0 \\ 0 & I_N - \frac{1}{m}\mathbf{1}_N\mathbf{1}_N^T \end{pmatrix}, \quad \mathbf{1}_B \in R^{n-m}, \mathbf{1}_N \in R^m.$$

The eigenspace is

$$\begin{aligned} E^+(M) &= \left\{h_M(d) : \mathbf{1}_B^T d_B = 0,\ \mathbf{1}_N^T d_N = 0\right\} + \left\{\begin{pmatrix} 0 & U_0 \\ U_0^T & 0 \end{pmatrix} : \mathbf{1}_B^T U_0 = 0,\ U_0 \mathbf{1}_N = 0\right\} \\ &= \left\{\begin{pmatrix} \mathbf{1}_B d_B^T + d_B \mathbf{1}_B^T & 0 \\ 0 & \mathbf{1}_N d_N^T + d_N \mathbf{1}_N^T \end{pmatrix} : \mathbf{1}_B^T d_B = 0,\ \mathbf{1}_N^T d_N = 0\right\} \\ &\quad + \left\{\begin{pmatrix} 0 & U_0 \\ U_0^T & 0 \end{pmatrix} : \mathbf{1}_B^T U_0 = 0,\ U_0 \mathbf{1}_N = 0\right\} \\ &= \left\{\begin{pmatrix} \mathbf{1}_B d_B^T + d_B \mathbf{1}_B^T & U_0 \\ U_0^T & \mathbf{1}_N d_N^T + d_N \mathbf{1}_N^T \end{pmatrix} : \mathbf{1}_B^T d_B = 0,\ \mathbf{1}_N^T d_N = 0,\ \mathbf{1}_B^T U_0 = 0,\ U_0 \mathbf{1}_N = 0\right\}. \end{aligned}$$

Remarks: We have completely identified the two sources of all attraction regions. One direction of study is to investigate the induced partition on these sources. We have also completely characterized all sinks of $(\bar{n}-1)$-dimensional boundaries. Since boundaries are the primary structure of the partition, this characterization lays a foundation for the constructive investigation of the partition.

7 Conclusions

In this paper we present three main results.

Section 4 shows that an equilibrium-eigenvector pair uniquely determines a path which converges to the equilibrium in the direction of the eigenvector. This defines an isomorphism between a stable/unstable manifold $\mathcal{W}^s(B, m_B)/\mathcal{W}^u(B, m_B)$ and a sink/source $\Sigma^-(B, m_B)/\Sigma_+(B, m_B)$. We are interested in these manifolds because attraction regions and their boundaries are stable manifolds. The structure of sinks/sources is much simpler than that of stable/unstable manifolds. Thus, characterization of attraction regions and their boundaries can be significantly simplified through sinks/sources.

Section 5 further shows that, for each equilibrium-eigenvector pair, we can construct an LP instance which defines the path corresponding to the given equilibrium and eigenvector. LP representations are important because they bridge $G(n,m)$ and $SLP(n,m)$. This LP representation is particularly useful because it is constructed in terms of equilibria and eigenvectors. Many applications of this LP representation will be seen in our coming papers.

Section 6 shows a method for finding complete descriptions of sources and sinks. We completely present $\bar{n}$- and $(\bar{n}-1)$-dimensional sources and sinks, because they represent the attraction regions and the major boundaries in the basis partition.

Attraction regions and their boundaries are isomorphic to sinks (as shown in Section 4) which are fully described in Section 6 and represented in Section 5 as sets of LP instances in $SLP(n,m)$. This approach will yield a simple characterization of the basis partition of $SLP(n,m)$. Details of this characterization will appear in our future papers.

Location relations between attraction regions, between an attraction region and its boundaries, and between boundaries, are another important aspect for characterizing the basis partition. The results in this paper will play a fundamental role in discovering these relations.

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