

CONVEXITY IN SEMI-ALGEBRAIC GEOMETRY AND POLYNOMIAL OPTIMIZATION

JEAN B. LASSERRE

ABSTRACT. We review several (and provide new) results on the theory of moments, sums of squares and basic semi-algebraic sets when convexity is present. In particular, we show that under convexity, the hierarchy of semidefinite relaxations for polynomial optimization simplifies and has finite convergence, a highly desirable feature as convex problems are in principle easier to solve. In addition, if a basic semi-algebraic set \mathbf{K} is convex but its defining polynomials are not, we provide two algebraic *certificate* of convexity which can be checked numerically. The second is simpler and holds if a sufficient (and almost necessary) condition is satisfied, it also provides a new condition for \mathbf{K} to have semidefinite representation. For this we use (and extend) some of recent results from the author and Helton and Nie [6]. Finally, we show that when restricting to a certain class of convex polynomials, the celebrated Jensen's inequality in convex analysis can be extended to linear functionals that are not necessarily probability measures.

1. INTRODUCTION

Motivation. This paper is a contribution to the new emerging field of convex semi-algebraic geometry, and its purpose is threefold: First we show that the moment approach for global polynomial optimization proposed in [13], and based on semidefinite programming (SDP), is consistent as it simplifies and/or has better convergence properties when solving convex problems. In other words, the SDP moment approach somehow "recognizes" convexity, a highly desirable feature for a general purpose method because, in principle, convex problems should be easier to solve.

We next review some recent results (and provide a new one) on the representation of convex basic semi-algebraic sets by linear matrix inequalities which show how convexity permits to derive relatively simple and *explicit* semidefinite representations. In doing so we also provide a *certificate* of convexity for \mathbf{K} when its defining polynomials are not convex.

Finally, we consider the important Jensen's inequality in convex analysis. When restricting its application to a class of convex polynomials, we provide an extension to a class of linear functionals that are not necessarily probability measures.

To do so, we use (and sometimes extend) some recent results of the author [16, 17] and Helton and Nie [6]. We hope to convince the reader that convex semi-algebraic geometry is indeed a very specific subarea of real algebraic geometry which should

1991 *Mathematics Subject Classification.* Primary 14P10, 90C22; Secondary 11E25 12D15 90C25.

Key words and phrases. Convex polynomials; sums of squares; basic semi-algebraic sets; convex sets; Jensen inequality; semidefinite programming.

Research partially supported by the (french) ANR grant NT05-3-41612.

deserve more attention from both the optimization and real algebraic geometry research communities.

Background. I. Relatively recent results in the theory of moments and its dual theory of positive polynomials have been proved useful in polynomial optimization as they provide the basis of a specific convergent numerical approximation scheme. Namely, one can define a hierarchy of semidefinite relaxations (in short SDP-relaxations) of the original optimization problem whose associated monotone sequence of optimal values converges to the global optimum. For a more detail account of this approach, the interested reader is referred to e.g. Lasserre [13, 14], Parrilo [21], Schweighofer [29], and the many references therein.

Remarkably, practice seems to reveal that convergence is often fast and even finite. However, the size of the SDP-relaxations grows rapidly with the rank in the hierarchy; typically the r -th SDP-relaxation in the hierarchy has $O(n^{2r})$ variables and semidefinite matrices of $O(n^r)$ sizes (where n is the number of variables in the original problem). On the other hand, it is well-known that a large class of convex optimization problems can be solved efficiently; see e.g. Ben Tal and Nemirovski [1]. Therefore, as the SDP-based moment approach is dedicated to solving difficult non convex (most of the time NP-hard) problems, it should have the highly desirable feature to somehow *recognize* "easy" problems like convex ones. That is, when applied to such easy problems it should show some significant improvement or a particular nice behavior not necessarily valid in the general case. Notice that this is *not* the case of the LP-based moment-approach described in [14, 15] for which only asymptotic (and *not* finite) convergence occurs in general (and especially for convex problems), a rather annoying feature. However, for SDP-relaxations, some results of [17] already show that indeed convexity helps as one provides specialized representation results for convex polynomials that are nonnegative on a basic semi-algebraic set.

II. Next, in view of the potential of semidefinite programming techniques, an important issue is the characterization of convex sets that are semidefinite representable (in short called SDr sets). A SDr set $\mathbf{K} \subset \mathbb{R}^n$ is the projection of a set defined by linear matrix inequalities (LMIs). That is,

$$\mathbf{K} := \{x \in \mathbb{R}^n : \exists y \in \mathbb{R}^s \text{ s.t. } A_0 + \sum_{i=1}^n x_i A_i + \sum_{j=1}^s y_j B_j \succeq 0\}$$

for some real symmetric matrices (A_i, B_j) (and where $A \succeq 0$ stands for A is positive semidefinite). For more details, the interested reader is referred to Ben Tal and Nemirovski [1], Lewis et al. [19], Parrilo [22], and more recently, Chua and Tuncel [2], Helton and Nie [6, 7], Henrion [8] and Lasserre [16]. For compact basic semi-algebraic sets

$$(1.1) \quad \mathbf{K} := \{x \in \mathbb{R}^n : g_j(x) \geq 0, \quad j = 1, \dots, m\},$$

recent results of Helton and Nie [6, 7] and the author [16] provide sufficient conditions on the defining polynomials $(g_j) \subset \mathbb{R}[X]$ for the convex hull $\text{co}(\mathbf{K})$ ($\equiv \mathbf{K}$ if \mathbf{K} is convex) to be SDr. Again, an interesting issue is to analyze whether convexity of \mathbf{K} (with or without concavity of the defining polynomials (g_j)) provides some additional insights and/or simplifications. Another interesting issue is how to detect whether a basic semi-algebraic set \mathbf{K} is convex, or equivalently, how to obtain an algebraic *certificate* of convexity of \mathbf{K} from its defining polynomials (g_j) . By

certificate we mean a mathematical statement that obviously implies convexity of \mathbf{K} , can be checked numerically and does not require infinitely many tests. So far, and to the best of our knowledge, such a certificate does not exist.

III. The celebrated Jensen's inequality is an important result in convex analysis which states that $E_\mu(f(x)) \geq f(E_\mu(x))$ for a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and a probability measure μ with $E_\mu(x) < \infty$. A third goal of this paper is to analyze whether when restricted to a certain class of convex polynomials, Jensen's inequality can be extended to a class of linear functionals larger than the class of probability measures.

Contribution. Concerning issue I: We first recall two previous results proved in [17]: (a) the cone of convex SOS is dense (for the l_1 -norm of coefficients) in the cone of nonnegative convex polynomials, and (b) a convex Positivstellensatz for convex polynomials nonnegative on \mathbf{K} (a specialization of Putinar's Positivstellensatz). We then analyze the role of convexity for the polynomial optimization problem

$$(1.2) \quad \mathbf{P} : \quad f^* = \min_x \{ f(x) : x \in \mathbf{K} \}$$

with \mathbf{K} as in (1.1), and show that indeed convexity helps and makes the SDP-relaxations more efficient. In particular, when \mathbf{K} is convex and Slater's condition¹ holds, by using some recent results of Helton and Nie [6], we show that

(i) If the polynomials $f, (-g_j)$ are all convex and $\nabla^2 f$ is positive definite (and so f is strictly convex) on \mathbf{K} , then the hierarchy of SDP-relaxations has *finite* convergence.

(ii) If f and $(-g_j)$ are all SOS-convex (i.e. their Hessian is a SOS matrix polynomial), then \mathbf{P} reduces to solving a *single* SDP whose index in the hierarchy is readily available.

Concerning II: Under certain sufficient conditions on the (g_j) (typically some second order positive curvature conditions) Helton and Nie [6, 7] have proved that $\text{co}(\mathbf{K})$ (or \mathbf{K} if convex) has a semidefinite representation that uses Schmüdgen or Putinar SOS representation of polynomials positive on \mathbf{K} ; see [6, 17]. Yet, in general its dimension depends on an unknown degree parameter in Schmüdgen (or Putinar) SOS representation. Our contribution is to provide a new sufficient condition for existence of a SDR when \mathbf{K} is compact with nonempty interior and its boundary satisfies some nondegeneracy assumption. It translates the geometric property of convexity of \mathbf{K} into a SOS Putinar representation of some appropriate polynomial obtained from each g_j . When satisfied, this representation provides an algebraic certificate of convexity for \mathbf{K} and it is almost necessary in the sense that it always holds true when relaxed by an arbitrary $\epsilon > 0$. It also contains as special cases Helton and Nie [6] sufficient conditions of SOS-convexity or strict convexity on $\partial\mathbf{K}$ of the $-g_j$'s, and leads to an explicit semidefinite representation of \mathbf{K} . We also provide a more general algebraic certificate based on Stengle's Positivstellensatz, but more complex and heavy to implement and so not very practical. In practice both certificates are obtained by solving a semidefinite program. Therefore, because of unavoidable numerical inaccuracies, the certificate is valid only up to machine precision.

¹Slater's condition holds for \mathbf{K} in (1.1) if for some $x_0 \in \mathbf{K}$, $g_j(x_0) > 0$, $j = 1, \dots, m$.

Concerning III, we prove that when restricting its application to the subclass of SOS-convex polynomials, Jensen's inequality can be extended to all linear functionals $L_{\mathbf{y}}$ (with $L_{\mathbf{y}}(1) = 1$) in the dual cone of SOS polynomials, hence *not* necessarily probability measures.

Some of the results already obtained in [6, 16] and in the present paper strongly suggest that the class of SOS-convex polynomials introduced in Helton and Nie [6] is particularly nice and should deserve more attention.

2. NOTATION, DEFINITIONS AND PRELIMINARY RESULTS

Let $\mathbb{R}[X]$ be the ring of real polynomials in the variables $X = (X_1, \dots, X_n)$, and let $\Sigma^2[X] \subset \mathbb{R}[X]$ be the subset of sums of squares (SOS) polynomials. Denote $\mathbb{R}[X]_d \subset \mathbb{R}[X]$ be the set of polynomials of degree at most d , which forms a vector space of dimension $s(d) = \binom{n+d}{d}$. If $f \in \mathbb{R}[X]_d$, write $f(X) = \sum_{\alpha \in \mathbb{N}^n} f_{\alpha} X^{\alpha}$ in the usual canonical basis (X^{α}) , and denote by $\mathbf{f} = (f_{\alpha}) \in \mathbb{R}^{s(d)}$ its vector of coefficients. Also write $\|f\|_1 (= \|\mathbf{f}\|_1 := \sum_{\alpha} |f_{\alpha}|)$ the l_1 -norm of f . Finally, denote by $\Sigma^2[X]_d \subset \Sigma^2[X]$ the subset of SOS polynomials of degree at most $2d$.

We use the notation X for the variable of a polynomial $X \mapsto f(X)$ and x when x is a point of \mathbb{R}^n , as for instance in $\{x \in \mathbb{R}^n : f(x) \geq 0\}$.

Moment matrix. With $\mathbf{y} = (y_{\alpha})$ being a sequence indexed in the canonical basis (X^{α}) of $\mathbb{R}[X]$, let $L_{\mathbf{y}} : \mathbb{R}[X] \rightarrow \mathbb{R}$ be the linear functional

$$f \quad (= \sum_{\alpha} f_{\alpha} X^{\alpha}) \quad \mapsto \quad L_{\mathbf{y}}(f) = \sum_{\alpha} f_{\alpha} y_{\alpha},$$

and let $M_d(\mathbf{y})$ be the symmetric matrix with rows and columns indexed in the canonical basis (X^{α}) , and defined by:

$$M_d(\mathbf{y})(\alpha, \beta) := L_{\mathbf{y}}(X^{\alpha+\beta}) = y_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{N}_d^n$$

with $\mathbb{N}_d^n := \{\alpha \in \mathbb{N}^n : |\alpha| (= \sum_i \alpha_i) \leq d\}$.

Localizing matrix. Similarly, with $\mathbf{y} = (y_{\alpha})$ and $g \in \mathbb{R}[X]$ written

$$X \mapsto g(X) = \sum_{\gamma \in \mathbb{N}^n} g_{\gamma} X^{\gamma},$$

let $M_d(g\mathbf{y})$ be the symmetric matrix with rows and columns indexed in the canonical basis (X^{α}) , and defined by:

$$M_d(g\mathbf{y})(\alpha, \beta) := L_{\mathbf{y}}(g(X) X^{\alpha+\beta}) = \sum_{\gamma} g_{\gamma} y_{\alpha+\beta+\gamma},$$

for every $\alpha, \beta \in \mathbb{N}_d^n$.

Putinar Positivstellensatz. Let $Q(g) \subset \mathbb{R}[X]$ be the quadratic module generated by the polynomials $(g_j) \subset \mathbb{R}[X]$, that is,

$$(2.1) \quad Q(g) := \left\{ \sigma_0 + \sum_{j=1}^m \sigma_j g_j : (\sigma_j) \subset \Sigma^2[X] \right\}.$$

Assumption 2.1. $\mathbf{K} \subset \mathbb{R}^n$ is a compact basic semi-algebraic set defined as in (1.1) and the quadratic polynomial $X \mapsto M - \|X\|^2$ belongs to $Q(g)$.

Assumption 2.1 is not very restrictive. For instance, it holds if every g_j is affine (i.e., \mathbf{K} is a convex polytope) or if the level set $\{x : g_j(x) \geq 0\}$ is compact for some $j \in \{1, \dots, m\}$. In addition, if $M - \|x\| \geq 0$ for all $x \in \mathbf{K}$, then it suffices to add the redundant quadratic constraint $M^2 - \|x\|^2 \geq 0$ to the definition (1.1) of \mathbf{K} and Assumption 2.1 will hold true.

Theorem 2.2 (Putinar's Positivstellensatz [24]). *Let Assumption 2.1 hold. If $f \in \mathbb{R}[X]$ is (strictly) positive on \mathbf{K} , then $f \in Q(g)$. That is:*

$$(2.2) \quad f = \sigma_0 + \sum_{j=1}^m \sigma_j g_j,$$

for some SOS polynomials $(\sigma_j) \subset \Sigma^2[X]$.

2.1. A hierarchy of semidefinite relaxations (SDP-relaxations). Let \mathbf{P} be the optimization problem (1.2) with \mathbf{K} as in (1.1) and let $r_j = \lceil (\deg g_j)/2 \rceil$, $j = 1, \dots, m$. With $f \in \mathbb{R}[X]$ and $2r \geq \max[\deg f, \max_j 2r_j]$, consider the hierarchy of semidefinite relaxations (\mathbf{Q}_r) defined by:

$$(2.3) \quad \mathbf{Q}_r : \begin{cases} \inf_{\mathbf{y}} & L_{\mathbf{y}}(f) \\ \text{s.t.} & M_r(\mathbf{y}) \succeq 0 \\ & M_{r-r_j}(g_j \mathbf{y}) \succeq 0, \quad j = 1, \dots, m \\ & y_0 = 1 \end{cases},$$

with optimal value denoted by $\inf \mathbf{Q}_r$. One says that \mathbf{Q}_r is solvable if it has an optimal solution (in which case one writes $\inf \mathbf{Q}_r = \min \mathbf{Q}_r$). The dual of \mathbf{Q}_r reads

$$(2.4) \quad \mathbf{Q}_r^* : \begin{cases} \sup & \lambda \\ \text{s.t.} & f - \lambda = \sigma_0 + \sum_{j=1}^m \sigma_j g_j \\ & \sigma_j \in \Sigma^2[X], \quad j = 0, 1, \dots, m \\ & \deg \sigma_0, \deg \sigma_j + \deg g_j \leq 2r, \quad j = 1, \dots, m \end{cases},$$

with optimal value denoted by $\sup \mathbf{Q}_r^*$ (or $\max \mathbf{Q}_r^*$ if the sup is attained).

By weak duality $\sup \mathbf{Q}_r^* \leq \inf \mathbf{Q}_r$ for every $r \in \mathbb{N}$ and under Assumption 2.1, $\inf \mathbf{Q}_r \uparrow f^*$ as $r \rightarrow \infty$. For a more detailed account see e.g. [13].

2.2. Convexity and SOS-convexity. We first briefly recall basic facts on a multivariate convex function. If $C \subseteq \mathbb{R}^n$ is a nonempty convex set, a function $f : C \rightarrow \mathbb{R}$ is convex on C if and only if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall \lambda \in (0, 1), x, y \in C.$$

Similarly, f is strictly convex on C if and only if the above inequality is strict for every $x, y \in C$, $x \neq y$, and all $\lambda \in (0, 1)$.

If $C \subseteq \mathbb{R}^n$ is an open convex set and f is twice differentiable on C , then f is convex on C if and only if its Hessian $\nabla^2 f$ is positive semidefinite on C (denoted $\nabla^2 f \succeq 0$ on C). Finally, if $\nabla^2 f$ is positive definite on C (denoted $\nabla^2 f \succ 0$ on C) then f is strictly convex on C .

SOS-convexity. Helton and Nie [6] have introduced the following interesting subclass of convex polynomials, called SOS-convex polynomials.

Definition 2.3 (Helton and Nie [6]). *A polynomial $f \in \mathbb{R}[X]_{2d}$ is said to be SOS-convex if $\nabla^2 f$ is SOS, that is, $\nabla^2 f = LL^T$ for some real matrix polynomial $L \in \mathbb{R}[X]^{n \times s}$ (for some $s \in \mathbb{N}$).*

As noted in [6], an important feature of SOS-convexity is that it can be checked numerically by solving a SDP. They have also proved the following important property:

Lemma 2.4 (Helton and Nie [6, Lemma 7]). *If a symmetric matrix polynomial $P \in \mathbb{R}[X]^{r \times r}$ is SOS then for any $u \in \mathbb{R}^n$, the double integral*

$$X \mapsto F(X, u) := \int_0^1 \int_0^t P(u + s(X - u)) ds dt$$

is also a symmetric SOS matrix polynomial in $\mathbb{R}[X]^{r \times r}$.

And also:

Lemma 2.5 (Helton and Nie [6, Lemma 8]). *For a polynomial $f \in \mathbb{R}[X]$ and every $x, u \in \mathbb{R}^n$:*

$$\begin{aligned} f(x) &= f(u) + \nabla f(u)^T (x - u) \\ &+ (x - u)^T \underbrace{\int_0^1 \int_0^t \nabla^2 f(u + s(x - u)) ds dt}_{F(x, u)} (x - u). \end{aligned}$$

And so if f is SOS-convex and $f(u) = 0, \nabla f(u) = 0$, then f is a SOS polynomial.

2.3. An extension of Jensen's inequality. Recall that if μ is a probability measure on \mathbb{R}^n with $E_\mu(x) < \infty$, Jensen's inequality states that if $f \in L_1(\mu)$ and f is convex, then

$$E_\mu(f(x)) \geq f(E_\mu(x)),$$

a very useful property in many applications.

We now provide an extension of Jensen's inequality when one restricts its application to the class of SOS-convex polynomials. Namely, we may consider the linear functionals $L_{\mathbf{y}} : \mathbb{R}[X]_{2d} \rightarrow \mathbb{R}$ in the dual cone of $\Sigma^2[X]_d$, that is, vectors $\mathbf{y} = (y_\alpha)$ such that $M_d(\mathbf{y}) \succeq 0$ and $y_0 = L_{\mathbf{y}}(1) = 1$; hence \mathbf{y} is *not* necessarily the (truncated) moment sequence of some probability measure μ . Crucial in the proof is Lemma 2.4 of Helton and Nie.

Theorem 2.6. *Let $f \in \mathbb{R}[X]_{2d}$ be SOS-convex, and let $\mathbf{y} = (y_\alpha)_{\alpha \in \mathbb{N}_{2d}^n}$ satisfy $y_0 = 1$ and $M_d(\mathbf{y}) \succeq 0$. Then:*

$$(2.5) \quad L_{\mathbf{y}}(f(X)) \geq f(L_{\mathbf{y}}(X)),$$

where $L_{\mathbf{y}}(X) = (L_{\mathbf{y}}(X_1), \dots, L_{\mathbf{y}}(X_n))$.

Proof. Let $z \in \mathbb{R}^n$ be fixed, arbitrary, and consider the polynomial $X \mapsto f(X) - f(z)$. Then,

$$(2.6) \quad f(X) - f(z) = \langle \nabla f(z), X - z \rangle + \langle (X - z), F(X)(X - z) \rangle,$$

with $F : \mathbb{R}^n \rightarrow \mathbb{R}[X]^{n \times n}$ being the matrix polynomial

$$X \mapsto F(X) := \int_0^1 \int_0^t \nabla^2 f(z + s(X - z)) ds dt.$$

As f is SOS-convex, by Lemma 2.4, F is a SOS matrix polynomial and so the polynomial $X \mapsto \Delta(X) := \langle (X - z), F(X)(X - z) \rangle$ is SOS, i.e., $\Delta \in \Sigma^2[X]$. Then applying $L_{\mathbf{y}}$ to the polynomial $X \mapsto f(X) - f(z)$ and using (2.6) yields (recall that $y_0 = 1$)

$$\begin{aligned} L_{\mathbf{y}}(f(X)) - f(z) &= \langle \nabla f(z), L_{\mathbf{y}}(X) - z \rangle + L_{\mathbf{y}}(\Delta(X)) \\ &\geq \langle \nabla f(z), L_{\mathbf{y}}(X) - z \rangle \quad [\text{because } L_{\mathbf{y}}(\Delta(X)) \geq 0]. \end{aligned}$$

As $z \in \mathbb{R}^n$ was arbitrary, taking $z := L_{\mathbf{y}}(X) (= (L_{\mathbf{y}}(X_1), \dots, L_{\mathbf{y}}(X_n)))$ yields the desired result. \square

As a consequence we also get:

Corollary 2.7. *Let f be a convex univariate polynomial, $g \in \mathbb{R}[X]$ (and so $f \circ g \in \mathbb{R}[X]$). Let $d := \lceil (\deg f \circ g)/2 \rceil$, and let $\mathbf{y} = (y_{\alpha})_{\alpha \in \mathbb{N}_{2d}^n}$ be such that $y_0 = 1$ and $M_d(\mathbf{y}) \succeq 0$. Then:*

$$(2.7) \quad L_{\mathbf{y}}[f(g(X))] \geq f(L_{\mathbf{y}}[g(X)]).$$

Proof. Again let $z \in \mathbb{R}^n$ be fixed, arbitrary, and consider the univariate polynomial $Y \mapsto f(Y) - f(z)$ so that (2.6) holds. That is,

$$f(Y) - f(z) = f'(z)(Y - z) + F(Y)(Y - z)^2,$$

with $F : \mathbb{R} \rightarrow \mathbb{R}[Y]$ being the univariate polynomial

$$Y \mapsto F(Y) := \int_0^1 \int_0^t f''(z + s(Y - z)) ds dt.$$

As f is convex, $f'' \geq 0$, and so the univariate polynomial $Y \mapsto F(Y)(Y - z)^2$ is nonnegative, and being univariate, is SOS. Therefore, with $Y := g(X)$,

$$f(g(X)) - f(z) = f'(z)(g(X) - z) + F(g(X))(g(X) - z)^2,$$

and so

$$\begin{aligned} L_{\mathbf{y}}[f(g(X))] - f(z) &= f'(z)(L_{\mathbf{y}}[g(X)] - z) + L_{\mathbf{y}}[F(g(X))(g(X) - z)^2] \\ &\geq f'(z)(L_{\mathbf{y}}[g(X)] - z) \end{aligned}$$

and taking $z := L_{\mathbf{y}}[g(X)]$ yields the desired result. \square

Hence the class of SOS-convex polynomials has the very interesting property to extend Jensen's inequality to some linear functionals that are not necessarily coming from a probability measure.

3. SEMIDEFINITE RELAXATIONS IN THE CONVEX CASE

3.1. A convex Positivstellensatz. Let \mathbf{K} be as in (1.1) and define $Q_c(g) \subset \mathbb{R}[X]$ to be the set:

$$(3.1) \quad Q_c(g) := \left\{ \sigma_0 + \sum_{j=1}^m \lambda_j g_j : \lambda \in \mathbb{R}_+^m; \sigma_0 \in \Sigma^2[X], \sigma_0 \text{ convex} \right\} \subset Q(g).$$

The set $Q_c(g)$ is a specialization of $Q(g)$ in (2.1) to the convex case, in that the weights associated with the g_j 's are nonnegative scalars, i.e., SOS polynomials of degree 0, and the SOS polynomial σ_0 is convex. In particular, every $f \in Q_c(g)$ is nonnegative on \mathbf{K} . Let $\mathcal{F}_{\mathbf{K}} \subset \mathbb{R}[X]$ be the convex cone of convex polynomials nonnegative on \mathbf{K} .

Theorem 3.1 (Lasserre [17]). *Let \mathbf{K} be as in (1.1), Slater's condition hold and g_j be concave for every $j = 1, \dots, m$.*

Then with $Q_c(g)$ as in (3.1), the set $Q_c(g) \cap \mathcal{F}_{\mathbf{K}}$ is dense in $\mathcal{F}_{\mathbf{K}}$ for the l_1 -norm $\|\cdot\|_1$. In particular, if $\mathbf{K} = \mathbb{R}^n$ (so that $\mathcal{F}_{\mathbb{R}^n} =: \mathcal{F}$ is now the set of nonnegative convex polynomials), then $\Sigma^2[X] \cap \mathcal{F}$ is dense in \mathcal{F} .

Theorem 3.1 states that if f is convex and nonnegative on \mathbf{K} (including the case $\mathbf{K} \equiv \mathbb{R}^n$) then one may approximate f by a sequence $\{f_{\epsilon r}\} \subset Q_c(g) \cap \mathcal{F}_{\mathbf{K}}$ with $\|f - f_{\epsilon r}\|_1 \rightarrow 0$ as $\epsilon \rightarrow 0$ (and $r \rightarrow \infty$). For instance, with $r_0 := \lfloor (\deg f)/2 \rfloor + 1$,

$$(3.2) \quad \begin{aligned} X \mapsto f_{\epsilon r}(X) &:= f + \epsilon(\theta_{r_0}(X) + \theta_r(X)), \quad \text{with} \\ X \mapsto \theta_r(X) &:= 1 + \sum_{k=1}^r \sum_{i=1}^n \frac{X_i^{2k}}{k!} \quad r \geq r_\epsilon, \end{aligned}$$

for some r_ϵ ; see Lasserre [17] for details. Observe that Theorem 3.1 provides f with a *certificate* of nonnegativity on \mathbf{K} . Indeed, let $x \in \mathbf{K}$ be fixed arbitrary. Then as $f_{\epsilon r} \in Q_c(g)$ one has $f_{\epsilon r}(x) \geq 0$. Letting $\epsilon \downarrow 0$ yields $0 \leq \lim_{\epsilon \rightarrow 0} f_{\epsilon r}(x) = f(x)$. And as $x \in \mathbf{K}$ was arbitrary, $f \geq 0$ on \mathbf{K} .

Theorem 3.1 is a convex (weak) version of Theorem 2.2 (Putinar's Positivstellensatz) where one replaces the quadratic module $Q(g)$ with its subset $Q_c(g)$. We call it a *weak* version of Theorem 2.2 because it invokes a density result (i.e. $f_{\epsilon r} \in Q_c(g)$ whereas f might not be an element of $Q_c(g)$). Notice that f is allowed to be nonnegative (instead of strictly positive) on \mathbf{K} and \mathbf{K} need *not* be compact; recall that extending Theorem 2.2 to non compact basic semi-algebraic sets \mathbf{K} and to polynomials f nonnegative on \mathbf{K} is hopeless in general; see Scheiderer [26].

Corollary 3.2. *Let \mathbf{K} be as in (1.1), $f \in \mathbb{R}[X]$ with $f^* := \min_x \{f(x) : x \in \mathbf{K}\}$ and let $d := \max[\lfloor (\deg f)/2 \rfloor, \max_j \lfloor (\deg g_j)/2 \rfloor]$. Consider the simplified SDP-relaxation*

$$(3.3) \quad \widehat{\mathbf{Q}} : \begin{cases} \inf_{\mathbf{y}} & L_{\mathbf{y}}(f) \\ \text{s.t.} & M_d(\mathbf{y}) \succeq 0 \\ & L_{\mathbf{y}}(g_j) \geq 0, \quad j = 1, \dots, m \\ & y_0 = 1 \end{cases}$$

and its dual

$$(3.4) \quad \widehat{\mathbf{Q}}^* : \begin{cases} \sup_{\gamma, \sigma_0, \lambda} & \gamma \\ \text{s.t.} & f - \gamma = \sigma_0 + \sum_{j=1}^m \lambda_j g_j \\ & \sigma_0 \in \Sigma^2[X]_d; \lambda_j \geq 0, \quad j = 1, \dots, m \end{cases}$$

(a) If $f - f^* \in Q_c(g)$ then the SDP-relaxation $\widehat{\mathbf{Q}}$ and its dual $\widehat{\mathbf{Q}}^*$ are exact.

(b) If $f, -g_j \in \mathbb{R}[X]$ are convex, $j = 1, \dots, m$, and if \mathbf{y} is an optimal solution of $\widehat{\mathbf{Q}}$ which satisfies

$$(3.5) \quad \text{rank } M_d(\mathbf{y}) = \text{rank } M_{d-1}(\mathbf{y}),$$

then $\widehat{\mathbf{Q}}$ is exact and $x^* := (L_{\mathbf{y}}(X_i)) \in \mathbf{K}$ is a (global) minimizer of f on \mathbf{K} .

Proof. (a) If $f - f^* \in Q_c(g)$, i.e., if $f - f^* = \sigma_0 + \sum_{j=1}^m \lambda_j g_j$, with $\sigma_0 \in \Sigma^2[X]_d$ and $\lambda \in \mathbb{R}_+^m$, the triplet (f^*, σ_0, λ) is a feasible solution of $\widehat{\mathbf{Q}}^*$ with value f^* . Therefore, as $\sup \widehat{\mathbf{Q}}^* \leq \inf \widehat{\mathbf{Q}} \leq f^*$, the SDP-relaxation $\widehat{\mathbf{Q}}$ and its dual $\widehat{\mathbf{Q}}^*$ are exact. In fact, (f^*, σ_0, λ) is an optimal solution of $\widehat{\mathbf{Q}}^*$.

(b) If \mathbf{y} satisfies the rank condition (3.5) then by the *flat extension* theorem of Curto and Fialkow [4], \mathbf{y} is the (truncated) moment sequence of an atomic probability measure μ on \mathbb{R}^n , say $\mu = \sum_{k=1}^s \lambda_k \delta_{x(k)}$ with $s = \text{rank } M_d(\mathbf{y})$, $0 < \lambda_k \leq 1$, $\sum_k \lambda_k = 1$, and $\delta_{x(k)}$ being the Dirac measure at $x(k) \in \mathbb{R}^n$, $k = 1, \dots, s$. Let $x^* := \sum_k \lambda_k x(k) = (L_{\mathbf{y}}(X_i)) \in \mathbb{R}^n$. Then $f^* \geq L_{\mathbf{y}}(f)$ and by convexity of f , $L_{\mathbf{y}}(f) = \sum_k \lambda_k f(x(k)) \geq f(\sum_k \lambda_k x(k)) = f(x^*)$. Similarly, by convexity of $-g_j$, $0 \leq L_{\mathbf{y}}(g_j) = \sum_k \lambda_k g_j(x(k)) \leq g_j(\sum_k \lambda_k x(k)) = g_j(x^*)$, $j = 1, \dots, m$. Therefore, $x^* \in \mathbf{K}$ and as $f(x^*) \leq f^*$, x^* is a global minimizer of f on \mathbf{K} . \square

Notice that \mathbf{K} in Corollary 3.2 need not be compact. Also, Corollary 3.2(b) has practical value because in general one does not know whether $f - f^* \in Q_c(g)$ (despite that in the convex case, $f - f^* \in \mathcal{F}_{\mathbf{K}}$ and $Q_c(g) \cap \mathcal{F}_{\mathbf{K}}$ is dense in $\mathcal{F}_{\mathbf{K}}$). However, one may still solve $\widehat{\mathbf{Q}}$ and check whether the rank condition (3.5) is satisfied. If in solving $\widehat{\mathbf{Q}}_r$, the rank condition (3.5) is not satisfied, then other sufficient conditions can be exploited as we next see.

3.2. The SOS-convex case. Part (a) of the following result is already contained in Lasserre [17, Cor. 2.5].

Theorem 3.3. *Let \mathbf{K} be as in (1.1) and Slater's condition hold. Let $f \in \mathbb{R}[X]$ be such that $f^* := \inf_x \{f(x) : x \in \mathbf{K}\} = f(x^*)$ for some $x^* \in \mathbf{K}$. If f is SOS-convex and $-g_j$ is SOS-convex for every $j = 1, \dots, m$, then:*

(a) $f - f^* \in Q_c(g)$.

(b) *The simplified SDP-relaxation $\widehat{\mathbf{Q}}$ in (3.3) and its dual (3.4) are exact and solvable. If \mathbf{y} is an optimal solution of $\widehat{\mathbf{Q}}$ then $x^* := (L_{\mathbf{y}}(X_i)) \in \mathbf{K}$ is a global minimizer of f on \mathbf{K} .*

Proof. (a) is proved in [17, Cor. 2.5]. (b) That $\widehat{\mathbf{Q}}$ is exact follows from (a) and Corollary 3.2(a). Hence it is solvable (e.g. take \mathbf{y} to be the moment sequence associated with the Dirac measure at a global minimizer $x^* \in \mathbf{K}$). So let \mathbf{y} be an

optimal solution of $\widehat{\mathbf{Q}}$, hence with $f^* = L_{\mathbf{y}}(f)$. As $-g_j$ is SOS-convex for every j , then by Theorem 2.6, $0 \leq L_{\mathbf{y}}(g_j) \leq g_j(x^*)$ with $x^* := (L_{\mathbf{y}}(X_i))$ and so $x^* \in \mathbf{K}$. Similarly, as f is SOS-convex, we also have $f^* = L_{\mathbf{y}}(f) \geq f(x^*)$ which proves that $f(x^*) = f^*$ and x^* is a global minimizer of f on \mathbf{K} . Finally, as by (a) $f - f^* \in Q_c(g)$ then $\widehat{\mathbf{Q}}^*$ is exact and solvable. \square

(Again notice that \mathbf{K} in Theorem 3.3 need not be compact.) So the class of SOS-convex polynomials is particularly interesting. Not only Jensen's inequality can be extended to some linear functionals that are not coming from a probability measure, but one may also solve SOS-convex optimization problems \mathbf{P} in (1.2) (i.e. with f and \mathbf{K} defined with SOS-convex polynomials) by solving the single semidefinite program (3.3).

Notice that a self-concordant² logarithmic barrier function exists for (3.3) whereas the logarithmic barrier function with barrier parameter μ :

$$(3.6) \quad x \mapsto \phi_{\mu}(x) := \mu f(x) - \sum_{j=1}^m \ln(-g_j(x)),$$

associated with \mathbf{P} , is not self-concordant in general. Therefore, despite (3.3) involves additional variables (a lifting), solving (3.3) via an interior point method might be more efficient than solving \mathbf{P} by using the logarithmic barrier function (3.6) with no lifting. In addition, all SOS-convex polynomials nonnegative on \mathbf{K} and which attain their minimum on \mathbf{K} , belong to $Q_c(g)$, a very specific version of Putinar Positivstellensatz (as f is only nonnegative and \mathbf{K} need not be compact).

3.3. The strictly convex case. If f or some of the $-g_j$'s is not SOS-convex but $\nabla^2 f \succ 0$ (so that f is strictly convex) and $-g_j$ is convex for every $j = 1, \dots, m$, then inspired by a nice argument from Helton and Nie [6] for existence of a semidefinite representation of convex sets, one obtains the following result.

Theorem 3.4. *Let \mathbf{K} be as in (1.1) and let Assumption 2.1 and Slater's condition hold. Assume that $f, -g_j \in \mathbb{R}[X]$ are convex, $j = 1, \dots, m$, with $\nabla^2 f \succ 0$ on \mathbf{K} .*

Then the hierarchy of SDP-relaxations defined in (2.3) has finite convergence. That is, $f^ = \sup \mathbf{Q}_r^* = \inf \mathbf{Q}_r$ for some index r . In addition, \mathbf{Q}_r and \mathbf{Q}_r^* are solvable so that $f^* = \max \mathbf{Q}^* = \min \mathbf{Q}_r$.*

Proof. Let $x^* \in \mathbf{K}$ be a global minimizer (i.e. $f^* = f(x^*)$). As Slater's condition holds, there exists a vector of Karush-Kuhn-Tucker (KKT) multipliers $\lambda \in \mathbb{R}_+^m$ such that the (convex) Lagrangian $L_f \in \mathbb{R}[X]$ defined by

$$(3.7) \quad X \mapsto L_f(X) := f(X) - f^* - \sum_{j=1}^m \lambda_j g_j(X)$$

has a global minimum at $x^* \in \mathbf{K}$, i.e., $\nabla L_f(x^*) = 0$. In addition, $\lambda_j g_j(x^*) = 0$ for every $j = 1, \dots, m$ and $L_f(x^*) = 0$. Then, by Lemma 2.5,

$$L_f(X) = \langle (X - x^*), F(X, x^*)(X - x^*) \rangle$$

²The self-concordance property introduced in [20] is fundamental in the design and efficiency of interior point methods for convex programming.

with

$$F(X, x^*) := \left(\int_0^1 \int_0^t \nabla^2 L_f(x^* + s(X - x^*)) ds dt \right).$$

Next, let I_n be the $n \times n$ identity matrix. As $\nabla^2 f \succ 0$ on \mathbf{K} , continuity of the (strictly positive) smallest eigenvalue of $\nabla^2 f$ and compactness of \mathbf{K} yield that $\nabla^2 f \succeq \delta I_n$ on \mathbf{K} , for some $\delta > 0$. Next, as $-g_j$ is convex for every j , and in view of the definition (3.7) of L_f , $\nabla^2 L_f \succeq \nabla^2 f \succeq \delta I_n$ on \mathbf{K} . Hence for every $\xi \in \mathbb{R}^n$, $\xi^T F(x, x^*) \xi \geq \delta \int_0^1 \int_0^t \xi^T \xi ds dt = \frac{\delta}{2} \xi^T \xi$, and so $F(x, x^*) \succeq \frac{\delta}{2} I_n$ for every $x \in \mathbf{K}$. Therefore, by the matrix polynomial version of Putinar Positivstellensatz,

$$F(X, x^*) = F_0(X) + \sum_{j=1}^m F_j(X) g_j(X),$$

for some real SOS matrix polynomials $X \mapsto F_j(X) = L_j(X) L_j(X)^T$ (for some appropriate $L_j \in \mathbb{R}[X]^{n \times p_j}$), $j = 0, \dots, m$. See Helton and Nie [6], Kojima and Maramatsu [10], Hol and Scherer [11]. But then

$$X \mapsto \langle (X - x^*), F_j(X, x^*) (X - x^*) \rangle = \sigma_j(X) \in \Sigma^2[X], \quad j = 0, \dots, m$$

and so

$$\begin{aligned} f(X) - f^* &= L_f(X) + \sum_{j=1}^m \lambda_j g_j(X) \\ &= \sigma_0(X) + \sum_{j=1}^m (\lambda_j + \sigma_j(X)) g_j(X). \end{aligned}$$

Let $2s$ be the maximum degree of the SOS polynomials (σ_j) . Then $(f^*, \{\sigma_j + \lambda_j\})$ is a feasible solution of the SDP-relaxation \mathbf{Q}_r^* in (2.4) with $r := s + \max_j r_j$. Therefore, as $\sup \mathbf{Q}_r^* \leq \inf \mathbf{Q}_r \leq f^*$, the SDP-relaxations \mathbf{Q}_r and \mathbf{Q}_r^* are exact, finite convergence occurs and \mathbf{Q}_r^* is solvable. But this also implies that \mathbf{Q}_r is solvable (take \mathbf{y} to be the moment sequence of the Dirac measure δ_{x^*} at any global minimizer $x^* \in \mathbf{K}$). \square

When compared to Theorem 3.3 for the SOS-convex case, in the strictly convex case the simplified SDP-relaxation $\widehat{\mathbf{Q}}$ in (3.3) is not guaranteed to be exact. However, finite convergence still occurs for the SDP-relaxations (\mathbf{Q}_r) in (2.3).

Remark 3.5. It is worth emphasizing that in general, the hierarchy of LP-relaxations (as opposed to SDP-relaxations) defined in [15] and based on Krivine's representation [12, 30] for polynomials positive on \mathbf{K} , *cannot* have finite convergence, especially in the convex case! For more details, the interested reader is referred to [14, 15]. Therefore, and despite LP software packages can solve LP problems of very large size, using LP-relaxations does not seem a good idea even for solving a convex polynomial optimization problem.

4. CONVEXITY AND SEMIDEFINITE REPRESENTATION OF CONVEX SETS

We now consider the semidefinite representation of convex sets. First recall the following result.

Theorem 4.1 (Lasserre [16]). *Let \mathbf{K} in (1.1) be compact with g_j concave, $j = 1, \dots, m$, and assume that Slater's condition holds. If the Lagrangian polynomial L_f in (3.7) associated with every linear polynomial $f \in \mathbb{R}[X]$ is SOS, then with $d := \max_j \lceil (\deg g_j)/2 \rceil$, the set*

$$(4.1) \quad \Omega := \left\{ (x, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^{s(2d)} : \begin{cases} M_d(\mathbf{y}) & \succeq 0 \\ L_{\mathbf{y}}(g_j) & \geq 0, \quad j = 1, \dots, m \\ L_{\mathbf{y}}(X_i) & = x_i, \quad i = 1, \dots, n \\ y_0 & = 1 \end{cases} \right.$$

is a semidefinite representation of \mathbf{K} .

Next, Helton and Nie [6, 7] have provided several interesting second-order positive curvature (sufficient and necessary) conditions on the defining polynomials (g_j) for \mathbf{K} (or its convex hull $\text{co}(\mathbf{K})$) to have a SDr. In particular (recall that $r_j = \lceil (\deg g_j)/2 \rceil$ for every $j = 1, \dots, m$):

Theorem 4.2 (Helton and Nie [6]). *Let \mathbf{K} in (1.1) be convex, Assumption 2.1 hold, and assume that Slater's condition holds and g_j is concave on \mathbf{K} , $j = 1, \dots, m$.*

(a) *If $-g_j$ is SOS-convex for every $j = 1, \dots, m$, then for every linear $f \in \mathbb{R}[X]$, the associated Lagrangian L_f (3.7) is SOS and the set Ω in (4.1) is a semidefinite representation of \mathbf{K} .*

(b) *If every $-g_j$ is either SOS-convex or satisfies $-\nabla^2 g_i \succ 0$ on $\mathbf{K} \cap \{x : g_j(x) = 0\}$, then there exists $r \in \mathbb{N}$ such that the set*

$$(4.2) \quad \Omega := \left\{ (x, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^{s(2r)} : \begin{cases} M_r(\mathbf{y}) & \succeq 0 \\ M_{r-r_j}(g_j \mathbf{y}) & \succeq 0, \quad j = 1, \dots, m \\ L_{\mathbf{y}}(X_i) & = x_i, \quad i = 1, \dots, n \\ y_0 & = 1 \end{cases} \right.$$

is a semidefinite representation of \mathbf{K} .

See [6, Theor. 6, and 9]. This follows from the fact that the Hessian $\nabla^2 L_f$ associated with a linear $f \in \mathbb{R}[X]$ has a Putinar representation in terms of SOS matrix polynomials, and with degree of the weights bounded uniformly in f . In principle, the degree parameter d in Theorem 4.2(b) may be computed by solving a hierarchy of semidefinite programs. Some other (more technical) weaker second-order positive curvature sufficient conditions (merely for existence of a SDr) are also provided in [6, 7] but the semidefinite representation is not explicit any more in terms of the defining polynomials (g_j). Notice that if \mathbf{K} is compact but Assumption 2.1 does not hold, then one still obtains a semidefinite representation for \mathbf{K} but more complicated as it is now based on Schmüdgen's representation [27] instead of Putinar's representation; see [6, Theor. 5].

We next provide a sufficient condition in the case where \mathbf{K} is convex but its defining polynomials ($-g_j$) are *not* necessarily convex. Among its distinguishing features, it is checkable numerically, contains Theorem 4.2 as a special case and leads to the explicit semidefinite representation (4.2) of \mathbf{K} .

4.1. Algebraic certificate of convexity. We first present the following characterization of convexity when \mathbf{K} is closed, satisfies a nondegeneracy assumption on its boundary, and Slater's condition holds.

Lemma 4.3. *Let \mathbf{K} be as in (1.1) (hence closed), Slater's condition hold and assume that for every $j = 1, \dots, m$, $\nabla g_j(y) \neq 0$ if $y \in \mathbf{K}$ and $g_j(y) = 0$. Then \mathbf{K} is convex if and only if for every $j = 1, \dots, m$,*

$$(4.3) \quad \langle \nabla g_j(y), x - y \rangle \geq 0, \quad \forall x \in \mathbf{K} \text{ and } \forall y \in \mathbf{K} \text{ with } g_j(y) = 0.$$

Proof. The *only if* part is obvious. Indeed if $\langle \nabla g_j(y), x - y \rangle < 0$ for some $x \in \mathbf{K}$ and $y \in \mathbf{K}$ with $g_j(y) = 0$, then there is some $\bar{t} > 0$ such that $g_j(y + t(x - y)) < 0$ for all $t \in (0, \bar{t})$ and so the point $x' := tx + (1 - t)y$ does not belong to \mathbf{K} , which in turn implies that \mathbf{K} is not convex.

For the *if* part, (4.3) implies that at every point of the boundary, there exists a supporting hyperplane for \mathbf{K} . As \mathbf{K} is closed with nonempty interior, the result follows from [28, Theor. 1.3.3]³. \square

The nondegeneracy assumption is crucial as demonstrated in the following simple example kindly provided by an anonymous referee:

Example 1. Consider the non convex set $\mathbf{K} \subset \mathbb{R}^2$ defined by:

$$\mathbf{K} := \{x \in \mathbb{R}^2 : (1 - x_1^2 + x_2^2)^3 \geq 0, 10 - x_1^2 - x_2^2 \geq 0\}$$

Then it is straightforward to see that (4.3) is satisfied. This is because ∇g_1 vanishes on the piece of boundary determined by $g_1(x) = 0$.

Next, using the above characterization (4.3), we provide an algebraic certificate of convexity.

Corollary 4.4 (Algebraic certificate of convexity). *Let \mathbf{K} be as in (1.1), Slater's condition hold and assume that for every $j = 1, \dots, m$, $\nabla g_j(y) \neq 0$ if $y \in \mathbf{K}$ and $g_j(y) = 0$. Then \mathbf{K} is convex if and only if for every $j = 1, \dots, m$,*

$$(4.4) \quad h_j(X, Y) \langle \nabla g_j(Y), X - Y \rangle = \langle \nabla g_j(Y), X - Y \rangle^{2l} + \theta_j(X, Y) + \varphi_j(X, Y) g_j(Y),$$

for some integer $l \in \mathbb{N}$, some polynomial $\varphi_j \in \mathbb{R}[X, Y]$ and some polynomials h_j, θ_j in the preordering⁴ of $\mathbb{R}[X, Y]$ generated by the family of polynomials $(g_k(X), g_p(Y))$, $k, p \in \{1, \dots, m\}$, $p \neq j$.

Proof. By Lemma 4.3, \mathbf{K} is convex if and only if for every $j = 1, \dots, m$, the polynomial $(X, Y) \mapsto \langle \nabla g_j(Y), X - Y \rangle$ is nonnegative on the set Ω_j defined by:

$$(4.5) \quad \Omega_j := \{(x, y) \in \mathbf{K} \times \mathbf{K} : g_j(y) = 0\}.$$

Equivalently, \mathbf{K} is convex if and only if for every $j = 1, \dots, m$:

$$\emptyset = \{(x, y) \in \mathbb{R}^n : \quad (x, y) \in \mathbf{K} \times \mathbf{K}; \quad g_j(y) = 0; \\ \langle \nabla g_j(y), x - y \rangle \leq 0; \langle \nabla g_j(y), x - y \rangle \neq 0\}.$$

Then (4.4) follows from Stengle's Positivstellensatz [25, Theor. 4.4.2, p. 92]. \square

³The author is grateful to L. Tuncel for providing us with the reference [28].

⁴The preordering of $\mathbb{R}[X]$ generated by a family $(g_1, \dots, g_m) \subset \mathbb{R}[X]$ is the set of polynomials $\{p : p = \sum_{J \subseteq \{1, \dots, m\}} \sigma_J (\prod_{j \in J} g_j), \text{ with } \sigma_J \in \Sigma^2[X]\}$.

Observe that Corollary 4.4 provides an algebraic certificate of convexity when \mathbf{K} is closed with nonempty interior and a nondegeneracy assumption holds on its boundary. If one fixes an a priori bound s on $l \in \mathbb{N}$ and on the degree of h_j, θ_j and φ_j , then checking whether (4.4) holds reduces to solving a semidefinite program. If \mathbf{K} is convex, by increasing s , eventually one would obtain such a certificate if one could solve semidefinite programs exactly. In practice, and because of unavoidable numerical inaccuracies, one only obtains a numerical approximation of the optimal value and so, a certificate valid *up to machine precision* only.

However, implementing such a procedure is extremely costly because one has potentially 2×2^m unknown SOS polynomials to define h_j and θ_j in (4.4)! Therefore, it is highly desirable to provide a less costly certificate but with no guarantee to hold for every \mathbf{K} as in Corollary 4.4.

In particular one only considers compact sets \mathbf{K} . Indeed, if \mathbf{K} is compact, one has the following result (recall that $g_0 \equiv 1$).

Lemma 4.5. *Let \mathbf{K} be convex, Assumption 2.1 and Slater's condition hold. Assume that for every $j = 1, \dots, m$, $\nabla g_j(y) \neq 0$ if $y \in \mathbf{K}$ and $g_j(y) = 0$. Then for every $\epsilon > 0$ and every $j = 1, \dots, m$:*

$$(4.6) \quad \langle \nabla g_j(Y), X - Y \rangle + \epsilon = \sum_{k=0}^m \sigma_{jk}(X, Y) g_k(X) + \sum_{k=0, k \neq j}^m \psi_{jk}(X, Y) g_k(Y) + \psi_j(X, Y) g_j(Y),$$

for some SOS polynomials (σ_{jk}) and $(\psi_{jk})_{k \neq j} \subset \Sigma^2[X, Y]$, and some polynomial $\psi_j \in \mathbb{R}[X, Y]$.

Proof. By Lemma 4.3, for every $j = 1, \dots, m$, and every $x, y \in \mathbf{K}$ such that $g_j(y) = 0$, (4.3) holds and therefore, for every $j = 1, \dots, m$,

$$(4.7) \quad \langle \nabla g_j(y), x - y \rangle + \epsilon > 0 \quad \forall (x, y) \in \Omega_j,$$

where Ω_j has been defined in (4.5). As \mathbf{K} satisfies Assumption 2.1 then so does Ω_j for every $j = 1, \dots, m$. Hence (4.6) follows from (4.7) and Theorem 2.2. \square

Therefore, inspired by Lemma 4.5, introduce the following condition:

Assumption 4.6 (Certificate of convexity). *For every $j = 1, \dots, m$, (4.6) holds with $\epsilon = 0$. Then let $d_j \in \mathbb{N}$ be such that $2d_j$ is larger than the maximum degree of the polynomials $\sigma_{jk}g_k, \psi_{jk}g_k, \psi_jg_j \in \mathbb{R}[X, Y]$ in (4.6), $j = 1, \dots, m$.*

When \mathbf{K} is closed (and not necessarily compact), Slater's condition holds and the nondegeneracy assumption on the boundary holds (i.e., $\nabla g_j(y) \neq 0$ if $y \in \mathbf{K}$ and $g_j(y) = 0$) Assumption 4.6 is indeed a certificate of convexity because then (4.3) holds for every $x, y \in \mathbf{K}$ with $g_j(y) = 0$, and by Lemma 4.3, \mathbf{K} is convex. It translates the geometric property of convexity of \mathbf{K} into an algebraic SOS Putinar representation of the polynomial $(X, Y) \mapsto \langle \nabla g_j(Y), X - Y \rangle$ nonnegative on Ω_j , $j = 1, \dots, m$. On the other hand, if \mathbf{K} is convex and Assumption 2.1, Slater's condition and the nondegeneracy assumption all hold, then Assumption 4.6 is almost necessary as, by Lemma 4.5, (4.6) holds with $\epsilon > 0$ arbitrary.

With d_j fixed a priori, checking whether (4.6) hold with $\epsilon = 0$ can be done numerically. (However, again it provides a certificate of convexity valid *up to machine precision* only.) For instance, for every $j = 1, \dots, m$, it suffices to solve the semidefinite program (recall that $r_k = \lceil (\deg g_k)/2 \rceil$, $k = 1 \dots, m$)

$$(4.8) \quad \begin{cases} \rho_j := \min_{\mathbf{z}} L_{\mathbf{z}}(\langle \nabla g_j(Y), X - Y \rangle) \\ \text{s.t.} & M_{d_j}(\mathbf{z}) \succeq 0 \\ & M_{d_j - r_k}(g_k(X) \mathbf{z}) \succeq 0, \quad k = 1, \dots, m \\ & M_{d_j - r_k}(g_k(Y) \mathbf{z}) \succeq 0, \quad k = 1, \dots, m; k \neq j \\ & M_{d_j - r_j}(g_j(Y) \mathbf{z}) = 0 \\ & y_0 = 1 \end{cases}.$$

If $\rho_j = 0$ for every $j = 1, \dots, m$, then Assumption 4.6 holds. This is in contrast to the PP-BDR property in [17] that cannot be checked numerically as it involves infinitely many linear polynomials f .

Remark 4.7. Observe that the usual rank condition (3.5) used as a stopping criterion to detect whether (4.8) is exact (i.e. $\rho_1 = 0$), cannot be satisfied in solving (4.8) with primal dual interior point methods (as in the SDP-solvers used by GloptiPoly) because one tries to find an optimal solution \mathbf{z}^* in the *relative interior* of the feasible set of (4.8) and this gives maximum rank to the moment matrix $M_{d_j}(\mathbf{z}^*)$. Therefore, in the context of (4.8), if indeed $\rho_j = 0$ then \mathbf{z}^* corresponds to the moment vector of some probability measure μ supported on the set of points $(x, x) \in \mathbf{K} \times \mathbf{K}$ that satisfy $g_j(x) = 0$ (as indeed $L_{\mathbf{z}^*}(\langle \nabla g_j(Y), X - Y \rangle) = 0 = \rho_j$). Therefore $\rho_j = 0$ as d_j increases but the rank of $M_{d_j}(\mathbf{z}^*)$ does not stabilize because μ is not finitely supported. In particular, a good candidate \mathbf{z}^* for optimal solution is the moment vector of the probability measure uniformly distributed on the set $\{(x, x) \in \mathbf{K} \times \mathbf{K} : g_j(x) = 0\}$.

Alternatively, if $\rho_j \approx 0$ and the dual of (4.8) has an optimal solution $(\sigma_{jk}, \psi_{jk}, \psi_j)$, then in some cases one may check if (4.6) holds exactly after appropriate rounding of coefficients of the solution. But in general, obtaining an exact certificate (i.e., $\rho_j = 0$ in the primal or (4.6) with $\epsilon = 0$ in the dual) numerically is hopeless.

Example 2. Consider the following simple illustrative example in \mathbb{R}^2 :

$$(4.9) \quad \mathbf{K} := \{x \in \mathbb{R}^2 : x_1 x_2 - 1/4 \geq 0; 0.5 - (x_1 - 0.5)^2 - (x_2 - 0.5)^2 \geq 0\}$$

Obviously \mathbf{K} is convex but its defining polynomial $x \mapsto g_1(x) := x_1 x_2 - 1/4$ is not concave whereas $x \mapsto g_2(x) := 0.5 - (x_1 - 0.5)^2 - (x_2 - 0.5)^2$ is.

With $d_1 = 3$, solving (4.8) using GloptiPoly 3⁵ yields the optimal value $\rho_1 \approx -4.58 \cdot 10^{-11}$ which, in view of the machine precision for the SDP solvers used in GloptiPoly, could be considered to be zero, but of course with no guarantee. However, and according to Remark 4.7, we could check that (again up to machine precision) for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2d_j$, $z_{\alpha, \alpha}^* = z_{2\alpha, 0}^*$ and $z_{\alpha, 0}^* = z_{0, \alpha}^*$. In addition, because of symmetry, $z_{\alpha, \beta} = z_{\alpha', \beta'}$ whenever $\alpha'_1 = \alpha_2$ and $\alpha'_2 = \alpha_1$ (and similarly for β and β'). Indeed for moments of order 1 we have $z_{\alpha, \beta}^* = (0.5707, 0.5707, 0.5707, 0.5707)$ and for moments of order 2,

$$z_{\alpha, \beta}^* = (0.4090, 0.25, 0.4090, 0.25, 0.4090, 0.25, 0.4090, 0.4090, 0.25, 0.4090).$$

⁵GloptiPoly 3 (a Matlab based public software) is an extension of GloptiPoly [9] to solve the generalized problem of moments described in [18]. For more details see www.laas.fr/~henrion/software/.

For $j = 2$ there is no test to perform because $-g_2$ being quadratic and convex yields

$$(4.10) \quad \langle \nabla g_2(Y), X - Y \rangle = g_2(X) - g_2(Y) + \underbrace{(X - Y)^T (-\nabla^2 g_2(Y)) (X - Y)}_{SOS}$$

which is in the form (4.6) with $d_2 = 1$.

We next show the role of Assumption 4.6 in obtaining a semidefinite representation of \mathbf{K} .

Theorem 4.8. *Let Assumption 2.1 and Slater's condition hold. Moreover, assume that for every $j = 1, \dots, m$, $\nabla g_j(y) \neq 0$ whenever $y \in \mathbf{K}$ and $g_j(y) = 0$. If Assumption 4.6 holds then \mathbf{K} is convex and Ω in (4.2) with $d := \max_j d_j$, is a semidefinite representation of \mathbf{K} .*

Proof. That \mathbf{K} is convex follows from Lemma 4.3. We next prove that the PP-BDR property defined in Lasserre [16] holds for \mathbf{K} . Let $f \in \mathbb{R}[X]$ be a linear polynomial with coefficient vector $\mathbf{f} \in \mathbb{R}^n$ (i.e., $X \mapsto f(X) = \mathbf{f}^T X$) and consider the optimization problem $\mathbf{P} : \min \{ \mathbf{f}^T x : x \in \mathbf{K} \}$. As \mathbf{K} is compact, let $x^* \in \mathbf{K}$ be a global minimizer of f . The Fritz-John optimality conditions state that there exists $0 \neq \lambda \in \mathbb{R}_+^{m+1}$ such that

$$(4.11) \quad \lambda_0 \mathbf{f} = \sum_{j=1}^m \lambda_j \nabla g_j(x^*); \quad \lambda_j g_j(x^*) = 0 \quad \forall j = 1, \dots, m.$$

(See e.g. [3].) We first prove by contradiction that if Slater's condition and the nondegeneracy assumption hold then $\lambda_0 > 0$. Suppose that $\lambda_0 = 0$ and let $J := \{j \in \{1, \dots, m\} : \lambda_j > 0\}$; hence J is nonempty as $\lambda \neq 0$. With $x_0 \in \mathbf{K}$ such that $g_j(x_0) > 0$ (as Slater's condition holds, one such x_0 exists), let $B(x_0, \rho) := \{z : \|z - x_0\| \leq \rho\}$. For ρ sufficiently small, $B(x_0, \rho) \subset \mathbf{K}$ and $g_j(z) > 0$ for all $z \in B(x_0, \rho)$ and every $j = 1, \dots, m$. Then by (4.11) and $\lambda_0 = 0$,

$$0 = \sum_{j=1}^m \lambda_j \langle \nabla g_j(x^*), z - x^* \rangle, \quad \forall z \in B(x_0, \rho),$$

which in turn implies (by nonnegativity of each term in the above sum)

$$\langle \nabla g_j(x^*), z - x^* \rangle = 0, \quad \forall z \in B(x_0, \rho), \quad j \in J.$$

But this clearly implies $\nabla g_j(x^*) = 0$ for every $j \in J$, in contradiction with the nondegeneracy assumption. Hence $\lambda_0 > 0$ and by homogeneity, we may and will take $\lambda_0 = 1$.

Therefore, letting $Y := x^*$ in (4.6), the polynomial $X \mapsto f(X) - f^*$ can be written

$$\begin{aligned} \mathbf{f}^T X - f^* &= \sum_{j=1}^m \lambda_j [\langle \nabla g_j(x^*), X - x^* \rangle] \\ &= \sum_{j=1}^m \lambda_j \left[\sum_{k=0}^m \sigma_{jk}(X, x^*) g_k(X) + \sum_{k=0, k \neq j}^m \psi_{jk}(X, x^*) g_k(x^*) \right. \\ &\quad \left. + \psi_j(X, x^*) g_j(x^*) \right] \end{aligned}$$

where we have used (4.6) with $Y = x^*$ and $\epsilon = 0$. Next, observe that :

$$\begin{aligned} X \mapsto \sigma_{jk}(X, x^*) &\in \Sigma^2[X] && [\text{as } \sigma_{jk} \in \Sigma^2[X, Y]] \\ X \mapsto \psi_{jk}(X, x^*) g_k(x^*) &\in \Sigma^2[X] && [\text{as } \psi_{jk} \in \Sigma^2[X, Y] \text{ and } g_j(x^*) \geq 0] \\ \lambda_j g_j(x^*) &= 0 && j = 1, \dots, m. \end{aligned}$$

And so, as $\lambda \in \mathbb{R}_+^m$,

$$(4.12) \quad X \mapsto \mathbf{f}^T X - f^* = \Delta_0(X) + \sum_{j=1}^m \Delta_j(X) g_j(X),$$

for SOS polynomials $(\Delta_j)_{j=0}^m \subset \Sigma^2[X]$ defined by

$$\begin{aligned} X \mapsto \Delta_0(X) &= \sum_{j=1}^m \lambda_j \left(\sum_{k=0, k \neq j}^m \psi_{jk}(X, x^*) g_k(x^*) \right) \\ X \mapsto \Delta_j(X) &= \sum_{l=1}^m \lambda_l \sigma_{lj}(X, x^*), \quad j = 1, \dots, m. \end{aligned}$$

Write every affine polynomial $f \in \mathbb{R}[X]$ as $\mathbf{f}^T X + f_0$ for some $\mathbf{f} \in \mathbb{R}^n$ and $f_0 = f(0)$. If f is nonnegative on \mathbf{K} then from (4.12),

$$\begin{aligned} f(X) = \mathbf{f}^T X - f^* + f^* + f_0 &= f^* + f_0 + \Delta_0(X) + \sum_{j=1}^m \Delta_j(X) g_j(X) \\ &= \widehat{\Delta}_0(X) + \sum_{j=1}^m \Delta_j(X) g_j(X) \quad \forall X, \end{aligned}$$

with $\widehat{\Delta}_0 \in \Sigma^2[X]$ (because $f^* + f_0 \geq 0$) and so, the PP-BDR property holds for \mathbf{K} with order d . By [16, Theor. 2], \mathbf{K} is SDr with the semidefinite representation (4.2). \square

We next show that the two sufficient conditions of strict convexity and SOS-convexity of Helton and Nie [6] in Theorem 4.2 both imply that Assumption 4.6 holds and so Theorem 4.8 contains Theorem 4.2 as a special case.

Corollary 4.9. *Let \mathbf{K} in (1.1) be convex and both Assumption 2.1 and Slater's condition hold. Assume that either $-g_j$ is SOS-convex or $-g_j$ is convex on \mathbf{K} and $-\nabla^2 g_j \succ 0$ on $\mathbf{K} \cap \{x : g_j(x) = 0\}$, for every $j = 1, \dots, m$. Then Assumption 4.6 holds and so Theorem 4.8 applies.*

Proof. By Lemma 2.5, for every $j = 1, \dots, m$, write

$$\begin{aligned} (X, Y) \mapsto g_j(X) - g_j(Y) - \langle \nabla g_j(Y), X - Y \rangle &= \\ \left\langle (X - Y), \underbrace{\left(\int_0^1 \int_0^t \nabla^2 g_j(Y + s(X - Y)) ds dt \right)}_{F_j(X, Y)} (X - Y) \right\rangle. \end{aligned}$$

If $-\nabla^2 g_j \succ 0$ on $y \in \mathbf{K}$ with $g_j(y) = 0$, then from the proof of [6, Lemma 19], $-F_j(x, y) \succ 0$ for all $x, y \in \mathbf{K}$ with $g_j(y) = 0$. In other words, $-F_j(x, y) \succeq \delta I_n$ on

Ω_j (defined in (4.5)) for some $\delta > 0$. Therefore, by the matrix polynomial version of Putinar Positivstellensatz in [6, Theor. 29],

(4.13)

$$-F_j(X, Y) = \sum_{k=0}^m \widehat{\sigma}_{jk}(X, Y) g_k(X) + \sum_{k=0, k \neq j}^m \widehat{\psi}_{jk}(X, Y) g_k(Y) + \widehat{\psi}_j(X, Y) g_j(Y)$$

for some SOS matrix polynomials $(\widehat{\sigma}_{jk}(X, Y))$, $(\widehat{\psi}_{jk}(X, Y))$ and some matrix polynomial $\widehat{\psi}_j(X, Y)$.

On the other hand, if $-g_j$ is SOS-convex then by Lemma 2.4, $-F_j(X, Y)$ is SOS and therefore (4.13) also holds (take $\widehat{\sigma}_{jk} \equiv 0$ for all $k \neq 0$, $\widehat{\psi}_{jk} \equiv 0$ for all k and $\widehat{\psi}_j \equiv 0$). But then

$$\begin{aligned} g_j(X) - g(Y) - \langle \nabla g_j(Y), X - Y \rangle &= \langle (X - Y), F_j(X, Y)(X - Y) \rangle \\ &= - \sum_{k=0}^m \langle (X - Y), \widehat{\sigma}_{jk}(X, Y)(X - Y) \rangle g_k(X) \\ &\quad - \sum_{k=0, k \neq j}^m \langle (X - Y), \widehat{\psi}_{jk}(X, Y)(X - Y) \rangle g_k(Y) \\ &\quad - \langle (X - Y), \widehat{\psi}_j(X, Y)(X - Y) \rangle g_j(Y) \\ &= - \sum_{k=0}^m \sigma_{jk}(X, Y) g_k(X) - \\ &\quad \sum_{k=0, k \neq j}^m \psi_{jk}(X, Y) g_k(Y) - \psi_j(X, Y) g_j(Y) \end{aligned}$$

for all X, Y and for some SOS polynomials $\sigma_{jk}, \psi_{jk} \in \mathbb{R}[X, Y]$ and some polynomial $\psi_j \in \mathbb{R}[X, Y]$. Equivalently,

$$\begin{aligned} \langle \nabla g_j(Y), X - Y \rangle &= g_j(X) - g_j(Y) + \sum_{k=0}^m \sigma_{jk}(X, Y) g_k(X) \\ &\quad + \sum_{k=0, k \neq j}^m \psi_{jk}(X, Y) g_k(Y) + \psi_j(X, Y) g_j(Y) \\ &= \sum_{k=0}^m \sigma'_{jk}(X, Y) g_k(X) + \sum_{k=0, k \neq j}^m \psi_{jk}(X, Y) g_k(Y) \\ &\quad + \psi'_j(X, Y) g_j(Y) \end{aligned}$$

for some SOS polynomials $\sigma'_{jk}, \psi_{jk} \in \Sigma^2[X, Y]$ and some polynomial $\psi'_j \in \mathbb{R}[X, Y]$. In other words, Assumption 4.6 holds, which concludes the proof. \square

Hence if each $-g_j$ is SOS-convex or convex on \mathbf{K} with $-\nabla^2 g_j \succ 0$ on $\mathbf{K} \cap \{x : g_j(x) = 0\}$, one obtains a numerical scheme to obtain the parameter d in Theorem 4.8 as well as the semidefinite representation (4.2) of \mathbf{K} . Solve the semidefinite programs (4.8) with degree parameter d_j . Eventually, $\rho_j = 0$ for every $j = 1, \dots, m$.

Example 3. Consider the convex set \mathbf{K} in (4.9) of Example 2 for which the defining polynomial g_1 of \mathbf{K} is not concave. We have seen that Assumption 4.6 holds (up to $\rho_1 \approx 10^{-11}$, close to machine precision) and $\max[d_1, d_2] = 3$. By Theorem 4.8, if ρ_1 would be exactly 0, the set

$$(4.14) \quad \Omega := \left\{ (x, \mathbf{y}) \in \mathbb{R}^n \times \mathbb{R}^{s(6)} : \begin{cases} M_3(\mathbf{y}) & \geq 0 \\ M_2(g_j \mathbf{y}) & \geq 0, \quad j = 1, 2 \\ L_{\mathbf{y}}(X_i) & = x_i, \quad i = 1, 2 \\ y_0 & = 1 \end{cases} \right\}.$$

would be a semidefinite representation of \mathbf{K} .

At least in practice, for every linear polynomial $f \in \mathbb{R}[X]$, minimizing $L_{\mathbf{y}}(f)$ over Ω yields the desired optimal value $f^* := \min_{x \in \mathbf{K}} f(x)$, up to $\rho_1 \approx -10^{-11}$.

Indeed, let $f \in \mathbb{R}[X]$ be $\mathbf{f}^T X$ for some vector $\mathbf{f} \in \mathbb{R}^n$. In minimizing f over \mathbf{K} , one has $\mathbf{f} = \lambda_1 \nabla g_1(x^*) + \lambda_2 \nabla g_2(x^*)$ for some $\lambda \in \mathbb{R}_+^2$, some $x^* \in \mathbf{K}$ with $\lambda_i g_i(x^*) = 0$, $i = 1, 2$, and $f^* = \lambda_1 \langle \nabla g_1(x^*), x^* \rangle + \lambda_2 \langle \nabla g_2(x^*), x^* \rangle = \min_{x \in \mathbf{K}} \mathbf{f}^T x$. Let x be as in (4.14), arbitrary. Then

$$\mathbf{f}^T x - f^* = L_{\mathbf{y}}(f(X) - f^*) = \sum_{i=1}^2 \lambda_i L_{\mathbf{y}}(\langle \nabla g_i(x^*), X - x^* \rangle).$$

If $\lambda_1 > 0$ so that $g_1(x^*) = 0$, use (4.12) to obtain

$$L_{\mathbf{y}}(\langle \nabla g_1(x^*), X - x^* \rangle) = L_{\mathbf{y}}(\rho_1 + \Delta_0(X) + \sum_{j=1}^2 \Delta_j(X) g_j(X)) \geq \rho_1,$$

because $L_{\mathbf{y}}(\Delta_0) \geq 0$ follows from $M_3(\mathbf{y}) \geq 0$, and $L_{\mathbf{y}}(\Delta_j g_j) \geq 0$, $j = 1, 2$, follows from $M_2(g_1 \mathbf{y}), M_2(g_2 \mathbf{y}) \geq 0$. If $\lambda_2 > 0$ so that $g_2(x^*) = 0$, then from (4.10)

$$L_{\mathbf{y}}(\langle \nabla g_2(x^*), X - x^* \rangle) = L_{\mathbf{y}}(g_2(X) - \langle (X - x^*), \nabla^2 g_2(x^*)(X - x^*) \rangle) \geq 0,$$

because $L_{\mathbf{y}}(g_2) \geq 0$ follows from $M_2(g_2 \mathbf{y}) \geq 0$ whereas the second term is nonnegative as $\langle (X - x^*), -\nabla^2 g_2(x^*)(X - x^*) \rangle$ is SOS and $M_3(\mathbf{y}) \geq 0$. Hence $\mathbf{f}^T x - f^* \geq \rho_1$. On the other hand, from $\mathbf{K} \subseteq \{x : (x, y) \in \Omega\}$, one finally obtains the desired result

$$f^* + \rho_1 \leq \min \{\mathbf{f}^T x : (x, y) \in \Omega\} \leq f^*.$$

5. CONCLUSION

As well-known, convexity is a highly desirable property in optimization. We have shown that it also has important specific consequences in polynomial optimization. For instance, for polynomial optimization problems with SOS-convex or strictly convex polynomial data, the basic SDP-relaxations of the moment approach [13] *recognizes* convexity and finite convergence occurs. Similarly, the set \mathbf{K} has a semidefinite representation, explicit in terms of the defining polynomials (g_j).

The class of SOS-convex polynomials introduced in Helton and Nie [6] is particularly interesting because the semidefinite constraint to handle in the semidefinite relaxation only involves the Hankel-like moment matrix which does *not* depend on the problem data! Hence one might envision a dedicated SDP solver that would take into account this peculiarity as Hankel-like or Toeplitz-like matrices enjoy very specific properties. Moreover, if restricted to this class of polynomials, Jensen's inequality can be extended to linear functionals in the dual cone of SOS polynomials (hence not necessarily probability measures).

Therefore, a topic of further research is to evaluate how *large* is the subclass of SOS-convex polynomials in the class of convex polynomials, and if possible, to also provide simple sufficient conditions for SOS-convexity.

ACKNOWLEDGEMENTS

The author wishes to thank L. Tunçel and Y. Nesterov for helpful discussions on various characterizations of convex sets, and also two anonymous referees for several corrections as well as suggestions and remarks to improve a first version of this paper.

REFERENCES

- [1] A. BEN TAL AND A. NEMIROVSKI, *Lectures on Modern Convex Optimization*, SIAM, Philadelphia, 2001.
- [2] C. BENG CHUA AND L. TUNCEL, *Invariance and efficiency of convex representations*, Math. Prog., 111 (2008), pp. 113–140.
- [3] S.I. BIRBIL, J. B. G. FRENK, AND G. J. STILL, *An elementary proof of the Fritz-John and Karush-Kuhn-Tucker conditions in nonlinear programming*, Eur. J. Oper. Res., 180 (2007), pp. 479–484.
- [4] R. E. CURTO AND L. A. FIALKOW, *Recursiveness, positivity, and truncated moment problems*, Houston J. Math., 17 (1991), pp. 603–635.
- [5] D. HANDELMAN, *Representing polynomials by positive linear functions on compact convex polyhedra*, Pac. J. Math., 132 (1988), pp. 35–62.
- [6] J. W. HELTON AND J. NIE, *Semidefinite representation of convex sets*, Math. Program., to appear.
arXiv:0705.4068v5
- [7] J. W. HELTON AND J. NIE, *Sufficient and necessary condition for semidefinite representation of sets*, SIAM J. Optim., to appear.
- [8] D. HENRION, *On semidefinite representations of plane quartics*, Research Report # 08444, LAAS-CNRS, Toulouse, France, 2008. Submitted.
- [9] D. HENRION AND J.B. LASSERRE, *GloptiPoly : Global Optimization over Polynomials with Matlab and SeDuMi*, ACM Trans. Math. Soft., 29 (2003), pp. 165–194.
- [10] M. KOJIMA AND M. MARAMATSU, *An extension of sums of squares relaxations to polynomial optimization problems over symmetric cones*, Math. Program., 110 (2007), pp. 315–336.
- [11] C. W. J. HOL AND C. W. SCHERER, *A sum-of-squares approach to fixed order H_∞ -synthesis*, in Positive Polynomials in Control, Garulli and Henrion (Eds.), Springer-Verlag, Berlin, 2005.
- [12] J. L. KRIVINE, *Anneaux préordonnés*, J. Anal. Math., 12 (1964), pp. 307–326.
- [13] J. B. LASSERRE, *Global optimization with polynomials and the problem of moments*, SIAM J. Optim., 11 (2001), pp. 796–817.
- [14] J. B. LASSERRE, *Semidefinite programming vs. LP relaxations for polynomial programming*, Math. Oper. Res., 27 (2002), pp. 347–360.
- [15] J. B. LASSERRE, *Polynomial programming: LP-relaxations also converge*, SIAM J. Optim., 15 (2004), pp. 383–393.
- [16] J. B. LASSERRE, *Convex sets with semidefinite representation*, Math. Program., to appear.
- [17] J. B. LASSERRE, *Representation of nonnegative convex polynomials*, Arch. Math., 91 (2008), pp. 126–130.
- [18] J. B. LASSERRE, *A Semidefinite programming approach to the generalized problem of moments*, Math. Program., 112 (2008), pp. 65–92.
- [19] A. S. LEWIS, P. PARRILO, AND M. V. RAMANA, *The Lax conjecture is true*, Proc. Amer. Math. Soc., 133 (2005), pp. 2495–2499.
- [20] Y. E. NESTEROV AND A. S. NEMIROVSKI, *Self-concordant functions and polynomial time methods in convex programming*, Report, Central Economical and Mathematical Institute, USSR Academy of Sciences, Moscow, 1989.
- [21] P. A. PARRILO, *Semidefinite programming relaxations for semialgebraic problems*, Math. Prog. Ser. B, 96 (2003), pp. 293–320.
- [22] P. PARRILO, *Exact semidefinite representations for genus zero curves*, talk at the Banff workshop "Positive Polynomials and Optimization", Banff, Canada, October 2006.
- [23] M. D. PERLMAN, *Jensen's inequality for a convex vector-valued function on an infinite dimensional space*, J. Mult. Variat. Anal., 4 (1974), pp. 52–65.
- [24] M. PUTINAR, *Positive polynomials on compact semi-algebraic sets*, Ind. Univ. Math. J., 42 (1993), pp. 969–984.

- [25] J. BOCHNAK, M. COSTE, AND M-F. ROY, *Real Algebraic Geometry*, Springer Verlag, Berlin, 1998.
- [26] C. SCHEIDERER, *Positivity and sums of squares: A guide to recent results*, in Emerging Applications of Algebraic Geometry, M. Putinar and S. Sullivant (eds.), IMA Volumes, Institute of Mathematics and Its Applications, Minneapolis, Springer, 2008, pp. 271–324.
- [27] K. SCHMÜDGEN, *The K -moment problem for compact semi-algebraic sets*, Math. Ann. 289 (1991), pp. 203–206.
- [28] R. SCHNEIDER, *Convex Bodies: The Brunn–Minkowski Theory*, Cambridge University Press, Cambridge, United Kingdom, 1994.
- [29] M. SCHWEIGHOFER, *Optimization of polynomials on compact semialgebraic sets*, SIAM J. Optim. 15 (2005), pp. 805–825.
- [30] F.-H. VASILESCU, *Spectral measures and moment problems*, Spectral Theory and Its Applications, Theta 2003, pp. 173–215.

LAAS-CNRS AND INSTITUTE OF MATHEMATICS, UNIVERSITY OF TOULOUSE, LAAS, 7 AVENUE
DU COLONEL ROCHE, 31077 TOULOUSE CÉDEX 4, FRANCE
E-mail address: `lasserre@laas.fr`