Closed-form solutions to static-arbitrage upper bounds on basket options

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Abstract

We provide a closed-form solution to the problem of computing the sharpest static-arbitrage upper bound on the price of a European basket option, given the prices of vanilla call options in the underlying securities. Unlike previous approaches to this problem, our solution technique is entirely based on linear programming. This also allows us to obtain an efficient (linear-size) linear programming formulation for the more realistic problem of computing sharp static arbitrage upper bounds taking into consideration bid-ask spreads in the given option prices and other transaction costs.

1 Introduction

We provide a closed-form solution to the problem of computing the sharpest upper bound on the price of a European basket option, given the only assumption of absence of arbitrage, and information on the prices of vanilla European call options on the same underlying assets and with the same maturity. Bounds of this type are called *static-arbitrage bounds.* These kinds of bounds can be seen as *robust* bounds that any sound pricing model must satisfy [5, 9]. They provide a mechanism for checking consistency of prices, as well as an initial price estimate for options regardless of any model specifics.

The computation of sharp static-arbitrage upper bounds can be formulated as the problem of finding an underlying asset price distribution that maximizes the discounted expected payoff of the basket option, and *replicates* the given option prices [3]. Under reasonably mild assumptions (see, e.g., Proposition 1 in Section 2), an equivalent dual formulation is to find the least expensive portfolio of cash and the options with known prices whose combined payoff super-replicates the payoff of the new basket option of interest (see, e.g. [3, 9]). This *semiparametric* approach can be seen as an alternative to *parametric* techniques (such as Monte Carlo simulations) that determine the price of an option as the discounted expected option's payoff under an appropriate risk-neutral measure. Semiparametric techniques are especially useful in incomplete market conditions, or

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when the number of underlying assets makes the computation of parametric prices numerically challenging.

The problem of computing static-arbitrage bounds has received a fair amount of attention in recent years. Of particular relevance to our work are the recent articles by d'Aspremont and El Ghaoui [3], and by Hobson, Laurence, and Wang [8, 9]. D'Aspremont and El Ghaoui [3] derive linear programming relaxations for static-arbitrage bounds based on an integral transformation of the options' price functions. They show that their bounds are tight in some special cases when only one or two vanilla options prices per underlying asset are given. On the other hand, Hobson, Laurence and Wang [8, 9] derive more general results for computing sharp static-arbitrage upper bounds when vanilla option prices on every asset are given for a continuum of strikes. Their approach relies on a Lagrangian programming formulation and the fact that the continuum of options determines the full marginal distributions of each of the assets.

We undertake a fairly different and novel approach to the static arbitrage upper bound problem based entirely on linear programming techniques. The foundational block of our work is the construction of an efficient (linear-size) polyhedral description for the set of *super-replicating portfolios*, that is, the set of portfolios of cash and the given options whose payoff super-replicates the basket option's payoff. We show that the set of super-replicating portfolios is a projection of a polyhedron whose description only requires a number of variables and constraints that is linear in the number of given options (see Lemma 6). The polyhedral description yields an efficient linear programming formulation for the static-arbitrage upper bound problem that can be solved in closed form (see Theorem 3). The latter generalizes a result of d'Aspremont and El Ghaoui [3] that was previously known only for basket options with positive weights in the special case when one or two vanilla option prices per underlying asset are given. Furthermore, the polyhedral description readily allows us to obtain efficient models that incorporate for the first time some important features in the static bounds' computations. Some of these features are basket options with negative weights, bid-ask spreads, limits on the sizes of long/short positions, and transaction costs. Moreover, the closed-form formula provided by Theorem 3 has an interesting application in the derivation of a robust portfolio allocation model [6].

Although it is intuitively clear that the set of super-replicating portfolios admits a polyhedral description, straightforward attempts to do so yield intractable descriptions that require a number of constraints and variables that is exponentially large in the number of given option prices. By contrast, we provide a polyhedral description whose number of variables and constraints is only linear in the number of given option prices. We note that the computation of static arbitrage *lower* bounds poses a different set of challenges as the nature of *sub-replicating portfolios* is fundamentally different from that of the super-replicating portfolios. The different nature of the upper and lower bound computation has been recognized previously, as it was apparent that the computation of the upper bounds was more tractable [3, 8, 9]. A separate article [11] presents some results for the computation of static arbitrage lower bounds that are similar in spirit to those discussed herein.

We note in passing that an interesting and active related area of research is the computation of option price bounds given moment information about the underlying risk-neutral asset price distribution. These kinds of bounds have been investigated in the work of Bertsimas and Popescu [1], Boyle and Lin [2], and Zuluaga and Peña [14] among others. This *moment* approach to compute option price bounds differs from the computation of arbitrage bounds studied here in two main aspects: First, in the moment approach an underlying model for the risk neutral asset prices distribution is assumed in order to *sample* (obtain) the necessary moment information. By contrast, the arbitrage approach relies solely on option price information observed in the market. Second, the computation of moment-based bounds in general requires the use of semidefinite programming solvers whereas arbitrage bounds can be computed via linear programming.

The paper is organized as follows. Section 2 presents the basic notation and closed-form solution to the static arbitrage upper bound on a basket option, given the prices of vanilla call options on the underlying assets. In Section 3 we present the main building block of our approach, namely an efficient polyhedral description of the super-replicating portfolios. The latter yields the first efficient linear programming formulation for the computation of static arbitrage bounds that incorporate bid-ask spreads. In Section 4 we provide numerical experiments to illustrate some of our results. Finally, Section 5 presents the proofs of the results in the paper.

2 Static-arbitrage bounds

Consider the problem of computing a sharp upper *static-arbitrage* bound on the price of a European basket option, given information on the prices of other similar options, without making any assumptions other than the absence of arbitrage. This problem can be formulated as the following optimization problem (see, e.g., [3]):

$$\begin{array}{ll} \sup\limits_{\pi} & \mathbb{E}_\pi[(\omega \cdot S - \kappa)^+] \\ \text{s.t.} & \mathbb{E}_\pi[1] = 1 \\ & \mathbb{E}_\pi[(w^j \cdot S - K_j)^+] = p_j, \quad j = 1, ..., r \\ & \pi \quad \text{a distribution in } \mathbf{R}^n_+. \end{array} \tag{U}$$

Above, the multivariate random variable S with probability distribution π represents the prices of the n underlying assets (at maturity) in the basket. The vectors $w^j \in \mathbf{R}^n$, and constants $K_j \in \mathbf{R}$, $j = 1, \dots, r$, represent the weights of the underlying assets and the strike price of the basket options whose prices are given. The vector $\omega \in \mathbf{R}^n$ and constant $\kappa \in \mathbf{R}$ represent the weights and strike of the basket option whose price we want to bound. Problem (U) maximizes the expected payoff of the basket option $(\omega \cdot S - \kappa)^+$ over all underlying asset price distributions π in $\mathbf{R}^n_+$ that *replicate* the price p_j of the basket option $(w^j \cdot S - K_j)^+$ for $j = 1, \dots, r$.

Following [3], we implicitly assume that all the options have the same maturity, and that without loss of generality, the risk-free interest rate is zero; or equivalently, we compare the prices in the forward market (at maturity).

Problem (U) has the following associated dual (see, [7]):

$$\begin{array}{ll} \inf\limits_{z,y} & z + \sum\limits_{j=1}^{r} p_j y_j \\ \text{s.t.} & z + \sum\limits_{j=1}^{r} y_j (w^j \cdot s - K_j)^+ \geq (\omega \cdot s - \kappa)^+ \quad \text{for all } s \in \mathbf{R}^n_+ \\ & y \in \mathbf{R}^r, \;\; z \in \mathbf{R}. \end{array} \tag{DU}$$

The dual problem has a natural financial interpretation: Find the cheapest portfolio of positions in cash and in the basket options $(w^j \cdot S - K_j)^+$, $j = 1, \dots, r$ that super-replicates the payoff of the basket option $(\omega \cdot S - \kappa)^+$. It is easy to see that weak duality holds between (U) and (DU). Furthermore, under reasonably mild assumptions, strong duality holds as well. For instance, Proposition 1 states two generic conditions that ensure strong duality in our context.

Proposition 1. *The optimal values of* (U) *and* (DU) *coincide —i.e., strong duality holds between* (U) *and* (DU)— *if at least one of the following two conditions holds.*

(i) *Strong primal feasibility:*

$$\begin{bmatrix} 1 \\ p \end{bmatrix} \in \operatorname{int}\left(\left\{\begin{bmatrix} \mathbb{E}_\pi[1] \\ \left(\mathbb{E}_\pi[(w^j \cdot S - K_j)^+)\right)_{j=1,\ldots,r}] \end{bmatrix} : \pi \quad \textit{is a distribution in } \mathbf{R}^n_+ \right\}\right).$$

In particular, strong duality holds provided the prices p are arbitrage-free and remain arbitrage-free after slight perturbations.

(ii) *Strong dual feasibility: There exists* $(\hat{z}, \hat{y}) \in \mathbf{R}^{r+1}$ *such that*

$$(\hat{z}, \hat{y}) \in \operatorname{int}\left(\left\{(z, y) \in \mathbf{R}^{r+1} : z + \sum_{j=1}^{r} y_j (w^j \cdot s - K_j)^+ \geq (\omega \cdot s - \kappa)^+ \textit{ for all } s \in \mathbf{R}^n_+ \right\}\right).$$

In particular, strong duality holds provided that for each asset at least one vanilla option price is known.

Proposition 1 follows from general convex duality results [12, 13]. A detailed discussion can be found in [14, Sec. 3].

The lower static-arbitrage bound on the price of a basket option corresponding to (U) can be obtained by changing sup to inf in (U). For some recent advances on this problem, see [8, 11].

2.1 Upper bound given calls on single assets

We next consider the computation of the upper bound on the price of the basket $(\omega \cdot S - \kappa)^+$ in the special but important case when prices of m calls $p_i^j = \mathbb{E}_\pi[(S_i - K_i^j)^+]$, $j = 1, \ldots, m$ and a forward $p_i^0 = \mathbb{E}_\pi[S_i]$ for each asset $i = 1, \ldots, n$ are known. Notice that the assumption on the same number of options m per asset can be made without loss of generality: If one of the assets has fewer than m options, we can artificially increase the number of known options to m by repeating one of the options.

In this case the static upper bound problem (U) is

$$\begin{array}{ll} \sup\limits_{\pi} & \mathbb{E}_\pi[(\omega \cdot S - \kappa)^+] \\ \text{s.t.} & \mathbb{E}_\pi[1] = 1 \\ & \mathbb{E}_\pi[S] = p^0 \\ & \mathbb{E}_\pi[(S - K^j)^+] = p^j, \qquad j = 1, \ldots, m \\ & \pi \quad \text{a distribution in } \mathbf{R}^n_+. \end{array} \tag{1}$$

Here and throughout the sequel we use the following convenient vector notation: $\mathbb{E}_\pi[(S - K^j)^+] = p^j$ is a shorthand for $\mathbb{E}_\pi[(S_i - K_i^j)^+] = p_i^j$, for $i = 1, \ldots, n$.

Without loss of generality assume:

$$\vec{0} \leq K^1 \leq \cdots \leq K^m \in \mathbf{R}^n.$$

It is convenient to put $K^0 := \vec{0}$ so that the dual (DU) of (1) can be written as

$$\begin{array}{ll} \inf\limits_{z, y^0, \ldots, y^m} & z + \sum\limits_{j=0}^{m} p^j \cdot y^j \\ \text{s.t.} & z + \sum\limits_{j=0}^{m} y^j \cdot (s - K^j)^+ \geq (\omega \cdot s - \kappa)^+ \quad \text{for all } s \in \mathbf{R}^n_+ \\ & y^j \in \mathbf{R}^n, \ j = 0, \ldots, m \\ & z \in \mathbf{R}. \end{array} \tag{2}$$

By Proposition 1(ii), the optimal values of (1) and (2) are the same. We shall assume that for each asset the given forward and call prices are arbitrage-free. This can be done without loss of generality as otherwise the static-arbitrage problem (1) is infeasible and hence uninteresting.

Theorem 3 below provides a closed-form expression for this bound under the arbitrage-free assumption. In addition, Theorem 5 provides a closed-form expression for the optimal super-replicating strategy that solves (2). We note that the closed-form formula (5) generalizes the closed-form formula derived by d'Aspremont and El Ghaoui [3, Prop.4]. The latter corresponds to (5) for the special case $m = 1$ and $\omega \geq 0$.

It is well-known [1] that a set of vanilla options $(S_i - K_i^j)^+, \; j = 0, 1, \ldots, m$ in asset i is arbitrage-free if and only if the following convexity condition holds:

$$0 \leq \frac{p_i^{m-1} - p_i^m}{K_i^m - K_i^{m-1}} \leq \frac{p_i^{m-2} - p_i^{m-1}}{K_i^{m-1} - K_i^{m-2}} \leq \cdots \leq \frac{p_i^1 - p_i^2}{K_i^2 - K_i^1} \leq \frac{p_i^0 - p_i^1}{K_i^1} \leq 1. \tag{3}$$

Thus, the arbitrage-free assumption can be stated as follows.

Assumption 2. Condition (3) holds for each $i = 1, \ldots, n$.

We note that Davis and Hobson [4] provide an interesting generalization of the arbitrage-free condition (3) to the multiperiod setting.

Before stating the closed-form expression for the bound (1), we introduce some convenient notation. Define

$$I^+ := \{i \in \{1, \ldots, n\} : \omega_i \geq 0\}, \quad I^- := \{i \in \{1, \ldots, n\} : \omega_i < 0\},$$

and let $\nu : [0,1] \to \mathbf{R}^n$ be defined as

$$\nu(\tau)_i = \begin{cases} \min\limits_{j=0,\ldots,m} \left\{ p_i^j + \tau K_i^j \right\} & \text{if } i \in I^+ \\ p_i^0 - \min\limits_{j=0,\ldots,m} \left\{ p_i^j + (1-\tau) K_i^j \right\} & \text{if } i \in I^-. \end{cases} \tag{4}$$

Theorem 3. *Suppose Assumption 2 holds. Then the optimal value of* (1) *is*

$$\max_{\tau \in [0,1]} (\omega \cdot \nu(\tau) - \tau\kappa). \tag{5}$$

Remark 4. Because the components of $\nu(\tau)$ are piecewise linear, the value (5) can be found by evaluating the *breakpoints* where the components of $\nu(\tau)$ change slopes, that is, the $mn + 2$ points:

$$\tau = 0, 1, \quad \tau = \frac{p_i^{j-1} - p_i^j}{K_i^j - K_i^{j-1}}, \; i \in I^+, \quad \text{and} \quad \tau = 1 - \frac{p_i^{j-1} - p_i^j}{K_i^j - K_i^{j-1}}, \; i \in I^-, \; j = 1, \ldots, m. \tag{6}$$

Furthermore, because each $\omega_i \cdot \nu(\tau)_i$ is concave, it follows that the function $\omega \cdot \nu(\tau) - \tau\kappa$ is concave as well. Thus once the $mn + 2$ breakpoints in (6) are sorted, the one-dimensional optimization problem (5) can be solved via a binary search by evaluating $\omega \cdot \nu(\tau) - \tau\kappa$ at $2\log(nm + 2)$ points.

We next describe the optimal super-replicating strategy that solves (2) and yields the optimal bound (5). To that end, we introduce some additional convenient notation. Assume $\tau \in [0,1]$ is given. For each $i = 1, \ldots, n$ let $j_i[\tau], j_i'[\tau]$ denote the indices where the relevant minimum in (4) is attained. More precisely, for each $i \in I^+$ let $j_i[\tau]$ and $j_i'[\tau]$ be such that

$$\nu(t)_i = p_i^{j_i[\tau]} + tK_i^{j_i[\tau]} \text{ for } t \downarrow \tau, \;\; \text{and } \nu(t)_i = p_i^{j_i'[\tau]} + tK_i^{j_i'[\tau]} \text{ for } t \uparrow \tau.$$

Likewise, for each $i \in I^-$ let $j_i[\tau]$ and $j'_i[\tau]$ be such that

$$\nu(t)_i = p_i^0 - (p_i^{j_i[\tau]} + (1-t)K_i^{j_i[\tau]}) \text{ for } t \downarrow \tau, \quad \text{and } \nu(t)_i = p_i^0 - (p_i^{j'_i[\tau]} + (1-t)K_i^{j'_i[\tau]}) \text{ for } t \uparrow \tau.$$

Notice that $j'_i[\tau] = j_i[\tau] \pm 1$ if τ is one of the relevant breakpoints (6) for $\nu(\tau)_i$, and otherwise $j'_i[\tau] = j_i[\tau]$.

Theorem 5. *Suppose Assumption 2 holds. Let $\overline{\tau}$ be the maximizer of* (5)*.*

(a) *If $\overline{\tau} \in (0,1)$ then $\sum_{i=1}^n \omega_i K_i^{j_i[\overline{\tau}]} \le \kappa \le \sum_{i=1}^n \omega_i K_i^{j'_i[\overline{\tau}]}$. Consequently, there exists $\lambda \in [0,1]$ such that*

$$\sum_{i=1}^n \omega_i \left(\lambda K_i^{j_i[\overline{\tau}]} + (1-\lambda) K_i^{j'_i[\overline{\tau}]}\right) = \kappa, \tag{7}$$

and an optimal solution to (2) *is*

$$\begin{array}{ll} z = \sum_{i \in I^-} \omega_i \left(-\lambda K_i^{j_i[\overline{\tau}]} - (1-\lambda) K_i^{j'_i[\overline{\tau}]}\right) & \\ y_i^{j_i[\overline{\tau}]} = \lambda|\omega_i|, \; y_i^{j'_i[\overline{\tau}]} = (1-\lambda)|\omega_i|, & \text{for } \; i = 1, \ldots, n \\ y_i^0 = \omega_i & \text{for } \; i \in I^- \\ y_i^j = 0 & \text{for all other } i, j. \end{array} \tag{8}$$

In the case $j_i[\overline{\tau}] = j'_i[\overline{\tau}] = j$, the second equation in (8) *is to be interpreted as $y_i^j = y_i^{j_i[\overline{\tau}]} + y_i^{j'_i[\overline{\tau}]} = |\omega_i|$.*

(b) *If $\overline{\tau} = 0$ then an optimal solution to* (2) *is*

$$\begin{array}{ll} z = -\sum_{i \in I^-} \omega_i K_i^{j_i[\overline{\tau}]} & \\ y_i^{j_i[\overline{\tau}]} = |\omega_i| & \text{for } \; i = 1, \ldots, n \\ y_i^0 = \omega_i & \text{for } \; i \in I^- \\ y_i^j = 0 & \text{for all other } i, j. \end{array}$$

(c) *If $\overline{\tau} = 1$ then an optimal solution to* (2) *is*

$$\begin{array}{ll} z = -\sum_{i \in I^+} \omega_i K_i^{j'_i[\overline{\tau}]} - \kappa & \\ y_i^{j'_i[\overline{\tau}]} = |\omega_i| & \text{for } \; i = 1, \ldots, n \\ y_i^0 = \omega_i & \text{for } \; i \in I^- \\ y_i^j = 0 & \text{for all other } i, j. \end{array}$$

3 Super-replicating portfolios

In this section we describe the main foundational block of our approach, namely an efficient polyhedral description for the set of super-replicating portfolios, i.e., the set of portfolios $(y^0, y^1, \ldots, y^m, z)$ that satisfy the constraints in (2).

3.1 Super-replication of a linear payoff

Assume $K = \begin{bmatrix} K^0 & K^1 & \cdots & K^m \end{bmatrix} \in \mathbf{R}^{n\times(m+1)}$, $b \in \mathbf{R}^n$ and $c \in \mathbf{R}$ are given. Define the set of *super-replicating strategies* $SR(K,b,c)$ as follows

$$SR(K,b,c) := \{(y,z) = (y^0,y^1,\ldots,y^m,z) \in \mathbf{R}^{n\times(m+1)}\times\mathbf{R} : z+\sum_{j=0}^{m} y^j\cdot(s-K^j)^+ \geq b\cdot s - c \text{ for all } s \in \mathbf{R}^n_+\}.$$

The set $SR(K,b,c)$ is the set of combinations of the call options $(s-K_i^j)^+$ and cash that super-replicate the linear payoff $b\cdot s - c$.

As Lemma 6 below states, the set $SR(K,b,c)$ is a projection of the *lifted* polyhedron $LSR(K,b,c)$. The latter is a set in a higher dimensional space with an efficient polyhedral description. Define $LSR(K,b,c)$ as the set of points $(y,z,\gamma,\beta,\xi) \in \mathbf{R}^{n\times(m+1)} \times \mathbf{R} \times \mathbf{R}_+^{n\times(m+1)} \times \mathbf{R}_+^{n\times m} \times \mathbf{R}^n$ that satisfy

$$\begin{aligned}
&\sum_{j=0}^{i} y^j - b = \gamma^i - \beta^i, && i = 0,\ldots,m-1\\
&\sum_{j=0}^{m} y^j - b = \gamma^m \\
&\sum_{j=0}^{i} K^j \circ y^j \leq \xi + K^i\circ\gamma^i - K^{i+1}\circ\beta^i, && i = 0,\ldots,m-1\\
&\sum_{j=0}^{m} K^j \circ y^j \leq \xi + K^m\circ\gamma^m \\
&-z - c \leq -e\cdot\xi.
\end{aligned} \tag{9}$$

Here $u\circ v \in \mathbf{R}^n$ denotes the *Hadamark product* of $u,v\in\mathbf{R}^n$, i.e., $(u\circ v)_i = u_i v_i,\ i = 1,\ldots,n$, and $e \in \mathbf{R}^n$ is the vector of all ones. Note that the number of variables and constraints in the description of $LSR(K,b,c)$ is proportional to mn, i.e., to the number of known option prices.

Lemma 6. *Assume* $\vec{0} = K^0 \leq K^1 \leq \cdots \leq K^m \in \mathbf{R}^m$ *and* $b \in \mathbf{R}^n$, $c\in\mathbf{R}$ *are given. Then* $(y,z) \in SR(K,b,c)$ *if and only if there exist* $\gamma \in \mathbf{R}_+^{n\times(m+1)}$, $\beta\in\mathbf{R}_+^{n\times m}$, *and* $\xi\in\mathbf{R}^n$ *such that* $(y,z,\gamma,\beta,\xi)\in LSR(K,b,c)$.

3.2 Static arbitrage upper bounds with bid-ask spreads

As we detail in Section 5, both Theorem 3 and Theorem 5 follow from Lemma 6. Furthermore, Lemma 6 readily yields efficient (linear-size) linear programming formulations for variations of the problem (2) that incorporate important additional features that have not been treated before. In particular, we next present an efficient linear programming formulation for the static arbitrage bound problem that takes into account bid-ask spreads in the prices of the known options. Previous approaches to the computation of arbitrage bounds [3, 8, 9] ignore this important feature and simply assume that the known options can be bought and sold at a mid-market price. This constitutes a major practical limitation as these mid-market prices are rarely arbitrage-free. One of our numerical examples in Section 4 illustrates this phenomenon.

Assume the vector of ask (buying) and bid (selling) prices of the options $(S-K^j)^+$ are $p_+^j \geq p_-^j$

respectively. In this case the optimal super-replication problem becomes

$$
\begin{array}{lll}
\inf\limits_{z,y,y_+,y_-} & z+\sum\limits_{j=0}^{m}(p_+^j\cdot y_+^j - p_-^j\cdot y_-^j) & \\
\text{s.t.} & z+\sum\limits_{j=0}^{m} y^j\cdot(s-K^j)^+ \geq (\omega\cdot s-\kappa)^+ \quad \text{for all } s\in\mathbf{R}_+^n & \\
 & y = y_+ - y_- & \\
 & y\in\mathbf{R}^{n\times(m+1)} & \\
 & y_+, y_- \in \mathbf{R}_+^{n\times(m+1)} & \\
 & z\in\mathbf{R}. &
\end{array} \tag{10}
$$

This gives the lowest upper bound on the bid price of the option $(\omega\cdot S-\kappa)^+$ that is implied by the options $(S_i-K_i^j)^+$, $i=1,\dots,n$, $j=0,\dots,m$, taking into consideration the bid-ask spreads.

Although problem (10) does not have a closed-form expression, Lemma 6 enables us to recast it as a linear program whose number of variables and constraints is proportional to mn, i.e., to the number of known option prices.

Theorem 7. *The optimal super-replication problem* (10) *can be rewritten as*

$$
\begin{array}{ll}
\min\limits_{z,y,y_+,y_-,\gamma,\beta,\xi,\tilde{\gamma},\tilde{\beta},\tilde{\xi}} & z+\sum\limits_{j=0}^{m}\left(p_+^j\cdot y_+^j - p_-^j\cdot y_-^j\right) \\
\text{s.t.} & (y,z,\gamma,\beta,\xi)\in LSR(K,\omega,\kappa) \\
 & (y,z,\tilde{\gamma},\tilde{\beta},\tilde{\xi})\in LSR(K,0,0) \\
 & y = y_+ - y_- \\
 & y_+, y_- \in \mathbf{R}_+^{n\times(m+1)}.
\end{array} \tag{11}
$$

Remark 8. The polyhedral description given by Lemma 6 allows the efficient modeling of other features in the optimal super-replication problem such as other types of transaction costs as well as restrictions on the positions of the super-replication strategy. Some of these features are illustrated in our numerical experiments in Section 4.

4 Some numerical results

We next present computational experiments that illustrate some of our results. The experiments focus on the new features that can be incorporated as a result of our linear programming approach. In particular, we present a numerical example that takes into account the presence of bid-ask spreads in option prices. We also discuss the possibility of adding diversification constraints to the super-replication strategy problem. Although these features are prevalent in real pricing problems, they were beyond the scope of previous approaches to static-arbitrage bounds. Finally, we consider an example with an *exchange* option, to illustrate how our approach allow us to consider basket options with negative weights.

Related numerical results are presented in [3, 9], where the authors provide extensive numerical experiments comparing static-arbitrage pricing techniques and parametric pricing techniques (such as Monte Carlo simulations) for basket options.

4.1 Bid-ask prices

In the formulation of the upper static-arbitrage bound (2), it is assumed that the options can be bought and sold at the same price. In practice, the price at which an investor buys the option, i.e., the *ask* price, is higher that the price at which the investor can sell the option, i.e., the *bid* price. This gives rise to the so-called bid-ask spread as can be observed in Table 1, which lists the prices of vanilla options on stocks in the DJX index as traded on May 17th, 2004 on the June contracts with maturity on June 18th, 2004. This dataset is similar to that of [9, Section 6.2, Table 2]. However, we have only included traded contracts (with volume greater than zero), for liquidity considerations. With the data in Table 1, we can use the linear programming formulation (11) in Section 3.2 to compute the cheapest super-replicating strategy for the DJX basket call option with strike price 80.00 taking into account the bid-ask spread. We obtain the super-replication strategy given in Table 2, which yields an upper bound of 19.8872. From market data, the best bid price for this option was 18.7, and the best ask price was 19.5. Table 2 provides the long (buy) positions on the call options with position different from zero in the super-replicating portfolio. In this particular experiment, the super-replicating portfolio does not contain any short (sell) positions.

Using bid and ask prices in the computation of the super-replicating strategy gives a more practical value to the static-arbitrage pricing approach. In particular, this resolves a major limitation in previous approaches [3, 9] that used mid-market prices (e.g., the average of the bid and ask prices) as the “nominal” option prices. Such approximation systematically underestimates the actual buying prices and overestimates the actual selling prices. It is then not surprising that the market data used in [3, 9] requires a fair amount of “cleaning” to rule out apparent arbitrage opportunities created by these estimates (see [9, Section 6.2]). By contrast, the model herein that takes into account bid-ask spreads does not suffer from this limitation.

We note that although the super-replicating strategy in Table 2 contains only long positions, this does not mean that the bid-ask DJX option price upper bound of 19.8872 could be found by simply using the ask (buy) prices as the option prices in the linear programming formulation of the problem (2). If this naive approach were attempted, the linear program would be unbounded, since the ask prices alone do not satisfy the arbitrage-free condition (3).

4.2 Diversifying the super-replicating strategy

Consider an investor looking at the strategy in Table 2, who wishes to create a super-replicating strategy that contains more positions in possible options, that is, a more diversified strategy. To obtain such a super-replicating strategy we add the following diversifying linear constraints to the LP formulation (11), which ensure that the portfolio will have a position in each *tier* of options:

$$e \cdot y^j \geq 0.05, j = 0, \ldots, m. \tag{12}$$

Above e represents the vector of all-ones. The solution to this *diversified* super-replicating strategy gives a portfolio whose cost is 19.9022, just 0.08% more expensive than the cheapest super-replicating strategy computed in Table 2. As Table 3 shows, such a strategy has the desired investor’s diversification preference.

4.3 Basket options with negative weights

Consider the problem of finding static-arbitrage bounds for a European *exchange* option, given the prices of vanilla options on the two assets involved. This corresponds to (1) with $n = 2$, $\omega = \begin{bmatrix} 1 & -1 \end{bmatrix}^{\mathrm{T}}$, and $\kappa = 0$. We will consider the case in which information about the forward prices of the two assets, and the following $m = 5$ call options is given:

$$K^1 = \begin{bmatrix} 0.85 \\ 0.85 \end{bmatrix}, K^2 = \begin{bmatrix} 0.90 \\ 0.90 \end{bmatrix}, K^3 = \begin{bmatrix} 0.95 \\ 0.95 \end{bmatrix}, K^4 = \begin{bmatrix} 1.00 \\ 1.00 \end{bmatrix}, K^5 = \begin{bmatrix} 1.05 \\ 1.05 \end{bmatrix}. \tag{13}$$

To set up the problem, we sample the values of p^j, $j = 0, \ldots, 5$ in (1) corresponding to the strikes in (13), by assuming that the underlying asset prices distribution follows a correlated multivariate lognormal distribution (see, e.g., eq. (15) in [2]). In particular we use a riskless interest rate $r = 0$, option maturity $T = 1$, current prices $S_1(0) = 0.95$, $S_2(0) = 0.90$, volatilities $\delta_1 = 0.2$, $\delta_2 = 0.22$. Thus, we obtain (using Black-Scholes formula) the following given option prices for the problem:

$$\begin{aligned} p^0 &= \begin{bmatrix} 0.9500 \\ 0.9000 \end{bmatrix}, \; p^1 = \begin{bmatrix} 0.1324 \\ 0.1042 \end{bmatrix}, \; p^2 = \begin{bmatrix} 0.1013 \\ 0.0788 \end{bmatrix}, \\ p^3 &= \begin{bmatrix} 0.0757 \\ 0.0584 \end{bmatrix}, \; p^4 = \begin{bmatrix} 0.0552 \\ 0.0425 \end{bmatrix}, \; p^5 = \begin{bmatrix} 0.0394 \\ 0.0304 \end{bmatrix}. \end{aligned} \tag{14}$$

The arbitrage condition (3) follows from computing the prices using the lognormal model. Thus, using Theorem 3, we obtain an upper static-arbitrage bound on the exchange option of 0.1802. Clearly, from the choice of the option prices in (14), it follows that the upper bound of 0.1802 must be higher than any *exact* price of the option computed by assuming that the underlying asset prices follow a lognormal distribution that replicates the option prices in (14). Indeed, using Magrabe's formula [10], the exact price of the exchange option would ranges between $[0.1361, 0.1801]$ if the correlation of the Wiener processes driving the lognormal distribution are negatively correlated, and between $[0.0500, 0.1361]$ if they are positively correlated.

5 Proofs

5.1 Proof of Lemma 6.

The high level idea of the proof is to divide $\mathbf{R}^n_+$ in regions where each of the functions $(s - K^j)^+$ is linear. Using Farkas' Lemma the set of super-replicating strategies in each region is a polyhedron. Thus the set of super-replicating strategies $SR(K, b, c)$ is just the intersection of all these polyhedra, again a polyhedron. Since the number of regions is exponential, the number of variables and constraints needed in a naive description of $SR(K, b, c)$ is exponential. The main point of the proof is that we can "collapse" first the variables and then the constrains to obtain an efficient description, using only a linear number of variables and constraints.

Throughout this section we rely on the following convenient notation: Given a vector $v \in \mathbf{R}^n$ and a set of indices $I \subseteq \{1, \ldots, n\}$, we let v_I denote the subvector of v obtained by selecting the components of v indexed by I.

Proof of Lemma 6. Define the set of partitions $\mathcal{P}(n, m)$ of $\{1, \ldots, n\}$ as follows:

$$\mathcal{P}(n,m) := \left\{ \left(J^0, J^1, \ldots, J^m\right) : \bigcup_{i=0}^{m} J^i = \{1, \ldots, n\}, J^i \cap J^j = \emptyset \text{ for } i \neq j \right\}.$$

Given $J \in \mathcal{P}(n, m)$, define

$$P_J := \left\{ s : K^i_{J^i} \leq s_{J^i} \leq K^{i+1}_{J^i} \text{ for } i = 0, 1, \ldots, m-1, \;\text{ and } s_{J^m} \geq K^m_{J^m} \right\}.$$

Since $\{P_J : J \in \mathcal{P}(n, m)\}$ is a partition of $\mathbf{R}^n_+$, it follows that $(y, z) \in SR(K, b, c)$ if and only if

$$z + \sum_{j=0}^{m} y^j \cdot (s - K^j)^+ \geq b \cdot s - c \text{ for all } s \in P_J \;\text{ for all } J \in \mathcal{P}(n, m). \tag{15}$$

From the construction of $\{P_J : J \in \mathcal{P}(n,m)\}$, it follows that each $(s_i - K_i^j)^+$ is linear on each P_J. Indeed, for $s \in P_J$ we have

$$\sum_{j=0}^{m} y^j \cdot (s - K^j)^+ = \sum_{i=0}^{m}\sum_{j=0}^{i} y^j_{J^i} \cdot (s_{J^i} - K^j_{J^i}).$$

Therefore, (15) is equivalent to

$$\sum_{i=0}^{m}\left(-b_{J^i} + \sum_{j=0}^{i} y^j_{J^i}\right) \cdot s_{J^i} \geq \sum_{i=0}^{m}\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i} - z - c \text{ for all } s \in P_J \text{ for all } J \in \mathcal{P}(n,m).$$

By Farkas Lemma, the latter holds if and only if for each $J \in \mathcal{P}(n,m)$ there exist $\gamma^{i,J}, \beta^{i,J} \in \mathbf{R}_+^{J^i}$, $i = 0,\dots,m-1$, $\gamma^{m,J} \in \mathbf{R}_+^{J^m}$ such that

$$\begin{aligned}
&-b_{J^i} + \sum_{j=0}^{i} y^j_{J^i} = \gamma^{i,J} - \beta^{i,J}, && i = 0,\dots,m-1\\
&-b_{J^m} + \sum_{j=0}^{m} y^j_{J^m} = \gamma^{m,J}\\
&\sum_{i=0}^{m}\left(\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i}\right) - z - c \leq \sum_{i=0}^{m-1}\left(K^i_{J^i} \cdot \gamma^{i,J} - K^{i+1}_{J^i} \cdot \beta^{i,J}\right) + K^m_{J^m} \cdot \gamma^{m,J}, && J \in \mathcal{P}(n,m).
\end{aligned} \tag{16}$$

We will first reduce the number of variables used in this description of $SR(K,b,c)$:

Claim 9. *Assume $(y_0,\dots,y_m,z) \in \mathbf{R}^n \times \dots \times \mathbf{R}^n \times \mathbf{R}$ is given. Then there exist $\gamma^{i,J}, \beta^{i,J} \in \mathbf{R}_+^{J^i}$, $i = 0,\dots,m-1$, $\gamma^{m,J} \in \mathbf{R}_+^{J^m}$ for each $J \in \mathcal{P}(n,m)$ such that (16) holds if and only if there exist $\gamma^i, \beta^i \in \mathbf{R}^n_+$, $i = 0,\dots,m-1$ and $\gamma^m \in \mathbf{R}^n_+$ such that*

$$\begin{aligned}
&-b + \sum_{j=0}^{i} y^j = \gamma^i - \beta^i, && i = 0,\dots,m-1\\
&-b + \sum_{j=0}^{m} y^j = \gamma^m\\
&\sum_{i=0}^{m}\left(\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i}\right) - z - c \leq \sum_{i=0}^{m-1}\left(K^i_{J^i} \cdot \gamma^i_{J^i} - K^{i+1}_{J^i} \cdot \beta^i_{J^i}\right) + K^m_{J^m} \cdot \gamma^m_{J^m}, && J \in \mathcal{P}(n,m).
\end{aligned} \tag{17}$$

Then, we will reduce the number of constrains used in (17):

Claim 10. *Assume $(y,z,\gamma,\beta) \in \mathbf{R}^{n\times(m+1)} \times \mathbf{R} \times \mathbf{R}_+^{n\times(m+1)} \times \mathbf{R}_+^{n\times m}$ is given. Then (17) holds if and only if there exists $\xi \in \mathbf{R}^n$ such that (9) holds.*

Claim 9 and Claim 10 show that $(y,z) \in SR(K,b,c)$ if and only if there exist $\gamma \in \mathbf{R}_+^{n\times(m+1)}$, $\beta \in \mathbf{R}_+^{n\times m}$, and $\xi \in \mathbf{R}^n$ such that $(y,z,\gamma,\beta,\xi) \in LSR(K,b,c)$. □

Now we present the proofs of Claim 9 and Claim 10.

Proof of Claim 9. It is straightforward to check (17) implies (16). Assume (16) holds. Define $\gamma^m \in \mathbf{R}^n_+$ and $\gamma^i, \beta^i \in \mathbf{R}^n_+$, $i = 0, \ldots, m-1$ as follows. Let $\overline{J} = (\emptyset, \ldots, \emptyset, \{1, \ldots, n\})$ and put

$$\gamma^m := \gamma^{m,\overline{J}}.$$

Then the second equation in (17) holds. Consequently, for any $J \in \mathcal{P}(n,m)$ we have

$$\gamma^m_{J^m} = -b_{J^m} + \sum_{j=0}^{m} y^j_{J^m} = \gamma^{m,J}. \tag{18}$$

Next, fix $i \in \{0, \ldots, m-1\}$. For each $\ell \in \{1, \ldots, n\}$ define the partition $J[i,\ell]$ by

$$J[i,\ell] := \underset{\{J \in \mathcal{P}(n,r) : l \in J^i\}}{\operatorname{argmax}} \left(K^i_\ell \gamma^{i,J}_\ell - K^{i+1}_\ell \beta^{i,J}_\ell \right).$$

Let $\gamma^i, \beta^i \in \mathbf{R}^n_+$ be defined by $\gamma^i_\ell = \gamma^{i,J[i,\ell]}_\ell$ and $\beta^i_\ell = \beta^{i,J[i,\ell]}_\ell$, $\ell \in \{1, \ldots, n\}$. From the first identity in (16), applied to $J = J[i,\ell]$, we get

$$-b_{J[i,\ell]^i} + \sum_{j=0}^{i} y^j_{J[i,l]^i} = \gamma^{i,J[i,l]} - \beta^{i,J[i,l]}.$$

In particular,

$$-b_\ell + \sum_{j=0}^{i} y^j_\ell = \gamma^{i,J[i,\ell]}_\ell - \beta^{i,J[i,\ell]}_\ell = \gamma^i_\ell - \beta^i_\ell.$$

This holds for $i \in \{0, \ldots, m-1\}$ and $\ell \in \{1, \ldots, n\}$ thus the first equation in (17) follows. It only remains to prove the last inequality in (17). To that end, fix $J \in \mathcal{P}(n,m)$. For $i = 0, \ldots, m-1$ and $\ell \in J^i$, the construction of $J[i,\ell]$ implies that

$$K^i_\ell \gamma^{i,J}_\ell - K^{i+1}_\ell \beta^{i,J}_\ell \le K^i_\ell \gamma^{i,J[i,\ell]}_\ell - K^{i+1}_\ell \beta^{i,J[i,\ell]}_\ell = K^i_\ell \gamma^i_\ell - K^{i+1}_\ell \beta^i_\ell.$$

Thus

$$K^i_{J^i} \cdot \gamma^{i,J} - K^{i+1}_{J^i} \cdot \beta^{i,J} \le K^i_{J^i} \cdot \gamma^i_{J^i} - K^{i+1}_{J^i} \cdot \beta^i_{J^i}.$$

Hence from the last inequality in (16) and (18) we get

$$\begin{aligned} \sum_{i=0}^{m} \left(\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i} \right) - z - c \quad &\le \quad \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^{i,J} - K^{i+1}_{J^i} \cdot \beta^{i,J} \right) + K^m_{J^m} \cdot \gamma^{m,J} \\ &\le \quad \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^i_{J_i} - K^{i+1}_{J^i} \cdot \beta^i_{J_i} \right) + K^m_{J^m} \cdot \gamma^m_{J^m}, \end{aligned}$$

completing the proof of the claim. □

Proof of Claim 10. Assume (9) holds. Let $J \in \mathcal{P}(n,m)$ be given. From the third inequality in (9) we have

$$\sum_{j=0}^{i} K^j_{J^i} \cdot y^j_{J^i} \le e_{J^i} \cdot \xi_{J^i} + K^i_{J^i} \cdot \gamma^i_{J^i} - K^{i+1}_{J^i} \cdot \beta^i_{J^i}, \ i = 0, \ldots, m-1.$$

Likewise, form the fourth inequality in (9) we have

$$\sum_{j=0}^{m} K^j_{J^m} \cdot y^j_{J^m} \leq e_{J^m} \cdot \xi_{J^m} + K^m_{J^m} \cdot \gamma^m_{J^m}.$$

Adding all of these inequalities and rearranging terms, we get

$$\sum_{i=0}^{m} \left(\sum_{j=0}^{i} y^j_{J^i} \cdot K^j_{J^i} \right) - e \cdot \xi \leq \sum_{i=0}^{m-1} \left(K^i_{J^i} \cdot \gamma^i_{J^i} - K^{i+1}_{J^i} \cdot \beta^i_{J^i} \right) + K^m_{J^m} \cdot \gamma^m_{J^m}.$$

From (9) $-z - c \leq -e \cdot \xi$, and we get (17).

Now assume that (17) holds. For $\ell = 1, \ldots, n$ let

$$\xi_\ell := \max \left\{ \sum_{j=0}^{m} K^j_\ell y^j_\ell - K^m_\ell \gamma^m_\ell, \sum_{j=0}^{i} K^j_\ell y^j_\ell - K^i_\ell \gamma^i_\ell + K^{i+1}_\ell \beta^i_\ell : i = 0, \ldots, m-1 \right\}.$$

This choice of ξ ensures that the first four constraints in (9) hold. Let $\overline{J} \in \mathcal{P}(n, m)$ be such that

$$\xi_{\overline{J}^m} = \sum_{j=0}^{m} K^j_{\overline{J}^m} \circ y^j_{\overline{J}^m} - K^m_{\overline{J}^m} \circ \gamma^m_{\overline{J}^m},$$

and

$$\xi_{\overline{J}^i} = \sum_{j=0}^{i} K^j_{\overline{J}^i} \circ y^j_{\overline{J}^i} - K^i_{\overline{J}^i} \gamma^i_{\overline{J}^i} + K^{i+1}_{\overline{J}^i} \beta^i_{\overline{J}^i} \quad \text{for } i = 0, \ldots, m-1.$$

Then, from (17) (applied to the partition $\overline{J}$) we have

$$\begin{aligned}
-z - c &\leq -\sum_{i=0}^{m} \left(\sum_{j=0}^{i} y^j_{\overline{J}^i} \cdot K^j_{\overline{J}^i} \right) + \sum_{i=0}^{m-1} \left(K^i_{\overline{J}^i} \cdot \gamma^i_{\overline{J}^i} - K^{i+1}_{\overline{J}^i} \cdot \beta^i_{\overline{J}^i} \right) + K^m_{\overline{J}^m} \cdot \gamma^m_{\overline{J}^m} \\
&= -\sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} y^j_{\overline{J}^i} \cdot K^j_{\overline{J}^i} - K^i_{\overline{J}^i} \cdot \gamma^i_{\overline{J}^i} + K^{i+1}_{\overline{J}^i} \cdot \beta^i_{\overline{J}^i} \right) - \sum_{j=0}^{m} y^j_{\overline{J}^m} \cdot K^j_{\overline{J}^m} + K^m_{\overline{J}^m} \cdot \gamma^m_{\overline{J}^m} \\
&= -\sum_{i=0}^{m} \sum_{\ell \in \overline{J}^i} \xi_\ell \\
&= -e \cdot \xi.
\end{aligned}$$

Hence the last constraint in (9) holds as well. This completes the equivalence between (9) and (17).

□

5.2 A technical lemma

The proof of Theorem 3 relies on Lemma 6 and on the following technical notation and lemma. Given $K = \begin{bmatrix} K^0 & K^1 & \cdots & K^m \end{bmatrix} \in \mathbf{R}^n \times \cdots \times \mathbf{R}^n$, define the set cone$(K)$ as

$$\text{cone}(K) = \left\{ (T, v, \tau) \in \mathbf{R}_+^{n \times (m+1)} \times \mathbf{R}^{n \times (m+1)} \times \mathbf{R}_+ : \begin{array}{l} T^j \circ K^j \leq v^j \leq T^j \circ K^{j+1},\ j = 0, \ldots, m-1, \\ T^m \circ K^m \leq v^m, \\ \sum_{j=0}^{m} T^j = \tau e \end{array} \right\}.$$

Lemma 11. *Assume $\tau \in [0,1]$ is fixed, and the strikes K and prices p satisfy Assumption 2. Then for each $i = 1, \ldots, n$ the optimal value of the linear program*

$$
\begin{array}{lll}
\max\limits_{T_i^j, v_i^j, \tilde{T}_i^j, \tilde{v}_i^j} & \sum\limits_{j=0}^{m} v_i^j & \\
\text{s.t.} & \sum\limits_{\ell=j}^{m} \left(v_i^\ell + \tilde{v}_i^\ell - (T_i^\ell + \tilde{T}_i^\ell) K_i^j \right) = p_i^j, \quad j = 0, \ldots, m & (19) \\
& (T_i, v_i, \tau) \in \text{cone}(K_i) & \\
& (\tilde{T}_i, \tilde{v}_i, 1-\tau) \in \text{cone}(K_i) &
\end{array}
$$

is

$$\min_{j=0,\ldots,m} \left\{ p_i^j + \tau K_i^j \right\}. \tag{20}$$

On the other hand, the optimal value of the linear program obtained by replacing max *with* min *in* (19) *is*

$$p_i^0 - \min_{j=0,\ldots,m} \left\{ p_i^j + (1-\tau) K_i^j \right\}. \tag{21}$$

Proof. To simplify notation we shall drop the subindex i. Note if $(T^j, v^j, \tilde{T}^j, \tilde{v}^j)$, $j = 0, \ldots, m$ is feasible for (19) then for each $j = 0, \ldots, m$ we have

$$\sum_{\ell=j}^{m} v^\ell = p^j + \sum_{\ell=j}^{m} ((T^\ell + \tilde{T}^\ell) K^j - \tilde{v}^\ell) \leq p^j + \sum_{\ell=j}^{m} T^\ell K^j,$$

and

$$\sum_{\ell=0}^{j-1} v^\ell \leq \sum_{\ell=0}^{j-1} T^\ell K^{\ell+1} \leq \sum_{\ell=0}^{j-1} T^\ell K^j.$$

Thus for each $j = 0, \ldots, m$

$$\sum_{\ell=0}^{m} v^\ell \leq p^j + \sum_{\ell=0}^{m} T^\ell K^j = p^j + \tau K^j.$$

Therefore the optimal value of (19) is at most $\min\limits_{j=0,\ldots,m} \left\{ p^j + \tau K^j \right\}$.

To complete the proof, we next construct a feasible solution to (19) whose objective for (19) attains this value. Put

$$\sigma^0 := 1; \; \sigma^\ell := \frac{p^\ell - p^{\ell+1}}{K^{\ell+1} - K^\ell}, \; \ell = 1, \ldots, m-1; \; \sigma^m := 0,$$

and construct the point $(T, v, \tilde{T}, \tilde{v})$ as follows

$$
\begin{array}{l}
T^m = 0, \\
T^\ell = \min(\tau, \sigma^\ell) - \min(\tau, \sigma^{\ell+1}), \; \ell = 0, \ldots, m-1 \\
v^m = p^m, \\
v^\ell = T^\ell K^{\ell+1}, \; \ell = 1, \ldots, m-1, \\
v^0 = \min \left\{ p^0 - p^1 - \sigma^1 K^1, T^0 K^1 \right\},
\end{array}
$$

and

$$\begin{aligned}
&\tilde{T}^\ell = \sigma^\ell - \sigma^{\ell+1} - T^\ell,\ \ell = 0,\dots,m-1,\\
&\tilde{T}^m = 0,\\
&\tilde{v}^0 = p^0 - p^1 - \sigma^1 K^1 - v^0,\\
&\tilde{v}^\ell = \tilde{T}^\ell K^{\ell+1},\ \ell = 1,\dots,m-1\\
&\tilde{v}^m = 0.
\end{aligned}$$

By construction $\sum_{j=\ell}^m T^j = \min(\tau,\sigma^\ell),\ \ell = 0,\dots,m$. Hence $(T, v, \tau) \in \text{cone}(K)$. Likewise, $(\tilde{T}, \tilde{v}, 1-\tau) \in \text{cone}(K)$. The rest of the proof relies on the following identity, which is clearly valid for $j = 0, 1, \dots, m-1$:

$$\begin{aligned}
\sum_{\ell=j}^{m-1}(\sigma^\ell - \sigma^{\ell+1})K^{\ell+1} + p^m &= \sigma^j K^{j+1} + \sum_{\ell=j+1}^{m-1}\sigma^\ell(K^{\ell+1} - K^\ell) + p^m\\
&= \sigma^j K^{j+1} + p^{j+1} - p^m + p^m\\
&= \sigma^j K^{j+1} + p^{j+1}\\
&= \sigma^j K^j + p^j.
\end{aligned} \tag{22}$$

Notice that the identity between the first and last quantities also holds for $j = m$.

From (22), for $j \geq 1$ we get

$$\begin{aligned}
\sum_{\ell=j}^{m}\left(v^\ell + \tilde{v}^\ell - (T^\ell + \tilde{T}^\ell)K^j\right) &= \sum_{\ell=j}^{m-1}(T^\ell + \tilde{T}^\ell)(K^{\ell+1} - K^j) + p^m\\
&= \sum_{\ell=j}^{m-1}(\sigma^\ell - \sigma^{\ell+1})(K^{\ell+1} - K^j) + p^m\\
&= \sum_{\ell=j}^{m-1}(\sigma^\ell - \sigma^{\ell+1})K^{\ell+1} + p^m - \sigma^j K^j\\
&= \sigma^j K^j + p^j - \sigma^j K^j\\
&= p^j.
\end{aligned}$$

From (22) we also get

$$\begin{aligned}
\sum_{\ell=0}^{m}\left(v^\ell + \tilde{v}^\ell - (T^\ell + \tilde{T}^\ell)K^0\right) &= \sum_{\ell=0}^{m}\left(v^\ell + \tilde{v}^\ell\right)\\
&= v^0 + \tilde{v}^0 + \sum_{\ell=1}^{m-1}(\sigma^\ell - \sigma^{\ell+1})K^{\ell+1} + p^m\\
&= p^0 - p^1 - \sigma^1 K^1 + \sum_{\ell=1}^{m-1}\sigma^\ell(K^{\ell+1} - K^\ell) + \sigma^1 K^1 + p^m\\
&= p^0.
\end{aligned}$$

Hence $(T, v, \tilde{T}, \tilde{v})$ is indeed a feasible solution for (19). To finish, we next show that the objective value of this point is $\min\limits_{j=0,\dots,m}\left\{p^j + \tau K^j\right\}$. To that end, let $j^* := \operatorname*{argmin}\limits_{j=0,\dots,m}\left\{p^j + \tau K^j\right\}$.

If $1 \leq j^* \leq m$ then $\sigma^{j^*} \leq \tau \leq \sigma^{j^*-1}$, and consequently $T^\ell = 0,\ \ell < j^* - 1,\ T^{j^*-1} = \tau - \sigma^{j^*}$, and $T^\ell = \sigma^\ell - \sigma^{\ell+1},\ j^* \leq \ell \leq m-1$. Thus from (22) we get

$$\begin{aligned}
\sum_{\ell=0}^{m} v^\ell &= (\tau - \sigma^{j^*})K^{j^*} + \sum_{\ell=j^*}^{m-1}(\sigma^\ell - \sigma^{\ell+1})K^{\ell+1} + p^m\\
&= \tau K^{j^*} - \sigma^{j^*}K^{j^*} + \sigma^{j^*}K^{j^*} + p^{j^*}\\
&= \tau K^{j^*} + p^{j^*}.
\end{aligned}$$

If $j^* = 0$ then $\tau \geq \frac{p^0 - p^1}{K^1} \geq \sigma^1$ and consequently $T^0 = \tau - \sigma^1$, $v^0 = p^0 - p^1 - \sigma^1 K^1$, and $T^\ell = \sigma^\ell - \sigma^{\ell+1}$, $\ell = 1, \ldots, m-1$. Thus, from (22) we get

$$\sum_{\ell=0}^{m} v^\ell = p^0 - p^1 - \sigma^1 K^1 + \sum_{\ell=1}^{m-1} (\sigma^\ell - \sigma^{\ell+1}) K^{\ell+1} + p^m = p^0.$$

In either case $\sum_{\ell=0}^{m} v^\ell = p^{j^*} + \tau K^{j^*} = \min_{j=0,\ldots,m} \left\{ p^j + \tau K^j \right\}$.

For the second part of the lemma, observe that the constraints in (19) imply that

$$\sum_{j=0}^{m} v_i^j = p_i^0 - \sum_{j=0}^{m} \tilde{v}_i^j.$$

Thus the problem obtained by replacing max with min in problem (19) is the same as that obtained by replacing the objective function with

$$p_i^0 - \min \sum_{j=0}^{m} \tilde{v}_i^j.$$

The resulting problem is identical to (19) with the roles of (T, v, τ) and $(\tilde{T}, \tilde{v}, 1-\tau)$ reversed. Thus (21) follows from (20).

$\square$

5.3 Proof of Theorem 3.

The optimal super-replication problem (2) can be written as:

$$\begin{array}{ll} \min\limits_{y,z} & z + \sum\limits_{j=0}^{m} p^j \cdot y^j \\ \text{s.t.} & (y, z) \in SR(K, \omega, \kappa) \\ & (y, z) \in SR(K, 0, 0) \end{array} \tag{23}$$

From Lemma 6 it follows that (23) is equivalent to the following linear program

$$\begin{array}{ll} \min\limits_{y,z,\gamma,\beta,\xi,\tilde{\gamma},\tilde{\beta},\tilde{\xi}} & z + \sum\limits_{j=0}^{m} p^j \cdot y^j \\ \text{s.t.} & (y, z, \gamma, \beta, \xi) \in LSR(K, \omega, \kappa) \\ & (y, z, \tilde{\gamma}, \tilde{\beta}, \tilde{\xi}) \in LSR(K, 0, 0). \end{array}$$

By linear programming duality, the optimal value of the latter is the same as that of its dual:

$$\begin{array}{ll} \max\limits_{v,T,\tau,\tilde{v},\tilde{T},\tilde{\tau}} & \omega \cdot \sum\limits_{j=0}^{m} v^j - \kappa\tau \\ \text{s.t.} & \tau + \tilde{\tau} = 1 \\ & \sum\limits_{\ell=j}^{m} \left(v^\ell + \tilde{v}^\ell - (T^\ell + \tilde{T}^\ell) \circ K^j \right) = p^j, \quad j = 0, \ldots, m \\ & (T, v, \tau) \in \text{cone}(K) \\ & (\tilde{T}, \tilde{v}, \tilde{\tau}) \in \text{cone}(K) \end{array} \tag{24}$$

To finish, observe that for fixed $\tau \in [0,1]$ the linear program (24) can be decoupled into n linear programs, each one of the form (19) or (19) with min instead of max. The expression (5) for the optimal value then follows from Lemma 11. □

5.4 Proof of Theorem 5.

(a) From the definition of $j_i[\overline{\tau}]$ and $j_i'[\overline{\tau}]$ it follows that

$$\omega \cdot \nu(\tau) - \tau\kappa = \sum_{i=1}^{n} |\omega_i| p_i^{j_i[\overline{\tau}]} + \sum_{i \in I^-} \omega_i (p_i^0 - K_i^{j_i[\overline{\tau}]}) + \tau \left(\sum_{i=1}^{n} \omega_i K_i^{j_i[\overline{\tau}]} - \kappa \right) \quad \text{for } \tau \downarrow \overline{\tau}, \tag{25}$$

and

$$\omega \cdot \nu(\tau) - \tau\kappa = \sum_{i=1}^{n} |\omega_i| p_i^{j_i'[\overline{\tau}]} + \sum_{i \in I^-} \omega_i (p_i^0 - K_i^{j_i'[\overline{\tau}]}) + \tau \left(\sum_{i=1}^{n} \omega_i K_i^{j_i'[\overline{\tau}]} - \kappa \right) \quad \text{for } \tau \uparrow \overline{\tau}, \tag{26}$$

Optimality of $\overline{\tau}$ implies that the slope in (25) is non-positive and the slope in (26) is non-negative. Thus $\sum_{i=1}^{n} \omega_i K_i^{j_i[\overline{\tau}]} \leq \kappa \leq \sum_{i=1}^{n} \omega_i K_i^{j_i'[\overline{\tau}]}$ and so there exists some $\lambda \in [0,1]$ such that (7) holds. Using (7), (25) and (26), we obtain

$$\omega \cdot \nu(\overline{\tau}) - \overline{\tau}\kappa = \sum_{i=1}^{n} \left(\lambda |\omega_i| p_i^{j_i[\overline{\tau}]} + (1-\lambda) |\omega_i| p_i^{j_i'[\overline{\tau}]} \right) + \sum_{i \in I^-} \omega_i \left(p_i^0 - \lambda K_i^{j_i'[\overline{\tau}]} - (1-\lambda) K_i^{j_i[\overline{\tau}]} \right).$$

Thus the point z, y_i^j given by (8) has objective value equal to (5). Therefore to finish it suffices to show that it is feasible for (2). Indeed, since $\omega_{I^+}, -\omega_{I^-} \geq 0$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}
&\sum_{j=0}^{m} y^j \cdot (S - K^j)^+ + z \\
&= \sum_{i=1}^{n} |\omega_i| \left(\lambda (s_i - K_i^{j_i[\overline{\tau}]})^+ + (1-\lambda)(s_i - K_i^{j_i'[\overline{\tau}]})^+ \right) \\
&\qquad\qquad + \sum_{i \in I^-} \omega_i s_i - \sum_{i \in I^-} \omega_i \left(\lambda K_i^{j_i'[\overline{\tau}]} + (1-\lambda) K_i^{j_i[\overline{\tau}]} \right) \\
&\geq \left(\sum_{i \in I^+} \omega_i \left(s_i - \lambda K_i^{j_i[\overline{\tau}]} - (1-\lambda) K_i^{j_i'[\overline{\tau}]} \right) \right)^+ + \left(- \sum_{i \in I^-} \omega_i \left(s_i - \lambda K_i^{j_i'[\overline{\tau}]} - (1-\lambda) K_i^{j_i[\overline{\tau}]} \right) \right)^+ \\
&\qquad\qquad + \sum_{i \in I^-} \omega_i \left(s_i - \lambda K_i^{j_i'[\overline{\tau}]} - (1-\lambda) K_i^{j_i[\overline{\tau}]} \right) \\
&\geq (\omega \cdot s - \kappa)^+ .
\end{aligned}$$

Above the first inequality follows from $a^+ + b^+ \geq (a+b)^+$, the second from $a^+ + b^+ - b \geq (a-b)^+$ and (7).

(b,c) These follow via similar (but simpler) arguments to that in (a). □

5.5 Proof of Theorem 7.

The optimal super-replication problem (10) can be written as:

$$\begin{array}{ll} \min\limits_{z,y,y_+,y_-} & z + \sum_{j=0}^{m} (p_+^j \cdot y_+^j - p_-^j \cdot y_-^j) \\ \text{s.t.} & (y,z) \in SR(K,\omega,\kappa) \\ & (y,z) \in SR(K,0,0) \\ & y = y_+ - y_- \\ & y \in \mathbf{R}^{n\times(m+1)} \\ & y_+, y_- \in \mathbf{R}_+^{n\times(m+1)}. \end{array} \tag{27}$$

From Lemma 6 it follows that (27) is equivalent to the following linear program

$$\begin{array}{ll} \min\limits_{z,y,y_+,y_-\gamma,\beta,\xi,\tilde{\gamma},\tilde{\beta},\tilde{\xi}} & z + \sum_{j=0}^{m} (p_+^j \cdot y_+^j - p_-^j \cdot y_-^j) \\ \text{s.t.} & (y,z,\gamma,\beta,\xi) \in LSR(K,\omega,\kappa) \\ & (y,z,\tilde{\gamma},\tilde{\beta},\tilde{\xi}) \in LSR(K,0,0) \\ & y = y_+ - y_- \\ & y \in \mathbf{R}^{n\times(m+1)} \\ & y_+, y_- \in \mathbf{R}_+^{n\times(m+1)}. \end{array}$$

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Table 1: CBOE data from May 17th, 2004 on June 2004 contracts expiring June 18. The table gives prices of call options traded (volume greater than zero) on May 17th for the 30 stocks underlying the DJX index basket option. For every stock, the first row corresponds to the different strike prices, and the second and third rows correspond to the respective ask and bid prices. The entry of 0.00 for each stock gives the close price of the stock, which can be considered as the forward option (call option with strike price zero) price.

	0.00	20.00	22.50	25.00	27.50				
MSFT	25.79	5.70	3.20	1.15	0.20				
	25.42	5.50	3.10	1.05	0.15				
	0.00	25.00	27.50	30.00	32.50	35.00			
AA	29.70	4.10	2.20	0.95	0.30	0.15			
	28.60	3.90	2.05	0.85	0.20	0.05			
	0.00	65.00	70.00	75.00					
AIG	70.15	5.40	2.10	0.45					
	69.22	5.30	2.00	0.40					
	0.00	47.50	50.00						
AXP	49.30	2.20	0.80						
	48.20	2.05	0.70						
	0.00	40.00	42.50	45.00					
BA	43.61	3.10	1.35	0.40					
	42.49	2.90	1.25	0.30					
	0.00	35.00	37.50						
VZ	36.74	1.50	0.40						
	35.68	1.40	0.30						
	0.00	60.00	70.00	75.00	80.00	85.00			
CAT	74.45	13.80	4.90	1.95	0.60	0.20			
	72.70	13.60	4.80	1.90	0.50	0.10			
	0.00	40.00	42.50	45.00					
DD	41.48	2.00	0.70	0.15					
	41.01	1.80	0.55	0.10					
	0.00	20.00	22.50	25.00	27.50				
DIS	22.99	3.00	1.05	0.20	0.10				
	22.69	2.95	0.90	0.15	0.00				
	0.00	25.00	27.50	30.00	32.50	35.00	37.50		
GE	30.06	5.10	2.70	0.85	0.15	0.05	0.05		
	29.68	4.90	2.60	0.75	0.10	0.00	0.00		
	0.00	47.50	50.00	55.00	60.00	65.00	70.00		
WMT	55.25	7.40	5.10	1.45	0.15	0.05	0.05		
	54.14	7.20	4.90	1.30	0.10	0.00	0.00		
	0.00	35.00	40.00	42.50	45.00	47.50	50.00	55.00	
GM	43.90	8.70	4.20	2.30	1.05	0.40	0.15	0.05	
	42.88	8.60	4.00	2.20	0.95	0.30	0.10	0.00	
	0.00	30.00	32.50	35.00	37.50	40.00			
HD	33.75	3.80	1.85	0.60	0.15	0.05			
	33.07	3.60	1.70	0.55	0.10	0.00			
	0.00	30.00	32.50	35.00	37.50	40.00			
HON	33.43	2.85	1.15	0.30	0.10	0.05			
	32.44	2.70	1.00	0.20	0.00	0.00			
	0.00	15.00	17.50	20.00	22.50				
HPQ	19.70	4.60	2.30	0.70	0.15				
	19.21	4.50	2.20	0.65	0.10				
	0.00	80.00	85.00	90.00	95.00	100.00			
IBM	86.03	6.30	2.65	0.70	0.20	0.05			
	85.15	6.10	2.50	0.65	0.15	0.00			
	0.00	27.50	30.00	32.50	35.00	37.50	40.00	42.50	45.00
JPM	35.47	8.00	5.60	3.30	1.45	0.45	0.10	0.10	0.05
	34.75	7.80	5.40	3.10	1.35	0.35	0.05	0.00	0.00
	0.00	47.50	50.00	55.00					
KO	50.12	2.70	1.00	0.05					
	49.51	2.55	0.85	0.00					
	0.00	40.00	42.50	45.00					
XOM	43.54	3.40	1.45	0.40					
	43.01	3.20	1.35	0.30					
	0.00	20.00	22.50	25.00	27.50	30.00			
INTC	27.30	6.90	4.50	2.30	0.80	0.20			
	26.44	6.80	4.30	2.25	0.70	0.10			
	0.00	50.00	55.00						
JNJ	55.10	5.00	1.10						
	54.13	4.80	1.05						
	0.00	80.00	85.00	90.00					
UTX	82.80	3.60	1.20	0.30					
	81.50	3.40	1.10	0.20					
	0.00	80.00	85.00	90.00					
MMM	83.89	4.20	1.30	0.25					
	82.75	4.00	1.15	0.15					
	0.00	45.00	47.50	50.00	55.00	60.00			
MO	50.00	4.90	2.80	1.20	0.15	0.10			
	48.50	4.70	2.65	1.15	0.10	0.00			
	0.00	45.00	47.50	50.00					
MRK	46.89	2.15	0.70	0.15					
	46.00	1.95	0.60	0.10					
	0.00	30.00	35.00	37.50	40.00	42.50			
PFE	35.91	5.70	1.30	0.30	0.10	0.05			
	35.00	5.50	1.20	0.25	0.05	0.00			
	0.00	90.00	95.00	100.00	105.00	110.00	115.00		
PG	107.15	16.50	11.70	7.10	3.30	1.00	0.25		
	105.81	16.30	11.40	6.90	3.10	0.90	0.20		
	0.00	25.00							
SBC	24.49	0.40							
	24.11	0.35							
	0.00	20.00	25.00	27.50	30.00				
MCD	26.05	5.90	1.40	0.35	0.05				
	25.50	5.80	1.30	0.25	0.05				
	0.00	30.00	35.00	40.00	42.50	45.00	47.50	50.00	55.00
C	45.30	15.00	10.00	5.20	3.00	1.30	0.40	0.15	0.05
	44.83	14.80	9.80	5.10	2.90	1.25	0.35	0.05	0.00

Table 2: Strike 80.00 DJX basket super-replicating strategy from LP formulation (eq. (11)). Option price upper bound = 19.8872. For every asset, the first row gives the strikes of the asset's call options with a position greater than zero in the super-replicating strategy. The second row gives the corresponding long position. In this particular experiment, the super-replicating portfolio does not contain any short positions.

	22.50		47.50		50.00	
MSFT	0.071	WMT	0.071	JNJ	0.071	
	25.00		35.00		0.00	80.00
AA	0.071	GM	0.071	UTX	0.054	0.017
	65.00		30.00		80.00	
AIG	0.071	HD	0.071	MMM	0.071	
	47.50		30.00		45.00	
AXP	0.071	HON	0.071	MO	0.071	
	40.00		15.00		45.00	
BA	0.071	HPQ	0.071	MRK	0.071	
	35.00		80.00		30.00	
VZ	0.071	IBM	0.071	PFE	0.071	
	60.00		27.50		90.00	
CAT	0.071	JPM	0.071	PG	0.071	
	0.00		47.50		0.00	
DD	0.071	KO	0.071	SBC	0.071	
	20.00		40.00		20.00	
DIS	0.071	XOM	0.071	MCD	0.071	
	25.00		20.00		35.00	
GE	0.071	INTC	0.071	C	0.071	

Table 3: Strike 80.00 DJX basket super-replicating strategy from LP formulation (eq. (11)) plus diversification constraints (eq. (12)). Option price upper bound = 19.9022. For every asset, the first row gives the strikes of the asset's call options with a position greater than zero in the super-replicating strategy. The second row gives the corresponding long position. In this particular experiment, the super-replicating portfolio does not contain any short positions.

	22.50		
MSFT	0.071		
	25.00		
AA	0.071		
	65.00		
AIG	0.071		
	47.50		
AXP	0.071		
	40.00		
BA	0.071		
	35.00		
VZ	0.071		
	60.00		
CAT	0.071		
	0.00		
DD	0.071		
	20.00		
DIS	0.071		
	25.00	35.00	37.50
GE	0.071	0.011	0.025

	47.50	65.00	70.00
WMT	0.071	0.010	0.025
	35.00	55.00	
GM	0.071	0.050	
	30.00	40.00	
HD	0.071	0.007	
	30.00	40.00	
HON	0.071	0.007	
	15.00		
HPQ	0.071		
	80.00	100.00	
IBM	0.071	0.007	
	27.50	45.00	
JPM	0.071	0.025	
	47.50	55.00	
KO	0.071	0.050	
	40.00		
XOM	0.071		
	20.00		
INTC	0.071		

	50.00	
JNJ	0.071	
	0.00	80.00
UTX	0.054	0.017
	80.00	
MMM	0.071	
	45.00	
MO	0.071	
	45.00	
MRK	0.071	
	30.00	42.50
PFE	0.071	0.007
	90.00	
PG	0.071	
	0.00	
SBC	0.071	
	20.00	30.00
MCD	0.071	0.050
	35.00	55.00
C	0.071	0.025