Full Nesterov-Todd Step Primal-Dual Interior-Point Methods for Second-Order Cone Optimization*

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Abstract

After a brief introduction to Jordan algebras, we present a primal-dual interior-point algorithm for second-order conic optimization that uses full Nesterov-Todd-steps; no line searches are required. The number of iterations of the algorithm is $O(\sqrt{N}\log(N/\varepsilon)$, where N stands for the number of second-order cones in the problem formulation and ε is the desired accuracy. The bound coincides with the currently best iteration bound for second-order conic optimization. We also generalize an infeasible interior-point method for linear optimization [26] to second-order conic optimization. As usual for infeasible interior-point methods the starting point depends on a positive number ζ. The algorithm either finds an ε-solution in at most $O\left(N\log(N/\varepsilon)\right)$ steps or determines that the primal-dual problem pair has no optimal solution with vanishing duality gap satisfying a condition in terms of ζ.

1 Introduction

Second-order conic optimization (SOCO) problems are convex optimization problems that minimize a linear objective function over the intersection of an affine linear manifold and the Cartesian product of a finite number of second-order (or Lorentz or ice-cream) cones. Mathematically, a typical second-order cone in $\mathbf{R}^n$ has the form

$$\mathcal{L} = \left\{(x_1,\, x_2\,;\,\ldots\,;\,x_n) \in \mathbf{R}^n : x_1^2 \geq \sum_{i=2}^{n} x_i^2,\, x_1 \geq 0\right\}, \tag{1}$$

where $n \geq 2$ is some natural number.

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Let $\mathcal{K} \subseteq \mathbf{R}^n$ be the Cartesian product of several second-order cones, i.e.,

$$\mathcal{K} = \mathcal{L}^1 \times \mathcal{L}^2 \ldots \times \mathcal{L}^N, \tag{2}$$

where $\mathcal{L}^j \subseteq \mathbf{R}^{n_j}$ for each j, $j = 1, 2, \ldots, N$. A second-order conic optimization (SOCO) problem has the form

$$(P) \quad \min \left\{ c^T x : Ax = b,\, x \in \mathcal{K} \right\},$$

where $A \in \mathbf{R}^{m \times n}$, $c \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$, and $n = \sum_{j=1}^N n_j$. Without loss of generality we assume that A has full row rank, i.e. rank $(A) = m$. Due to the fact that $\mathcal{K}$ is self-dual, the dual problem of (P) is given by

$$(D) \quad \max \left\{ b^T y : A^T y + s = c,\, s \in \mathcal{K} \right\}.$$

SOCO problems are nonlinear convex problems that include linear optimization (LO) problems, convex quadratic optimization problems and quadratically constrained convex quadratic optimization problems as special cases, and arise in many engineering problems [10, 32, 34].

On the other hand, SOCO problems are essentially a specific case of Semidefinite Optimization (SDO) problems. Thus SOCO problems can be solved via the algorithms for SDO problems. However, it has been pointed out [18] that an interior-point method (IPM) that solves the SOCO problem directly has much better complexity than an IPM applied to the semidefinite formulation of the SOCO problem.

Several authors have discussed IPMs for SOCO. Nesterov and Todd [19, 20] considered linear cone optimization problems in which the cone is *self-scaled*. They presented a primal-dual IPM for optimization over such cones. It has become clear later that self-scaled cones are precisely the cones of squares in Jordan algebras. Adler and Alizadeh [1] studied the relationship between SDO and SOCO problems and presented a unified approach to these problems. Alizadeh and Goldfarb [2] and Schmieta and Alizadeh [28, 29] showed that Euclidean Jordan algebras underly the analysis of IPMs for optimization over symmetric cones. Faybusovich [5] used Euclidean Jordan Algebras to analyze when the Nesterov-Todd direction is well-defined.

Peng et al. [21, 22] presented primal-dual feasible IPMs by using self-regular proximity functions for LO, SDO and SOCO. They obtained the complexity bounds $O(\sqrt{N}\log(N/\varepsilon))$ for small-update and $O(\sqrt{N}\log N \log(N/\varepsilon)$ for large-update methods, which are currently the best known iteration bounds for SOCO problems. Recently, Bai et al. [3] designed a primal-dual feasible IPM for SOCO problems based on a kernel function. They obtained the same complexity bounds as in [22].

In so-called *feasible* IPMs it is assumed that the starting point is feasible and lies in the interior of the cone. Such a starting point is called strictly feasible. All the points generated by feasible IPMs are also strictly feasible. In practice, however, it is sometimes difficult to obtain an initial strictly feasible point. Infeasible IPMs (IIPMs) do not require that the starting point is feasible, but only that it is in the interior of the cone. IIPMs are used in most practical implementations. Global convergence of a primal-dual IIPM for LO was first established by Kojima et al. [8]. Subsequently, Zhang [37], Mizuno [15] and Potra [23, 24] presented polynomial iteration complexity results for variants of this algorithm. Later, Zhang [38] extended it to SDO. Rangarajan [25] established polynomial-time convergence of IIPMs for conic programs over symmetric cones using a wide neighborhood of the central path. Recently, Roos [26] established a new IIPM which uses full Newton steps. Its complexity bound is $O(n\log(n/\varepsilon))$. Later, Mansouri [11] generalized it to SDO.

The aim of this paper is to generalize the IIPM for LO of Roos to SOCO. Since its analysis requires a quadratic convergence result for the feasible case we first present a primal-dual (feasible) IPM with full NT-steps for SOCO and its analysis. To our knowledge this is the first time that a full NT-step IPM for SOCO is considered. We use the Nesteorv-Todd (NT) direction. We obtain the same complexity bound as in [3, 22] which is the currently best bound. Then we extend Roos's IIPM for LO to SOCO. We prove that the complexity bound of our IIPM is $O(N \log(N/\varepsilon))$.

The paper is organized as follows. In Section 2 we briefly review some properties of the second-order cone and its associated Euclidean Jordan algebra, focussing on what is needed in the rest of the paper. We derive some new inequalities that are crucial for the analysis of our algorithms. Then, in Section 3 we present a feasible IPM for SOCO, and in Section 4 our IIPM. Section 5 contains some conclusions and topics for further research.

Some notations used throughout the paper are as follows. The superscript ^{T}is used to denote the transpose of a vector or matrix. $\mathbf{R}^n$, $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ denote the set of real vectors with n components, the set of nonnegative vectors and the set of positive vectors, respectively. We follow the convention of some high level programming languages, such as MATLAB, and use ";" for adjoining vectors in a column. Thus for instance for column vectors x, y and z we have:

$$(x\,;\,y\,;\,z) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Superscripted vectors such as x^j usually represent the j^{th} block of x. It should be noted that sometimes the notation x^j refers to the j-th power of x. The meaning is always clear from the context. $\mathbf{R}^{m\times n}$ is the space of all $m \times n$ matrices. $\mathbf{S}^n$, $\mathbf{S}^n_+$ and $\mathbf{S}^n_{++}$ denote the cone of symmetric, symmetric positive semidefinite and symmetric positive definite $n \times n$ matrices, respectively. For any symmetric matrix A, $\lambda_{\min}(A)$ $(\lambda_{\max}(A))$ denotes the minimal (maximal) eigenvalue of A. As usual, $\|\cdot\|$ denotes the $2-$norm for vectors and matrices. We denote the trace of a matrix as $\mathbf{Tr}(\cdot)$ and the trace of a vector as $\mathbf{tr}\,(\cdot)$. The Löwner partial ordering $\succeq_{\mathcal{K}}$ of $\mathbf{R}^n$ defined by a cone $\mathcal{K}$ is defined by $x \succeq_{\mathcal{K}} s$ if $x - s \in \mathcal{K}$. The interior of $\mathcal{K}$ is denoted as $\mathcal{K}_+$. We write $x \succ_{\mathcal{K}} s$ if $x - s \in \mathcal{K}_+$. $\mathcal{P}$ and $\mathcal{D}$ denote the feasible sets of the primal and the dual problem, respectively. In this paper we assume that both the primal problem and its dual are feasible. Finally, E_n denotes the $n \times n$ identity matrix.

2 Preliminaries

2.1 Euclidean Jordan Algebras

We recall certain basic notions and well-known facts concerning Jordan algebras. For omitted proofs we refer to the given references and also to [4, 7, 14].

Definition 2.1 *Let $\mathcal{J}$ be an n-dimensional vector space over* $\mathbf{R}$. *A map $h : \mathcal{J} \times \mathcal{J} \longmapsto \mathcal{J}$ is called bilinear if for all x, y, $z \in \mathcal{J}$ and α, $\beta \in \mathbf{R}$:*

(i) $h(\alpha x + \beta y,\, z) = \alpha h(x,\, z) + \beta h(y,\, z)$;

(ii) $h(z,\, \alpha x + \beta y) = \alpha h(z,\, x) + \beta h(z,\, y)$.

Definition 2.2 *Let $\mathcal{J}$ be an n-dimensional vector space over $\mathbf{R}$ along with a bilinear map $\circ$: $(x, y) \mapsto x \circ y \in \mathcal{J}$. Then $(\mathcal{J}, \circ)$ is called a Euclidean Jordan algebra if for all $x, y \in \mathcal{J}$:*

(i) *$x \circ y = y \circ x$ (commutativity);*

(ii) *$x \circ \left(x^2 \circ y\right) = x^2 \circ (x \circ y)$, where $x^2 = x \circ x$ (Jordan identity);*

(iii) *there exists an inner product, denoted by $\langle x, y\rangle$, such that $\langle x \circ y, z\rangle = \langle x, y \circ z\rangle$ (associativity).*

We call $x \circ y$ the Jordan product of x and y. In addition, we assume that there is an element $e \in \mathcal{J}$ such that $e \circ x = x \circ e = x$ for all $x \in \mathcal{J}$, which is called the identity element in $\mathcal{J}$. The Jordan product is not necessarily associative, but it is power associative, i.e., the subalgebra generated by a single element $x \in \mathcal{J}$ is associative (Proposition II.1.2 of [4]).

For $x \in \mathcal{J}$, let r be the smallest number such that the set $\left\{e,\, x,\, x^2,\, \ldots,\, x^r\right\}$ is linearly dependent. Then r is called the degree of x and is denoted by $\deg(x)$. The rank of $\mathcal{J}$, denoted by $\operatorname{rank}(\mathcal{J})$, is defined as the maximum of $\deg(x)$ over all $x \in \mathcal{J}$. An element $x \in \mathcal{J}$ is called *regular* if $\deg(x) = \operatorname{rank}(\mathcal{J})$.

For an element x of degree d, since $\left\{e,\, x,\, x^2,\, \ldots,\, x^d\right\}$ is linearly dependent, there exist real numbers $a_1(x),\, a_2(x),\, \ldots,\, a_d(x)$ such that

$$x^d - a_1(x)x^{d-1} + a_2(x)x^{d-2} + \ldots + (-1)^d a_d(x) = 0,$$

where 0 is the zero vector. Then the polynomial $\lambda^d - a_1(x)\lambda^{d-1} + a_2(x)\lambda^{d-2} + \ldots + (-1)^d a_d(x) = 0$ is called the *minimum polynomial* of x. The minimum polynomial of a regular element x is called the *characteristic polynomial* of x.

The characteristic polynomial is a polynomial of degree r in λ, where r is the rank of $\mathcal{J}$. The roots $\lambda_1,\, \ldots,\, \lambda_r$ of the characteristic polynomial of x are called the eigenvalues (spectral values) of x [4].

Definition 2.3 (Definition 2.6 in [25]) *Let $x \in \mathcal{J}$ and $\lambda_1,\, \ldots,\, \lambda_r$ be the eigenvalues of x. Then,*

(i) *$\mathbf{tr}\,(x) := \lambda_1 + \ldots + \lambda_r$, is called the trace of x.*

(ii) *$\mathbf{det}\,(x) := \lambda_1\ \ldots\ \lambda_r$ is called the determinant of x.*

Recall that a nonzero element c of $\mathcal{J}$ is called idempotent if $c^2 = c$. A complete system of orthogonal idempotent is a set $\{c_1,\, \ldots,\, c_k\}$ of idempotents where $c_i \circ c_j = 0$ for all $i \neq j$ and $c_1 + \ldots + c_k = e$. An idempotent is called primitive if it is not the sum of two other orthogonal idempotents. A complete system of orthogonal primitive idempotents is called a Jordan frame. Jordan frames always contain r primitive idempotents, where r is the rank of $\mathcal{J}$ [4].

The spectral decomposition theorem (Theorem III.1.2 of [4]) of an Euclidean Jordan algebra $\mathcal{J}$ states that for $x \in \mathcal{J}$ there exists a Jordan frame $c_1, \ldots, c_r$ (r is the rank of $\mathcal{J}$) and real numbers $\lambda_1, \ldots, \lambda_r$ (the eigenvalues of x) such that

$$x = \lambda_1 c_1 + \ldots + \lambda_r c_r.$$

Using this, for each $x \in \mathbf{R}^n$ we can define the following [2]:

square root: $x^{\frac{1}{2}} := \lambda_1^{\frac{1}{2}} c_1 + \ldots + \lambda_r^{\frac{1}{2}} c_r$, whenever all $\lambda_i \geq 0$, and undefined otherwise.

inverse: $x^{-1} := \lambda_1^{-1} c_1 + \ldots + \lambda_r^{-1} c_r$, whenever all $\lambda_i \neq 0$, and undefined otherwise.

square: $x^2 := \lambda_1^2 c_1 + \ldots + \lambda_r^2 c_r$.

Indeed, one has $x^2 = x \circ x$ and $(x^{\frac{1}{2}})^2 = x$. If x^{-1} is defined, then $x \circ x^{-1} = e$, and we call x invertible. Also note that since e has eigenvalue 1, with multiplicity r, $\mathbf{tr}(e) = r$ and $\mathbf{det}(e) = 1$.

We consider the set of squares in $\mathcal{J}$:

$$\mathcal{K}_{\mathcal{J}} := \left\{ x^2 \ : \ x \in \mathcal{J} \right\}$$

It is well-known that this set is a convex cone with nonempty interior. It is called the *cone of squares* in $\mathcal{J}$. Below we denote this cone simply as $\mathcal{K}$. We have $x \in \mathcal{K}$ ($x \in \mathcal{K}_+$) if and only if all eigenvalues of x are nonnegative (positive).

For an element x in $\mathcal{J}$, let $L(x)$ be the linear map of $\mathcal{J}$ defined by

$$L(x)y := x \circ y, \tag{3}$$

and let

$$P(x) := 2L(x)^2 - L\left(x^2\right), \tag{4}$$

where $L(x)^2 = L(x)L(x)$. The map P is called the quadratic representation of $\mathcal{J}$. Due to Definition 2.2.(ii) the maps $L(x)$ and $L(x^2)$ commute. Hence, also $P(x)$ commutes with $L(x)$.

The automorphism group of (any convex cone) $\mathcal{K}$ is defined by

$$\mathrm{Aut}(\mathcal{K}) = \left\{ g \in \mathrm{Gl}(\mathcal{K}) : \ g(\mathcal{K}) = \mathcal{K} \right\},$$

where $\mathrm{Gl}(\mathcal{K})$ is the set of invertible linear maps g from $\mathcal{J}$ into itself. The cone $\mathcal{K}$ is called homogeneous if $\mathrm{Aut}(\mathcal{K})$ acts transitively on the interior of $\mathcal{K}$, i.e., for all x, y in $\mathcal{K}_+$ there exists $g \in \mathrm{Aut}(\mathcal{K})$ such that $gx = y$. The cone $\mathcal{K}$ is symmetric if it is homogeneous and self-dual. The next two results imply that the cone of squares $\mathcal{K} = \mathcal{K}_{\mathcal{J}}$ is symmetric.

Proposition 2.4 (Proposition 2.2 in [35]) *For each $x \in \mathcal{K}_+$, $P(x)$ is an automorphism of $\mathcal{K}$ and $P(x)\mathcal{K}_+ = \mathcal{K}_+$. Furthermore, $P(x)$ is positive definite for each $x \in \mathcal{K}_+$.*

Proposition 2.5 (Proposition 2.4 in [35]) *Suppose that $a, b \in \mathcal{K}_+$. Then there exists a unique $x \in \mathcal{K}_+$ such that*

$$P(x)\, a = b.$$

Moreover,

$$x = P(a^{-\frac{1}{2}}) \left(P(a^{\frac{1}{2}}) b \right)^{\frac{1}{2}} \qquad \left[= P(b^{\frac{1}{2}}) \left(P(b^{-\frac{1}{2}}) a^{-1} \right)^{\frac{1}{2}} \right].$$

For the equality of the two different expressions for x we refer to, e.g., [35, Theorem 2.8].

We recall a few more results that will be needed in the sequel. Recall that two matrices are similar if they share the same set of eigenvalues; in this case, we write $A \sim B$. Analogously, we say that two elements x and y in $\mathcal{J}$ are similar, denoted as $x \sim y$, if and only if x and y share the same set of eigenvalues. For more details we refer to [30].

Proposition 2.6 (Proposition 19 in [29]) *Two elements x and y of an Euclidean Jordan algebra are similar if and only if $L(x)$ and $L(y)$ are similar.*

Proposition 2.7 (Corollary 20 in [29]) *Let x and y be two elements in $\mathcal{K}_+$. Then x and y are similar if and only if $P(x)$ and $P(y)$ are similar.*

Proposition 2.8 (Proposition 2.1 in [35]) *The following holds for any $x, s \in \mathbf{R}^n$.*

(*i*) *x is invertible if and only if $P(x)$ is invertible. In this case:*

$$P(x)\, x^{-1} = x,\ P(x)^{-1} = P\left(x^{-1}\right),\ P(x)\, e = P(x^{\frac{1}{2}})x = x^2.$$

(*ii*) *If x and s are invertible, then $P(x)s$ is invertible and $(P(x)\, s)^{-1} = P\left(x^{-1}\right) s^{-1}$.*

(*iii*) *For any two elements x and s:*

$$P\left(P(x)s\right) = P(x)P(s)P(x).$$

(*iv*) *If $x, s \in \mathcal{K}_+$, then $P(x^{\frac{1}{2}})s \sim P(s^{\frac{1}{2}})x$.*

The third identity is far from trivial; it is known as the *fundamental formula* for Jordan algebras. Since $P(e) = E_n$, taking $s = e$ it gives

$$P\left(x^2\right) = P(x)^2.$$

The fourth item follows from the *fundamental formula*. The proof is simple. It also uses Proposition 2.7 and goes as follows:

$$P(P(x^{\frac{1}{2}})s) = P(x^{\frac{1}{2}})P(s)P(x^{\frac{1}{2}}) \sim P(x)P(s) \sim P(s^{\frac{1}{2}})P(x)P(s^{\frac{1}{2}}) = P(P(s^{\frac{1}{2}})x).$$

A for our goal very important generalization is the following result. Because of its importance we include the proof.

Lemma 2.9 (Proposition 21 in [29]) *Let $x,\, s,\, p \in \mathcal{K}_+$. Defining $\tilde{x} = P(p)x$ and $\tilde{s} = P(p^{-1})s$, one has*

$$P(\tilde{x}^{\frac{1}{2}})\tilde{s} \sim P(x^{\frac{1}{2}})s.$$

Proof: Since $P(P(x^{\frac{1}{2}})s) \sim P(x)P(s)$, and similarly, $P(P(\tilde{x}^{\frac{1}{2}})\tilde{s}) \sim P(\tilde{x})P(\tilde{s})$, it suffices to show that $P(\tilde{x})P(\tilde{s}) \sim P(x)P(s)$. Using the fundamental formula we obtain

$$P(\tilde{x})P(\tilde{s}) = P(P(p)x)P(P(p^{-1}s) = P(p)P(x)P(p)P(p^{-1})P(s)P(p^{-1}) = P(p)P(x)P(s)P(p^{-1}).$$

The last matrix is similar to $P(x)P(s)$. Hence the proof is complete. $\square$

The next lemma depends on Proposition 2.8(*ii*) and the fundamental formula.

Lemma 2.10 (Proposition 3.2.4 in [33]) *Let $x,\, s \in \mathcal{K}_+$. If w is the scaling point of x and s, then*

$$\left(P(x^{\frac{1}{2}})s\right)^{\frac{1}{2}} \sim P(w)^{\frac{1}{2}}\, s,$$

2.2 Algebraic properties of second-order cones

In this section we briefly review some algebraic properties of the second-order cone $\mathcal{L}$ as defined by (1) and its associated Euclidean Jordan algebra. For more details and proofs we refer to, e.g., [3, 17, 22, 28, 31].

For $x,\, s \in \mathbf{R}^n$, we define the bilinear operator $\circ$ as follows:

$$x \circ s := \left(x^T s\,;\, x_1 s_2 + s_1 x_2\,;\, \ldots\,;\, x_1 s_n + s_1 x_n\right) = \left(x^T s\,;\, x_1 \bar{s} + s_1 \bar{x}\right),$$

where $\bar{x} = (x_2\,;\, \ldots\,;\, x_n)$. One easily checks that $(\mathbf{R}^n, \circ)$ is an Euclidean Jordan algebra, with the vector

$$e = (1\,;\, 0\,;\, \ldots\,;\, 0) \in \mathbf{R}^n$$

as identity element. In the sequel we denote the vector $(x_2; \ldots; x_n)$ shortly as $\bar{x}$. So $x = (x_1; \bar{x})$. One easily verifies that each $x \in \mathbf{R}^n$ satisfies the quadratic equation

$$x^2 - 2x_1 x + \left(x_1^2 - \|\bar{x}\|^2\right) e = 0.$$

This means that $\lambda^2 - 2x_1\lambda + \left(x_1^2 - \|\bar{x}\|^2\right) = 0$ is the characteristic polynomial of x. Hence the rank of this Jordan algebra is 2 and the two eigenvalues of x are

$$\lambda_{\max}(x) = x_1 + \|\bar{x}\|, \qquad \lambda_{\min}(x) = x_1 - \|\bar{x}\|. \tag{5}$$

Therefore, the trace and the determinant of $x \in \mathbf{R}^n$ are

$$\begin{aligned} \mathbf{tr}\,(x) &= \lambda_{\max}(x) + \lambda_{\min}(x) = 2x_1, \\ \mathbf{det}\,(x) &= \lambda_{\max}(x)\,\lambda_{\min}(x) = x_1^2 - \|\bar{x}\|^2. \end{aligned}$$

Lemma 2.11 *For all $x, s \in \mathbf{R}^n$ one has*

(*i*) $\mathbf{tr}\,(x \circ s) = 2x^T s$;

(*ii*) $\mathbf{det}\,(x \circ s) \le \mathbf{det}(x)\mathbf{det}(s)$; *equality holds iff* $\bar{x} = \alpha\bar{s}$, $\alpha > 0$;

Proof: The relation (*i*) is obvious. For (*ii*) we refer to (its elementary proof in) [22, Lemma 6.2.3]. □

It is worth pointing out that the fact that (*ii*) does not always hold with equality is related to the fact that the second-order cone is not closed under the Jordan product.

The spectral decomposition of vector $x \in \mathbf{R}^n$ is given by

$$x = \lambda_{\max}(x)\, c_1 + \lambda_{\min}(x)\, c_2,$$

where the Jordan frame $\{c_1, c_2\}$ is given by

$$c_1 = \frac{1}{2}\left(1\,;\, \frac{\bar{x}}{\|\bar{x}\|}\right), \qquad c_2 := \frac{1}{2}\left(1\,;\, \frac{-\bar{x}}{\|\bar{x}\|}\right).$$

Here by convention $\frac{-\bar{x}}{\|\bar{x}\|} = 0$ if $\bar{x} = 0$. Note that c_1 and c_2 belong to $\mathcal{L}$ (but not to $\mathcal{L}_+$).

Since $x^2 = (\|x\|^2 ; 2x_1\bar{x})$, one easily understands that $\{c_1, c_2\}$ is also a Jordan frame for x^2. This implies that the matrices $L(x)$ and $L(x^2)$ commute. See, e.g., [29, Theorem 27]. (It also confirms Definition 2.2.(ii).)

The natural inner product is given by

$$\langle x, s\rangle := \mathbf{tr}\,(x \circ s) = 2x^T s, \qquad x, s \in \mathbf{R}^n.$$

Hence, the norm induced by this inner product, which is denoted as $\|\cdot\|_F$ (cf. [2]), satisfies

$$\|x\|_F = \sqrt{\langle x, x\rangle} = \sqrt{\mathbf{tr}\,(x^2)} = \left(\lambda_{\max}(x)^2 + \lambda_{\min}(x)^2\right)^{\frac{1}{2}} = \sqrt{2}\,\|x\|\,. \tag{6}$$

We proceed with some simple properties of this inner product and the induced norm.

Lemma 2.12 *Let $x \in \mathbf{R}^n$ and $s \in \mathcal{K}$. Then*

$$\lambda_{\min}(x)\ \mathbf{tr}(s) \le \mathbf{tr}(x \circ s) \le \lambda_{\max}(x)\ \mathbf{tr}(s).$$

Proof: For any $x \in \mathbf{R}^n$ we have $\lambda_{\max}(x)e - x \in \mathcal{K}$. Since also $s \in \mathcal{K}$, it follows that $\mathbf{tr}\,((\lambda_{\max}(x)e - x) \circ s) \ge 0$. Hence the second inequality in the lemma follows by writing

$$\mathbf{tr}(x \circ s) \le \mathbf{tr}\,(\lambda_{\max}(x)e \circ s) = \lambda_{\max}(x)\mathbf{tr}(e \circ s) = \lambda_{\max}(x)\mathbf{tr}(s),$$

The proof of the first inequality goes in the same way. $\square$

Lemma 2.13 *For all $x, s \in \mathbf{R}^n$ one has*

(i) $\left\|x^2\right\|_F \le \|x\|_F^2$; *equality holds if and only if* $|x_1| = \|\bar{x}\|$;

(ii) $\mathbf{tr}\left[(x \circ s)^2\right] \le \mathbf{tr}\left(x^2 \circ s^2\right)$;

(iii) $\|x \circ s\|_F^2 \le \lambda_{\max}\left(x^2\right)\|s\|_F^2 \le \|x\|_F^2\,\|s\|_F^2$.

Proof: Using $x^2 = \left(\|x\|^2 ; 2x_1\bar{x}\right)$ we may write, also using $2ab \le a^2 + b^2$,

$$\left\|x^2\right\|_F^2 = 2\left(\|x\|^4 + (2x_1\,\|\bar{x}\|)^2\right) \le 2\left(\|x\|^4 + \left(x_1^2 + \|\bar{x}\|^2\right)^2\right) = 4\,\|x\|^4 = \|x\|_F^4\,,$$

which implies (i). Using the Cauchy-Schwarz inequality (in the third line below), we may write

$$\begin{aligned}
\tfrac{1}{2}\mathbf{tr}\left((x \circ s)^2\right) &= \|x \circ s\|^2 = \left(x^T s\right)^2 + \|x_1\bar{s} + s_1\bar{x}\|^2 = \left(x_1 s_1 + \bar{x}^T\bar{s}\right)^2 + \|x_1\bar{s} + s_1\bar{x}\|^2 \\
&= x_1^2 s_1^2 + \left(\bar{x}^T\bar{s}\right)^2 + x_1^2\,\|\bar{s}\|^2 + s_1^2\,\|\bar{x}\|^2 + 4x_1 s_1\bar{x}^T\bar{s} \\
&\le x_1^2 s_1^2 + \|\bar{x}\|^2\,\|\bar{s}\|^2 + x_1^2\,\|\bar{s}\|^2 + s_1^2\,\|\bar{x}\|^2 + 4x_1 s_1\bar{x}^T\bar{s} \\
&= \left(x_1^2 + \|\bar{x}\|^2\right)\left(s_1^2 + \|\bar{s}\|^2\right) + 4x_1 s_1\bar{x}^T\bar{s} \\
&= \|x\|^2\,\|s\|^2 + 4x_1 s_1\bar{x}^T\bar{s} = \tfrac{1}{2}\mathbf{tr}\left(x^2 \circ s^2\right),
\end{aligned}$$

which proves (ii). Finally, using part (ii) we may write

$$\|x \circ s\|_F^2 = \mathbf{tr}\left((x \circ s)^2\right) \leq \mathbf{tr}\left(x^2 \circ s^2\right).$$

Due to Lemma 2.12 part (i) this implies

$$\|x \circ s\|_F^2 \leq \lambda_{\max}\left(x^2\right) \mathbf{tr}\left(s^2\right) = \lambda_{\max}\left(x^2\right) \left\|s^2\right\|_F \leq \lambda_{\max}\left(x^2\right) \|s\|_F^2,$$

which is the first inequality in (iii). The second inequality in (iii) follows by applying (6). This completes the proof. □

As we mentioned before, the Jordan product is not associative. However, remarkably enough, the trace function is associative (which confirms Definition 2.2.(iii)). We have (cf. Proposition II.4.3 in [4])

$$\mathbf{tr}\left((x \circ y) \circ z\right) = \mathbf{tr}\left(x \circ (y \circ z)\right). \tag{7}$$

An important consequence of the associativity of the trace function is that $L(x)$ is self-adjoint with respect to the above inner product:

$$\langle L(x)y, z\rangle = \mathbf{tr}\left((x \circ y) \circ z\right) = \mathbf{tr}\left((y \circ x) \circ z\right) = \mathbf{tr}\left(y \circ (x \circ z)\right) = \langle y, L(x)z\rangle.$$

Since $P(x)$ is a linear combination of the self-adjoint matrices $L(x)^2$ and $L(x^2)$, $P(x)$ is self-adjoint as well (cf. [25, page 1214]).

It easily can be verified that the cone of squares of the current Jordan algebra is given by (1), and that

$$x \in \mathcal{L} \Leftrightarrow \lambda_{\min}(x) \geq 0, \quad x \in \mathcal{L}_+ \Leftrightarrow \lambda_{\min}(x) > 0.$$

For each $x \in \mathbf{R}^n$, the matrices of $L(x)$ and $P(x)$ with respect to the natural basis will be denoted with the same notations as the maps themselves. As a consequence we have

$$L(x) = \begin{bmatrix} x_1 & \bar{x}^T \\ \bar{x} & x_1 \mathbf{E}_{n-1} \end{bmatrix}, \qquad P(x) = \begin{bmatrix} \|x\|^2 & 2x_1\bar{x}^T \\ 2x_1\bar{x} & \mathbf{det}(x)\mathbf{E}_{n-1} + 2\bar{x}\bar{x}^T \end{bmatrix}.$$

The eigenvalues of $L(x)$ are $\lambda_{\max}(x)$ and $\lambda_{\min}(x)$, both with multiplicity 1, and x_1, with multiplicity $n-2$, and those of $P(x)$ are $\lambda_{\max}(x)^2$ and $\lambda_{\min}(x)^2$, both with multiplicity 1, and $\mathbf{det}(x)$, with multiplicity $n-2$ (cf. [2, Theorem 3]).[1] This implies the following two important facts.

(i) $x \in \mathcal{L}$ $(x \in \mathcal{L}_+)$ if and only if $L(x)$ is positive semidefinite (positive definite);

(ii) if $x \in \mathcal{L}$ then $P(x)$ is positive semidefinite; if $x \in \mathcal{L}_+$ then $P(x)$ is positive definite.

The first property implies that SOCO is a special case of semidefinite optimization (SDO).

We conclude this section with three other useful results.

Lemma 2.14 *Let $x, s \in \mathcal{K}_+$. Then we have*

(i) $\lambda_{\min}\left(P(x)^{\frac{1}{2}} s\right) \geq \lambda_{\min}(x \circ s)$;

[1] Observe that this means that the determinant of $P(x)$, being the product of its eigenvalues, equals $\mathbf{det}(x)^n$.

(*ii*) $\lambda_{\max}\left(P(x)^{\frac{1}{2}}s\right) \leq \lambda_{\max}(x \circ s)$.

Proof: The proof of part (*i*) of the lemma is far from trivial. We omit this proof and refer to the literature [25, Lemma 3.5] (see also [28, Lemma 30]). We only show that (*ii*) is an almost immediate consequence of (*i*). We write

$$\mathbf{tr}(P(x)^{1/2}s) = \mathbf{tr}(P(x)^{1/2}s \circ e) = \mathbf{tr}(s \circ P(x)^{1/2}e) = \mathbf{tr}(x \circ s), \tag{8}$$

which means that

$$\lambda_{min}(P(x)^{1/2}s) + \lambda_{max}(P(x)^{1/2}s) = \lambda_{min}(x \circ s) + \lambda_{max}(x \circ s).$$

Due to (*i*) this implies (*ii*). $\square$

Lemma 2.15 *Let $x, s \in \mathcal{K}_+$, $u = P(x)^{\frac{1}{2}}s$ and $z = x \circ s \in \mathcal{K}_+$. Then we have*

$$\left\|u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right\|_F \leq \left\|z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right\|_F.$$

Proof: Let

$$\lambda_1 = \lambda_{\min}\left(P(x)^{\frac{1}{2}}s\right),\ \lambda_2 = \lambda_{\max}\left(P(x)^{\frac{1}{2}}s\right), \quad \mu_1 = \lambda_{\min}(x \circ s),\ \mu_2 = \lambda_{\max}(x \circ s).$$

Then, using Lemma 2.14 and (8) we get

$$\mu_1 \leq \lambda_1 \leq \lambda_2 \leq \mu_2, \quad \lambda_1 + \lambda_2 = \mu_1 + \mu_2 = \mathbf{tr}(x \circ s).$$

Since these eigenvalues are all nonnegative, there exist nonnegative numbers α, β and γ such that

$$\lambda_1 = \gamma - \alpha,\ \lambda_2 = \gamma + \alpha, \quad \mu_1 = \gamma - \beta,\ \mu_2 = \gamma + \beta, \quad 0 \leq \alpha \leq \beta < \gamma.$$

Note that since $w = x \circ s \in \mathcal{K}_+$, the eigenvalue μ_1 is positive, which explains $\beta < \gamma$. This also implies that $\gamma > 0$. We now have

$$\left\|u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right\|_F^2 = \left(\sqrt{\gamma-\alpha} - \frac{1}{\sqrt{\gamma-\alpha}}\right)^2 + \left(\sqrt{\gamma+\alpha} - \frac{1}{\sqrt{\gamma+\alpha}}\right)^2$$
$$\left\|w^{\frac{1}{2}} - w^{-\frac{1}{2}}\right\|_F^2 = \left(\sqrt{\gamma-\beta} - \frac{1}{\sqrt{\gamma-\beta}}\right)^2 + \left(\sqrt{\gamma+\beta} - \frac{1}{\sqrt{\gamma+\beta}}\right)^2.$$

Defining

$$h(\alpha) := \left(\sqrt{\gamma-\alpha} - \frac{1}{\sqrt{\gamma-\alpha}}\right)^2 + \left(\sqrt{\gamma+\alpha} - \frac{1}{\sqrt{\gamma+\alpha}}\right)^2 = 2\gamma + \frac{2\gamma}{\gamma^2-\alpha^2} - 4,$$

the inequality in the lemma reduces to $h(\alpha) \leq h(\beta)$. Since

$$h'(\alpha) = \frac{4\alpha\gamma}{(\gamma^2-\alpha^2)^2} \geq 0, \quad 0 \leq \alpha < \gamma,$$

$h(\alpha)$ is monotonically increasing for $0 \leq \alpha < \gamma$. Since $\alpha \leq \beta < \gamma$, the inequality follows. $\square$

Lemma 2.16 (Lemma 2.1 and Lemma 2.2 in [3]) *Let $x,\ s \in \mathbf{R}^n$. Then*

$$\lambda_{\min}(x+s) \geq \lambda_{\min}(x) + \lambda_{\min}(s) \geq \lambda_{\min}(x) - \|s\|_F.$$

2.3 Rescaling the cone $\mathcal{L}$

When defining the search direction in our algorithm, we need a rescaling of the space in which the cone lives. Let $x,\, s \in \mathcal{L}_+$. Since $\lambda_{\min}(x)$ and $\lambda_{\min}(s)$ are positive, x^{-1} and s^{-1} exist. By Proposition 2.5 there exists a unique $w \in \mathcal{L}_+$ such that

$$P(w)s = x,$$

namely

$$w = P(s^{-\frac{1}{2}})\left(P(s^{\frac{1}{2}})x\right)^{\frac{1}{2}} \qquad \left[= P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})s^{-1}\right)^{\frac{1}{2}}\right]. \tag{9}$$

Due to Proposition 2.4, $P(w)$ is an automorphism. The point w is called the scaling point of x and s (in this order). As a consequence there exists $\tilde{v} \in \mathcal{L}_+$ such that

$$\tilde{v} = P(w)^{-\frac{1}{2}}x = P(w)^{\frac{1}{2}}s.$$

We call this Nesterov-Todd (NT)-scaling of $\mathbf{R}^n$, after the inventors. In the following lemma we recall several properties of the NT-scaling scheme. Because of their importance we include their short proofs.

Lemma 2.17 (cf. Proposition 6.3.3 in [22]) *Let $W = P(w^{\frac{1}{2}})$ for some $w \in \mathcal{K}_+$. Then the following holds for any two vectors $x,\, s \in \mathbf{R}^n$.*

(i) $\mathbf{tr}(Wx \circ W^{-1}s) = \mathbf{tr}(x \circ s)$;

(ii) $\mathbf{det}\,(Wx) = \mathbf{det}(w)\mathbf{det}(x)$, $\mathbf{det}\left(W^{-1}s\right) = \mathbf{det}(w^{-1})\mathbf{det}(s)$;

(iii) if w is the scaling point of x and s then $\mathbf{det}\left(Wx \circ W^{-1}s\right) = \mathbf{det}(x)\mathbf{det}(s)$.

Proof: The proof of (i) is straightforward:

$$\mathbf{tr}\left(Wx \circ W^{-1}s\right) = 2\,(Wx)^T\left(W^{-1}s\right) = 2x^TW^TW^{-1}s = 2x^Ts = \mathbf{tr}(x \circ s).$$

For the proof of (ii) we need the matrix

$$Q = \operatorname{diag}(1,\, -1,\, \ldots,\, -1) \in \mathbf{R}^{n\times n}. \tag{10}$$

Obviously, $Q^2 = \mathbf{E}_n$ where $\mathbf{E}_n$ denotes the identity matrix of size $n \times n$. Moreover, $\mathbf{det}\,(x) = x^TQx$, for any x. It is well known that $WQW = \mathbf{det}(w)Q$ (cf. Proposition 3 in the appendix of [17]). Hence we may write

$$\mathbf{det}\,(Wx) = (Wx)^T\,Q\,(Wx) = x^TWQWx = \mathbf{det}(w)x^TQx = \mathbf{det}(w)\mathbf{det}(x).$$

In a similar way we can prove $\mathbf{det}\left(W^{-1}s\right) = \mathbf{det}(w^{-1})\mathbf{det}(s)$. Finally, for proving (iii) we use that if w is the scaling point of x and s then $Wx = W^{-1}s$. Hence, using Lemma 2.11 and part (ii) of the current lemma, we write

$$\mathbf{det}\left(Wx \circ W^{-1}s\right) = \mathbf{det}\,(Wx)\,\mathbf{det}\left(W^{-1}s\right) = \mathbf{det}(w)\mathbf{det}(x)\mathbf{det}(w^{-1})\mathbf{det}(s) = \mathbf{det}(x)\mathbf{det}(s),$$

where we used that $\mathbf{det}(w)\mathbf{det}(w^{-1}) = 1$. Hence the proof is complete. $\square$

2.4 Rescaling the cone $\mathcal{K}$

In this section we show how the definitions and properties in the previous sections can be adapted to the case where $N > 1$, when the cone underlying the given problems (P) and (D) is the Cartesian product of N cones $\mathcal{L}^j$, as given in (2).

First we partition any vector $x \in \mathbf{R}^n$ according to the dimensions of the successive cones $\mathcal{L}^j$, so

$$x = \left(x^1\,;\,\ldots\,;\,x^N\right), \qquad x^j \in \mathbf{R}^{n_j},$$

and we define the algebra $(\mathbf{R}^n,\,\circ)$ as a direct product of the Jordan algebras $(\mathbf{R}^{n_j},\,\circ)$, by defining

$$x \circ s := \left(x^1 \circ s^1\,;\,\ldots\,;\,x^N \circ s^N\right).$$

Obviously, if $e^j \in \mathcal{L}^j$ is the unit element in the Jordan algebra for the j-th cone, then the vector

$$e = \left(e^1\,;\,\ldots\,;\,e^N\right) \tag{11}$$

is the unit element in $(\mathbf{R}^n,\,\circ)$. Moreover, $\mathbf{tr}(e) = 2N$, which is the rank of $(\mathbf{R}^n,\,\circ)$. One easily verifies that $L(\cdot)$ and $P(\cdot)$ are given by [2]:

$$\begin{aligned} L(x) &:= \operatorname{diag}\,\left(L\left(x^1\right),\,\ldots,\,L\left(x^N\right)\right),\\ P\left(x\right) &:= \operatorname{diag}\,\left(P\left(x^1\right),\,\ldots,\,P\left(x^N\right)\right). \end{aligned}$$

The NT-scaling scheme for the general case can be obtained as follows. For $x^j,\,s^j \in \mathcal{L}^j_+$, let w^j be the scaling point in $\mathcal{L}^j$. Then

$$P\left(w^j\right)^{-\frac{1}{2}} x^j = P\left(w^j\right)^{\frac{1}{2}} s^j, \qquad 1 \le j \le N.$$

The scaling point of x and s in $\mathcal{K}$ is then defined by

$$w := \left(w^1\,;\,\ldots\,;\,w^N\right).$$

Since $P(w^j)$ is symmetric and positive definite for each j, the matrix

$$P(w) := \operatorname{diag}\,\left(P\left(w^1\right),\,\ldots,\,P\left(w^N\right)\right)$$

is symmetric and positive definite as well and represents an automorphism of $\mathcal{K}$ such that $P(w)s = x$. Therefore $P(w)$ can be used to rescale x and s to the same vector

$$v := \left(v^1\,;\,\ldots\,;\,v^N\right), \tag{12}$$

according to (22). Since $L(x) := \operatorname{diag}\,\left(L\left(x^1\right),\,\ldots,\,L\left(x^N\right)\right)$, one easily gets

$$\lambda_{\max}\left(x\right) = \lambda_{\max}\left(L(x)\right) = \max\left\{\lambda_{\max}\left(x^j\right)\,:\,1 \le j \le N\right\}, \tag{13}$$

$$\lambda_{\min}\left(x\right) = \lambda_{\min}\left(L(x)\right) = \min\left\{\lambda_{\min}\left(x^j\right)\,:\,1 \le j \le N\right\}. \tag{14}$$

Furthermore,

$$\mathbf{tr}\left(x\right) = \sum_{j=1}^{N} \mathbf{tr}\left(x^j\right) = \sum_{j=1}^{N} \left[\lambda_{\min}\left(x^j\right) + \lambda_{\max}\left(x^j\right)\right], \tag{15}$$

$$\|x\|_F^2 = \sum_{j=1}^{N} \left\|x^j\right\|_F^2 = \sum_{j=1}^{N} \left[\lambda_{\min}\left(x^j\right)^2 + \lambda_{\max}\left(x^j\right)^2\right], \tag{16}$$

$$\mathbf{det}\left(x\right) = \prod_{j=1}^{N} \mathbf{det}\left(x^j\right) = \prod_{j=1}^{N} \lambda_{\min}\left(x^j\right)\lambda_{\max}\left(x^j\right). \tag{17}$$

3 A feasible full NT-step algorithm

In this section we present a full NT-step feasible IPM and its analysis. The results of this section will be used later on, when dealing with the purpose of this paper, a full step infeasible IPM.

3.1 The central path for SOCO

We assume that both (P) and (D) satisfy the interior-point condition (IPC), i.e., there exists $\left(x^0,\, s^0,\, y^0\right)$ such that

$$Ax^0 = b,\ x^0 \in \mathcal{K}_+,\ A^T y^0 + s^0 = c,\ s^0 \in \mathcal{K}_+.$$

It is well known that the IPC can be assumed without loss generality [36] . Finding an optimal solution of (P) and (D) is equivalent to solving the following system [5].

$$\begin{aligned} Ax &= b, & x &\in \mathcal{K}, \\ A^T y + s &= c, & s &\in \mathcal{K}, \\ x \circ s &= 0. \end{aligned} \tag{18}$$

The basic idea of primal-dual IPMs is to replace the third equation in (18), the so-called complementary condition for (P) and (D), by the parameterized equation $x \circ s = \mu e$, with $\mu > 0$. Thus we consider the system

$$\begin{aligned} Ax &= b, & x &\in \mathcal{K}, \\ A^T y + s &= c, & s &\in \mathcal{K}, \\ x \circ s &= \mu e. \end{aligned} \tag{19}$$

For each $\mu > 0$ the parameterized system (19) has a unique solution $(x(\mu),\, y(\mu),\, s(\mu))$ and we call $x(\mu)$ and $(y(\mu),\, s(\mu))$ the μ-center of (P) and (D), respectively. Note that at the μ-center we have

$$x(\mu)^T s(\mu) = \frac{1}{2}\mathbf{tr}\,(x(\mu) \circ s(\mu)) = \frac{1}{2}\mathbf{tr}\,(\mu e) = \frac{\mu}{2}\ \mathbf{tr}\,(e) = \mu N,$$

where we used that $\mathbf{tr}\,(e) = 2N$. The set of μ-centers (with μ running through all positive real numbers) gives a homotopy path, which is called the central path of (P) and (D) [5]. If $\mu \to 0$ then the limit of the central path exists and since the limit points satisfy the complementarity condition, the limit yields optimal solutions for (P) and (D) [5].

3.2 The Nesterov-Todd search direction

The natural way to define a search direction is to follow the Newton approach and to linearize the third equation in (19), which leads to the system

$$\begin{aligned} A\Delta x &= 0, \\ A^T \Delta y + \Delta s &= 0, \\ x \circ \Delta s + s \circ \Delta x &= \mu e - x \circ s. \end{aligned} \tag{20}$$

Due to the fact that x and s do not operator commute in general (i.e., $L(x)L(s) \neq L(s)L(x)$) this system not always has a solution. For an example of this phenomenon we refer to [22, Section 6.3.1]. It is now well known that this difficulty can be solved by applying a *scaling scheme*. This goes as follows. Let $u \in \mathcal{K}_+$. Then we have

$$x \circ s = \mu e \quad \Leftrightarrow \quad P(u)x \circ P(u^{-1})s = \mu e.$$

This is an easy consequence of Proposition 2.8.(ii), as becomes clear when using that $x \circ s = \mu e$ holds if and only if $x = \mu s^{-1}$ (cf. Lemma 28 in [29]). Now replacing the third equation in (20) by $P(u)x \circ P(u^{-1})s = \mu e$, and then applying Newton's method, we obtain the system

$$\begin{aligned} A\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ P(u)x \circ P(u^{-1})\Delta s + P(u^{-1})s \circ P(u)\Delta x &= \mu e - P(u)x \circ P(u^{-1})s. \end{aligned} \tag{21}$$

By choosing u appropriately this system can be used to define search directions. In the literature the following choices are well known: $u = s^{\frac{1}{2}}$, $u = x^{-\frac{1}{2}}$ and $u = w^{-\frac{1}{2}}$, where w is the NT-scaling point of x and s. The first two choices lead to the so-called sx-direction and xs-direction, respectively. In this paper we focus on the third choice, which gives rise to the NT-direction. For that case we define

$$v := \frac{P(w)^{-\frac{1}{2}}x}{\sqrt{\mu}} \qquad \left[= \frac{P(w)^{\frac{1}{2}}s}{\sqrt{\mu}}\right], \tag{22}$$

and

$$\bar{A} := \sqrt{\mu}AP(w)^{\frac{1}{2}}, \quad d_x := \frac{P(w)^{-\frac{1}{2}}\Delta x}{\sqrt{\mu}}, \quad d_s := \frac{P(w)^{\frac{1}{2}}\Delta s}{\sqrt{\mu}}. \tag{23}$$

This enables us to rewrite the system (21) as follows:

$$\bar{A}d_x = 0, \tag{24}$$

$$\bar{A}^T\frac{\Delta y}{\mu} + d_s = 0, \tag{25}$$

$$d_s + d_x = v^{-1} - v. \tag{26}$$

That substitution of (22) and (23) into the first two equations of (21) yield the equations (24) and (25) is easy to verify. It is less obvious that the third equation in (21) yields (26). After the substitution we get, after dividing both sides by μ, $v \circ (d_s + d_x) = e - v^2$. This can be written as $L(v)\,(d_s + d_x) = e - v^2$. After multiplying of both sides from the left with $L(v)^{-1}$, while using $L(v)^{-1}e = v^{-1}$ and $L(v)^{-1}v^2 = v$, we obtain (26). It easily follows that the above system has unique solution. Since (24) requires that d_x belongs to the null space of $\bar{A}$, and (25) that d_s belongs to the row space of $\bar{A}$, it follows that system (24) – (26) determines d_x and d_s uniquely as the (mutually orthogonal) components of the vector $v^{-1} - v$ in these two spaces. From (26) and the orthogonality of d_x and d_s we obtain

$$\|d_x + d_s\|_F^2 = \|d_x\|_F^2 + \|d_s\|_F^2 = \left\|v^{-1} - v\right\|_F^2. \tag{27}$$

Therefore the displacements d_x, d_s (and since $\bar{A}$ has full row rank, also Δy) are zero if and only if $v^{-1} - v = 0$. In this case it easily follows that $v = e$, and that this implies that x, y and s coincide with the respective μ-centers.

To get the search directions Δx and Δs in the original we simply transform the scaled search directions back to the x- and s-space by using (23):

$$\Delta x = \sqrt{\mu} P(w)^{\frac{1}{2}} d_x, \quad \Delta s = \sqrt{\mu} P(w)^{-\frac{1}{2}} d_s. \tag{28}$$

The new iterates are obtained by taking a full step, as follows.

$$\begin{aligned} x_+ &= x + \Delta x, \\ y_+ &= y + \Delta y, \\ s_+ &= s + \Delta s. \end{aligned} \tag{29}$$

Using definition (22) and Lemma 2.17.(i), it readily follows that

$$\mu\, \mathbf{tr}\left(v^2\right) = \mathbf{tr}\left(x \circ s\right). \tag{30}$$

3.3 Proximity measure

In the analysis of the algorithm we need a measure for the distance of the iterates (x, y, s) to the current μ-center $(x(\mu), y(\mu), s(\mu))$. The aim of this section is to present such a measure and to show how it depends on the eigenvalues of the vector v.

The proximity measure that we are going to use is defined as follows.

$$\delta\left(x,\, s\,;\, \mu\right) \equiv \delta\left(v\right) := \frac{1}{2}\left\|v^{-1} - v\right\|_F = \frac{1}{2}\sqrt{\sum_{j=1}^{N}\left\|\left(v^j\right)^{-1} - v^j\right\|_F^2}. \tag{31}$$

According to (27) we have $\|d_x\|_F^2 + \|d_s\|_F^2 = \left\|v - v^{-1}\right\|_F^2$. Therefore, (31) implies that

$$\|d_x\|_F \le 2\delta(v), \quad \|d_s\|_F \le 2\delta(v). \tag{32}$$

In the sequel we will often use the following relation:

$$4\delta\left(v\right)^2 = \left\|v - v^{-1}\right\|_F^2 = \mathbf{tr}\left(v^2\right) + \mathbf{tr}\left(v^{-2}\right) - 4, \tag{33}$$

which expresses $\delta\left(v\right)^2$ in the eigenvalues of v^2 and its inverse.

3.4 The feasible algorithm

The full step feasible algorithm is given in Figure 1. We show below (cf. Lemma 3.3) that after a full NT-step the duality gap $x^T s$ gets its target value $N\mu$. Hence, if the algorithm stops then the duality gap equals $N\mu$, which by then is less than ε.

3.5 Analysis of the full NT-step

3.5.1 Feasibility of the full NT-step

Our aim is to find a condition that guarantees feasibility of the iterates after a full NT-step. As before, let $x,\, s \in \mathcal{K}_+$, $\mu > 0$ and let w be the scaling point of x and s. Using (22), (28) and (29), we obtain

$$x_+ = x + \Delta x = \sqrt{\mu} P(w)^{\frac{1}{2}}\left(v + d_x\right) \tag{34}$$

$$s_+ = s + \Delta s = \sqrt{\mu} P(w)^{-\frac{1}{2}}\left(v + d_s\right). \tag{35}$$

Primal-Dual Algorithm for SOCO

Input:
Accuracy parameter $\varepsilon > 0$;
a barrier update parameter θ, $0 < \theta < 1$;
a strictly feasible pair $\left(x^0,\, s^0\right)$ and $\mu^0 > 0$ such
that $x^{0^T} s^0 = N\mu^0$ and $\delta\left(x^0,\, s^0\,;\, \mu^0\right) \leq \tau$.
begin
$x := x^0\,;\; s := s^0\,;\; \mu := \mu^0;$
while $N\mu \geq \varepsilon$ **do**
begin
$(x,\, y,\, s) := (x,\, y,\, s) + (\Delta x,\, \Delta y,\, \Delta s);$
$\mu := (1-\theta)\mu;$
end
end

Figure 1: Feasible algorithm.

Since $P\,(w)^{\frac{1}{2}}$ and its inverse $P\,(w)^{-\frac{1}{2}}$ are automorphisms of $\mathcal{K}$, x_+ and s_+ will belong to $\mathcal{K}_+$ if and only if $v + d_x$ and $v + d_s$ belong to $\mathcal{K}_+$. For the proof of our main result in this section, which is Lemma 3.2, we need the following lemma.

Lemma 3.1 *If $\delta(v) \leq 1$ then $e + d_x \circ d_s \in \mathcal{K}$. Moreover, if $\delta(v) < 1$ then $e + d_x \circ d_s \in \mathcal{K}_+$.*

Proof: Since d_x and d_s are orthogonal, Lemma A.3 (i) implies that the absolute values of the eigenvalues of $d_x \circ d_s$ do not exceed $\frac{1}{4}\left\|d_x + d_s\right\|_F^2$. Since $d_x + d_s = v^{-1} - v$ and $\left\|v^{-1} - v\right\|_F^2 = 4\delta(v)^2$ it follows that the absolute values of the eigenvalues of $d_x \circ d_s$ do not exceed $\delta(v)^2$. This implies that $1 - \delta(v)^2$ is a lower bound for the eigenvalues of $e + d_x \circ d_s$. Hence, if $\delta(v) \leq 1$ then $e + d_x \circ d_s \in \mathcal{K}$ and if $\delta(v) < 1$, then $e + d_x \circ d_s \in \mathcal{K}_+$. This proves the lemma. $\Box$

Lemma 3.2 *The full NT-step is feasible if $\delta(v) \leq 1$ and strictly feasible if $\delta(v) < 1$.*

Proof: For the proof of the first statement we introduce a step length α, $0 \leq \alpha \leq 1$, and we define

$$v_x^\alpha = v + \alpha d_x, \quad v_s^\alpha = v + \alpha d_s.$$

We then have $v_x^0 = v, v_x^1 = v + d_x$ and $v_s^0 = v, v_s^1 = v + d_s$. Since $d_x + d_s = v^{-1} - v$, it follows that

$$\begin{aligned} v_x^\alpha \circ v_s^\alpha &= (v + \alpha d_x) \circ (v + \alpha d_s) = v^2 + \alpha v \circ (d_x + d_s) + \alpha^2 d_x \circ d_s \\ &= v^2 + \alpha v \circ \left(v^{-1} - v\right) + \alpha^2 d_x \circ d_s = (1-\alpha)v^2 + \alpha e + \alpha^2 d_x \circ d_s. \end{aligned}$$

Since $\delta(v) \le 1$, Lemma 3.1 implies that $d_x \circ d_s \succeq_{\mathcal{K}} -e$. Substitution gives

$$v_x^\alpha \circ v_s^\alpha \succeq_{\mathcal{K}} (1-\alpha)v^2 + \alpha e - \alpha^2 e = (1-\alpha)\left(v^2 + \alpha e\right).$$

If $0 \le \alpha < 1$, the last vector belongs to $\mathcal{K}_+$. Hence we then have $\mathbf{det}\,(v_x^\alpha \circ v_s^\alpha) > 0$. By Lemma 2.11.$(ii)$ this implies that $\mathbf{det}\,(v_x^\alpha)\,\mathbf{det}\,(v_s^\alpha) > 0$, for each $\alpha \in [0,1)$. It follows that $\mathbf{det}\,(v_x^\alpha)$ and $\mathbf{det}\,(v_s^\alpha)$ do not vanish for $\alpha \in [0,1)$. Since $\mathbf{det}\left(v_x^0\right) = \mathbf{det}\left(v_s^0\right) = \mathbf{det}(v) > 0$, by continuity, $\mathbf{det}\,(v_x^\alpha)$ and $\mathbf{det}\,(v_s^\alpha)$ stay positive for all $\alpha \in [0,1)$. Again by continuity, we also have that $\mathbf{det}\left(v_x^1\right)$ and $\mathbf{det}\left(v_s^1\right)$ are nonnegative. This proves that if $\delta(v) \le 1$ then $v + d_x \in \mathcal{K}$ and $v + d_s \in \mathcal{K}$. If $\delta(v) < 1$ then we have $d_x \circ d_s \succ_{\mathcal{K}} -e$ and the same arguments imply that $\mathbf{det}\,(v_x^\alpha)\,\mathbf{det}\,(v_s^\alpha) > 0$, for each $\alpha \in [0,1]$, whence $v + d_x \in \mathcal{K}_+$ and $v + d_s \in \mathcal{K}_+$. This proves the lemma. □

The next lemma shows that the target duality gap is attained after a full NT-step.

Lemma 3.3 *Let* $(x,\, s) \in \mathcal{K}$ *and* $\mu > 0$. *Then*

$$x_+^T s_+ = N\mu.$$

Proof: Due to (34) and (35) we may write

$$x_+^T s_+ = \left(\sqrt{\mu} P\,(w)^{\frac{1}{2}}\,(v + d_x)\right)^T \left(\sqrt{\mu} P\,(w)^{-\frac{1}{2}}\,(v + d_s)\right) = \mu\,(v + d_x)^T\,(v + d_s)\,.$$

Using the third equation in (25) we obtain

$$(v + d_x)^T\,(v + d_s) = v^T v + v^T\,(d_x + d_s) + d_x^T d_s = v^T v + v^T\left(v^{-1} - v\right) + d_x^T d_s = e^T e + d_x^T d_s.$$

Since d_x and d_s are orthogonal, and $e^T e = N$, the lemma follows. □

3.5.2 Quadratic convergence

In this section we prove quadratic convergence to the target point $(x\,(\mu)\,,\, s\,(\mu))$ when taking full NT-steps. According to (22), the v-vector after the step is given by:

$$v_+ := \frac{P\,(w_+)^{-\frac{1}{2}}\,x_+}{\sqrt{\mu}} \qquad \left[= \frac{P\,(w_+)^{\frac{1}{2}}\,s_+}{\sqrt{\mu}}\right], \tag{36}$$

where w_+ is the scaling point of x_+ and s_+.

Lemma 3.4 (Proposition 5.9.3 in [33]) *One has*

$$v_+ \sim \left(P\,(v + d_x)^{\frac{1}{2}}\,(v + d_s)\right)^{\frac{1}{2}}.$$

Proof: It readily follows from (36) and Lemma 2.10 that

$$\sqrt{\mu}\,v_+ = P\,(w_+)^{\frac{1}{2}}\,s_+ \sim \left(P\,(x_+)^{\frac{1}{2}}\,s_+\right)^{\frac{1}{2}}.$$

Due to (34), (35) and Lemma 2.9, with $p = w^{\frac{1}{2}}$, we may write

$$P(x_+)^{\frac{1}{2}} s_+ = \mu P \left(P(w)^{\frac{1}{2}} (v + d_x) \right)^{\frac{1}{2}} P(w)^{-\frac{1}{2}} (v + d_s) \sim \mu P (v + d_x)^{\frac{1}{2}} (v + d_s).$$

From this the lemma follows. □

The above lemma implies

$$v_+^2 \sim P(v + d_x)^{\frac{1}{2}} (v + d_s).$$

Now using Lemma 2.15, with $u = P(v + d_x)^{\frac{1}{2}} (v + d_s)$ and $z = (v + d_x) \circ (v + d_s)$ we obtain the following inequality:

$$4\delta (v_+)^2 = \left\| u^{\frac{1}{2}} - u^{-\frac{1}{2}} \right\|_F^2 \leq \left\| z^{\frac{1}{2}} - z^{-\frac{1}{2}} \right\|_F^2 = \mathbf{tr}(z) + \mathbf{tr}(z^{-1}) - 2\mathbf{tr}(e). \tag{37}$$

Using $d_x + d_s = v^{-1} - v$, we obtain

$$\begin{aligned} z &= (v + d_x) \circ (v + d_s) = v^2 + v \circ (d_x + d_s) + d_x \circ d_s = v^2 + v \circ (v^{-1} - v) + d_x \circ d_s \\ &= e + d_x \circ d_s. \end{aligned} \tag{38}$$

Lemma 3.5 *If $\delta := \delta(v) < 1$, then the full NT-step is strictly feasible and*

$$\delta(v_+) \leq \frac{\delta^2}{\sqrt{2(1 - \delta^2)}}.$$

Proof: Due to (37) and (38) we may write

$$4\delta (v_+)^2 \leq \left\| z^{\frac{1}{2}} - z^{-\frac{1}{2}} \right\|_F^2 = \left\| z^{-\frac{1}{2}} \circ (z - e) \right\|_F^2 \leq \lambda_{\max} \left(z^{-1} \right) \|z - e\|_F^2$$

The last inequality is due to Lemma 2.13 (iii). Now using that $z = e + d_x \circ d_s$ we get

$$4\delta (v_+)^2 \leq \frac{\|z - e\|_F^2}{\lambda_{\min}(z)} = \frac{\|d_x \circ d_s\|_F^2}{\lambda_{\min}(e + d_x \circ d_s)}.$$

Yet we apply Lemma A.3. Part (i) of Lemma A.3 implies that $1 - \delta^2$ is a lower bound for the eigenvalues of $e + d_x \circ d_s$, as we already established in the proof of Lemma 3.1. Also using part (ii) of Lemma A.3 we may now write

$$4\delta (v_+)^2 \leq \frac{\|d_x + d_s\|_F^4}{8(1 - \delta^2)} = \frac{16\delta^4}{8(1 - \delta^2)} = \frac{2\delta^4}{1 - \delta^2},$$

which implies the lemma. □

Remark 3.6 *It is possible to get a tight upper bound for $\delta(v_+)$. Recall that d_x and d_s are orthogonal. Hence $\mathbf{tr}(d_x \circ d_s) = 0$. Therefore we have*

$$\mathbf{tr}(w) = \mathbf{tr}(e) = 2N, \quad \mathbf{tr}(w^{-1}) = \sum_{j=1}^{N} \left(\frac{1}{1 + \lambda_{\max}\left(d_x^j \circ d_s^j\right)} + \frac{1}{1 + \lambda_{\min}\left(d_x^j \circ d_s^j\right)} \right).$$

Substitution into (37) yields

$$4\delta\left(v_{+}\right)^{2} \leq \sum_{i=1}^{2N}\left(\frac{1}{1+\lambda_{i}\left(d_{x}\circ d_{s}\right)}-1\right)=\sum_{i=1}^{2N}\frac{-\lambda_{i}\left(d_{x}\circ d_{s}\right)}{1+\lambda_{i}\left(d_{x}\circ d_{s}\right)},$$

where $\lambda_i\left(d_x \circ d_s\right)$ runs through all the eigenvalues of $d_x \circ d_s$. To simplify the notation we denote $\lambda_i\left(d_x \circ d_s\right)$ simply as λ_i. Then we may write

$$4\delta\left(v_{+}\right)^{2} \leq \sum_{\lambda_i>0}\frac{-\lambda_i}{1+\lambda_i}+\sum_{\lambda_i<0}\frac{-\lambda_i}{1+\lambda_i}. \tag{39}$$

We define

$$I_{+}:=\{i\ :\ \lambda_i>0\}, \quad I_{-}:=\{i\ :\ \lambda_i<0\}.$$

Then, since $\mathbf{tr}(d_x \circ d_s)=0$,

$$\sum_{i\in I_{+}}\lambda_i=-\sum_{i\in I_{-}}\lambda_i.$$

Let $\sigma \geq 0$ denote the value of the first sum. Since $\frac{-\lambda_i}{1+\lambda_i}$ is convex in λ_i, and vanishes if $\lambda_i=0$, we may apply Corollary A.2, which gives

$$\sum_{\lambda_i>0}\frac{-\lambda_i}{1+\lambda_i}\leq\frac{-\sigma}{1+\sigma}.$$

The second sum in (39) can be majorized in a similar way. We just write $z_i=-\lambda_i$ for $i\in I_{-}$. Since $\frac{z_i}{1-z_i}$ is convex in z_i, and vanishes if $z_i=0$ we obtain from Corollary A.2 that

$$\sum_{\lambda_i<0}\frac{-\lambda_i}{1+\lambda_i}=\sum_{z_i>0}\frac{z_i}{1-z_i}\leq\frac{\sigma}{1-\sigma}.$$

Substituting the above bounds in (39) we obtain

$$4\delta\left(v_{+}\right)^{2}\leq\frac{-\sigma}{1+\sigma}+\frac{\sigma}{1-\sigma}=\frac{2\sigma^{2}}{1-\sigma^{2}},$$

which implies that

$$\delta\left(v_{+}\right)\leq\frac{\sigma}{\sqrt{2(1-\sigma^{2})}}.$$

The last expression is monotonically increasing in σ. Hence we may replace it by an upper bound. By using $a+b\leq\sqrt{2(a^2+b^2)}$ we get

$$\begin{aligned}
\sigma &= \frac{1}{2}\sum_{i=1}^{2N}\left|\lambda_i\left(d_x\circ d_s\right)\right| = \frac{1}{2}\sum_{j=1}^{N}\left(\left|\lambda_{\max}\left(d_x^j\circ d_s^j\right)\right|+\left|\lambda_{\min}\left(d_x^j\circ d_s^j\right)\right|\right)\\
&\leq \frac{1}{\sqrt{2}}\sum_{j=1}^{N}\sqrt{\lambda_{\max}\left(d_x^j\circ d_s^j\right)^2+\lambda_{\min}\left(d_x^j\circ d_s^j\right)^2} = \frac{1}{\sqrt{2}}\sum_{j=1}^{N}\left\|d_x^j\circ d_s^j\right\|_F\\
&\leq \frac{1}{\sqrt{2}}\sum_{j=1}^{N}\left\|d_x^j\right\|_F\left\|d_s^j\right\|_F \leq \frac{1}{2\sqrt{2}}\sum_{j=1}^{N}\left(\left\|d_x^j\right\|_F^2+\left\|d_s^j\right\|_F^2\right)\\
&= \frac{1}{2\sqrt{2}}\left(\left\|d_x\right\|_F^2+\left\|d_s\right\|_F^2\right) = \frac{1}{2\sqrt{2}}\left\|d_x+d_s\right\|_F^2 = \sqrt{2}\,\delta(v)^2.
\end{aligned}$$

Substitution of this bound for σ yields that

$$\delta\left(v_{+}\right) \leq \frac{\sqrt{2}\,\delta(v)^2}{\sqrt{2(1-2\sigma^4)}}.$$

Can this be done better?

Corollary 3.7 *If $\delta(v) \leq \frac{1}{\sqrt{2}}$ then $\delta(v_{+}) \leq \delta(v)^2$. In other words, if $\delta(v) \leq \frac{1}{\sqrt{2}}$ then the NT-process converges quadratically fast to the μ-center.*

3.5.3 Updating the barrier parameter μ

In this section we establish a simple relation for our proximity measure just before and after a μ-update.

Lemma 3.8 *Let $(x, s) \in \mathcal{K}_{+}$, $x^T s = N\mu$, and $\delta = \delta(x, s; \mu)$. If $\mu^{+} = (1-\theta)\mu$ for some $0 < \theta < 1$, then*

$$\delta\left(x,\, s\,;\, \mu^{+}\right)^2 = \frac{\theta^2 N}{2(1-\theta)} + (1-\theta)\,\delta^2.$$

Proof: When updating μ to μ^{+} the vector v is divided by the factor $\sqrt{1-\theta}$. Hence we may write

$$4\delta\left(x,\, s\,;\, \mu^{+}\right)^2 = \left\|\sqrt{1-\theta}v^{-1} - \frac{v}{\sqrt{1-\theta}}\right\|_F^2 = \left\|-\frac{\theta\, v}{\sqrt{1-\theta}} + \sqrt{1-\theta}\left(v^{-1}-v\right)\right\|_F^2.$$

Yet we observe that the vectors v and $v^{-1} - v$ are orthogonal. This is due to $\mathbf{tr}\,(x \circ s) = 2N\mu$, which by (30) implies that $\mathbf{tr}\left(v^2\right) = 2N$. Hence we have

$$\mathbf{tr}\left(v \circ \left(v^{-1}-v\right)\right) = \mathbf{tr}\left(e - v^2\right) = \mathbf{tr}\,(e) - \mathbf{tr}\left(v^2\right) = 2N - 2N = 0.$$

Therefore we may proceed as follows:

$$4\delta\left(x,\, s\,;\, \mu^{+}\right)^2 = \frac{\theta^2}{1-\theta}\left\|v\right\|_F^2 + (1-\theta)\left\|v^{-1}-v\right\|_F^2 = \frac{2\theta^2 N}{1-\theta} + 4(1-\theta)\delta^2.$$

This implies the lemma. □

3.6 Iteration bound

We conclude this section with a theorem that gives us the complexity of the algorithm in Figure 1. Because the quadratic convergence lemma (i.e., Lemma 3.5) and, when we replace $2N$ by n, the lemma describing the effect of a barrier parameter update (i.e., Lemma 3.8) are exactly the same as in [26] (cf. [26, Lemma 2.2] and [26, Lemma 2.3]), and also after a full NT-step the target value of the duality gap is attained, we can use the same arguments as in [26] to prove the following result.

Theorem 3.9 *If $\theta = \frac{1}{2\sqrt{N}}$, then the number of iterations of the feasible primal-dual path-following algorithm with full NT-steps does not exceed*

$$2\sqrt{N}\log\frac{N\mu^0}{\varepsilon}.$$

4 An infeasible full NT-step algorithm

In this section we present our infeasible interior-point algorithm. As has become usual for infeasible IPMs we start the algorithm with a triple (x^0, y^0, s^0) and $\mu^0 > 0$ such that

$$x^0 = \zeta e, \quad y^0 = 0, \quad s^0 = \zeta e, \quad \mu^0 = \zeta^2, \tag{40}$$

where ζ is a (positive) number such that

$$x^* + s^* \preceq_{\mathcal{K}} \zeta e, \tag{41}$$

for some optimal solutions (x^*, y^*, s^*) of (P) and (D). The algorithm generates an ε-solution of (P) and (D), or it establishes that there do not exist optimal solutions satisfying (41).
The initial values of the primal and dual residual vectors are denoted as r_b^0 and r_c^0, respectively. So we have

$$r_b^0 := b - Ax^0, \tag{42}$$

$$r_c^0 := c - A^T y^0 - s^0. \tag{43}$$

In general we have $r_b^0 \neq 0$ and $r_c^0 \neq 0$. In other words, the initial iterates are not feasible. The iterates generated by the algorithm will (in general) be infeasible for (P) and (D) as well, but they will be feasible for perturbed versions of (P) and (D) that we introduce in the next subsection.

4.1 Perturbed problems

For any ν with $0 < \nu \leq 1$ we consider the perturbed problem (P_ν), defined by

$$(P_\nu) \quad \min \left\{ \left(c - \nu r_c^0\right)^T x \;:\; b - Ax = \nu r_b^0,\, x \in \mathcal{K} \right\},$$

and its dual problem (D_ν), which is given by

$$(D_\nu) \quad \max \left\{ \left(b - \nu r_b^0\right)^T y \;:\; c - A^T y - s = \nu r_c^0,\, s \in \mathcal{K} \right\}.$$

Note that these problems are defined in such a way that if (x, y, s) is feasible for (P_ν) and (D_ν), then $x \in \mathcal{K}$ and $s \in \mathcal{K}$ and

$$r_b := b - Ax = \nu r_b^0,$$

$$r_c := c - A^T y - s = \nu r_c^0.$$

In other words, the residual vectors for the given triple (x, y, s) with respect to the original problems (P) and (D) are νr_b^0 and νr_c^0, respectively.
If $\nu = 1$ then $x = x^0$ yields a strictly feasible solution of (P_ν), and $(y, s) = (y^0, s^0)$ a strictly feasible solution of (D_ν). This means that if $\nu = 1$ then (P_ν) and (D_ν) satisfy the IPC.

Lemma 4.1 *Let (P) and (D) be feasible and $0 < \nu \leq 1$. Then the perturbed problems (P_ν) and (D_ν) satisfy the IPC.*

Proof: Let $\bar{x}$ be a feasible solution of (P) and $(\bar{y}, \bar{s})$ a feasible solution of (D). Then $A\bar{x} = b$ and $A^T\bar{y} + \bar{s} = c$, with $\bar{x} \in \mathcal{K}$ and $\bar{s} \in \mathcal{K}$. Consider

$$x = (1-\nu)\,\bar{x} + \nu\, x^0, \quad y = (1-\nu)\,\bar{y} + \nu\, y^0, \quad s = (1-\nu)\,\bar{s} + \nu\, s^0.$$

Since x is the sum of the vectors $(1-\nu)\bar{x} \in \mathcal{K}$ and $\nu x^0 \in \mathcal{K}_+$ we have $x \in \mathcal{K}_+$. Moreover

$$b - Ax = b - A\left[(1-\nu)\,\bar{x} + \nu\, x^0\right] = b - (1-\nu)\,b - \nu A x^0 = \nu(b - Ax^0) = \nu r_b^0,$$

showing that x is strictly feasible for (P_ν). In precisely the same way one shows that (y, s) is strictly feasible for (D_ν). Thus we have shown that (P_ν) and (D_ν) satisfy the IPC. □

It should be mentioned that the problems (P_ν) and (D_ν) have been studied first in [16], and later also in [6].

4.2 The central path of the perturbed problems

Let (P) and (D) be feasible and $0 < \nu \leq 1$. Then Lemma 4.1 implies that the problems (P_ν) and (D_ν) satisfy the IPC, and therefore their central paths exist. This means that the system

$$\begin{aligned} b - Ax &= \nu r_b^0, & x &\in \mathcal{K} \quad (44)\\ c - A^T y - s &= \nu r_c^0, & s &\in \mathcal{K} \quad (45)\\ x \circ s &= \mu e. \end{aligned}$$

has a unique solution, for every $\mu > 0$. This unique solution is denoted as $(x(\mu,\, \nu),\, y(\mu,\, \nu),\, s(\mu,\, \nu))$. These are the μ-centers of the perturbed problems (P_ν) and (D_ν). In the sequel the parameters μ and ν will always be in a one-to-one correspondence, according to

$$\mu = \nu\mu^0 = \nu\zeta^2,$$

and, therefore, we feel free to denote $(x(\mu,\, \nu),\, y(\mu,\, \nu),\, s(\mu,\, \nu))$ simply as $(x(\nu),\, y(\nu),\, s(\nu))$.

Due to the choice of the initial iterates, according to (40), we have $x^0 \circ s^0 = \mu^0 e$. Hence x^0 is the μ^0-center of the perturbed problem (P_1) and (y^0, s^0) the μ^0-center of (D_1). In other words, $(x(1),\, y(1),\, s(1)) = (x^0, y^0, s^0)$.

4.3 An iteration of our algorithm

We just established that if $\nu = 1$ and $\mu = \mu^0$, then $x = x^0$ and $(y, s) = \left(y^0, s^0\right)$ are the μ-center of (P_ν) and (D_ν) respectively. These are our initial iterates.

We measure proximity to the μ-center of the perturbed problems by the quantity $\delta\,(x, s; \mu)$ as defined in (31). So, initially we have $\delta\,(x, s; \mu) = 0$. In the sequel we assume that at the start of each iteration, just before the μ-update, $\delta\,(x, s; \mu)$ is smaller than or equal to a (small) threshold value $\tau > 0$. Since we then have $\delta\,(x, s; \mu) = 0$, this condition is certainly satisfied at the start of the first iteration, and also $x^T s = N\mu^0$.

Now we describe one (main) iteration of our algorithm. Suppose that for some $\nu \in (0,\, 1]$ we have x, y and s satisfying the feasibility conditions (44) and (45) for $\mu = \nu\mu^0$, and such that $x^T s = N\mu$ and $\delta\,(x, s; \mu) \leq \tau$. We reduce ν to $\nu^+ = (1-\theta)\,\nu$, with $\theta \in (0,\, 1)$, and find new iterates x_+, y_+

and s_+ that satisfy (44) and (45), with ν replaced by ν^+ and μ by $\mu^+ = \nu^+\mu^0 = (1-\theta)\,\mu$, and such that $x^T s = N\mu^+$ and $\delta\left(x_+, s_+; \mu^+\right) \leq \tau$.

One (main) iteration consists of a feasibility step and a few centering steps. The feasibility step serves to get iterates $(x_f,\, y_f,\, s_f)$ that are strictly feasible for (P_{ν^+}) and (D_{ν^+}), and such that $\delta\left(x_f, s_f; \mu^+\right) \leq 1/\sqrt{2}$. In other words, $(x_f,\, y_f,\, s_f)$ belongs to the region of quadratic convergence of the μ^+-center $\left(x\left(\nu^+\right),\, y\left(\nu^+\right),\, s\left(\nu^+\right)\right)$ of (P_{ν^+}) and (D_{ν^+}). Hence, because the NT-step is quadratically convergent in that region, a few centering steps, starting at $(x_f,\, y_f,\, s_f)$ and targeting at the μ^+-centers of (P_{ν^+}) and (D_{ν^+}) will generate iterates $(x_+,\, y_+,\, s_+)$ that are feasible for (P_{ν^+}) and (D_{ν^+}) and that satisfy $\delta\left(x_+, s_+; \mu^+\right) \leq \tau$. Since each iteration reduces the duality gap $x^T s$ with the factor $1-\theta$, and the size of the residual vectors are reduced with the same factor, given θ we can easily compute the number of main iterations that is necessary to satisfy the stopping criteria in the algorithm. If our aim is to get the duality gap and the norms of the residual vectors less than or equal to some small number ε then this number is given by

$$\frac{1}{\theta}\log\frac{\max\left\{2N\zeta^2,\, \left\|r_b^0\right\|,\, \left\|r_c^0\right\|\right\}}{\varepsilon}. \tag{46}$$

During the centering steps the iterates stay feasible for (P_{ν^+}) and (D_{ν^+}). So from Section 3 we precisely know how to analyze these steps. If $\delta\left(x_f, s_f; \mu^+\right) \leq 1/\sqrt{2}$ then by Corollary 3.7, after k centering steps we will have iterates $(x_+,\, y_+,\, s_+)$ that are still feasible for (P_{ν^+}) and (D_{ν^+}) and such that

$$\delta\left(x_+,\, s_+,\, \mu^+\right) \leq \left(\frac{1}{\sqrt{2}}\right)^{2^k}.$$

From this one easily deduces that $\delta\left(x_+,\, s_+\,;\, \mu^+\right) \leq \tau$ will hold after at most

$$\log_2\left(\log_2\frac{1}{\tau^2}\right) \tag{47}$$

centering steps.

It follows from this that we only need to define and analyze the feasibility step. This is the most difficult part of the analysis. In essence we follow the same chain of arguments as in [26], but at several places the analysis is different and more elegant.

In the rest of this section we describe the feasibility step in detail. The analysis will follow in subsequent sections. Suppose we have strictly feasible iterates (x, y, s) for (P_ν) and (D_ν). This means that (x, y, s) satisfies (44) and (45), with $\mu = \nu\zeta^2$. We need displacements $\Delta^f x$, $\Delta^f y$ and $\Delta^f s$ such that such that

$$\begin{aligned} x_f &:= x + \Delta^f x,\\ y_f &:= y + \Delta^f y,\\ s_f &:= s + \Delta^f s, \end{aligned} \tag{48}$$

are feasible for (P_{ν^+}) and (D_{ν^+}). One may easily verify that (x_f, y_f, s_f) satisfies (44) and (45), with ν replaced by ν^+ and μ by $\mu^+ = \nu^+\mu^0 = (1-\theta)\,\mu$, only if the first two equations in the following system are satisfied.

$$\begin{aligned} A\Delta^f x &= \theta\nu r_b^0,\\ A^T\Delta^f y + \Delta^f s &= \theta\nu r_c^0\\ P(u)x \circ P(u^{-1})\Delta^f s + P(u^{-1})s \circ P(u)\Delta^f x &= (1-\theta)\mu e - P(u)x \circ P(u^{-1})s, \end{aligned} \tag{49}$$

The third equation is inspired by the third equation in the system (21) that we used to define search directions for the feasible case, except that we target at the μ^+-centers of (P_{ν^+}) and (D_{ν^+}). As in the feasible case, we use the NT-scaling scheme to guarantee that the above system has a unique solution. So we take $u = w^{-\frac{1}{2}}$, where w is the NT-scaling point of x and s. Then the third equation becomes

$$P(w)^{-\frac{1}{2}}x \circ P(w)^{\frac{1}{2}}\Delta^f s + P(w)^{\frac{1}{2}}s \circ P(w)^{-\frac{1}{2}}\Delta^f x = (1-\theta)\mu e - P(w)^{-\frac{1}{2}}x \circ P(w)^{\frac{1}{2}}s. \tag{50}$$

Due to this choice of u the coefficient matrix of the resulting system is exactly the same as in the feasible case, and hence it defines the feasibility step uniquely.

By its definition, after the feasibility step the iterates satisfy the affine equations in (44) and (45), with $\nu = \nu^+$. The hard part in the analysis will be to guarantee that $x_f,\ s_f \in \mathcal{K}_+$ and to show that the new iterates satisfy $\delta\left(x_f, s_f; \mu^+\right) \le 1/\sqrt{2}$.

4.4 The infeasible algorithm

A formal description of the algorithm is given in Figure 2. Recall that after each iteration the residuals and the duality gap are reduced by the factor $1-\theta$. The algorithm stops if the norms of the residuals and the duality gap are less than the accuracy parameter ε.

4.5 Analysis of the feasibility step

Let x, y and s denote the iterates at the start of an iteration with $x^T s = N\mu$ and $\delta\left(x, s; \mu\right) \le \tau$. Recall that at the start of the first iteration this is certainly true, because ${x^0}^T s^0 = N\mu^0$ and $\delta\left(x^0, s^0; \mu^0\right) = 0$.

We scale the matrix A and the search directions, just as we did in the feasible case (cf. (23)), by defining

$$\bar{A} := \sqrt{\mu} A P\left(w\right)^{\frac{1}{2}}, \quad d_x^f := \frac{P\left(w\right)^{-\frac{1}{2}}\Delta^f x}{\sqrt{\mu}}, \quad d_s^f := \frac{P\left(w\right)^{\frac{1}{2}}\Delta^f s}{\sqrt{\mu}}, \tag{51}$$

with w denoting the scaling point of x and s, as defined in (9). With the vector v as defined before (cf. (22)), the equation (50) can be restated as

$$\mu\, v \circ \left(d_x^f + d_s^f\right) = (1-\theta)\mu e - \mu\, v^2.$$

By multiplying both sides of this equation from left with $\mu^{-1}L(v)^{-1}$ this equation gets the form

$$d_x^f + d_s^f = (1-\theta)v^{-1} - v.$$

Thus we arrive at the following system for the scaled search directions for the feasibility step:

$$\begin{aligned} \bar{A} d_x^f &= \theta\nu r_b^0, \\ \frac{1}{\mu}\bar{A}^T\Delta^f y + d_s^f &= \frac{1}{\sqrt{\mu}}\theta\nu P\left(w\right)^{\frac{1}{2}} r_c^0, \\ d_x^f + d_s^f &= (1-\theta)v^{-1} - v. \end{aligned} \tag{52}$$

Primal-Dual Infeasible IPM

Input:
Accuracy parameter $\varepsilon > 0$;
barrier update parameter θ, $0 < \theta < 1$;
threshold parameter $\tau > 0$;
parameter $\zeta > 0$.
begin
$x^0 := \zeta e,\ s^0 := \zeta e,\ y^0 := 0\,;\ \mu^0 := \zeta^2$;
while $\max\left(x^T s,\ \|r_b\|,\ \|r_c\|\right) \geq \varepsilon$ **do**
begin
feasibility step:
$(x,\ y,\ s) := (x,\ y,\ s) + \left(\Delta^f x,\ \Delta^f y,\ \Delta^f s\right)$;
update of μ and ν:
$\nu := (1-\theta)\nu$;
$\mu := (1-\theta)\mu$;
centering steps:
while $\delta\,(x,\ s,\ \mu) \geq \tau$ **do**
begin
$(x,\ y,\ s) := (x,\ y,\ s) + (\Delta x,\ \Delta y,\ \Delta s)$;
end
end
end

Figure 2: Infeasible full NT-step algorithm

To get the search directions $\Delta^f x$ and $\Delta^f s$ in the x- and s-space we use (51), which gives

$$\Delta^f x = \sqrt{\mu} P\,(w)^{\frac{1}{2}}\, d_x^f, \quad \Delta^f s = \sqrt{\mu} P\,(w)^{-\frac{1}{2}}\, d_s^f. \tag{53}$$

The new iterates are obtained by taking a full step, as given by (48). Hence we have

$$x_f = x + \Delta^f x = \sqrt{\mu} P\,(w)^{\frac{1}{2}}\,(v + d_x^f), \tag{54}$$

$$s_f = s + \Delta^f s = \sqrt{\mu} P\,(w)^{-\frac{1}{2}}\,(v + d_s^f). \tag{55}$$

From the third equation in (52) we derive that

$$(v + d_x^f) \circ (v + d_s^f) = v^2 + v \circ \left[(1-\theta) v^{-1} - v\right] + d_x^f \circ d_s^f = (1-\theta)\, e + d_x^f \circ d_s^f. \tag{56}$$

As we mentioned before the analysis of the algorithm as presented below is much more difficult than in the feasible case. The main reason for this is that the scaled search directions d_x^f and d_s^f are not (necessarily) orthogonal.

4.5.1 Feasibility of the feasibility step

By the same arguments as in Section 3.5.1 it follows from (54) and (55) that x_f and s_f are strictly feasible if and only if $v + d_x^f$ and $v + d_s^f$ belong to $\mathcal{K}_+$. Using this we have the following result.

Lemma 4.2 *The iterates $\left(x^f, y^f, s^f\right)$ are strictly feasible if $(1-\theta)e + d_x^f \circ d_s^f \in \mathcal{K}_+$.*

Proof: Just as in the proof of Lemma 3.2 we introduce a step length α, $0 \leq \alpha \leq 1$, and we define

$$v_x^\alpha = v + \alpha d_x^f, \quad v_s^\alpha = v + \alpha d_s^f.$$

We then have $v_x^0 = v, v_x^1 = v + d_x^f$ and $v_s^0 = v, v_s^1 = v + d_s^f$. Since $d_x^f + d_s^f = (1-\theta)v^{-1} - v$, it follows that

$$\begin{aligned} v_x^\alpha \circ v_s^\alpha &= (v + \alpha d_x^f) \circ (v + \alpha d_s^f) = v^2 + \alpha v \circ (d_x^f + d_s^f) + \alpha^2 d_x^f \circ d_s^f \\ &= v^2 + \alpha v \circ \left[(1-\theta)v^{-1} - v\right] + \alpha^2 d_x \circ d_s = (1-\alpha)v^2 + \alpha(1-\theta)e + \alpha^2 d_x \circ d_s. \end{aligned}$$

The hypothesis in the lemma implies that $d_x^f \circ d_s^f \succ_{\mathcal{K}} -(1-\theta)e$. Substitution gives

$$v_x^\alpha \circ v_s^\alpha \succ_{\mathcal{K}} (1-\alpha)v^2 + \alpha(1-\theta)e - \alpha^2(1-\theta)e = (1-\alpha)\left(v^2 + \alpha(1-\theta)e\right). \tag{57}$$

Since $v^2 \in \mathcal{K}_+$ and $\alpha(1-\theta)e \in \mathcal{K}$, we have $v^2 + \alpha(1-\theta)e \in \mathcal{K}_+$. Hence, if $0 \leq \alpha < 1$, then $(1-\alpha)\left(v^2 + \alpha(1-\theta)e\right) \in \mathcal{K}_+$. Due to (57) this implies that $v_x^\alpha \circ v_s^\alpha \in \mathcal{K}_+$. Therefore, all eigenvalues of $v_x^\alpha \circ v_s^\alpha$ are positive, whence we have $\mathbf{det}\,(v_x^\alpha \circ v_s^\alpha) > 0$, for each $\alpha \in [0,1)$. By Lemma 2.11.(ii) this implies that $\mathbf{det}\,(v_x^\alpha)\,\mathbf{det}\,(v_s^\alpha) > 0$, for each $\alpha \in [0,1)$. It follows that $\mathbf{det}\,(v_x^\alpha)$ and $\mathbf{det}\,(v_s^\alpha)$ do not vanish for $\alpha \in [0,1)$. Since $\mathbf{det}\left(v_x^0\right) = \mathbf{det}\left(v_s^0\right) = \mathbf{det}(v) > 0$, by continuity, $\mathbf{det}\,(v_x^\alpha)$ and $\mathbf{det}\,(v_s^\alpha)$ stay positive for all $\alpha \in [0,1)$. Again by continuity, it follows that $\mathbf{det}\left(v_x^1\right)$ and $\mathbf{det}\left(v_s^1\right)$ are nonnegative. Since (57) also holds if $\alpha = 1$, we have $\mathbf{det}\,(v_x^\alpha)\,\mathbf{det}\,(v_s^\alpha) > 0$ for $\alpha = 1$. Hence it follows that $\mathbf{det}\left(v_x^1\right)$ and $\mathbf{det}\left(v_s^1\right)$ are positive. Since $\mathbf{det}\,(v_x^\alpha)$ and $\mathbf{det}\,(v_s^\alpha)$, do not vanish for all $\alpha \in [0,1]$, it follows that the eigenvalues of v_x^α and v_s^α stay positive for all $\alpha \in [0,1]$. In particular, the eigenvalues of v_x^1 and v_s^1 are positive, which means that $v + d_x^f$ and $v + d_s^f$ belong to $\mathcal{K}_+$. Hence the proof of the lemma is complete. $\square$

It is clear from the above lemma that the feasibility of the iterates $\left(x^f, y^f, s^f\right)$ highly depends on the eigenvalues of the vector $d_x^f \circ d_s^f$.

It will be convenient to denote the $2N$ eigenvalues of a vector $x \in \mathbf{R}^n$ as $\lambda_i(x)$, $1 \leq i \leq 2N$. Then it follows from Lemma 4.2 that $\left(x^f, y^f, s^f\right)$ is strictly feasible if

$$(1-\theta) + \lambda_i(d_x^f \circ d_s^f) > 0, \quad i = 1, \ldots, 2N.$$

We assume below that this inequality holds.

4.5.2 Proximity after the feasibility step

We proceed by deriving an upper bound for $\delta(x^f, s^f; \mu^+)$. Let w_f be the scaling point of x_f and s_f. When denoting the v-vector after the feasibility step, with respect to the μ^+-center, as v_f, according to (22) this vector is given by

$$v_f := \frac{P(w_f)^{-\frac{1}{2}} x_f}{\sqrt{\mu\,(1-\theta)}} \qquad \left[= \frac{P(w_f)^{\frac{1}{2}} s_f}{\sqrt{\mu\,(1-\theta)}}\right].$$

Lemma 4.3 *One has*

$$\sqrt{1-\theta}\, v_f \sim \left[P\left(v+d_x^f\right)^{\frac{1}{2}}\left(v+d_s^f\right)\right]^{\frac{1}{2}}.$$

Proof: It follows from (36) and Lemma 2.10 that

$$\sqrt{\mu\,(1-\theta)}\, v_f = P\,(w_f)^{\frac{1}{2}}\, s_f \sim \left(P\,(x_f)^{\frac{1}{2}}\, s_f\right)^{\frac{1}{2}}.$$

Due to (54), (55) and Lemma 2.9, with $p = w^{\frac{1}{2}}$, we may write

$$P\,(x_f)^{\frac{1}{2}}\, s_f = \mu P\left(P\,(w)^{\frac{1}{2}}\left(v+d_x^f\right)\right)^{\frac{1}{2}} P\,(w)^{-\frac{1}{2}}\left(v+d_s^f\right) \sim \mu P\left(v+d_x^f\right)^{\frac{1}{2}}\left(v+d_s^f\right).$$

Thus we obtain

$$\sqrt{\mu\,(1-\theta)}\, v_f \sim \sqrt{\mu}\left[P\left(v+d_x^f\right)^{\frac{1}{2}}\left(v+d_s^f\right)\right]^{\frac{1}{2}}.$$

From this the lemma follows. □

The above lemma implies that

$$v_f^2 \sim P\left(\frac{v+d_x^f}{\sqrt{1-\theta}}\right)^{\frac{1}{2}}\left(\frac{v+d_s^f}{\sqrt{1-\theta}}\right).$$

In the sequel we denote $\delta(x^f, s^f; \mu^+)$ also shortly by $\delta(v^f)$. By Lemma 2.15 (with $x = \frac{v+d_x^f}{\sqrt{1-\theta}}$, $s = \frac{v+d_s^f}{\sqrt{1-\theta}}$, $u = P(x)^{\frac{1}{2}} s$ and $z = x \circ s$) this implies the inequality below:

$$4\delta\,(v_f)^2 = \left\|v_f - v_f^{-1}\right\|_F^2 = \left\|u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right\|_F^2 \le \left\|z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right\|_F^2.$$

Since $d_x + d_s = (1-\theta)v^{-1} - v$, one has

$$\begin{aligned}(1-\theta)z &= (v+d_x)\circ(v+d_s) = v^2 + v\circ(d_x+d_s) + d_x\circ d_s \\ &= v^2 + v\circ((1-\theta)v^{-1} - v) + d_x\circ d_s = (1-\theta)e + d_x\circ d_s.\end{aligned}$$

So we have

$$4\delta\,(v_f)^2 \le \left\|z^{\frac{1}{2}} - z^{-\frac{1}{2}}\right\|_F^2 = \mathbf{tr}(z) + \mathbf{tr}(z^{-1}) - 2\mathbf{tr}(e), \quad z = e + \frac{d_x\circ d_s}{1-\theta}. \tag{58}$$

In what follows we denote the eigenvalues $\lambda_i(d_x\circ d_s)$ of $d_x\circ d_s$ simply as λ_i, $1 \le i \le 2N$, and λ will denote the vector in $\mathbf{R}^{2N}$ with the eigenvalues λ_i as entries (in some arbitrary order).

We can prove the following result. In this result $\|\lambda\|_1$ denotes the 1-norm of λ, i.e., the sum of the absolute values of the eigenvalues λ_i.

Lemma 4.4 *If* $(1-\theta)e + d_x^f \circ d_s^f \in \mathcal{K}_+$, *then*

$$4\delta(v^f)^2 \le f\left(\frac{\|\lambda\|_1}{1-\theta}\right),$$

where

$$f(t) := 1 - t + \frac{1}{1-t} - 2, \quad |t| < 1. \tag{59}$$

Proof: Since the eigenvalues of z are $1+\lambda_i/(1-\theta)$, it follows from (37) that

$$4\delta\left(v_f\right)^2 \leq \sum_{i=1}^{2N}\left(1+\frac{\lambda_i}{1-\theta}+\frac{1}{1+\frac{\lambda_i}{1-\theta}}-2\right)=\sum_{i=1}^{2N} f\left(\frac{-\lambda_i}{1-\theta}\right).$$

One easily verifies that $f(t) \leq f(|t|)$, for all $t \in (-1,1)$. Hence

$$4\delta\left(v_f\right)^2 \leq \sum_{i=1}^{2N} f\left(\frac{|\lambda_i|}{1-\theta}\right).$$

Since the function f is convex and $f(0)=0$ we may apply Corollary A.2, with $z_i=\frac{|\lambda_i|}{1-\theta} \geq 0$, which gives the inequality in the lemma. □

An upper bound for $\|\lambda\|_1$ can be obtained as follows.

$$\begin{aligned}
\|\lambda\|_1 &= \sum_{i=1}^{2N}\left|\lambda_i\left(d_x^f \circ d_s^f\right)\right| = \sum_{j=1}^{N}\left(\left|\lambda_{\max}\left((d_x^f)^j \circ (d_s^f)^j\right)\right| + \left|\lambda_{\min}\left((d_x^f)^j \circ (d_s^f)^j\right)\right|\right)\\
&\leq \sqrt{2}\sum_{j=1}^{N}\sqrt{\lambda_{\max}\left((d_x^f)^j \circ (d_s^f)^j\right)^2+\lambda_{\min}\left((d_x^f)^j \circ (d_s^f)^j\right)^2} = \sqrt{2}\sum_{j=1}^{N}\left\|(d_x^f)^j \circ (d_s^f)^j\right\|_F\\
&\leq \sqrt{2}\sum_{j=1}^{N}\left\|(d_x^f)^j\right\|_F\left\|(d_s^f)^j\right\|_F \leq \frac{1}{\sqrt{2}}\sum_{j=1}^{N}\left(\left\|(d_x^f)^j\right\|_F^2+\left\|(d_s^f)^j\right\|_F^2\right)\\
&= \frac{1}{\sqrt{2}}\left(\left\|d_x^f\right\|_F^2+\left\|d_s^f\right\|_F^2\right).
\end{aligned}$$

In the present case, contrary to the case of a feasible method, the scaled search directions d_x^f and d_s^f are not orthogonal. As has become clear in the case of LO, this fact complicates the analysis drastically [26]. To deal with this complication it will be convenient to define

$$\omega(v) := \frac{1}{2}\sqrt{\left\|d_x^f\right\|_F^2+\left\|d_s^f\right\|_F^2}.$$

Then it follows from Lemma 4.4 that

$$4\delta(v^f)^2 \leq f\left(\frac{2\sqrt{2}\,\omega(v)^2}{1-\theta}\right). \tag{60}$$

Because we need to have $\delta(v^f) \leq 1/\sqrt{2}$, it follows from (60) that it suffices to have

$$f\left(\frac{2\sqrt{2}\,\omega(v)^2}{1-\theta}\right) \leq 2.$$

One may easily verify that $f(t) \leq 2$ holds if $0 \leq t \leq \sqrt{3}-1 \approx 0.732051$. Hence we should find θ such that it is positive (and as large as possible) and such that it satisfies

$$\theta \leq 1-\frac{4\sqrt{2}\omega(v)^2}{\sqrt{3}-1} \approx 1-3.8637\omega(v)^2,$$

which certainly holds if

$$\theta \leq 1 - 4\omega(v)^2. \tag{61}$$

It should be noted that by its definition $\omega(v)$ depends on d_x^f and d_s^f, and hence on θ itself. In the next section we investigate this dependence.

4.5.3 Upper bound for $\omega(v)$

Recall that the scaled search directions d_x^f and d_s^f are determined by the system (52). Let us to define the linear space $\mathcal{S}$ as follows:

$$\mathcal{S} := \left\{\xi \in \mathbf{R}^n \;:\; \bar{A}\xi = 0\right\}.$$

It is clear from the first equation in (52) that the affine space

$$\left\{\xi \in \mathbf{R}^n \;:\; \bar{A}\xi = \theta\nu r_b^0\right\},$$

equals $d_x^f + \mathcal{S}$. Moreover, from linear algebra we know that the orthogonal complement of the linear space $\mathcal{S}$ is the row space of $\bar{A}$, i,e,

$$\mathcal{S}^\perp = \left\{\bar{A}^T\vartheta \;:\; \vartheta \in R^m\right\}.$$

From the second equation in (52), it is clear that the affine space

$$\left\{\frac{1}{\sqrt{\mu}}\theta\nu P\left(w\right)^{\frac{1}{2}} r_c^0 + \bar{A}^T\vartheta \;:\; \vartheta \in \mathbf{R}^m\right\}$$

equals $d_s^f + \mathcal{S}^\perp$. Since $\mathcal{S} \cap \mathcal{S}^\perp = \{0\}$, the spaces $d_x^f + \mathcal{S}$ and $d_s^f + \mathcal{S}^\perp$ meet in a unique point. We call this point q. So q is uniquely determined by the system

$$\bar{A}\,q = \theta\nu r_b^0, \tag{62}$$

$$\bar{A}^T\vartheta + q = \frac{1}{\sqrt{\mu}}\theta\nu P\left(w\right)^{\frac{1}{2}} r_c^0. \tag{63}$$

Lemma 4.5 *One has*

$$4\,\omega(v)^2 \leq \|q\|_F^2 + \left(\|q\|_F + \sqrt{4(1-\theta)^2\delta(v)^2 + 2\theta^2 N}\right)^2.$$

Proof: To simplify the notation in this proof we denote $r = (1-\theta)v^{-1} - v$. Using exactly the same arguments as in the proof of Lemma 4.6 in [26] one shows that

$$4\omega(v)^2 = \left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2 = \|q - r\|_F^2 + \|q\|_F^2. \tag{64}$$

From this moment on the proof differs, because in [26] we had $r = v^{-1} - v$. We may proceed as follows. One easily sees that $q = 0$ only if d_x^f and d_s^f are orthogonal, and then the lemma is trivial because then $\|d_x^f\|_F^2 + \|d_s^f\|_F^2 = \|d_x^f + d_s^f\|_F^2 = \|r\|_F^2$. Therefore, we may assume that $q \neq 0$. The right-hand side in (64) is maximal if $-r$ is a nonnegative multiple of q, i.e., if $r = -\|r\|_F\, q / \|q\|_F$. Thus we derive from (64) that

$$4\omega(v)^2 \leq \left\|\left(1 + \frac{\|r\|_F}{\|q\|_F}\right) q\right\|^2 + \|q\|_F^2 = \|q\|_F^2 + \left(\|q\|_F + \|r\|_F\right)^2. \tag{65}$$

Recall that v is the v-vector of vectors x and s that are feasible for (P_ν) and (D_ν). These vectors are obtained after a full-NT step for a feasible problem, whence $\mu\left\|v\right\|_F^2 = 2x^Ts = 2N\mu$. The latter means that v is orthogonal to $v - v^{-1}$. So we may write

$$\|r\|_F^2 = \left\|(1-\theta)v^{-1} - v\right\|_F^2 = \left\|(1-\theta)(v^{-1}-v) - \theta v\right\|_F^2 = (1-\theta)^2\left\|v^{-1}-v\right\|_F^2 + \theta^2\left\|v\right\|_F^2.$$

Since $\left\|v^{-1}-v\right\|_F^2 = 4\delta(v)^2$ and $\|v\|_F^2 = 2N$, we obtain

$$\|r\|_F^2 = 4(1-\theta)^2\delta(v)^2 + 2\theta^2 N.$$

Substitution into (65) yields the lemma. □

4.5.4 Upper bound for $\|q\|$

Recall that the vector q is determined by the equations (66) and (67), where $\bar{A} = \sqrt{\mu}AP(w)^{\frac{1}{2}}$, with w denoting the scaling point of x and s, as defined in (9). So we have

$$\sqrt{\mu}AP(w)^{\frac{1}{2}}\, q = \theta\nu r_b^0, \tag{66}$$

$$\sqrt{\mu}P(w)^{\frac{1}{2}}\, A^T\vartheta + q = \frac{1}{\sqrt{\mu}}\theta\nu P(w)^{\frac{1}{2}}\, r_c^0. \tag{67}$$

We proceed by proving the following upper bound for $\|q\|_F$.

Lemma 4.6 *If $\left(x^0, y^0, s^0\right)$, (x^*, y^*, s^*) and ζ are as defined in (40) and (41) then*

$$\|q\|_F \le \theta\sqrt{\nu\,\mathbf{tr}\left(w^2 + w^{-2}\right)}. \tag{68}$$

Proof: To keep de notation simple we use the notations

$$D := P(w)^{\frac{1}{2}}, \qquad r_b := \theta\nu r_b^0, \qquad r_c := \theta\nu r_c^0.$$

Then we have

$$\sqrt{\mu}AD\,q = r_b,$$

$$\sqrt{\mu}\,DA^T\xi + q = \frac{1}{\sqrt{\mu}}Dr_c.$$

Exactly the same system occurs in [26, Section 4.4]. There it has been shown (cf. [26, eqn. (4.14)]) that it implies the following inequality:

$$\sqrt{\mu}\,\|q\|_F \le \theta\nu\sqrt{\left\|D\left(s^0-s^*\right)\right\|_F^2 + \left\|D^{-1}\left(x^0-x^*\right)\right\|_F^2}, \tag{69}$$

where we used $\|.\|_F = \sqrt{2}\,\|.\|$. Since x^* is feasible for (P) we have $x^* \succeq_{\mathcal{K}} 0$. Also $s^* \in \mathcal{K}_+$. Hence we have $0 \preceq_{\mathcal{K}} x^* \preceq_{\mathcal{K}} x^* + s^* \preceq_{\mathcal{K}} \zeta e$. In a similar we derive for $s^* \succeq_{\mathcal{K}} 0$ that $0 \preceq_{\mathcal{K}} s^* \preceq_{\mathcal{K}} \zeta e$. It therefore follows that

$$0 \preceq_{\mathcal{K}} x^0 - x^* \preceq_{\mathcal{K}} \zeta e, \quad 0 \preceq_{\mathcal{K}} s^0 - s^* \preceq_{\mathcal{K}} \zeta e. \tag{70}$$

We first consider the term $\left\|D\left(s^0-s^*\right)\right\|^2$. Using that D is self-adjoint with respect to the inner product $\langle .,. \rangle$ and $D^2e = P\left(w\right)e = w^2$, we may write

$$\begin{aligned}\left\|D\left(s^0-s^*\right)\right\|_F^2 &= \left\langle D\left(s^0-s^*\right), D\left(s^0-s^*\right)\right\rangle = \left\langle D^2\left(s^0-s^*\right), s^0-s^*\right\rangle \\ &= \left\langle D^2\left(s^0-s^*\right), \zeta e\right\rangle - \left\langle D^2\left(s^0-s^*\right), \zeta e-\left(s^0-s^*\right)\right\rangle \\ &\le \left\langle D^2\left(s^0-s^*\right), \zeta e\right\rangle = \left\langle s^0-s^*, D^2\zeta e\right\rangle = \zeta\left\langle s^0-s^*, w^2\right\rangle \\ &= \zeta\left\langle \zeta e, w^2\right\rangle - \zeta\left\langle \zeta e-(s^0-s^*), w^2\right\rangle \\ &\le \zeta\left\langle \zeta e, w^2\right\rangle = \zeta^2\left\langle e, w^2\right\rangle = \zeta^2\mathbf{tr}\left(w^2\right).\end{aligned}$$

In the same way it follows that

$$\left\|D^{-1}\left(x^0-x^*\right)\right\|_F^2 \le \zeta^2\mathbf{tr}\left(w^{-2}\right).$$

Substitution of the last two inequalities into (69) gives

$$\sqrt{\mu}\left\|q\right\|_F \le \theta\nu\sqrt{\zeta^2\mathbf{tr}\left(w^2\right)+\zeta^2\mathbf{tr}\left(w^{-2}\right)} = \theta\nu\zeta\sqrt{\mathbf{tr}\left(w^2+w^{-2}\right)}.$$

Finally, by using $\mu = \nu\mu^0 = \nu\zeta^2$ the inequality in the lemma follows. □

Our next task is to find an upper bound for $\mathbf{tr}\left(w^2+w^{-2}\right)$. Before doing this we recall the following relations:

$$P(s^{\frac{1}{2}})x \sim P(x^{\frac{1}{2}})s \sim \left(P\left(w\right)^{\frac{1}{2}}s\right)^2 \sim \left(P\left(w\right)^{-\frac{1}{2}}x\right)^2 = \mu v^2; \tag{71}$$

where the similarities are due to Proposition 2.8 (iv) and Lemma 2.10, and the equality to (22). We now can prove the following result.

Lemma 4.7 *Let $x, s \in \mathcal{K}$ and w the scaling point of x and s. Then*

$$\left\|q\right\|_F \le \theta\frac{\mathbf{tr}\left(x+s\right)}{\zeta\lambda_{\min}\left(v\right)}. \tag{72}$$

Proof: For the moment, let $u := (P(x^{\frac{1}{2}})s)^{-\frac{1}{2}}$. Then, by (9), $w = P(x^{\frac{1}{2}})u$. Using that $P(x^{\frac{1}{2}})$ is self-adjoint, and also Lemma 2.12, we obtain

$$\mathbf{tr}\left(w^2\right) = \left\langle P(x^{\frac{1}{2}})u,\, P(x^{\frac{1}{2}})u\right\rangle = \langle u,\, P(x)u\rangle \le \lambda_{\max}(u)\mathbf{tr}(P(x)u).$$

By using the same arguments and also $P(x)e = x^2$ we may write

$$\mathbf{tr}(P(x)u) = \mathbf{tr}(P(x)u\circ e) = \langle P(x)u, e\rangle = \langle u, P(x)e\rangle = \left\langle u, x^2\right\rangle \le \lambda_{\max}(u)\mathbf{tr}(x^2),$$

where the last inequality follows from Lemma 2.12. Combining the above inequalities we obtain

$$\mathbf{tr}\left(w^2\right) \le \lambda_{\max}\left(P(x^{\frac{1}{2}})s\right)^{-1}\mathbf{tr}\left(x^2\right).$$

Due to (71) we have

$$\lambda_{\max}\left(P(x^{\frac{1}{2}})s\right)^{-1} = \frac{1}{\lambda_{\min}\left(P(x^{\frac{1}{2}})s\right)} = \frac{1}{\mu\lambda_{\min}\left(v\right)^2}.$$

Thus we obtain

$$\mathbf{tr}\left(w^2\right) \le \frac{\mathbf{tr}\left(x^2\right)}{\mu\lambda_{\min}\left(v\right)^2}.$$

By noting that w^{-1} is the scaling element of s and x, it follows from the above inequality, by interchanging the role of x and s, that

$$\mathbf{tr}\left(w^{-2}\right) \le \frac{\mathbf{tr}\left(s^2\right)}{\mu\lambda_{\min}\left(v\right)^2}.$$

By adding the last two inequalities we obtain

$$\mathbf{tr}\left(w^2 + w^{-2}\right) \le \frac{\mathbf{tr}\left(x^2\right) + \mathbf{tr}\left(s^2\right)}{\mu\lambda_{\min}\left(v\right)^2}. \tag{73}$$

Since $x,\, s \in \mathcal{K}$, we have $\mathbf{tr}\left(x \circ s\right) \ge 0$. Hence, also using that $\mathbf{tr}(z^2) \le \mathbf{tr}(z)^2$ for each $z \in \mathcal{K}$,

$$\mathbf{tr}\left(x^2\right) + \mathbf{tr}\left(s^2\right) \le \mathbf{tr}\left(x^2\right) + \mathbf{tr}\left(s^2\right) + 2\mathbf{tr}\left(x \circ s\right) = \mathbf{tr}\left(\left(x+s\right)^2\right) \le \mathbf{tr}\left(x+s\right)^2. \tag{74}$$

Substituting (73) and (74) into (75), also using that $\mu = \nu\zeta^2$, yields

$$\|q\|_F \le \theta\sqrt{\nu\frac{\mathbf{tr}\left(x^2\right) + \mathbf{tr}\left(s^2\right)}{\mu\lambda_{\min}\left(v\right)^2}} \le \theta\sqrt{\frac{\mathbf{tr}\left(x+s\right)^2}{\zeta^2\lambda_{\min}\left(v\right)^2}} = \theta\frac{\mathbf{tr}\left(x+s\right)}{\zeta\lambda_{\min}\left(v\right)}, \tag{75}$$

which completes the proof. $\square$

4.5.5 Upper bound for $\mathbf{tr}\left(x+s\right)$.

In this section we compute an upper bound for $\mathbf{tr}\left(x+s\right)$.

Lemma 4.8 *Let x and $(y,\, s)$ be feasible for the perturbed problems (P_ν) and (D_ν) respectively and $\left(x^0,\, y^0,\, s^0\right)$ as defined in (40) and ζ as in (41). We then have*

$$\nu\zeta\mathbf{tr}\left(x+s\right) \le \mathbf{tr}\left(x \circ s\right) + 2N\mu.$$

Proof: Let $(x^*,\, y^*,\, s^*)$ be optimal solutions satisfying (41). We define

$$\begin{aligned} x' &= x - \nu x^0 - \left(1-\nu\right)x^*, \\ y' &= y - \nu y^0 - \left(1-\nu\right)y^*, \\ s' &= s - \nu s^0 - \left(1-\nu\right)s^*. \end{aligned}$$

From the feasibility conditions (44) and (45) of the perturbed problems (P_ν) and (D_ν), it is easily seen that $(x',\, y',\, s')$ satisfies

$$\begin{aligned} Ax' &= 0, \\ A^T y' + s' &= 0. \end{aligned}$$

This implies that x' and s' are orthogonal, i.e.,

$$\mathbf{tr}\left(\left(x-\nu x^0-(1-\nu)\,x^*\right)\circ\left(s-\nu s^0-(1-\nu)\,s^*\right)\right)=0.$$

By expanding the last equality and using the fact that $\mathbf{tr}\,(x^*\circ s^*)=0$, since the triple $(x^*,\,y^*,\,s^*)$ is optimal, we obtain

$$\begin{aligned}\nu\,\mathbf{tr}\big(s^0\circ x+x^0\circ s\big)=\;&\mathbf{tr}\,(s\circ x)+\nu^2\mathbf{tr}\left(s^0\circ x^0\right)\\&+\nu\,(1-\nu)\,\mathbf{tr}\left(s^0\circ x^*+x^0\circ s^*\right)-(1-\nu)\,\mathbf{tr}\,(s\circ x^*+s^*\circ x)\,.\end{aligned}$$

Since $\left(x^0,\,y^0,\,s^0\right)$ are as in (40) we have

$$\mathbf{tr}\left(s^0\circ x+x^0\circ s\right)=\zeta\mathbf{tr}\,(x+s)\,,\quad \mathbf{tr}\left(s^0\circ x^0\right)=2N\zeta^2,\quad \mathbf{tr}\left(s^0\circ x^*+x^0\circ s^*\right)=\zeta\mathbf{tr}\,(x^*+s^*)\,.$$

Due to (41) we have $\mathbf{tr}\,(x^*+s^*)\leq\zeta\mathbf{tr}(e)=2N\zeta$. Substitution gives

$$\begin{aligned}\nu\,\zeta\mathbf{tr}\,(x+s)&=\mathbf{tr}\,(s\circ x)+2\nu^2N\zeta^2+\nu\,(1-\nu)\,\zeta\mathbf{tr}\,(x^*+s^*)-(1-\nu)\,\mathbf{tr}\,(s\circ x^*+s^*\circ x)\\&\leq\mathbf{tr}\,(s\circ x)+2\nu^2N\zeta^2+2\nu\,(1-\nu)\,N\zeta^2-(1-\nu)\,\mathbf{tr}\,(s\circ x^*+s^*\circ x)\\&=\mathbf{tr}\,(s\circ x)+2\nu N\zeta^2-(1-\nu)\,\mathbf{tr}\,(s\circ x^*+s^*\circ x)\\&\leq\mathbf{tr}\,(s\circ x)+2\nu N\zeta^2,\end{aligned}$$

where the last inequality is due to the fact that x, s, x^* and s^* belong to $\mathcal{K}$, which implies

$$\mathbf{tr}\,(s\circ x^*+s^*\circ x)=2\left(s^Tx^*+x^Ts^*\right)\geq 0.$$

Since $\nu\zeta^2=\mu$, this completes the proof. $\Box$

Lemma 4.9 *Let $\delta=\delta\,(v)$ be given by (31). Then for each $j\in\{1,\,\ldots,\,N\}$,*

$$\frac{1}{\rho\,(\delta)}\leq\lambda_{\min}\left(v^j\right)\leq\lambda_{\max}\left(v^j\right)\leq\rho\,(\delta)\,,$$

where

$$\rho\,(\delta):=\delta+\sqrt{1+\delta^2}. \tag{76}$$

Proof: Using (33), the proof is easy and similar to the proof of Lemma II.60 in [27]. $\Box$

Lemma 4.10 *With the same notations as in Lemma 4.8 and $p\,(\delta)$ as defined in (76), one has*

$$\mathbf{tr}\,(x+s)\leq\left(\rho(\delta)^2+1\right)2N\zeta.$$

Proof:

$$\nu\zeta\mathbf{tr}\,(x+s)\leq\mathbf{tr}\,(x\circ s)+2\mu N.$$

Dividing both sides of the inequality in Lemma 4.8 by $\nu\zeta$, while using that $\mu=\nu\zeta^2$, we get

$$\mathbf{tr}\,(x+s)\leq\mathbf{tr}\left(\frac{x\circ s}{\mu}\right)\zeta+2N\zeta.$$

Hence it suffices for the proof if we show that

$$\mathbf{tr}\left(\frac{x \circ s}{\mu}\right) \le 2N\rho(\delta)^2.$$

Since we have $\mathbf{tr}\left(\frac{x \circ s}{\mu}\right) = \mathbf{tr}\left(v^2\right) = \|v\|_F^2$, by (30), because of (16) the last inequality can be written as

$$\sum_{j=1}^{N} \left(\lambda_{\min}(v^j)^2 + \lambda_{\max}(v^j)^2\right) \le 2N\rho(\delta)^2.$$

This inequality is an immediate consequence of Lemma 4.9, whence the lemma follows. □

4.5.6 Putting all things together

We proved in Section 4.5.2 (cf. eqn. (61)) that in order to have $\delta(v^f) \le \frac{1}{\sqrt{2}}$ one should have

$$\theta \le 1 - 4\omega(v)^2. \tag{77}$$

Then, in Lemma 4.5 (Section 4.5.3) we showed that

$$4\,\omega(v)^2 \le \|q\|_F^2 + \left(\|q\|_F + \sqrt{4(1-\theta)^2\delta(v)^2 + 2\theta^2 N}\right)^2. \tag{78}$$

We may restate (77) as $4\omega(v)^2 \le 1 - \theta$. Due to (78) this holds if

$$\|q\|_F^2 + \left(\|q\|_F + \sqrt{4(1-\theta)^2\delta(v)^2 + 2\theta^2 N}\right)^2 \le 1 - \theta. \tag{79}$$

We also proved, in Lemma 4.7 (Section 4.5.4), that

$$\|q\|_F \le \theta \frac{\mathbf{tr}\,(x+s)}{\zeta \lambda_{\min}(v)}. \tag{80}$$

Furthermore, by Lemma 4.10 (Section 4.5.5)

$$\mathbf{tr}\,(x+s) \le \left(\rho(\delta)^2 + 1\right) 2N\zeta. \tag{81}$$

From (80) and (81) it follows that

$$\|q\|_F \le \theta \frac{\left(\rho(\delta)^2 + 1\right) 2N\zeta}{\zeta \lambda_{\min}(v)} = \theta \frac{\left(\rho(\delta)^2 + 1\right) 2N}{\lambda_{\min}(v)}.$$

Since $\lambda_{\min}(v) \ge \frac{1}{\rho(\delta)}$, by Lemma 4.9, we get

$$\|q\|_F \le \theta\rho(\delta)\left(\rho(\delta)^2 + 1\right) 2N.$$

Substitution into (79) yields the following condition:

$$\left[\theta\rho(\delta)\left(\rho(\delta)^2+1\right)2N\right]^2 + \left(\theta\rho(\delta)\left(\rho(\delta)^2+1\right)2N + \sqrt{4(1-\theta)^2\delta(v)^2 + 2\theta^2 N}\right)^2 \le 1 - \theta.$$

Since at the start of each main iteration we have $\delta(v) \leq \tau$, this holds if the parameter θ and τ satisfy

$$\left[\theta\rho(\tau)\left(\rho(\tau)^2+1\right)2N\right]^2 + \left[\theta\rho(\tau)\left(\rho(\tau)^2+1\right)2N + \sqrt{4(1-\theta)^2\tau^2+2\theta^2 N}\right]^2 \leq 1-\theta. \tag{82}$$

Dividing both sides by $4\theta^2N^2$ this becomes

$$\left[\rho(\tau)\left(\rho(\tau)^2+1\right)\right]^2 + \left[\rho(\tau)\left(\rho(\tau)^2+1\right) + \sqrt{\left(\frac{1-\theta}{\theta N}\right)^2\tau^2+\frac{1}{2N}}\right]^2 \leq \frac{1-\theta}{4\theta^2N^2}. \tag{83}$$

Since $\rho(\tau)\left(\rho(\tau)^2+1\right) \geq 1$, the left-hand side is larger than 1. Hence we must have $4\theta^2N^2 \leq 1$, which already implies $\theta \leq \frac{1}{2N}$. Taking

$$\theta = \frac{1}{9N}, \quad \tau = \frac{1}{16} \tag{84}$$

this inequality is satisfied, as we now show. For the right hand side of (83) we then have

$$\frac{1-\theta}{4\theta^2N^2} = \left(1-\frac{1}{9N}\right)\frac{81}{4} = \frac{81}{4} - \frac{9}{4N} \geq \frac{81}{4} - \frac{9}{4} = 18.$$

For the given value of τ one has $\rho(\tau)\left(\rho(\tau)^2+1\right) \doteq 2.27053 \leq \frac{5}{2}$. Hence the left-hand side does not exceed

$$\begin{aligned}
\frac{25}{4} + \left[\frac{5}{2} + \sqrt{\frac{1}{16^2}\left(9\left(1-\frac{1}{9N}\right)\right)^2+\frac{1}{2N}}\right]^2 &\leq \frac{25}{4} + \left[\frac{5}{2} + \sqrt{\frac{1}{16^2}\left(9-\frac{1}{N}\right)^2+\frac{1}{2N}}\right]^2 \\
&\leq \frac{25}{4} + \left[\frac{5}{2} + \sqrt{\frac{1}{16^2}8^2+\frac{1}{2}}\right]^2 \\
&= \frac{25}{4} + \left[\frac{5}{2} + \frac{1}{2}\sqrt{3}\right]^2 \doteq 17.5801 < 18.
\end{aligned}$$

4.6 Iteration bound

With τ as defined in (84), according to (47), after the feasibility step we need at most

$$\log_2\left(\log_2\frac{1}{\tau^2}\right) = \log_2\left(\log_2 256\right) = \log_2\left(\log_2 2^8\right) = \log_2 8 = 3$$

centering steps to get iterates that satisfy $\delta(x, s; \mu^+) \leq \tau$. So each main iteration consists of one feasibility step and at most three centering steps. In total we therefore have at most four inner iterations per main iteration. Hence, with θ as given in (84), the total number of inner iterations is bounded above by four times the number of main iterations. The number of main iterations being given by (46), we may state our main result without further proof as follows.

Theorem 4.11 *If (P) and (D) have optimal solution $(x^*,\, y^*,\, s^*)$ such that $x^*+s^* \preceq \zeta e$, then after at most*

$$36N\,\log\frac{\max\left\{2N\zeta^2,\,\left\|r_b^0\right\|,\,\left\|r_c^0\right\|\right\}}{\varepsilon}$$

inner iterations the algorithm finds an ε-solution of (P) and (D).

The above iteration bound has been derived under the assumption of the existence of optimal solutions of (P) and (D) such that $x^* + s^* \preceq \zeta e$. One might ask what happens if this condition is not satisfied. In that case, during the course of the algorithm it will certainly happen that after some feasibility step the proximity measure $\delta(x, s; \mu)$ exceeds the value $1/\sqrt{2}$, because otherwise there is no reason why the algorithm would not generate an ε-solution of (P) and (D). So, if this happens it tell us that either the problems (P) and (D) do not have optimal solutions (with zero duality gap) or the value of ζ has been chosen too small. In the latter case one might run the algorithm once more with a larger value of ζ.

5 Concluding remarks

The first contribution of this paper is the first primal-dual interior-point algorithm for solving SOCO problems that uses full NT-steps only. So no line searches are required. Then using the method proposed first in [26] (see also [13]) for LO, and that was extended to SDO in [11, 12], we extended our algorithm to an infeasible primal-dual interior-point method algorithm for SOCO that uses full NT-steps only. In both cases the iterations bounds coincide with the currently best known iteration bounds for SOCO.

In the current paper the feasibility step targets at the μ^+-center of the new pair of perturbed problems, whereas the feasibility step in [26] targeted at the μ-center of the new pair of perturbed problems. Different options for defining the feasibility step were also considered in [9, 11, 13]. It might be a topic for further research to analyze our algorithm for SOCO with such modified feasibility steps.

A more challenging task is to unify the analysis for LO, SOCO and SDO by considering optimization problems over general symmetric cones. Another topic for further research is to consider large-update variants of the algorithm, since such methods are much more efficient in practice. Finally, a question that might be considered is if full step methods (either of Newton or NT-type) can be made efficient by using dynamic updates of the barrier parameter.

A Technical lemmas

Lemma A.1 *For $i = 1, \ldots, n$, let $f_i : \mathbf{R}_+ \to \mathbf{R}$ denote a convex function. Then we have for any nonzero vector $z \in \mathbf{R}^n_+$ the following inequality:*

$$\sum_{i=1}^{n} f_i(z_i) \leq \frac{1}{e^T z} \sum_{j=1}^{n} z_j \left(f_j(e^T z) + \sum_{i \neq j} f_i(0) \right).$$

Proof: We define the function $F : \mathbf{R}^n_+ \to \mathbf{R}$ by

$$F(z) = \sum_{i=1}^{n} f_i(z_i), \quad z \geq 0.$$

Letting e_j denote the j-th unit vector in $\mathbf{R}^n$, we may write z as a convex combination of the vectors $\left(e^T z\right) e_j$, as follows.

$$z = \sum_{j=1}^{n} \frac{z_j}{e^T z} \left(e^T z\right) e_j,$$

Indeed, $\sum_{j=1}^{n} \frac{z_j}{e^T z} = 1$ and $z_j / e^T z \geq 0$ for each j. Since $F(z)$ is a sum of convex functions, $F(z)$ is convex in z, and hence we have

$$F(z) \leq \sum_{j=1}^{n} \frac{z_j}{e^T z} F\left(\left(e^T z\right) e_j\right) = \sum_{j=1}^{n} \frac{z_j}{e^T z} \sum_{i=1}^{n} f_i\left(\left(e^T z\right) (e_j)_i\right).$$

Since $(e_j)_i = 1$ if $i = j$ and zero if $i \neq j$, we obtain

$$F(z) \leq \sum_{j=1}^{n} \frac{z_j}{e^T z} \left(f_j(e^T z) + \sum_{i \neq j} f_i(0) \right).$$

Hence the inequality in the lemma follows. □

Corollary A.2 *Let $f : \mathbf{R}_+ \to \mathbf{R}$ be a convex function such that $f(0) = 0$. Then we have for any vector $z \in \mathbf{R}_+^n$ the following inequality:*

$$\sum_{i=1}^{n} f(z_i) \leq f\left(\sum_{i=1}^{n} z_i\right).$$

Proof: In Lemma A.1, take $f_i = f$ for each i, then the result follows. □

Lemma A.3 *Let $x, s \in \mathbf{R}^n$ and $x^T s = 0$, then one has*

(i) $-\frac{1}{4} \|x+s\|_F^2 \, e \preceq_{\mathcal{K}} x \circ s \preceq_{\mathcal{K}} \frac{1}{4} \|x+s\|_F^2 \, e$;

(ii) $\|x \circ s\|_F \leq \frac{1}{2\sqrt{2}} \|x+s\|_F^2$.

Proof: We write

$$x \circ s = \tfrac{1}{4}\left((x+s)^2 - (x-s)^2\right). \tag{85}$$

Since $(x+s)^2 \in \mathcal{K}$, we have

$$x \circ s + \tfrac{1}{4}(x-s)^2 \in \mathcal{K}.$$

Using Lemma 2.13 (i), we may write

$$(x-s)^2 \preceq \left\|(x-s)^2\right\|_F e \preceq \|x-s\|_F^2 \, e,$$

whence it follows that

$$x \circ s + \tfrac{1}{4} \|x-s\|_F^2 \, e \in \mathcal{K},$$

which means that $-\frac{1}{4} \|x-s\|_F^2 \, e \preceq_{\mathcal{K}} x \circ s$. In the same way one derives from (85) that

$$\tfrac{1}{4} \|x+s\|_F^2 \, e - x \circ s \in \mathcal{K},$$

whence $x \circ s \preceq_{\mathcal{K}} \frac{1}{4} \|x+s\|_F^2 \, e$. Since x and s are orthogonal, we have $\mathbf{tr}\,(x \circ s) = 2x^T s = 0$, whence $\|x+s\|_F = \|x-s\|_F$, part (i) of the lemma follows.

For the proof of (ii) we return to (85). Using $\|z\|_F^2 = \mathbf{tr}\left(z^2\right)$, we obtain

$$\|x \circ s\|_F^2 = \left\| \frac{1}{4}\left((x+s)^2 - (x-s)^2\right) \right\|_F^2 = \frac{1}{16}\mathbf{tr}\left[\left((x+s)^2 - (x-s)^2\right)^2\right]$$
$$= \frac{1}{16}\left[\mathbf{tr}\left((x+s)^4\right) + \mathbf{tr}\left((x-s)^4\right) - 2\mathbf{tr}\left((x+s)^2 \circ (x-s)^2\right)\right].$$

Since $(x+s)^2$ and $(x-s)^2$ belong to $\mathcal{K}$, the trace of their product is nonnegative. Thus we obtain

$$\|x \circ s\|_F^2 \leq \frac{1}{16}\left[\mathbf{tr}\left((x+s)^4\right) + \mathbf{tr}\left((x-s)^4\right)\right] = \frac{1}{16}\left[\left\|(x+s)^2\right\|_F^2 + \left\|(x-s)^2\right\|_F^2\right].$$

Using Lemma 2.13 (i) and $\|x+s\|_F = \|x-s\|_F$ again, we get

$$\|x \circ s\|_F^2 \leq \frac{1}{16}\left[\|x+s\|_F^4 + \|x-s\|_F^4\right] = \frac{1}{8}\|x+s\|_F^4.$$

This proves the lemma. □

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