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Convex Relaxations of Non-Convex Mixed Integer Quadratically Constrained Programs: Extended Formulations

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Abstract This paper addresses the problem of generating strong convex relaxations of Mixed Integer Quadratically Constrained Programming (MIQCP) problems. MIQCP problems are very difficult because they combine two kinds of non-convexities: integer variables and non-convex quadratic constraints. To produce strong relaxations of MIQCP problems, we use techniques from disjunctive programming and the lift-and-project methodology. In particular, we propose new methods for generating valid inequalities by using the equation $Y = xx^T$. We use the concave constraint $0 \succcurlyeq Y - xx^T$ to derive disjunctions of two types. The first ones are directly derived from the eigenvectors of the matrix $Y - xx^T$ with positive eigenvalues, the second type of disjunctions are obtained by combining several eigenvectors in order to minimize the *width* of the disjunction. We also use the convex SDP constraint $Y - xx^T \succcurlyeq 0$ to derive convex quadratic cuts, and we combine both approaches in a cutting plane algorithm. We present computational results to illustrate our findings.

1 Introduction

In this paper we study the mixed integer quadratically constrained program defined as follows:

$$
(\mathbf{MIQCP'}) \qquad \begin{array}{ll} \min & a_0^T x \\ s.t. & \\ & x^T A_i x + a_i^T x + b_i \leq 0 \ , \ i = 1 \ldots m \ ; \\ & x_j \in \mathbb{Z} \ , \ j \in N_I \ ; \\ & l \leq x \leq u \ , \end{array}
$$

where $N = \{1, \ldots, n\}$ denotes the set of variables, $N_I = \{1, \ldots, p\}$ denotes the set of integer constrained variables, A_i $(i = 1 \ldots m)$ are $n \times n$ symmetric (usually not positive semidefinite) matrices, a_i $(i = 0 \ldots m)$, l and u are n-dimensional vectors and b_i $(i = 1 \ldots m)$ are scalars. The decision variant of **MIQCP′** is well known to be undecidable, even in the pure integer case, when the variables are not bounded (see [16]). Many natural applications of **MIQCP′** can be found in the global-optimization

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literature. In this paper our focus is to derive tight convex relaxations for **MIQCP′** by using cutting plane approaches.

A standard approach to derive a convex relaxation of **MIQCP′** is to first introduce extra variables $Y_{ij} = x_i x_j$ in the formulation. Consequently, the following lifted reformulation of **MIQCP′** is obtained[1]

$$(\mathbf{MIQCP}) \qquad \begin{array}{lll} \min & a_0^T x & \\ s.t. & & \\ & A_i.Y + a_i^T x + b_i \leq 0 \ , & i = 1 \ldots m \ ; \\ & x_j \in \mathbb{Z} \ , & j \in N_I \ ; \\ & l \leq x \leq u \ ; & \\ & Y = xx^T \ . & \end{array}$$

Note that the only non-convex constraint in **MIQCP** is the set of non-linear equations $Y = xx^T$, which can be relaxed as a pair of SDP inequalities $Y - xx^T \succcurlyeq 0$ and $xx^T - Y \succcurlyeq 0$. The former of these inequalities can be expressed as a LMI (Linear Matrix Inequality) on the cone of positive semi-definite matrices, while treatment of the latter non-convex inequality constitutes the emphasis of this paper.

One of the common predicaments for non-convex problems is that they are composed of seemingly innocuous looking non-convex constraints (for example $x_i \in \{0, 1\}$) linked together through a set of (usually linear or convex) constraints. For instance, a mixed integer 0-1 linear program is composed of a linear program and $\{0, 1\}$-constraints on some of the variables. In these kind of problems, convexifying the non-convex constraints does not yield any significant improvement until the convexification process explicitly takes into account the (convex) constraints linking the non-convexities together. For instance, convexifying the 0-1 constraints on a set of variables in a mixed integer linear program (MILP) yields the unit hypercube, which obviously offers little help in solving the MILP.

Interestingly, most of the existing convexification-based approaches in Mixed Integer Non-Linear Programming (MINLP) fail to take the linking constraints into account and work exclusively with *simple* non-convex sets, and try to derive closed form expressions for the convexified sets [31,32]. Some other approaches try to perform local convexification and impose that by additional constraints. For instance, imposing the SDP constraint $Y - xx^T \succcurlyeq 0$ falls into this category of approaches. Naturally, we are interested in an approach which takes a holistic view of the problem and tries to capitalize on the interaction between the problem constraints. In this paper, we use the framework of disjunctive programming to accomplish this goal.

Classical disjunctive programming of Balas [2] requires a linear relaxation of the problem and a disjunction that is satisfied by all the feasible solution to the problem. As is now customary in the MINLP literature, we will use the outer-approximation (OA) of **MIQCP** as the quintessential convex relaxation. We use the phrase "suitably defined OA" in this paper to emphasize the dependence of OA under discussion on the solution $(\hat{x}, \hat{Y})$ to the convex relaxation of **MIQCP**.

As for the choice of disjunctions, we seek the sources of non-convexities in **MIQCP**. Evidently, **MIQCP** has two of these, namely, the integrality conditions on the x_j $(j \in N_I)$ variables and the non-linear equations $Y = xx^T$. Integrality constraints have been used to derive disjunctions in MILP for the past five decades, and we do not add anything new to this body of work (also see [29]) . Our main contribution lies in deriving valid disjunctions from $Y = xx^T$, by analyzing the eigenvectors of the matrix $\hat{Y} - \hat{x}\hat{x}^T$ (defined with respect to a solution $(\hat{x}, \hat{Y})$ of the current relaxation), and deriving "univariate expressions" of the form $Y.cc^T \leq (c^T x)^2$ which are subsequently used to derive disjunctive cuts.

The rest of the paper is organized as follows. In §2, we revisit some of the basic ideas in disjunctive programming and give a detailed description of our disjunctive cut generator. In §3, we derive a large class of valid disjunctions for **MIQCP** and establish interesting connections with elementary 0-1 disjunctions in MILP. In §4, we investigate the problem of designing disjunctions that use more problem information than is available from the eigenvectors of $\hat{Y} - \hat{x}\hat{x}^T$. We introduce the notion of the *width* of a disjunction, and show that disjunctions with smaller widths are likely to give rise to stronger disjunctive cuts. We build on this observation and design a MILP model to find better disjunctions. We also briefly discuss a scheme for diversifying the class of disjunctions based on the Grahm-Schmidt orthogonalization procedure. Finally, in §5, we report computational results on three types of instances: selected problems from GLOBALLib [17], some examples of **MIQCP** instances from [20] which arise in

[1] For symmetric matrices A and B of conformable dimensions, we define $A.B = tr(AB)$.

chemical engineering applications and some continuous boxed-constrained Quadratic Programs (QPs) from [34].

Proofs of most of the propositions presented in this paper are straightforword, and hence omitted for the sake of brevity. A preliminary version of this paper appeared in the 2008 IPCO Proceedings [23].

2 Disjunctive Programming

In this section we review some of the basic ideas from disjunctive programming and give a detailed description of our cut generator. Given a polytope $P = \{x \mid Ax \geq b\}$, a disjunction $D = \bigvee_{t=1}^{q} (D^t x \geq d^t)$ and a point $\hat{x} \in P$, a central question in disjunctive programming is to show that $\hat{x} \in Q = \mathrm{clconv}\ \cup_{t=1}^{q} \{\mathrm{x \in P \mid D^t x \geq d^t}\}$ or find a valid inequality $\alpha x \geq \beta$ for Q that is violated by $\hat{x}$.

This question arises in several areas of computational optimization where the specific form of the polytope P and disjunction D is governed by the underlying application. For instance, in the context of MILP, the polytope P usually represents the LP-relaxation of the MILP, while the disjunctions are obtained by exploiting the integrality constraints (see for example [3,5]). Similarly, in the context of probabilistic programming, P usually represents the deterministic variant of the problem, while the disjunction is derived from the so-called p-efficient frontier [26,25]. In the context of MIQCP, P will represent a suitably chosen outer-approximation of **MIQCP**, while the disjunction is obtained by exploiting the integrality constraints on the variables x_j $(j \in N_I)$ or from the eigenvectors of the matrix $Y - xx^T$ (see §3).

The theorem that follows formulates the separation problem mentioned above as a linear program. It follows immediately from the results presented in [2].

Theorem 1 *$\hat{x} \in Q$ if and only if the optimal value of the following Cut-Generation Linear Program (CGLP) is non-negative.*

$$\text{(CGLP)} \quad \begin{array}{lrl} \min & \alpha\hat{x} - \beta & \\ s.t. & & \\ & \alpha = u^t A + v^t D^t \ , & t = 1 \dots q \ ; \\ & \beta \leq u^t b + v^t d^t \ , & t = 1 \dots q \ ; \\ & u^t, v^t \geq 0 \ , & t = 1 \dots q \ ; \\ & \sum_{t=1}^{q} (u^t \xi + v^t \xi^t) = 1 \ , & \end{array}$$

where ξ, ξ^t $(t = 1 \dots q)$ are any non-negative vectors of conformable dimensions that satisfy $\xi^t > 0$ $(t = 1 \dots q)$. If the optimal value of the CGLP is negative, and $(\alpha, \beta, u^1, v^1, \dots, u^q, v^q)$ is an optimal solution of the CGLP, then $\alpha x \geq \beta$ is a valid inequality for Q which cuts off $\hat{x}$.

The constraint $\sum_{t=1}^{q} (u^t \xi + v^t \xi^t) = 1$ of the CGLP, referred to as the *normalization constraint*, plays a central role in determining the strength and numerical stability of the resulting cut (see [5]). In our computational results, we used the following normalization:

1. $\xi_i^t = 1$, $\forall\ i = 1 \dots m_t$, $t = 1 \dots q$, where m_t denotes the number of rows in the matrix D_t .
2. $$\xi_i = \begin{cases} 0 \ , & \text{for } i \in L \ ; \\ ||a_i||_1 \ , & \text{otherwise,} \end{cases}$$

   where L denotes the set of lower-bound constraints in $Ax \geq b$, while a_i denotes the i^{th} row of the matrix A .

The above normalization has two important characteristics. First, it implicitly scales the constraints in $Ax \geq b$ (other than the lower-bound constraints) so that all of them have a ℓ_1-norm of 1 , which in turn significantly improves the numerical properties of the resulting cut. Second, assigning a normalization coefficient of zero to the lower-bound constraints allows us to handle these constraints as bounds on variables associated with the dual of the CGLP, thereby speeding up the overall algorithm to solve the CGLP.

In order to use the machinery of disjunctive programming to strengthen the formulation of **MIQCP**, we need a class of disjunctions that are satisfied by every feasible solution to **MIQCP**. Note that

MIQCP has two sources of non-convexities, namely the integrality constraints on x_j $(j \in N_I)$ variables, and the equality constraints $Y = xx^T$. While the former can be used to derive split disjunctions, as is usually done in MILP, the latter need to be handled more carefully. The section that follows gives a novel way of deriving valid disjunctions from the constraints $Y = xx^T$.

3 Valid Disjunctions for MIQCP

Throughout the paper, unless otherwise stated, we denote by $(\hat{x}, \hat{Y})$ the solution to a convex relaxation of **MIQCP** which we want to cut off.

Note that for $c \in \mathbb{R}^n$, any feasible solution to **MIQCP** satisfies $(c^T x)^2 = Y.cc^T$, which in turn is equivalent to the following two inequalities $(c^T x)^2 \leq Y.cc^T$ and $(c^T x)^2 \geq Y.cc^T$. The former of these two inequalities is a convex quadratic constraint that can be readily added to the formulation. The second constraint $(c^T x)^2 \geq Y.cc^T$, on the other hand, gives rise to the following disjunction which is satisfied by every feasible solution to **MIQCP**:

$$(1) \qquad \begin{bmatrix} \eta_L(c) \leq c^T x \leq \theta \\ -(c^T x)(\eta_L(c) + \theta) + \theta\eta_L(c) \leq -Y.cc^T \end{bmatrix} \bigvee \begin{bmatrix} \theta \leq c^T x \leq \eta_U(c) \\ -(c^T x)(\eta_U(c) + \theta) + \theta\eta_U(c) \leq -Y.cc^T \end{bmatrix},$$

where $\eta_L(c) = \min\{c^T x \mid (x, Y) \in \tilde{P}\}$, $\eta_U(c) = \max\{c^T x \mid (x, Y) \in \tilde{P}\}$, $\tilde{P}$ is a suitably chosen relaxation of **MIQCP** and $\theta \in (\eta_L(c), \eta_U(c))$. In our computational experiments, we chose $\tilde{P}$ to be a suitably defined outer-approximation of **MIQCP** and $\theta = c^T \hat{x}$. The above disjunction can be derived by splitting the range $[\eta_L(c), \eta_U(c)]$ of the function $c^T x$ over $\tilde{P}$ into two intervals $[\eta_L(c), \theta]$ and $[\theta, \eta_U(c)]$ and constructing a secant approximation of the function $-(c^T x)^2$ in each of the intervals, respectively. The above disjunction can be used to derive disjunctive cuts by using the apparatus of CGLP. Furthermore, for any integer $q > 1$, a q-term disjunction can be obtained by splitting the $[\eta_L(c), \eta_U(c)]$ interval into q parts and constructing a secant approximation of $-(c^T x)^2$ in each one of the q intervals. Non-convex inequalities of the form $(c^T x)^2 \geq Y.cc^T$ are referred to as *univariate expressions* in the sequel.

From a computational standpoint, the only question that remains to be answered is, how can we judiciously choose a vector c that is likely to give rise to strong cuts. We describe two procedures for deriving such vectors; both of these procedures use the eigenvectors of the matrix $\hat{Z} = \hat{Y} - \hat{x}\hat{x}^T$. Let $c_1, \ldots, c_n$ denote a set of orthonormal eigenvectors of $\hat{Z}$, and let $\mu_1 \geq \mu_2 \ldots \geq \mu_n$ be the corresponding eigenvalues.

Let $k \in \{1, \ldots, n\}$, and let $c = c_k$. Note that if $\mu_k < 0$, then $(c^T x)^2 \leq Y.cc^T$ is a valid convex quadratic cut which cuts off $(\hat{x}, \hat{Y})$. If $\mu_k > 0$, then $(c^T x)^2 \geq Y.cc^T$ is a valid inequality (albeit non-convex) for **MIQCP** which cuts off $(\hat{x}, \hat{Y})$. Consequently, in this case the disjunction derived from $(c^T x)^2 \geq Y.cc^T$ is a good candidate for generating disjunctive cuts. In our computational experiments, we added a convex quadratic cut from every negative eigenvalue of $\hat{Z}$, and generated a disjunctive cut (if any) from every positive eigenvalue of $\hat{Z}$.

Two comments are in order. First, the relaxation of **MIQCP** obtained by replacing $Y = xx^T$ by $Y - xx^T \succcurlyeq 0$ has been studied by several other authors ([19, 28, 8, 1]). From an engineering viewpoint, incorporating the positive semi-definiteness condition $Y - xx^T \succcurlyeq 0$ as part of the relaxation poses a serious hurdle, since most general purpose solvers for nonlinear optimization (such as Ipopt [35] and FilterSQP [15]) are not designed to handle conic constraints of the form $Y - xx^T \succcurlyeq 0$, or equivalently

$$\begin{bmatrix} 1 & x \\ x^T & Y \end{bmatrix} \succcurlyeq 0 \, .$$

Special purpose software packages for conic programming (such as SeDuMi [30]), on the other hand, cannot handle arbitrary convex constraints. Since our solver for the convex relaxations Ipopt [35] is a general purpose solver, we incorporated the effect of $Y - xx^T \succcurlyeq 0$ by iteratively generating convex quadratic inequalities $(c^T x)^2 \leq Y.cc^T$ derived from eigenvectors c of $\hat{Z}$ associated with negative eigenvalues.

Second, our approach of strengthening the relaxation of **MIQCP** by generating disjunctive cuts can also be viewed as convexifying the feasible region of **MIQCP**. *Convexification* of non-convex feasible regions is an active research area in the MINLP community ([31,32,33,34]). Most of these convexification based approaches, however, aim to convexify non-convex problem constraints individually, and often fail to exploit the interaction across problem constraints to derive stronger cuts. A disjunctive programming based approach, such as the one presented in this paper, takes a holistic view of the problem and tries to draw stronger inferences *à la* disjunctive cuts by combining information from all of the problem constraints.

Balas [2] showed that mixed 0-1 linear programs (M01LP) are special cases of facial disjunctive programs which possess the sequential convexifiability property. Simply put, this means that under suitable qualification conditions, the closed convex hull of all feasible solutions to a M01LP can be obtained by imposing the 0-1 condition on the binary variables *sequentially*; i.e. by imposing the 0-1 condition on the first binary variable and convexifying the resulting set, followed by imposing the 0-1 condition on the second variable, and so on. The theorem that follows proves a similar result for **MIQCP**.

Theorem 2 *Suppose that the feasible region of* **MIQCP** *is bounded, and that all of the integer-constrained variables in* **MIQCP** *are also constrained to be binary. Let* $c_1, \ldots, c_n$ *denote a set of mutually-orthogonal unit vectors in* $\mathbb{R}^n$*, and let*

$$S_0 = \left\{ (x,Y) \;\middle|\; \begin{array}{c} A_i.Y + a_i^T x + b_i \leq 0 \;\; i = 1 \ldots m \\ l \leq x \leq u \\ Y - xx^T \succcurlyeq 0 \end{array} \right\}$$

$$S_j = clconv\left(S_{j-1} \cap \left\{(x,Y) \mid Y.c_j c_j^T \leq (c_j^T x)^2\right\}\right) \;\; for\ j = 1 \ldots n$$

$$S_{n+j} = clconv\left(S_{n+j-1} \cap \{(x,Y) \mid x_j \in \{0,1\}\}\right) \;\; for\ j = 1 \ldots p\ .$$

The following statements hold true:

$$S_n = clconv \left\{ (x,Y) \;\middle|\; \begin{array}{c} A_i.Y + a_i^T x + b_i \leq 0 \;\; i = 1 \ldots m \\ l \leq x \leq u \\ Y - xx^T = 0 \end{array} \right\}$$

$$S_{n+p} = clconv \left\{ (x,Y) \;\middle|\; \begin{array}{c} A_i.Y + a_i^T x + b_i \leq 0 \;\; i = 1 \ldots m \\ l \leq x \leq u \\ Y - xx^T = 0 \\ x_j \in \{0,1\} \;\; j = 1 \ldots p \end{array} \right\} .$$

□

The above theorem follows immediately from the results presented in [6] and the observation that $\{(x,Y) \mid Y - xx^T \succcurlyeq 0\} \cap \{(x,Y) \mid Y.c_j c_j \leq (c_j^T x)^2 \; \forall j = 1 \ldots n\} = \{(x,Y) \mid Y = xx^T\}$, where $c_1, \ldots c_n$ is any set of mutually orthogonal unit vectors in $\mathbb{R}^n$. Some remarks are in order.

First, the boundedness assumption in the above theorem can be relaxed by imposing the qualification condition discussed in [6]; the resulting theorem, however, is too technical and of limited interest in context of the current paper. Second, note that the above theorem holds true for *any* choice of mutually orthogonal unit vectors $c_1, \ldots c_n$ in $\mathbb{R}^n$. Alternatively, for any set of such n mutually orthogonal unit vectors, **MIQCP** can be reformulated as

$$\min \left\{ a_0^T x \;\middle|\; \begin{array}{c} A_i.Y + a_i^T x + b_i \leq 0 \;\; i = 1 \ldots m \\ l \leq x \leq u \\ Y - xx^T \succcurlyeq 0 \\ Y.c_j c_j^T \leq (c_j^T x)^2 \;\; j = 1 \ldots n \\ x_j \in \{0,1\} \;\; j = 1 \ldots p \end{array} \right\} .$$

For the purpose of cuts generation, we would like to use a reformulation that most effectively elucidates the infeasibility of the solution $(\hat{x}, \hat{Y})$ of the convex relaxation w.r.t **MIQCP**. In other words, we are interested in a set of mutually orthogonal unit vectors $\{c_1, \ldots, c_n\}$ that maximize the infeasibility $\max_{j=1 \ldots n} c_j^T (\hat{Y} - \hat{x}\hat{x}^T) c_j$ of $(\hat{x}, \hat{Y})$ w.r.t the corresponding reformulation of **MIQCP**. Clearly, the set of eigenvectors of $\hat{Y} - \hat{x}\hat{x}^T$ constitutes an optimal solution to the above problem. Thus our choice of using the eigenvectors of $\hat{Y} - \hat{x}\hat{x}^T$ to construct univariate expressions can be viewed as a dynamic reformulation scheme which rotates the coordinate axes so as to amplify the hidden infeasibilities of $\hat{Y} - \hat{x}\hat{x}^T$ w.r.t **MIQCP**.

Third, there is a distinct difference between M01LP and MIQCP in each step of the sequential convexification process. To see this, note that M01LP with a single binary variable is a polynomial-time solvable problem; in fact, Balas [2] gives a polynomial-sized lifted linear-programming formulation of this problem. On the other hand, a similar problem in the context of **MIQCP** involves minimizing a linear function over a non-convex set of the form,

$$\left\{ (x, Y) \;\middle|\; \begin{array}{c} A_i.Y + a_i^T x + b_i \le 0 \quad i = 1 \ldots m \\ l \le x \le u \\ Y - xx^T \succcurlyeq 0 \\ Y.cc^T \le (c^T x)^2 \end{array} \right\} ,$$

for some unit vector c. It is not immediately clear if this is a polynomial-time solvable problem; in fact, its likely to be a NP-hard problem itself (see [21]).

4 More disjunctions

Note that univariate expressions derived from eigenvectors of $\hat{Z}$ are oblivious to other constraints in the problem. In other words, these eigenvectors are not influenced by most of the problem constraints, and hence do not completely exploit the problem structure. In this section, we give a systematic procedure for generating univariate expressions that utilize all of the problem constraints, and are hence likely to give rise to stronger cuts (also see §5).

For $c \in \mathbb{R}^n$, let $\eta_L(c) = \min\{c^T x \mid (x, Y) \in P\}$ and $\eta_U(c) = \max\{c^T x \mid (x, Y) \in P\}$, for a suitably chosen outer approximation P of **MIQCP**. Let $\eta(c) = \eta_U(c) - \eta_L(c)$ denote the width of the interval $[\eta_L(c), \eta_U(c)]$. The following inequality represents the secant approximation of the function $-(c^T x)^2$ in the $[\eta_L(c), \eta_U(c)]$ interval, and is hence a valid disjunctive cut derived from the disjunction (1).

$$-c^T x(\eta_L(c) + \eta_U(c)) + \eta_L(c)\eta_U(c) \le -Y.cc^T \ . \tag{2}$$

The proposition that follows gives a closed-form expression for the maximum error incurred by the secant approximation of the negative square function in a bounded interval.

Proposition 1 *Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = -x^2$, and let $g(x) = -x(a+b)+ab$ represent the secant approximation of $f(x)$ in the $[a, b]$ $(a, b \in \mathbb{R})$ interval; then* $\max_{x\in[a,b]} (f(x) - g(x)) = (a - b)^2/4$.

As a direct consequence of the above proposition, it follows that secant-approximation error incurred by (2) is proportional to $\eta(c)^2$. Consequently, we can use $-\eta(c)$ as a proxy to measure the strength of the disjunctive cuts obtainable from $Y.cc^T \le (c^T x)^2$. The proposition that follows shows that $\eta(c)$ can be computed by solving a linear program.

Proposition 2 *Let $P = \{(x, Y) \mid Ax + BY \ge b\}$, where B is a tensor of conformable dimensions. Then*

$$\begin{aligned} \eta(c) = -\max\ & (u + v)^T b \\ s.t.\ & \\ & uA = c \ ; \\ & -vA = c \ ; \\ & u.B = 0 \ ; \\ & v.B = 0 \ ; \\ & u, v \ge 0 \ . \end{aligned}$$

To summarize, we are looking for vectors $c \in \mathbb{R}^n$ whose univariate expression $Y.cc^T \le (c^T x)^2$ is violated by $(\hat{x}, \hat{Y})$ and has a small width $\eta(c)$. Note that if we restrict our attention to the subspace spanned by eigenvectors of $\hat{Z}$ with positive eigenvalues, then the first condition is automatically satisfied. Thus, we can model the problem of determining a vector c that gives the best univariate expression as

$$
\begin{aligned}
&\min \eta(c) - \epsilon \left(\textstyle\sum_{j=1}^{n} |\lambda_j| \mu_j \right) \\
&s.t. \\
&\qquad c = \textstyle\sum_{j=1}^{n} \lambda_j c_j \\
&\qquad \textstyle\sum_{j=1}^{n} |\lambda_j| = 1 \ ; \\
&\qquad \lambda_j = 0 \ , \qquad \forall j \in \{1 \dots n\} \ s.t. \ \mu_j \leq 0 \ .
\end{aligned}
$$

In the above model, the constraint $\sum_{j=1}^{n} |\lambda_j| = 1$ enforces that the ℓ_1-norm of c expressed in the basis defined by $(c_1, \dots, c_n)$ is equal to 1 . The constraint $\lambda_j = 0 \ \forall j$ s.t $\mu_j \leq 0$ ensures that c lies in the subspace spanned by eigenvectors of $\hat{Z}$ with positive eigenvalues. The penalty term $\epsilon \left(\sum_{j=1}^{n} |\lambda_j| \mu_j \right)$ $(\epsilon = 10^{-4})$ expresses our desire to bias c toward eigenvectors with large eigenvalues. The above model can be easily recast as the following mixed integer program, referred to as Univariate-expression Generating Mixed Integer Program (UGMIP):

$$
\text{(UGMIP)} \quad
\begin{aligned}
&\min -(u+v)^T b - \textstyle\sum_{j=1}^{n} \lambda_j^+ \mu_j \epsilon \\
&s.t. \\
&\qquad uA = c \ ; \\
&\qquad -vA = c \ ; \\
&\qquad u.B = 0 \ ; \\
&\qquad v.B = 0 \ ; \\
&\qquad u, v \geq 0 \ ; \\
&\qquad c = \textstyle\sum_{j=1}^{n} \lambda_j c_j \ ; \\
&\qquad z_j - 1 \leq \lambda_j \ , \ \forall j = 1 \dots n \ ; \\
&\qquad \lambda_j \leq z_j \ , \ \forall j = 1 \dots n \ ; \\
&\qquad \lambda_j^+ \leq \lambda_j + 2(1 - z_j) \ , \ \forall j = 1 \dots n \ ; \\
&\qquad \lambda_j^+ \leq -\lambda_j + 2 z_j \ , \ \forall j = 1 \dots n \ ; \\
&\qquad \lambda_j^+ \geq 0 \ , \ \forall j = 1 \dots n \ ; \\
&\qquad \textstyle\sum_{j=1}^{n} \lambda_j^+ = 1 \ ; \\
&\qquad \lambda_j = 0 \ , \ \forall j \in \{1 \dots n\} \ s.t. \ \mu_j \leq 0 \ ; \\
&\qquad z_j \in \{0, 1\} \ j \in \{1, \dots, n\}.
\end{aligned}
$$

Another idea that has played a significant role in the successful application of general-purpose cutting planes in MILP is that of *cut diversification* [14,5]. Cut diversification refers to the strategy of adding a batch of cuts, each of which affects a different part of the solution of the convex relaxation, thereby triggering a collaborative action and yielding improvements that cannot be obtained by a single cut. For instance, the tremendous practical performance of Mixed Integer Gomory Cuts is often attributed to their well-diversified nature (see [9]). Interestingly, the above UGMIP can be easily augmented to generate a set of diversified vectors instead of a single vector. To see this, suppose that a set of vectors $\tilde{c}^k = \sum_{j=1}^{n} \lambda_j^{(k)} c_j$ $(i = 1 \dots K)$ has already been generated, and we are interested in finding a vector c that is different from $\tilde{c}^k$ $(k = 1 \dots K)$. This can be accomplished by appending the following constraints to UGMIP and resolving the resulting mixed integer program:

$$
\sum_{j=1}^{n} \lambda_j \lambda_j^k = 0 \ \forall k = 1 \dots K \ .
$$

This diversification scheme is motivated by the well known Grahm-Schmidt orthogonalization procedure for generating an orthogonal basis of a finite-dimensional vector space. To see this, observe that at the end of each step of the above diversification procedure, the new vector c is orthogonal to each one of the vectors $\tilde{c}^k$ for $k = 1 \dots K$.

In our computational experiments, we solved UGMIP using CPLEX 10.1, enumerating at most 2000 branch-and-bound nodes. Furthermore, the diversification scheme mentioned above was used iteratively until the resulting UGMIP became infeasible or CPLEX was unable to find a feasible solution within the stipulated node limit.

5 Computational Results

We report computational results in this section. Since the aim of these experiments was to assess the performance of different classes of cutting planes and their relative strengths, we report the percentage duality gap closed by each one of them at the root node. All of the experiments described in this section used the following general setup:

1. Solve the convex relaxation of **MIQCP**.
2. Generate cutting planes to cut off $(\hat{x}, \hat{Y})$.
3. If a violated cut was generated, then goto step (1), else STOP.

The above loop was repeated until a time-limit of 60 minutes was reached or the code was unable to find any violated cut. We implemented the following three variants of cutting planes discussed in the previous sections.

– **Variant** 1: Only convex quadratic cuts derived from eigenvectors associated with negative eigenvalues of $\hat{Y} - \hat{x}\hat{x}^T$ were used.
– **Variant** 2: Same as Variant 1, except that disjunctive cuts from univariate expression derived from eigenvectors of $\hat{Z}$ with positive eigenvalues were also used.
– **Variant** 3: Same as Variant 2 except that disjunctive cuts from additional univariate expressions found by using the UGMIP machinery and diversification scheme were also used.

The three variants were implemented using the open-source framework Bonmin [7] from COIN-OR. The nonlinear solver used is Ipopt [35], the eigenvalue problems are solved using Lapack and the cut generation linear programs are solved using CPLEX 10.1. A few comments are in order. First, we strengthen the initial convex relaxation of **MIQCP** by adding the following well-known RLT inequalities [22,27], for $i, j \in \{1 \dots n\}$ such that $i \leq j$,

$$\begin{aligned} y_{ij} - l_i x_j - u_j x_i + l_i u_j &\leq 0 \\ y_{ij} - l_j x_i - u_i x_j + l_j u_i &\leq 0 \\ y_{ij} - l_j x_i - l_i x_j + l_j l_i &\geq 0 \\ y_{ij} - u_j x_i - u_i x_j + u_j u_i &\geq 0 \ . \end{aligned}$$

Second, while generating the disjunctive cuts, we remove all of the RLT inequalities from the outer approximation except those which are binding at the solution of the convex relaxation. While solving the CGLP we use a column-generation based approach to generate u_i^t variables corresponding to non-binding RLT inequalities. Because there is a huge number $O(n^2)$ of RLT inequalities, we found it to be more efficient to use a column generation based approach to handle them while solving the CGLPs, thereby exploiting the reoptimization capabilities of the CPLEX linear-programming solver. Since Ipopt (as well as other interior-point methods) has very limited support for warm-starting, we found it more suitable to supply all of the RLT inequalities simultaneously while solving the convex relaxations.

Third, in order to control the size of the convex relaxation we used the following cut-purging strategy. We check every third iteration if the optimal value of the convex relaxation has improved over the last three iterations; if an improvement is detected, then we remove all of the cuts from the current formulation that are not binding at the solution to the convex relaxation. Fourth, we used the following mechanism to control the rank of the disjunctive cuts.[2] At every third iteration, we make a copy of the current convex relaxation, and use it to derive outer-approximation and disjunctive cuts in the subsequent three iterations. Consequently, we generate only rank-1 cuts in the first three iterations, only rank-2 cuts in the next three iterations and so on. Preliminary experimentation clearly suggests that such a rank-control mechanism significantly improves the numerical properties of the cuts, and delays the eventual tailing off behavior which often occurs in cutting-plane procedures.

Next we describe our computational results on the following three test-beds: GLOBALLib [17], instances from Lee and Grossmann [20], and Box-QP instances from [34].

GLOBALLib is a repository of 413 global optimization instances of widely varying types and sizes. Of these 413 instances, we selected all problems with at most 50 variables which can be easily

[2] See [5] and [14] for importance of low-rank cuts in cutting plane procedures.

	V1	V2	V3
>99.99 % gap closed	16	23	23
98-99.99 % gap closed	1	44	52
75-98 % gap closed	10	23	21
25-75 % gap closed	11	22	20
0-25 % gap closed	91	17	13
Total Number of Instances	129	129	129
Average Gap Closed	24.80%	76.49%	80.86%

Table 1 Summary Results: GLOBALLib instances with non-zero Duality Gap

converted into instances of **MIQCP**. For instance, some of the problems have product-of-powers terms ($x_1x_2x_3x_4x_5$, x_1^3, $x^{0.75}$, etc.) which can be converted into quadratic expressions by introducing additional variables. Additionally, some of the problems do not have explicit upper bounds on the variables; for such problems we used linear programming techniques to determine valid upper bounds thereby making them amenable to techniques discussed in this paper. The final set of selected problems comprised 153 instances.[3]

Among the 153 instances, 24 instances have zero duality gap[4]; in other words the RLT relaxation already closes 100% of the gap on these instances. Tables 6, 7 and 8 report the computational results on the remaining 129 instances, while Table 1 reports the same in summarized form. The second column of Tables 6, 7 and 8 reports the optimal value of the RLT relaxation of **MIQCP**, while the third column reports the value of the best known solution. Note that either Variant 2 or Variant 3 closes more than 99% of the duality gap on some of the instances (st_qpc-m3a, st_ph13, st_ph11, ex3_1_4, st_jcbpaf2, ex2_1_9 etc) on which Variant 1 is unable to close any gap. Furthermore, Variant 3 closes 10% more duality gap than Variant 2 on some of the instances (ex2_1_1, ex3_1_4, ex5_2_4, ex7_3_1, ex9_2_3, st_pan2 etc) showing the interest of disjunctions obtained from solution of the UGMIP problem.

Finally, in order to assess the performance of our code on the 24 instances with no duality gap, we report the spectral norm of $\hat{Y} - \hat{x}\hat{x}^T$ in Table 9, where $(\hat{x}, \hat{Y})$ denotes the solution of the convex relaxation at the last iteration of the respective variant. Note that we were able to generate almost-feasible solutions (i.e spectral-norm $\leq 10^{-4}$) on 17 out of 24 instances.

The ex9$\star$ instances in the GLOBALLib repository contain the linear-complementarity constraints (LCC) $x_ix_j = 0$ on a subset of variables. These constraints give rise to the following disjunction, $(x_i = 0) \vee (x_j = 0)$, which in turn can be embedded within the CGLP framework to generate disjunctive cuts. In order to test the effectiveness of these cuts, we modified our code to automatically detect linear-complementarity constraints, and use the corresponding disjunctions along with the default medley of disjunctions to generate disjunctive cuts. Table 10 reports our computational results. It is worth observing that while the default version of our code is unable to close any significant gap on the ex9_1_4 instance, when augmented with disjunctive cuts from the linear-complementarity constraints, it closes 100% of the duality gap.

Next we present our computational results on the **MIQCP** instances proposed in [20]. These problems have both continuous and integer variables and quadratic constraints. They are of relatively small size with between 10 and 54 variables. Table 2 summarizes the experiment. RLT is the value of the RLT relaxation, Opt is the value of the global optimum of the problem and V1, V2 and V3 give the strengthened bound obtained by each of the three variants. As can be seen from the results Variants 2 and 3 close almost all the gap for the secondx and third instance. For the first and fourth example, the gap closed is not as much, but in all cases Variant 2 and 3 close substantially more gap than Variant 1.

Next, we present our results on box-constrained QPs. The test bed consists of a subset the test problems used in [34]. These problems are randomly generated box QPs with A_0 of various densities. For this experiment, we ran the three variants of our cut-generation procedures on the 42 problems with 20, 30 and 40 variables. We found that the RLT relaxation of these problem when strengthened

[3] These instances can be downloaded in AMPL .mod format from www.andrew.cmu.edu/user/anureets/MIQCP

[4] We define the duality gap closed by a relaxation $\mathcal{I}$ of **MIQCP** as, $\frac{opt(\mathcal{I})-RLT}{opt-RLT} \times 100$, where opt($\mathcal{I}$), RLT, and opt denote the optimal value of $\mathcal{I}$, the RLT relaxation of **MIQCP** and **MIQCP**, respectively.

Instance	RLT	Opt	V1	V2	V3
Example 1	-58.70	-11	-58.70	-37.44	-37.44
Example 2	-414.94	-14	-93.19	-14.26	-14.26
Example 3	-819.66	-510.08	-793.15	-513.61	- 511.10
Example 4	-499282.59	-116,575	-472,727.49	-363,487.69	-359,618.10

Table 2 Summary of results on the Lee-Grossmann examples.

	V1	V2	V3	V2-SA	V3-SA
>99.99 %	16	23	23	24	27
98-99.99 %	1	44	52	4	6
75-98 %	10	23	21	17	25
25-75 %	11	22	20	26	22
0-25 %	91	17	13	58	49
Average Gap Closed	24.80%	76.49%	80.86%	44.40%	52.56%

Table 3 Marginal Value of Disjunctive Programming

with the convex-quadratic cuts, already closes around 95% of the duality gap. Hence, in order to better evaluate the performance of our cutting planes, we weakened the initial RLT relaxation (referred to as wRLT in the sequel) by removing the inequalities $y_{ii} \leq x_i$; these inequalities are envelope inequalities associated with the product term $y_{ii} = x_i x_i$.

Table 11 summarizes the experiments. The second column of the table reports the optimal value of the wRLT relaxation, whereas the third column of the table gives the value of the optimal solution as reported in [34]. Overall, Variant 1 closes substantially less gap than Variants 2 and 3. On average the amount of gap closed by Variant 1 is 46.81% while Variant 2 closes 65.28% and Variant 3 closes 71.51% .

Note that there are two ways of deriving a cut form a univariate expression $(c^T x)^2 \geq Y.cc^T$. First, we can relax $(c^T x)^2 \geq Y.cc^T$ to a disjunction (1), and embed the resulting disjunction in the framework of CGLP to derive a disjunctive cut, as we currently do. Second, we can directly use the secant inequality (2) to cut off the solution $(\hat{x}, \hat{Y})$ of the convex relaxation, if $(\hat{x}, \hat{Y})$ violates (2). While the former approach takes a holistic view of the problem, it is also computationally more expensive than the latter. Naturally, we are interested in the marginal value of disjunctive programming; i.e., how much do we gain by using disjunctive cuts derived from a computationally-expensive CGLP machinery, as compared to using the readily available secant inequality? In order to answer this question we conducted the following experiment on 129 GLOBALLib instances with non-zero duality gap. We modified our code so that once the univariate expression $(c^T x)^2 \geq Y.cc^T$ has been generated, we use the secant inequality (2) instead of invoking our disjunctive cut generator. Tables 12, 13 and 14 report the computational results, while Table 3 reports the same in summarized form. A suffix of "SA" indicates that the corresponding version of our code was modified to use the secant inequality (2) instead of invoking the disjunctive cut generator.

Two comments are in order. First, versions of the code that use the secant inequality do close a significant proportion of the gap, namely 44.40% and 52.56% with Variants 2 and 3, respectively. Second, using disjunctive cuts improves these numbers to 76.49% and 80.86%, respectively, thereby demonstrating the marginal benefits of disjunctive programming.

Note that Variants 2 and 3 also use the convex-quadratic cuts derived from eigenvectors of $\hat{Y} - \hat{x}\hat{x}^T$ with negative eigenvalues. Similar to the above experiment, we designed the following experiment to evaluate the marginal value of these convex-quadratic cuts in Variants 2 and 3. We modified our code so that convex quadratic cuts were not added in each iteration, and only disjunctive cuts were used to strengthen the initial formulation. Tables 15, 16 and 17 report the computational results, while Table 4 reports the same in summarized form. A suffix of "Dsj" indicates that the corresponding version of our code was modified to use only disjunctive cuts, and not the convex-quadratic cuts. Note that the absence of the convex-quadratic cuts severely affects the performance of variants 2 and 3.

The basic premise of our paper lies in generating valid cutting planes for **MIQCP** from the spectrum of $\hat{Y} - \hat{x}\hat{x}^T$. The results presented so far discuss the contribution of these cutting planes in

	V1	V2	V3	V2-Dsj	V3-Dsj
>99.99 %	16	23	23	1	1
98-99.99 %	1	44	52	29	33
75-98 %	10	23	21	10	10
25-75 %	11	22	20	29	24
0-25 %	91	17	13	60	61
Average Gap Closed	24.80%	76.49%	80.86%	41.54%	42.90%

Table 4 Marginal Value of Convex Quadratic Cuts

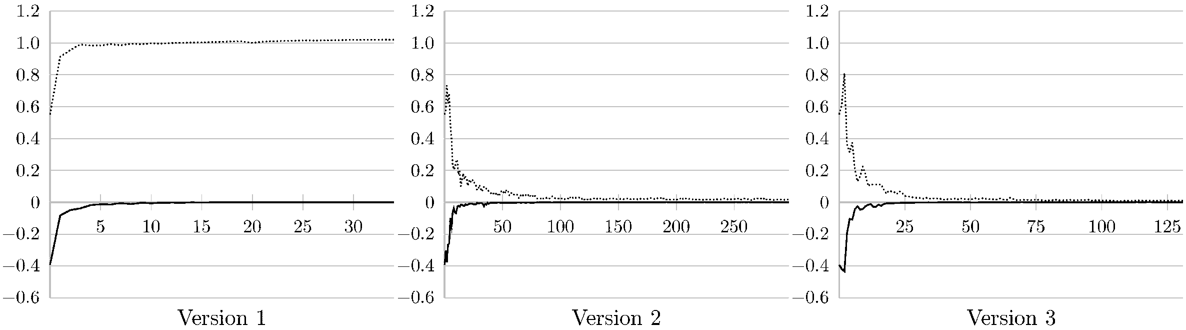

Figure 1 Plot of the sum of the positive and negative eigenvalues for st_jcbpaf2 with versions 1,2 and 3.

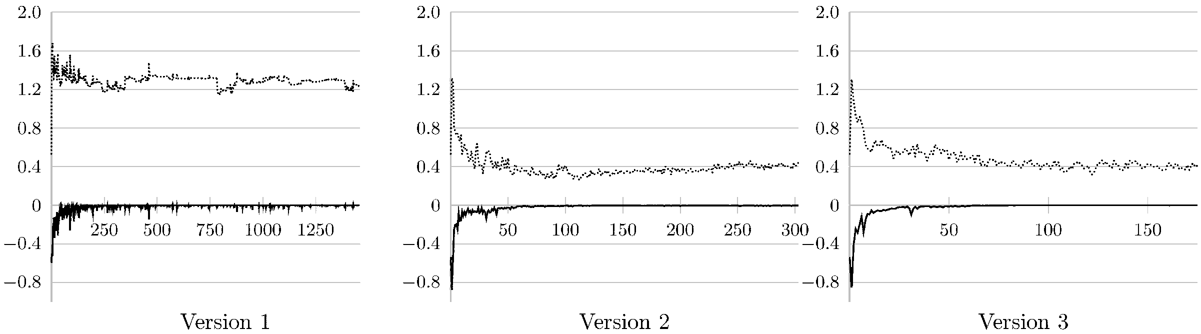

Figure 2 Plot of the sum of the positive and negative eigenvalues for ex_9_2_7 with versions 1,2 and 3.

reducing the duality gap; next we present detailed results on three instances to highlight their impact on the spectrum itself. For each one of the three instances, we report the sum of the positive and negative eigenvalues of $\hat{Y} - \hat{x}\hat{x}^T$ in each iteration of Variant 1, 2 and 3 of our algorithm. We chose one instance for each one the three characterizations listed in Table 5.

% Duality Gap closed by		
V1	V2	Instance Chosen
< 10 %	> 90 %	st_jcbpaf2
> 40%	< 60%	ex9_2_7
< 10%	< 10%	ex7_3_1

Table 5 Selection Criteria

Figures 1, 3 and 2 report the key results. The horizontal axis represents the number of iterations while the vertical axis reports the sum of the positive (broken line) and negative (solid line) eigenvalues of $\hat{Y} - \hat{x}\hat{x}^T$. Some remarks are in order.

First, the graph of the sum of negative eigenvalues converges to zero much faster than the corresponding graph for positive eigenvalues. This is not surprising since the problem of eliminating the

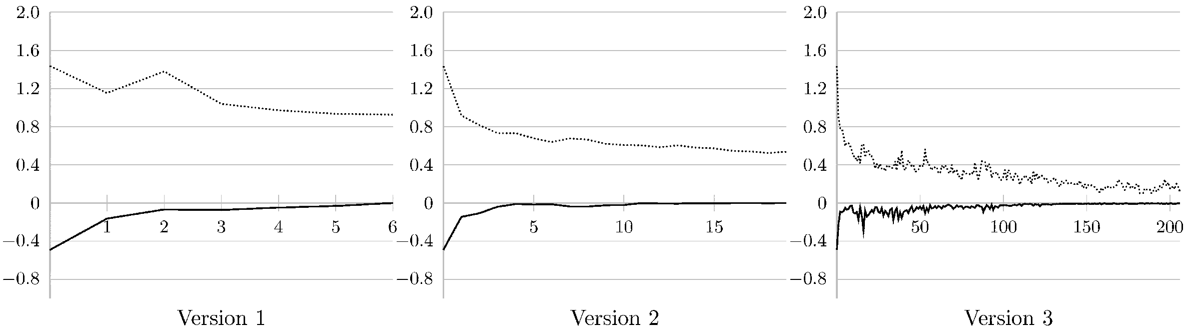

Figure 3 Plot of the sum of the positive and negative eigenvalues for ex_7_3_1 with versions 1,2 and 3.

negative eigenvalues is a convex programming problem, namely a SDP; our approach of adding convex-quadratic cuts is just an iterative cutting-plane based technique to impose the $Y - xx^T \succcurlyeq 0$ condition. Second, Variant 1 has a widely varying effect on the sum of positive eigenvalues of $Y - xx^T$. This is to be expected since the $Y - xx^T \succcurlyeq 0$ condition imposes no constraint on the positive eigenvalues of $Y - xx^T$. Furthermore, the sum of positive eigenvalues represents the part of the non-convexity of **MIQCP** that is not captured by the SDP relaxation. Third, consider the graphs corresponding to Variants 2 and 3 for the st_jcbpaf2 instance. Note that for both of these variants, the sum of positive eigenvalues decays to zero — albeit, the rate of decay is much higher for Variant 3 than for Variant 2. This lends support to the observation that Variant 3 is able to close a higher fraction of the duality gap in a fewer number of iterations, as compared to Variant 2. The same inference can be obtained by a careful examination of tables 1, 6, 7 and 8; despite being more computationally demanding, Variant 3 is able to close more duality gap than Variants 1 and 2 (on average) while operating under the common time-limit of 60 minutes.

6 Conclusion

Since the mid 90's, SDP relaxations of certain combinatorial problems have received considerable attention (for example, see [10,11,12,13,18]). Subsequently, the SDP relaxation of MIQCP has been extensively studied, both in the theoretical and computational communities. While researchers concentrated on exploring the strengths and weaknesses of the convex constraint $Y - xx^T \succcurlyeq 0$, a detailed investigation of its non-convex alter ego $xx^T - Y \succcurlyeq 0$ had remained an unchartered territory.

In this paper, we have described novel techniques for combining ideas from disjunctive programming, lift-and-project methodology and spectral theory to generate cutting planes for **MIQCP** that exploit the non-convex constraint $xx^T - Y \succcurlyeq 0$. We introduced the notion of univariate expressions and discussed techniques for deriving them from eigenvectors of $\hat{Y} - \hat{x}\hat{x}^T$ with positive eigenvalues. These univariate expressions were used to derive valid disjunctions which in turn were embedded in the CGLP machinery to derive disjunctive cuts. We noticed the importance of the width of a univariate expression and designed a MIP model to extract those such expressions having smaller width by taking combinations of eigenvectors. All of the ideas presented in this paper were tested on a test-bed of **MIQCP** problem instances comprising more than 200 instances. While the computational results corroborated the usefulness of these ideas, the discussion on marginal contribution of disjunctive programming and convex quadratic cuts further deepened our understanding of relaxations for **MIQCP**.

Interestingly, many of the ideas presented in this paper can be used to derive disjunctive cuts for arbitrary non-convex MINLPs. For instance, consider a MINLP that includes a univariate non-convex inequality of the form $y \leq f(x)$, where $f : \mathbb{R} \to \mathbb{R}$ is a convex function, $x, y \in \mathbb{R}$ and $L \leq x \leq U$. In this case, one can derive valid disjunctions by splitting the range of x into two intervals, say $[L, \theta]$ and $[\theta, U]$ (where $\theta \in (L, U)$), and deriving secant approximations of $f(x)$ in each one of the intervals. The resulting pair of secant inequalities represent a valid disjunction for the MINLP, which in turn can be used with a linear outer-approximation of the MINLP to derive disjunctive cuts via the CGLP machinery. The key challenge lies in choosing a univariate inequality $y \leq f(x)$ that is likely to give rise to strong disjunctive cuts; as we have demonstrated in this paper, at least for the case of **MIQCP**, univariate inequalities derived from eigenvectors of $\hat{Y} - \hat{x}\hat{x}^T$ seem to work well.

Finally, we would like to emphasize that all of the relaxations of **MIQCP′** derived in this paper are defined in the extended space obtained by introducing the y_{ij} variables. While these additional y_{ij} variables enhance the expressive power of our cutting planes, they also increase the size of the formulation drastically resulting in a huge computational overhead which is incurred at every node of the branch-and-bound tree. Ideally, we would like to extract the strength of these extended reformulations in the form of cutting planes that are defined only in the space of the x variables. Systematic approaches for constructing such convex relaxations of **MIQCP′** constitute the topic of our forthcoming paper [24].

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References

1. K.M. Anstreicher, Semidefinite Programming versus the Reformulation-Linearization Technique for Nonconvex Quadratically Constrained Quadratic Programming. Pre-print, Optimization Online, May 2007.
2. E. Balas, Disjunctive programming: properties of the convex hull of feasible points. *Discrete Applied Mathematics*, 89(1-3), 1998, 3–44.
3. E. Balas, S. Ceria, and G. Cornuéjols. A lift-and-project cutting plane algorithm for mixed 0-1 programs. *Mathematical Programming*, 58, 1993, 295–324.
4. E. Balas. Projection and Lifting in Combinatorial Optimization. In: M. Juenger and D. Naddef (editors), "Computational Combinatorial Optimization: Optimal or Provably Near-Optimal Solutions," Springer, *Lecture Notes in Computer Science,* volume 2241, 2001, 26–56.
5. E. Balas and A. Saxena. Optimizing over the split closure. *Mathematical Programming*, 113(2), 2008, 219–240.
6. E. Balas, J. Tama and J. Tind. Sequential convexification in reverse convex and disjunctive programming. *Mathematical Programming*, 44, 1989, 337–350.
7. P. Bonami, L.T. Biegler, A.R. Conn, G. Cornuéjols, I.E. Grossmann, C.D. Laird, J. Lee, A. Lodi, F. Margot, N. Sawaya and A. Wächter. An Algorithmic Framework for Convex Mixed-integer Nonlinear Programs. *Discrete Optimization* 5, 2008, 186-204.
8. S. Burer and D. Vandenbussche. A finite branch-and-bound algorithm for nonconvex quadratic programming via semidefinite relaxations. To appear in *Mathematical Programming.*
9. G. Cornuéjols. Revival of the Gomory cuts in the 1990's. *Annals of Operations Research*, 149(1), 2007, 63–66.
10. C. Delorme and S. Poljak. Laplacian eigenvalues and the maximum cut problem. *Mathematical Programming, Series A*, 62(3), 1993, 557–574.
11. C. Helmberg, S. Poljak, F. Rendl and H. Wolkowicz, Combining semidefinite and polyhedral relaxations for integer programs. "Integer programming and combinatorial optimization (Copenhagen, 1995)," 124–134, *Lecture Notes in Comput. Sci.*, 920, Springer, Berlin, 1995.
12. M. Laurent and S. Poljak. On a positive semidefinite relaxation of the cut polytope. Special issue honoring Miroslav Fiedler and Vlastimil Ptk. *Linear Algebra and its Applications*, 223/224, 1995, 439–461.
13. M. Laurent and S. Poljak. On the facial structure of the set of correlation matrices. *SIAM J. Matrix Anal. Appl.*, 17(3), 1996, 530–547.
14. M. Fischetti and A. Lodi. Optimizing over the first Chvatal closure. *Mathematical Programming*, 110(1), 2007, 2007.
15. R. Fletcher and S. Leyffer. User Manual for FilterSQP. *Numerical Analysis Report NA/181*, Dundee University, 1998.
16. R.G. Jeroslow. There cannot be any algorithm for integer programming with quadratic constraints. *Operations Research*, 21(1), 1973, 221–224.
17. GLOBALLib, `www.gamsworld.org/global/globallib/globalstat.htm`
18. M.X. Goemans and D.P. Williamson. Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming. *J. ACM*, 42, 1995, 1115–1145.
19. S. Kim and M. Kojima. Second order cone programming relaxation of nonconvex quadratic optimization problems. *Optimization Methods and Software*, 15, 2001, 201-204.

20. S. Lee and I.E. Grossmann. A global optimization algorithm for nonconvex generalized disjunctive programming and applications to process systems. *Computers and Chemical Engineering*, 25, 2001, 1675–1697.
21. T. Matsui. NP-hardness of linear multiplicative programming and related problems. *Journal of Global Optimization*, 9, 1996, 113–119.
22. G.P. McCormick. Computability of global solutions to factorable nonconvex programs: Part I Convex underestimating problems. *Mathematical Programming*, 10, 1976, 147–175.
23. A. Saxena, P. Bonami and J. Lee. Disjunctive Cuts for Non-Convex Mixed Integer Quadratically Constrained Problems. In: Integer programming and combinatorial optimization (Bertinoro, 2008)," A. Lodi, A. Panconesi, and G. Rinaldi, Eds., *Lecture Notes in Computer Science,* volume 5035, pp. 17-33. Springer-Verlag Berlin Heidelberg, 2008.
24. A. Saxena, P. Bonami and J. Lee. Convex Relaxations of Mixed Integer Quadratically Constrained Programs: Projected Formulations. In Preparation.
25. A. Saxena, V. Goyal and M. Lejeune. MIP Reformulations of the Probabilistic Set Covering Problem, To appear in *Mathematical Programming*.
26. S. Sen. Relaxations for probabilistically constrained programs with discrete random variables, *Operations Research Letters* 11(2), 1992, 81–86.
27. H.D. Sherali and W.P. Adams. A reformulation-linearization technique for solving discrete and continuous nonconvex problems. Kluwer, Dordecht 1998.
28. H.D. Sherali and B.M.P. Fraticelli. Enhancing RLT relaxations via a new class of semidefinite cuts. *J. Global Optimization* 22, 2002, 233–261.
29. R. Stubbs and S. Mehrotra. A branch-and-cut method for 0-1 mixed convex programming, *Mathematical Programming*, 86, 515-532 (1999).
30. J.F. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11–12, 1999, 625–653.
31. M. Tawarmalani and N.V. Sahinidis. "Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming: Theory, Algorithms, Software, and Applications." Kluwer Academic Publishers, Boston MA, 2002.
32. M. Tawarmalani and N.V. Sahinidis. Global optimization of mixed integer nonlinear programs: A theoretical and computational study. *Mathematical Programming*, 99(3), 2004, 563–591.
33. D. Vandenbussche and G.L. Nemhauser. A polyhedral study of nonconvex quadratic programs with box constraints. *Mathematical Programming*, 102(3), 2005, 531–556.
34. D. Vandenbussche and G.L. Nemhauser. A branch-and-cut algorithm for nonconvex quadratic programs with box constraints. *Mathematical Programming*, 102(3), 2005, 559–575.
35. A. Wächter and L.T. Biegler. On the implementation of a primal-dual interior point filter line search algorithm for large-scale nonlinear programming. *Mathematical Programming*, 106(1), 2006, 25–57.

Appendix

			% Duality Gap Closed			Time(sec)		
Instance	**RLT**	**OPT**	**V1**	**V2**	**V3**	**V1**	**V2**	**V3**
alkyl	-2.7634	-1.7650	0.00	55.83	63.75	10.621	3619.874	3693.810
circle	0.0000	4.5742	45.74	99.89	99.84	0.218	0.456	0.664
dispatch	3101.2805	3155.2879	100.00	100.00	100.00	0.044	0.052	0.066
ex2_1_1	-18.9000	-17.0000	0.00	72.62	99.92	0.009	704.400	17.835
ex2_1_10	39668.0556	49318.0180	22.05	99.37	99.82	6.719	29.980	70.168
ex2_1_5	-269.4528	-268.0146	0.00	99.98	99.99	0.020	0.173	0.188
ex2_1_6	-44.4000	-39.0000	0.00	99.95	99.97	0.023	3397.650	54.326
ex2_1_7	-6031.9026	-4150.4101	0.00	41.17	45.58	0.188	3607.439	3763.506
ex2_1_8	-82460.0000	15639.0000	0.00	84.70	92.75	0.491	3632.275	3627.700
ex2_1_9	-2.2000	-0.3750	0.00	98.79	99.73	0.140	1587.940	3615.766
ex3_1_1	2533.2008	7049.2480	0.00	15.94	22.13	1.391	3600.268	3681.021
ex3_1_2	-30802.7563	-30665.5387	49.74	99.99	99.99	0.035	0.083	0.108
ex3_1_3	-440.0000	-310.0000	0.00	99.99	99.99	0.013	0.064	0.096
ex3_1_4	-6.0000	-4.0000	0.00	86.31	99.57	0.009	21.261	581.295
ex4_1_1	-173688.7998	-7.4873	100.00	100.00	100.00	0.287	0.310	0.444
ex4_1_3	-7999.4583	-443.6717	56.40	93.54	99.86	0.080	0.285	0.552
ex4_1_4	-200.0000	0.0000	100.00	100.00	100.00	0.247	0.243	0.532
ex4_1_6	-24075.0002	7.0000	100.00	100.00	100.00	0.185	0.308	0.508
ex4_1_7	-206.2500	-7.5000	100.00	100.00	100.00	0.128	0.114	0.165
ex4_1_8	-29.0000	-16.7389	100.00	100.00	100.00	0.043	0.059	0.103
ex4_1_9	-6.9867	-5.5080	0.00	43.59	37.48	0.008	1.307	1.273
ex5_2_2_case1	-599.8996	-400.0000	0.00	0.00	0.00	0.011	0.016	0.935
ex5_2_2_case2	-1200.0000	-600.0000	0.00	0.00	0.00	0.021	0.047	0.511
ex5_2_2_case3	-875.0000	-750.0000	0.00	0.36	0.31	0.016	0.358	0.474
ex5_2_4	-2933.3334	-450.0000	0.00	79.31	99.92	0.046	68.927	1044.400
ex5_2_5	-9700.0001	-3500.0001	0.00	6.27	6.37	1.825	3793.169	3618.084
ex5_3_2	0.9979	1.8642	0.00	7.27	21.00	0.355	245.821	3672.529
ex5_3_3	1.6313	3.2340	0.00	0.21	0.18	3764.946	3693.758	7511.839
ex5_4_2	2598.2452	7512.2301	0.00	27.57	26.41	1.141	3614.376	3866.626
ex7_3_1	0.0000	0.3417	0.00	0.00	85.43	0.313	5.582	3622.223
ex7_3_2	0.0000	1.0899	0.00	59.51	70.26	0.788	3609.704	3614.759
ex8_1_3	-7.7486E+12	1.0000	0.04	0.04	0.00	0.509	0.494	0.641
ex8_1_4	-13.0000	0.0000	100.00	100.00	100.00	0.020	0.038	0.051
ex8_1_5	-3.3333	0.0000	68.30	68.97	68.96	0.839	1.246	100.476
ex8_1_7	-757.5775	0.0293	77.43	77.43	95.79	75.203	75.203	3615.517
ex8_1_8	-0.8466	-0.3888	0.00	76.49	90.88	7.722	3607.682	3628.366
ex8_4_1	-5.0000	0.6186	91.84	91.09	86.49	3659.232	3642.131	4180.427
ex8_4_2	-5.0000	0.4852	94.07	93.04	87.87	3641.875	3606.071	3757.098
ex9_1_4	-63.0000	-37.0000	0.00	0.00	1.55	0.077	0.603	244.126
ex9_2_1	-16.0000	17.0000	54.54	60.04	92.02	3603.428	2372.638	3622.960

Table 6 GLOBALLib Instances with non-zero Duality Gap (Part 1)

			% Duality Gap Closed			Time(sec)		
Instance	**RLT**	**OPT**	**V1**	**V2**	**V3**	**V1**	**V2**	**V3**
ex9_2_2	-50.0000	100.0000	70.37	88.29	98.06	1227.898	3606.357	3610.411
ex9_2_3	-30.0000	0.0000	0.00	0.00	47.17	0.125	3.819	3625.114
ex9_2_4	-396.0000	0.5000	99.87	99.87	99.89	2.801	8.897	5.258
ex9_2_6	-406.0000	-1.0000	87.23	87.93	62.00	851.127	2619.018	1058.376
ex9_2_7	-9.0000	17.0000	42.31	51.47	86.25	3602.364	3628.249	3627.920
himmel11	-30802.7566	-30665.5387	49.74	99.99	99.99	0.053	0.082	0.120
house	-5230.5433	-4500.0000	0.00	86.93	97.92	0.435	12.873	149.678
hydro	4019717.9291	4366944.1597	100.00	100.00	100.00	8.354	20.668	191.447
mathopt1	-912909.0091	1.0000	100.00	100.00	100.00	1.727	2.448	3.770
mathopt2	-11289.0001	0.0000	100.00	100.00	100.00	0.351	0.229	0.400
meanvar	0.0000	5.2434	100.00	100.00	100.00	0.179	0.276	0.657
nemhaus	0.0000	31.0000	53.97	100.00	100.00	0.836	0.198	0.355
prob05	0.3151	0.7418	0.00	99.78	99.49	0.007	0.165	0.173
prob06	1.0000	1.1771	100.00	100.00	100.00	0.023	0.024	0.031
prob09	-100.0000	0.0000	100.00	99.99	100.00	0.582	0.885	1.689
process	-2756.5935	-1161.3366	7.68	88.05	95.03	6.379	3620.085	3611.299
qp1	-1.4313	0.0008	85.76	89.12	81.23	3659.085	3897.521	3700.918
qp2	-1.4313	0.0008	86.13	89.15	83.06	3643.188	4047.592	4255.863
rbrock	-659984.0066	-5.6733	100.00	100.00	100.00	0.353	3.194	5.611
st_bpaf1a	-46.0058	-45.3797	0.00	81.73	88.52	0.049	0.894	3.790
st_bpaf1b	-43.1255	-42.9626	0.00	90.73	92.86	0.047	3.299	12.166
st_bpv2	-11.2500	-8.0000	0.00	99.99	99.99	0.033	0.029	0.034
st_bsj2	-0.6260	1.0000	0.00	99.98	99.96	0.009	1.974	2.235
st_bsj3	-86768.5509	-86768.5500	0.00	0.00	0.00	0.012	0.011	0.011
st_bsj4	-72700.0507	-70262.0500	0.00	99.86	99.80	0.014	1.715	1.384
st_e02	171.4185	201.1591	0.00	99.88	99.95	0.008	0.095	0.118
st_e03	-2381.8947	-1161.3366	29.58	91.63	92.82	715.006	3639.297	3613.883
st_e05	3826.3885	7049.2493	0.00	50.43	58.38	0.194	16.217	41.354
st_e06	0.0000	0.1609	0.00	0.00	0.00	0.215	0.726	1.911
st_e07	-500.0000	-400.0000	0.00	99.97	99.97	0.042	0.350	0.383
st_e08	0.3125	0.7418	0.00	99.81	99.89	0.008	0.208	0.171
st_e09	-0.7500	-0.5000	0.00	92.58	92.58	0.012	0.014	0.018
st_e10	-29.0000	-16.7389	100.00	100.00	100.00	0.036	0.045	0.069
st_e18	-3.0000	-2.8284	100.00	100.00	100.00	0.015	0.018	0.022
st_e19	-879.7500	-86.4222	93.50	95.21	95.18	0.373	0.613	0.991
st_e20	-0.8466	-0.3888	0.00	76.38	90.88	7.409	3610.271	3623.275
st_e23	-3.0000	-1.0833	0.00	98.40	98.40	0.011	0.087	0.108
st_e24	0.0000	3.0000	0.00	99.81	99.81	0.007	0.501	0.657
st_e25	0.2473	0.8902	87.20	100.00	100.00	0.312	0.161	0.247
st_e26	-513.0000	-185.7792	0.00	99.96	99.96	0.006	0.036	0.050

Table 7 GLOBALLib Instances with non-zero Duality Gap (Part 2)

			% Duality Gap Closed			Time(sec)		
Instance	**RLT**	**OPT**	**V1**	**V2**	**V3**	**V1**	**V2**	**V3**
st_e28	-30802.7566	-30665.5387	49.74	99.99	99.99	0.051	0.088	0.118
st_e30	-3.0000	-1.5811	0.00	0.00	0.00	0.014	0.035	6.489
st_e33	-500.0000	-400.0000	0.00	99.94	99.95	0.047	0.457	0.382
st_fp1	-18.9000	-17.0000	0.00	72.62	99.92	0.009	658.824	18.013
st_fp5	-269.4528	-268.0146	0.00	99.98	99.99	0.018	0.175	0.180
st_fp6	-44.4000	-39.0000	0.00	99.92	99.97	0.025	3603.767	54.613
st_fp7a	-435.5237	-354.7506	0.00	45.13	53.58	0.151	806.493	1801.106
st_fp7b	-715.5237	-634.7506	0.00	22.06	55.51	0.153	11.941	3610.617
st_fp7c	-10310.4738	-8695.0122	0.00	44.26	57.10	0.181	3621.180	3672.666
st_fp7d	-195.5237	-114.7506	0.00	50.03	55.53	0.111	3627.749	3734.806
st_fp8	7219.4999	15639.0000	0.00	0.83	3.17	0.331	4.911	88.867
st_glmp_fp2	7.0681	7.3445	0.00	45.70	49.74	0.009	0.732	1.170
st_glmp_kk92	-13.3548	-12.0000	0.00	99.98	99.98	0.023	0.038	0.053
st_glmp_kky	-3.0000	-2.5000	0.00	99.80	99.71	0.011	0.133	0.248
st_glmp_ss1	-38.6667	-24.5714	0.00	89.30	89.30	0.031	0.556	0.736
st_ht	-2.8000	-1.6000	0.00	99.81	99.89	0.006	0.142	0.451
st_iqpbk1	-1722.3760	-621.4878	97.99	99.86	99.99	3.825	5.086	286.844
st_iqpbk2	-3441.9520	-1195.2257	97.93	100.00	100.00	2.515	31.614	243.169
st_jcbpaf2	-945.4511	-794.8559	0.00	99.47	99.61	2.650	3622.733	3636.491
st_jcbpafex	-3.0000	-1.0833	0.00	98.40	98.40	0.012	0.085	0.114
st_kr	-104.0000	-85.0000	0.00	99.93	99.95	0.008	0.090	0.131
st_m1	-505191.3385	-461356.9389	0.00	99.96	99.96	0.222	368.618	756.237
st_m2	-938513.6772	-856648.8187	0.00	70.19	58.99	1.226	3641.449	3876.446
st_pan1	-5.6850	-5.2837	0.00	99.72	99.92	0.007	0.926	0.771
st_pan2	-19.4000	-17.0000	0.00	68.54	99.91	0.009	3038.430	26.401
st_ph1	-243.8112	-230.1173	0.00	99.98	99.98	0.011	0.225	0.059
st_ph11	-11.7500	-11.2813	0.00	99.46	98.19	0.007	0.910	0.337
st_ph12	-23.5000	-22.6250	0.00	99.49	99.62	0.006	0.353	0.311
st_ph13	-11.7500	-11.2813	0.00	99.38	98.80	0.009	0.751	0.703
st_ph14	-231.0000	-229.7222	0.00	99.85	99.86	0.010	0.051	0.131
st_ph15	-434.7346	-392.7037	0.00	99.83	99.81	0.009	0.476	0.541
st_ph2	-1064.4960	-1028.1173	0.00	99.98	99.98	0.014	0.159	0.062
st_ph20	-178.0000	-158.0000	0.00	99.98	99.98	0.007	0.036	0.049
st_ph3	-447.8488	-420.2348	0.00	99.98	99.98	0.011	0.031	0.039
st_phex	-104.0000	-85.0000	0.00	99.96	99.96	0.007	0.088	0.088
st_qpc-m0	-6.0000	-5.0000	0.00	99.96	99.96	0.007	0.015	0.023
st_qpc-m1	-612.2714	-473.7778	0.00	99.99	99.98	0.009	0.223	0.233
st_qpc-m3a	-725.0518	-382.6950	0.00	98.10	99.16	0.025	3615.442	3727.123
st_qpc-m3b	-24.6757	0.0000	0.00	100.00	100.00	0.021	0.566	1.648
st_qpk1	-11.0000	-3.0000	0.00	99.98	99.98	0.007	0.110	0.053
st_qpk2	-21.0000	-12.2500	0.00	71.34	83.33	0.025	3599.788	3622.692
st_qpk3	-66.0000	-36.0000	0.00	33.53	50.04	0.077	3621.930	3778.200
st_rv1	-64.2359	-59.9439	0.00	96.19	98.44	0.023	3607.723	3602.339
st_rv2	-73.0007	-64.4807	0.00	88.79	81.85	0.079	3601.528	44.550
st_rv3	-38.5155	-35.7607	0.00	40.40	72.68	0.108	112.028	3807.828
st_rv7	-148.9816	-138.1875	0.00	45.43	62.28	0.269	3640.861	3880.783
st_rv8	-143.5829	-132.6616	0.00	29.90	45.80	0.663	3696.452	3874.801
st_rv9	-134.9131	-120.1164	0.00	20.56	31.64	1.019	3920.213	3675.654
st_z	-0.9674	0.0000	0.00	99.96	99.95	0.009	2.749	0.790

Table 8 GLOBALLib Instances with non-zero Duality Gap (Part 3)

			Spectral Norm of $Y - xx^T$		
Instance	**RLT**	**Opt**	**V1**	**V2**	**V3**
st_e17	0.0019	0.0019	0.000000	0.000000	0.000000
st_qpc-m3c	0.0000	0.0000	0.000000	0.000000	0.000000
st_qpc-m4	0.0000	0.0000	0.000000	0.000000	0.000000
ex2_1_2	-213.0000	-213.0000	0.000000	0.000000	0.000000
ex2_1_4	-11.0000	-11.0000	0.000000	0.000000	0.000000
st_e42	18.7842	18.7842	0.000000	0.000000	0.000000
st_fp2	-213.0000	-213.0000	0.000000	0.000000	0.000000
st_fp4	-11.0000	-11.0000	0.000000	0.000000	0.000000
st_bpk1	-13.0000	-13.0000	0.000000	0.000000	0.000000
st_bpk2	-13.0000	-13.0000	0.000000	0.000000	0.000000
st_glmp_fp1	10.0000	10.0000	0.000000	0.000000	0.000000
st_ph10	-10.5000	-10.5000	0.000000	0.000000	0.000000
st_bpv1	10.0000	10.0000	0.027262	0.000007	0.000007
st_glmp_ss2	3.0000	3.0000	0.043577	0.000021	0.000021
st_glmp_kk90	3.0000	3.0000	0.021689	0.000022	0.000022
st_e34	0.0156	0.0156	0.064299	0.000030	0.000029
st_e01	-6.6667	-6.6667	0.056653	0.000046	0.000046
st_fp3	-15.0000	-15.0000	0.293089	0.302328	0.000139
ex2_1_3	-15.0000	-15.0000	0.297487	0.000962	0.000150
st_glmp_fp3	-12.0000	-12.0000	0.000637	0.000235	0.000235
ex14_1_2	0.0000	0.0000	0.171873	0.171873	0.001654
ex14_1_5	0.0000	0.0000	0.146196	0.229286	0.103878
ex14_1_6	0.0000	0.0000	0.182808	0.208698	0.219895
st_robot	0.0000	0.0000	0.230963	0.227246	0.215491

Table 9 GLOBALLib Instances with zero Duality Gap

			% Duality Gap Closed		Time (sec)	
Instance	**RLT**	**Opt**	**V2**	**V3**	**V2**	**V3**
ex9_1_4	-63.0000	-37.0000	100.00	99.97	2.462	22.418
ex9_2_1	-16.0000	17.0000	99.95	99.95	3609.323	2351.308
ex9_2_2	-50.0000	100.0000	100.00	100.00	401.642	743.086
ex9_2_3	-30.0000	0.0000	99.99	99.99	27.718	522.123
ex9_2_4	-396.0000	0.5000	99.99	100.00	3.547	5.136
ex9_2_6	-406.0000	-1.0000	80.22	92.09	338.001	3652.873
ex9_2_7	-9.0000	17.0000	99.97	99.95	3607.258	3478.207

Table 10 GLOBALLib Instances with Linear Complementarity Constraints

			% Duality Gap Closed			Time (sec)		
Instance	wRLT	OPT	V1	V2	V3	V1	V2	V3
spar020-100-1	-1137	-706.5	58.66	95.40	99.64	3635.459	3638.200	3646.691
spar030-090-3	-2619.5	-1494	60.25	86.37	92.68	3730.348	3701.849	3607.885
spar040-060-2	-3011	-2004.23	43.05	55.79	61.63	3813.728	3707.992	3879.912
spar020-100-2	-1328.5	-856.5	70.36	93.08	97.81	3629.580	3636.665	3634.559
spar030-100-1	-2683.5	-1227.13	59.97	81.10	87.48	3647.126	3692.504	3624.834
spar040-060-3	-3532	-2454.5	56.60	72.63	79.30	3688.747	3764.079	3716.242
spar020-100-3	-1224	-772	70.70	97.47	99.97	3609.973	3632.560	3621.301
spar030-100-2	-2870.5	-1260.5	50.56	72.87	82.52	3662.868	3697.329	3753.816
spar040-070-1	-3194.5	-1605	53.82	64.03	70.28	3716.161	3642.681	3929.653
spar030-060-1	-1472.5	-706	32.55	60.00	73.32	3685.753	3823.051	3742.955
spar030-100-3	-2831.5	-1511.05	63.32	84.10	90.29	3712.164	3606.496	3682.094
spar040-070-2	-3446.5	-1867.5	45.84	57.91	63.86	3695.329	3756.377	3767.655
spar030-060-2	-1741	-1377.17	62.19	91.16	93.04	3731.242	3715.979	3748.334
spar040-030-1	-1162	-839.5	14.16	31.05	42.21	3694.667	3719.223	3874.422
spar040-070-3	-3833.5	-2436.5	50.57	62.94	69.89	3783.908	3693.666	3656.632
spar030-060-3	-2073.5	-1293.5	53.27	77.41	85.36	3666.710	3696.495	3702.028
spar040-030-2	-1695	-1429	13.92	27.74	31.29	3814.827	3937.898	3910.581
spar040-080-1	-3969	-1838.5	42.80	58.37	64.47	3710.865	3808.258	3811.056
spar030-070-1	-1647	-654	30.74	57.39	70.49	3685.224	3786.025	3679.571
spar040-030-3	-1322	-1086	2.35	28.00	34.74	3639.965	3798.683	4079.434
spar040-080-2	-3902.5	-1952.5	51.27	66.96	71.16	3667.295	4062.433	3845.179
spar030-070-2	-1989.5	-1313	61.19	86.60	92.26	3642.745	3708.212	3653.440
spar040-040-1	-1641	-837	17.42	33.31	37.70	3689.320	3817.844	3883.183
spar040-080-3	-4440	-2545.5	61.18	72.31	77.20	3703.711	4057.149	3806.478
spar030-070-3	-2367.5	-1657.4	73.58	88.66	92.85	3680.997	3744.044	3731.627
spar040-040-2	-1967.5	-1428	24.27	35.19	39.92	3839.449	3968.111	3667.330
spar040-090-1	-4490	-2135.5	54.63	66.64	72.50	3715.925	3781.044	3977.672
spar030-080-1	-2189	-952.729	41.71	69.67	78.41	3706.572	3600.777	3715.601
spar040-040-3	-2089	-1173.5	14.76	26.71	30.88	3718.280	3972.902	4002.336
spar040-090-2	-4474	-2113	55.86	66.46	70.59	3815.415	3931.349	3615.504
spar030-080-2	-2316	-1597	53.96	86.25	92.48	3690.453	3627.132	3702.961
spar040-050-1	-2204	-1154.5	23.12	36.72	43.34	3750.454	3819.720	3619.095
spar040-090-3	-4641	-2535	61.08	73.49	78.86	3808.143	4003.706	3777.561
spar030-080-3	-2504.5	-1809.78	69.28	91.42	95.70	3642.447	3666.392	3735.913
spar040-050-2	-2403.5	-1430.98	27.17	40.87	48.62	3738.085	3610.640	3757.075
spar040-100-1	-5118	-2476.38	65.26	76.24	79.10	3848.559	3853.573	3631.410
spar030-090-1	-2521	-1296.5	54.64	81.15	89.47	3702.696	3676.815	3657.596
spar040-050-3	-2715	-1653.63	20.75	33.95	43.11	3709.104	3639.977	3865.383
spar040-100-2	-5043	-2102.5	54.47	63.89	70.40	3759.668	3658.261	3771.344
spar030-090-2	-2755	-1466.84	56.33	82.66	88.79	3658.607	3646.756	3663.516
spar040-060-1	-2934	-1322.67	35.83	47.75	54.57	3648.720	3760.964	3724.381
spar040-100-3	-5196.5	-1866.07	52.41	59.92	65.08	3712.925	3842.685	3950.384

Table 11 Box QP Instances

			% Duality Gap Closed				Time(sec)			
Instance	**RLT**	**OPT**	**V2**	**V3**	**V2-SA**	**V3-SA**	**V2**	**V3**	**V2-SA**	**V3-SA**
alkyl	-2.7634	-1.7650	55.83	63.75	7.24	47.84	3619.874	3693.810	2171.990	3674.346
circle	0.0000	4.5742	99.89	99.84	99.96	99.72	0.456	0.664	0.306	0.391
dispatch	3101.2805	3155.2879	100.00	100.00	100.00	100.00	0.052	0.066	0.044	0.046
ex2_1_1	-18.9000	-17.0000	72.62	99.92	0.00	0.00	704.400	17.835	0.009	0.010
ex2_1_10	39668.0556	49318.0180	99.37	99.82	28.74	64.97	29.980	70.168	12.681	45.972
ex2_1_5	-269.4528	-268.0146	99.98	99.99	28.31	49.71	0.173	0.188	0.095	0.112
ex2_1_6	-44.4000	-39.0000	99.95	99.97	38.94	64.77	3397.650	54.326	0.066	0.297
ex2_1_7	-6031.9026	-4150.4101	41.17	45.58	0.00	0.72	3607.439	3763.506	0.672	410.503
ex2_1_8	-82460.0000	15639.0000	84.70	92.75	17.50	95.74	3632.275	3627.700	1.774	3675.957
ex2_1_9	-2.2000	-0.3750	98.79	99.73	99.97	99.98	1587.940	3615.766	3.962	39.335
ex3_1_1	2533.2008	7049.2480	15.94	22.13	0.03	0.68	3600.268	3681.021	45.468	1499.937
ex3_1_2	-30802.7563	-30665.5387	99.99	99.99	49.74	49.74	0.083	0.108	0.032	0.045
ex3_1_3	-440.0000	-310.0000	99.99	99.99	100.00	100.00	0.064	0.096	0.036	0.047
ex3_1_4	-6.0000	-4.0000	86.31	99.57	0.00	0.00	21.261	581.295	0.007	0.012
ex4_1_1	-173688.7998	-7.4873	100.00	100.00	100.00	100.00	0.310	0.444	0.275	0.294
ex4_1_3	-7999.4583	-443.6717	93.54	99.86	82.74	97.42	0.285	0.552	0.193	0.359
ex4_1_4	-200.0000	0.0000	100.00	100.00	100.00	100.00	0.243	0.532	0.199	0.401
ex4_1_6	-24075.0002	7.0000	100.00	100.00	100.00	100.00	0.308	0.508	0.193	0.267
ex4_1_7	-206.2500	-7.5000	100.00	100.00	100.00	100.00	0.114	0.165	0.136	0.207
ex4_1_8	-29.0000	-16.7389	100.00	100.00	100.00	100.00	0.059	0.103	0.051	0.080
ex4_1_9	-6.9867	-5.5080	43.59	37.48	3.82	3.82	1.307	1.273	0.029	0.042
ex5_2_2_case1	-599.8996	-400.0000	0.00	0.00	0.00	0.00	0.016	0.935	0.013	0.068
ex5_2_2_case2	-1200.0000	-600.0000	0.00	0.00	0.00	0.00	0.047	0.511	0.020	0.089
ex5_2_2_case3	-875.0000	-750.0000	0.36	0.31	0.00	0.02	0.358	0.474	0.074	0.204
ex5_2_4	-2933.3334	-450.0000	79.31	99.92	63.22	82.02	68.927	1044.400	1.536	9.487
ex5_2_5	-9700.0001	-3500.0001	6.27	6.37	0.00	1.59	3793.169	3618.084	3691.723	3692.598
ex5_3_2	0.9979	1.8642	7.27	21.00	6.15	15.13	245.821	3672.529	118.851	1127.986
ex5_3_3	1.6313	3.2340	0.21	0.18	0.00	0.00	3693.758	7511.839	3856.422	6574.641
ex5_4_2	2598.2452	7512.2301	27.57	26.41	0.00	1.67	3614.376	3866.626	3.387	2873.420
ex7_3_1	0.0000	0.3417	0.00	85.43	0.00	0.00	5.582	3622.223	0.303	4.036
ex7_3_2	0.0000	1.0899	59.51	70.26	0.00	0.00	3609.704	3614.759	0.911	108.323
ex8_1_3	-7.7486E+12	1.0000	0.04	0.00	0.00	0.04	0.494	0.641	0.207	0.503
ex8_1_4	-13.0000	0.0000	100.00	100.00	100.00	100.00	0.038	0.051	0.018	0.022
ex8_1_5	-3.3333	0.0000	68.97	68.96	68.30	68.30	1.246	100.476	0.861	3.062
ex8_1_7	-757.5775	0.0293	77.43	95.79	81.76	85.22	75.203	3615.517	91.659	1711.753
ex8_1_8	-0.8466	-0.3888	76.49	90.88	60.57	84.87	3607.682	3628.366	48.869	2072.270
ex8_4_1	-5.0000	0.6186	91.09	86.49	91.84	87.63	3642.131	4180.427	3655.223	4449.738
ex8_4_2	-5.0000	0.4852	93.04	87.87	94.07	89.48	3606.071	3757.098	3629.070	4433.259
ex9_1_4	-63.0000	-37.0000	0.00	1.55	0.00	36.55	0.603	244.126	0.168	448.719
ex9_2_1	-16.0000	17.0000	60.04	92.02	54.54	95.01	2372.638	3622.960	788.499	1005.569
ex9_2_2	-50.0000	100.0000	88.29	98.06	78.47	99.55	3606.357	3610.411	194.058	621.763
ex9_2_3	-30.0000	0.0000	0.00	47.17	0.00	46.51	3.819	3625.114	0.199	3600.542

Table 12 Disjunctive Cuts versus Secant Inequalities (Part 1)

			% Duality Gap Closed				Time(sec)			
Instance	**RLT**	**OPT**	**V2**	**V3**	**V2-SA**	**V3-SA**	**V2**	**V3**	**V2-SA**	**V3-SA**
ex9_2_4	-396.0000	0.5000	99.87	99.89	99.87	100.00	8.897	5.258	3.240	2.627
ex9_2_6	-406.0000	-1.0000	87.93	62.00	85.02	70.84	2619.018	1058.376	1297.364	3652.779
ex9_2_7	-9.0000	17.0000	51.47	86.25	42.31	98.62	3628.249	3627.920	741.771	3176.606
himmel11	-30802.7566	-30665.5387	99.99	99.99	49.74	49.74	0.082	0.120	0.047	0.068
house	-5230.5433	-4500.0000	86.93	97.92	81.68	93.76	12.873	149.678	2.052	33.518
hydro	4019717.9291	4366944.1597	100.00	100.00	100.00	100.00	20.668	191.447	8.756	114.413
mathopt1	-912909.0091	1.0000	100.00	100.00	100.00	100.00	2.448	3.770	1.318	2.492
mathopt2	-11289.0001	0.0000	100.00	100.00	100.00	100.00	0.229	0.400	0.236	0.270
meanvar	0.0000	5.2434	100.00	100.00	100.00	100.00	0.276	0.657	0.200	0.264
nemhaus	0.0000	31.0000	100.00	100.00	100.00	100.00	0.198	0.355	0.071	0.090
prob05	0.3151	0.7418	99.78	99.49	68.01	50.82	0.165	0.173	0.095	1.183
prob06	1.0000	1.1771	100.00	100.00	100.00	100.00	0.024	0.031	0.022	0.028
prob09	-100.0000	0.0000	99.99	100.00	100.00	100.00	0.885	1.689	0.593	0.862
process	-2756.5935	-1161.3366	88.05	95.03	51.81	80.87	3620.085	3611.299	55.385	3603.806
qp1	-1.4313	0.0008	89.12	81.23	92.63	85.73	3897.521	3700.918	3674.194	4776.069
qp2	-1.4313	0.0008	89.15	83.06	92.75	82.88	4047.592	4255.863	3823.257	6236.168
rbrock	-659984.0066	-5.6733	100.00	100.00	100.00	100.00	3.194	5.611	0.510	0.556
st_bpaf1a	-46.0058	-45.3797	81.73	88.52	0.00	0.00	0.894	3.790	0.104	0.506
st_bpaf1b	-43.1255	-42.9626	90.73	92.86	0.00	0.00	3.299	12.166	0.276	4.002
st_bpv2	-11.2500	-8.0000	99.99	99.99	100.00	100.00	0.029	0.034	0.023	0.026
st_bsj2	-0.6260	1.0000	99.98	99.96	55.43	85.32	1.974	2.235	0.027	0.476
st_bsj3	-86768.5509	-86768.5500	0.00	0.00	0.00	0.00	0.011	0.011	0.010	0.011
st_bsj4	-72700.0507	-70262.0500	99.86	99.80	0.00	0.00	1.715	1.384	0.014	0.016
st_e02	171.4185	201.1591	99.88	99.95	100.00	100.00	0.095	0.118	0.094	0.084
st_e03	-2381.8947	-1161.3366	91.63	92.82	66.62	75.89	3639.297	3613.883	461.562	3614.941
st_e05	3826.3885	7049.2493	50.43	58.38	6.23	10.63	16.217	41.354	0.661	0.795
st_e06	0.0000	0.1609	0.00	0.00	0.00	0.00	0.726	1.911	0.299	0.727
st_e07	-500.0000	-400.0000	99.97	99.97	25.83	25.97	0.350	0.383	0.236	0.268
st_e08	0.3125	0.7418	99.81	99.89	46.87	50.05	0.208	0.171	0.060	0.087
st_e09	-0.7500	-0.5000	92.58	92.58	78.95	78.95	0.014	0.018	0.009	0.013
st_e10	-29.0000	-16.7389	100.00	100.00	100.00	100.00	0.045	0.069	0.039	0.050
st_e18	-3.0000	-2.8284	100.00	100.00	100.00	100.00	0.018	0.022	0.016	0.020
st_e19	-879.7500	-86.4222	95.21	95.18	93.50	93.51	0.613	0.991	0.382	0.517
st_e20	-0.8466	-0.3888	76.38	90.88	60.57	84.87	3610.271	3623.275	48.902	2118.320
st_e23	-3.0000	-1.0833	98.40	98.40	97.10	97.10	0.087	0.108	0.065	0.083
st_e24	0.0000	3.0000	99.81	99.81	0.00	0.00	0.501	0.657	0.008	0.010
st_e25	0.2473	0.8902	100.00	100.00	100.00	100.00	0.161	0.247	0.119	0.117
st_e26	-513.0000	-185.7792	99.96	99.96	96.07	96.08	0.036	0.050	0.040	0.050
st_e28	-30802.7566	-30665.5387	99.99	99.99	49.74	49.74	0.088	0.118	0.051	0.068
st_e30	-3.0000	-1.5811	0.00	0.00	0.00	0.00	0.035	6.489	0.008	0.069
st_e33	-500.0000	-400.0000	99.94	99.95	25.68	27.05	0.457	0.382	0.269	0.248
st_fp1	-18.9000	-17.0000	72.62	99.92	0.00	0.00	658.824	18.013	0.008	0.014

Table 13 Disjunctive Cuts versus Secant Inequalities (Part 2)

			% Duality Gap Closed				Time(sec)			
Instance	**RLT**	**OPT**	**V2**	**V3**	**V2-SA**	**V3-SA**	**V2**	**V3**	**V2-SA**	**V3-SA**
st_fp5	-269.4528	-268.0146	99.98	99.99	28.31	49.71	0.175	0.180	0.091	0.114
st_fp6	-44.4000	-39.0000	99.92	99.97	38.94	64.77	3603.767	54.613	0.070	0.297
st_fp7a	-435.5237	-354.7506	45.13	53.58	0.00	0.01	806.493	1801.106	0.251	41.874
st_fp7b	-715.5237	-634.7506	22.06	55.51	0.00	9.04	11.941	3610.617	0.248	999.777
st_fp7c	-10310.4738	-8695.0122	44.26	57.10	0.00	0.01	3621.180	3672.666	0.327	14.381
st_fp7d	-195.5237	-114.7506	50.03	55.53	0.00	1.86	3627.749	3734.806	0.195	123.754
st_fp8	7219.4999	15639.0000	0.83	3.17	0.00	0.00	4.911	88.867	0.314	37.445
st_glmp_fp2	7.0681	7.3445	45.70	49.74	0.00	0.00	0.732	1.170	0.008	0.017
st_glmp_kk92	-13.3548	-12.0000	99.98	99.98	100.00	100.00	0.038	0.053	0.013	0.015
st_glmp_kky	-3.0000	-2.5000	99.80	99.71	0.11	16.66	0.133	0.248	0.215	0.942
st_glmp_ss1	-38.6667	-24.5714	89.30	89.30	74.82	74.82	0.556	0.736	0.084	0.099
st_ht	-2.8000	-1.6000	99.81	99.89	0.00	0.00	0.142	0.451	0.008	0.009
st_iqpbk1	-1722.3760	-621.4878	99.86	99.99	99.88	99.75	5.086	286.844	30.682	376.196
st_iqpbk2	-3441.9520	-1195.2257	100.00	100.00	100.00	99.98	31.614	243.169	20.452	593.524
st_jcbpaf2	-945.4511	-794.8559	99.47	99.61	71.12	81.90	3622.733	3636.491	35.329	1730.322
st_jcbpafex	-3.0000	-1.0833	98.40	98.40	97.10	97.10	0.085	0.114	0.066	0.083
st_kr	-104.0000	-85.0000	99.93	99.95	62.70	62.70	0.090	0.131	0.025	0.034
st_m1	-505191.3385	-461356.9389	99.96	99.96	0.00	75.52	368.618	756.237	3.147	3600.500
st_m2	-938513.6772	-856648.8187	70.19	58.99	0.00	0.11	3641.449	3876.446	8.252	3758.175
st_pan1	-5.6850	-5.2837	99.72	99.92	19.04	37.59	0.926	0.771	0.033	0.148
st_pan2	-19.4000	-17.0000	68.54	99.91	0.00	0.00	3038.430	26.401	0.009	0.012
st_ph1	-243.8112	-230.1173	99.98	99.98	0.00	100.00	0.225	0.059	0.012	0.033
st_ph11	-11.7500	-11.2813	99.46	98.19	0.00	0.00	0.910	0.337	0.006	0.008
st_ph12	-23.5000	-22.6250	99.49	99.62	0.00	0.00	0.353	0.311	0.008	0.011
st_ph13	-11.7500	-11.2813	99.38	98.80	0.00	0.00	0.751	0.703	0.004	0.011
st_ph14	-231.0000	-229.7222	99.85	99.86	0.00	0.00	0.051	0.131	0.010	0.010
st_ph15	-434.7346	-392.7037	99.83	99.81	0.00	3.39	0.476	0.541	0.010	0.034
st_ph2	-1064.4960	-1028.1173	99.98	99.98	0.00	100.00	0.159	0.062	0.015	0.034
st_ph20	-178.0000	-158.0000	99.98	99.98	75.00	75.00	0.036	0.049	0.014	0.017
st_ph3	-447.8488	-420.2348	99.98	99.98	0.00	0.00	0.031	0.039	0.011	0.014
st_phex	-104.0000	-85.0000	99.96	99.96	62.70	62.70	0.088	0.088	0.026	0.035
st_qpc-m0	-6.0000	-5.0000	99.96	99.96	0.00	0.00	0.015	0.023	0.007	0.008
st_qpc-m1	-612.2714	-473.7778	99.99	99.98	86.12	100.00	0.223	0.233	0.064	0.111
st_qpc-m3a	-725.0518	-382.6950	98.10	99.16	79.52	95.76	3615.442	3727.123	10.939	413.257
st_qpc-m3b	-24.6757	0.0000	100.00	100.00	100.00	100.00	0.566	1.648	0.064	0.133
st_qpk1	-11.0000	-3.0000	99.98	99.98	97.04	97.04	0.110	0.053	0.161	0.261
st_qpk2	-21.0000	-12.2500	71.34	83.33	0.00	1.02	3599.788	3622.692	0.024	10.685
st_qpk3	-66.0000	-36.0000	33.53	50.04	0.00	0.00	3621.930	3778.200	0.073	450.765
st_rv1	-64.2359	-59.9439	96.19	98.44	0.00	0.39	3607.723	3602.339	0.064	0.865
st_rv2	-73.0007	-64.4807	88.79	81.85	0.00	1.32	3601.528	44.550	0.079	20.013
st_rv3	-38.5155	-35.7607	40.40	72.68	0.00	13.74	112.028	3807.828	0.102	3618.842
st_rv7	-148.9816	-138.1875	45.43	62.28	0.00	4.19	3640.861	3880.783	0.256	3639.194
st_rv8	-143.5829	-132.6616	29.90	45.80	0.00	0.32	3696.452	3874.801	0.616	3636.131
st_rv9	-134.9131	-120.1164	20.56	31.64	0.00	2.96	3920.213	3675.654	2.832	3605.675
st_z	-0.9674	0.0000	99.96	99.95	12.51	93.36	2.749	0.790	0.025	0.277

Table 14 Disjunctive Cuts versus Secant Inequalities (Part 3)

			% Duality Gap Closed				Time(sec)			
Instance	**RLT**	**OPT**	**V2**	**V3**	**V2-Dsj**	**V3-Dsj**	**V2**	**V3**	**V2-Dsj**	**V3-Dsj**
alkyl	-2.7634	-1.7650	55.83%	63.75%	0.00%	0.00%	3619.874	3693.810	1.591	32.800
circle	0.0000	4.5742	99.89%	99.84%	0.00%	0.00%	0.456	0.664	0.011	0.011
dispatch	3101.2805	3155.2879	100.00%	100.00%	0.00%	0.00%	0.052	0.066	0.009	0.008
ex2_1_1	-18.9000	-17.0000	72.62%	99.92%	70.27%	99.94%	704.400	17.835	316.184	15.284
ex2_1_10	39668.0556	49318.0180	99.37%	99.82%	93.61%	93.58%	29.980	70.168	223.946	262.330
ex2_1_5	-269.4528	-268.0146	99.98%	99.99%	99.86%	99.98%	0.173	0.188	0.207	0.203
ex2_1_6	-44.4000	-39.0000	99.95%	99.97%	99.93%	99.97%	3397.650	54.326	2026.656	91.494
ex2_1_7	-6031.9026	-4150.4101	41.17%	45.58%	41.71%	24.69%	3607.439	3763.506	3630.345	17.136
ex2_1_8	-82460.0000	15639.0000	84.70%	92.75%	88.73%	94.00%	3632.275	3627.700	3631.907	3706.784
ex2_1_9	-2.2000	-0.3750	98.79%	99.73%	91.84%	92.65%	1587.940	3615.766	3601.199	3608.268
ex3_1_1	2533.2008	7049.2480	15.94%	22.13%	1.12%	1.22%	3600.268	3681.021	245.717	682.051
ex3_1_2	-30802.7563	-30665.5387	99.99%	99.99%	0.00%	0.00%	0.083	0.108	0.044	0.064
ex3_1_3	-440.0000	-310.0000	99.99%	99.99%	99.99%	99.99%	0.064	0.096	0.119	0.177
ex3_1_4	-6.0000	-4.0000	86.31%	99.57%	96.52%	97.02%	21.261	581.295	720.933	36.999
ex4_1_1	-173688.7998	-7.4873	100.00%	100.00%	19.92%	20.17%	0.310	0.444	1.770	1.440
ex4_1_3	-7999.4583	-443.6717	93.54%	99.86%	81.46%	81.26%	0.285	0.552	0.271	0.339
ex4_1_4	-200.0000	0.0000	100.00%	100.00%	33.12%	33.12%	0.243	0.532	0.056	0.071
ex4_1_6	-24075.0002	7.0000	100.00%	100.00%	34.98%	31.98%	0.308	0.508	3.487	0.288
ex4_1_7	-206.2500	-7.5000	100.00%	100.00%	51.27%	51.27%	0.114	0.165	0.385	0.529
ex4_1_8	-29.0000	-16.7389	100.00%	100.00%	0.00%	0.00%	0.059	0.103	0.008	0.007
ex4_1_9	-6.9867	-5.5080	43.59%	37.48%	26.67%	27.17%	1.307	1.273	0.351	0.305
ex5_2_2_case1	-599.8996	-400.0000	0.00%	0.00%	0.00%	0.00%	0.016	0.935	0.016	0.052
ex5_2_2_case2	-1200.0000	-600.0000	0.00%	0.00%	0.00%	0.00%	0.047	0.511	0.066	0.086
ex5_2_2_case3	-875.0000	-750.0000	0.36%	0.31%	0.00%	0.00%	0.358	0.474	0.036	0.051
ex5_2_4	-2933.3334	-450.0000	79.31%	99.92%	18.52%	18.57%	68.927	1044.400	30.096	1.207
ex5_2_5	-9700.0001	-3500.0001	6.27%	6.37%	0.00%	0.00%	3793.169	3618.084	29.801	549.897
ex5_3_2	0.9979	1.8642	7.27%	21.00%	0.00%	0.00%	245.821	3672.529	0.474	1.838
ex5_3_3	1.6313	3.2340	0.21%	0.18%	0.00%	0.00%	3693.758	7511.839	3668.947	4086.645
ex5_4_2	2598.2452	7512.2301	27.57%	26.41%	1.51%	1.63%	3614.376	3866.626	353.722	677.163
ex7_3_1	0.0000	0.3417	0.00%	85.43%	0.00%	0.00%	5.582	3622.223	0.618	3.761
ex7_3_2	0.0000	1.0899	59.51%	70.26%	0.00%	0.00%	3609.704	3614.759	0.400	1.727
ex8_1_3	-7.7486E+12	1.0000	0.04%	0.00%	0.00%	0.00%	0.494	0.641	0.321	0.662
ex8_1_4	-13.0000	0.0000	100.00%	100.00%	18.19%	18.19%	0.038	0.051	0.362	0.543
ex8_1_5	-3.3333	0.0000	68.97%	68.96%	10.82%	10.02%	1.246	100.476	24.090	21.245
ex8_1_7	-757.5775	0.0293	77.43%	95.79%	49.33%	49.31%	75.203	3615.517	3609.473	3620.105
ex8_1_8	-0.8466	-0.3888	76.49%	90.88%	27.15%	28.65%	3607.682	3628.366	3602.924	3608.082
ex8_4_1	-5.0000	0.6186	91.09%	86.49%	0.00%	0.00%	3642.131	4180.427	0.918	2.201
ex8_4_2	-5.0000	0.4852	93.04%	87.87%	0.00%	0.00%	3606.071	3757.098	0.922	2.232
ex9_1_4	-63.0000	-37.0000	0.00%	1.55%	0.00%	0.00%	0.603	244.126	0.097	0.956
ex9_2_1	-16.0000	17.0000	60.04%	92.02%	0.00%	0.00%	2372.638	3622.960	0.137	1.097

Table 15 Marginal Value of Convex Quadratic Cuts (Part 1)

			% Duality Gap Closed				Time(sec)			
Instance	**RLT**	**OPT**	**V2**	**V3**	**V2-Dsj**	**V3-Dsj**	**V2**	**V3**	**V2-Dsj**	**V3-Dsj**
ex9_2_2	-50.0000	100.0000	88.29%	98.06%	0.00%	5.05%	3606.357	3610.411	0.980	3601.772
ex9_2_3	-30.0000	0.0000	0.00%	47.17%	0.00%	0.00%	3.819	3625.114	0.599	4.895
ex9_2_4	-396.0000	0.5000	99.87%	99.89%	0.00%	0.00%	8.897	5.258	0.257	0.398
ex9_2_6	-406.0000	-1.0000	87.93%	62.00%	0.00%	0.00%	2619.018	1058.376	0.664	1.131
ex9_2_7	-9.0000	17.0000	51.47%	86.25%	0.00%	0.00%	3628.249	3627.920	0.272	0.813
himmel11	-30802.7566	-30665.5387	99.99%	99.99%	0.00%	0.00%	0.082	0.120	0.050	0.071
house	-5230.5433	-4500.0000	86.93%	97.92%	5.43%	5.93%	12.873	149.678	28.838	150.973
hydro	4019717.9291	4366944.1597	100.00%	100.00%	0.00%	0.00%	20.668	191.447	0.218	0.405
mathopt1	-912909.0091	1.0000	100.00%	100.00%	1.27%	1.35%	2.448	3.770	0.783	2.119
mathopt2	-11289.0001	0.0000	100.00%	100.00%	32.54%	32.68%	0.229	0.400	0.627	0.944
meanvar	0.0000	5.2434	100.00%	100.00%	0.00%	0.00%	0.276	0.657	0.009	0.009
nemhaus	0.0000	31.0000	100.00%	100.00%	0.00%	0.00%	0.198	0.355	0.049	0.089
prob05	0.3151	0.7418	99.78%	99.49%	90.89%	90.62%	0.165	0.173	0.216	0.226
prob06	1.0000	1.1771	100.00%	100.00%	0.00%	0.00%	0.024	0.031	0.007	0.005
prob09	-100.0000	0.0000	99.99%	100.00%	0.00%	0.00%	0.885	1.689	0.008	0.012
process	-2756.5935	-1161.3366	88.05%	95.03%	7.97%	7.70%	3620.085	3611.299	3612.928	3640.484
qp1	-1.4313	0.0008	89.12%	81.23%	59.58%	60.95%	3897.521	3700.918	3688.640	3720.181
qp2	-1.4313	0.0008	89.15%	83.06%	61.22%	61.37%	4047.592	4255.863	3642.175	3681.558
rbrock	-659984.0066	-5.6733	100.00%	100.00%	0.00%	0.00%	3.194	5.611	0.010	0.009
st_bpaf1a	-46.0058	-45.3797	81.73%	88.52%	0.00%	0.00%	0.894	3.790	0.124	0.238
st_bpaf1b	-43.1255	-42.9626	90.73%	92.86%	0.00%	0.00%	3.299	12.166	0.301	0.356
st_bpv2	-11.2500	-8.0000	99.99%	99.99%	69.16%	69.16%	0.029	0.034	0.374	0.536
st_bsj2	-0.6260	1.0000	99.98%	99.96%	99.94%	99.95%	1.974	2.235	0.987	0.732
st_bsj3	-86768.5509	-86768.5500	0.00%	0.00%	0.00%	0.00%	0.011	0.011	0.010	0.012
st_bsj4	-72700.0507	-70262.0500	99.86%	99.80%	99.73%	99.92%	1.715	1.384	1.250	0.731
st_e02	171.4185	201.1591	99.88%	99.95%	99.74%	99.83%	0.095	0.118	0.084	0.120
st_e03	-2381.8947	-1161.3366	91.63%	92.82%	8.26%	8.61%	3639.297	3613.883	3615.422	3613.065
st_e05	3826.3885	7049.2493	50.43%	58.38%	32.44%	40.00%	16.217	41.354	4.733	37.922
st_e06	0.0000	0.1609	0.00%	0.00%	0.00%	0.00%	0.726	1.911	0.079	0.119
st_e07	-500.0000	-400.0000	99.97%	99.97%	0.00%	0.00%	0.350	0.383	0.085	0.095
st_e08	0.3125	0.7418	99.81%	99.89%	91.18%	90.87%	0.208	0.171	0.193	0.242
st_e09	-0.7500	-0.5000	92.58%	92.58%	28.95%	28.95%	0.014	0.018	0.041	0.048
st_e10	-29.0000	-16.7389	100.00%	100.00%	0.00%	0.00%	0.045	0.069	0.006	0.007
st_e18	-3.0000	-2.8284	100.00%	100.00%	0.00%	0.00%	0.018	0.022	0.013	0.016
st_e19	-879.7500	-86.4222	95.21%	95.18%	0.99%	0.99%	0.613	0.991	0.997	0.786
st_e20	-0.8466	-0.3888	76.38%	90.88%	27.15%	28.65%	3610.271	3623.275	3600.675	3599.440
st_e23	-3.0000	-1.0833	98.40%	98.40%	0.00%	0.00%	0.087	0.108	0.016	0.019
st_e24	0.0000	3.0000	99.81%	99.81%	99.82%	99.82%	0.501	0.657	0.563	0.749
st_e25	0.2473	0.8902	100.00%	100.00%	0.00%	0.00%	0.161	0.247	0.009	0.008
st_e26	-513.0000	-185.7792	99.96%	99.96%	99.96%	99.96%	0.036	0.050	0.035	0.067

Table 16 Marginal Value of Convex Quadratic Cuts (Part 2)

			% Duality Gap Closed				Time(sec)			
Instance	**RLT**	**OPT**	**V2**	**V3**	**V2-Dsj**	**V3-Dsj**	**V2**	**V3**	**V2-Dsj**	**V3-Dsj**
st_e28	-30802.7566	-30665.5387	99.99%	99.99%	0.00%	0.00%	0.088	0.118	0.052	0.073
st_e30	-3.0000	-1.5811	0.00%	0.00%	0.00%	0.00%	0.035	6.489	0.032	0.413
st_e33	-500.0000	-400.0000	99.94%	99.95%	0.00%	0.00%	0.457	0.382	0.039	0.078
st_fp1	-18.9000	-17.0000	72.62%	99.92%	70.27%	99.94%	658.824	18.013	311.516	15.417
st_fp5	-269.4528	-268.0146	99.98%	99.99%	99.86%	99.98%	0.175	0.180	0.201	0.208
st_fp6	-44.4000	-39.0000	99.92%	99.97%	99.93%	99.97%	3603.767	54.613	2166.810	95.078
st_fp7a	-435.5237	-354.7506	45.13%	53.58%	52.80%	59.79%	806.493	1801.106	3603.973	2125.128
st_fp7b	-715.5237	-634.7506	22.06%	55.51%	51.25%	64.63%	11.941	3610.617	3355.367	3643.094
st_fp7c	-10310.4738	-8695.0122	44.26%	57.10%	49.16%	59.84%	3621.180	3672.666	3646.248	3623.625
st_fp7d	-195.5237	-114.7506	50.03%	55.53%	53.38%	61.72%	3627.749	3734.806	3630.742	3657.918
st_fp8	7219.4999	15639.0000	0.83%	3.17%	3.16%	3.90%	4.911	88.867	3629.451	251.935
st_glmp_fp2	7.0681	7.3445	45.70%	49.74%	0.00%	0.00%	0.732	1.170	0.033	0.041
st_glmp_kk92	-13.3548	-12.0000	99.98%	99.98%	44.42%	44.42%	0.038	0.053	0.114	0.147
st_glmp_kky	-3.0000	-2.5000	99.80%	99.71%	0.00%	0.00%	0.133	0.248	0.072	0.123
st_glmp_ss1	-38.6667	-24.5714	89.30%	89.30%	40.55%	40.55%	0.556	0.736	0.254	0.359
st_ht	-2.8000	-1.6000	99.81%	99.89%	99.90%	99.87%	0.142	0.451	0.213	0.500
st_iqpbk1	-1722.3760	-621.4878	99.86%	99.99%	0.00%	0.00%	5.086	286.844	0.008	0.012
st_iqpbk2	-3441.9520	-1195.2257	100.00%	100.00%	0.00%	0.00%	31.614	243.169	0.009	0.009
st_jcbpaf2	-945.4511	-794.8559	99.47%	99.61%	32.35%	34.94%	3622.733	3636.491	3602.511	3617.253
st_jcbpafex	-3.0000	-1.0833	98.40%	98.40%	0.00%	0.00%	0.085	0.114	0.017	0.015
st_kr	-104.0000	-85.0000	99.93%	99.95%	99.71%	99.94%	0.090	0.131	0.165	0.058
st_m1	-505191.3385	-461356.9389	99.96%	99.96%	99.59%	99.52%	368.618	756.237	105.344	222.951
st_m2	-938513.6772	-856648.8187	70.19%	58.99%	80.23%	53.62%	3641.449	3876.446	3650.336	3885.121
st_pan1	-5.6850	-5.2837	99.72%	99.92%	99.91%	99.93%	0.926	0.771	0.414	0.181
st_pan2	-19.4000	-17.0000	68.54%	99.91%	57.48%	99.93%	3038.430	26.401	15.446	21.076
st_ph1	-243.8112	-230.1173	99.98%	99.98%	99.98%	99.70%	0.225	0.059	0.231	0.099
st_ph11	-11.7500	-11.2813	99.46%	98.19%	99.46%	99.68%	0.910	0.337	0.166	0.393
st_ph12	-23.5000	-22.6250	99.49%	99.62%	99.08%	99.65%	0.353	0.311	0.319	0.245
st_ph13	-11.7500	-11.2813	99.38%	98.80%	96.98%	99.52%	0.751	0.703	0.118	1.096
st_ph14	-231.0000	-229.7222	99.85%	99.86%	99.84%	99.88%	0.051	0.131	0.055	0.114
st_ph15	-434.7346	-392.7037	99.83%	99.81%	99.85%	99.41%	0.476	0.541	0.712	0.304
st_ph2	-1064.4960	-1028.1173	99.98%	99.98%	99.94%	99.97%	0.159	0.062	0.128	0.184
st_ph20	-178.0000	-158.0000	99.98%	99.98%	99.98%	99.98%	0.036	0.049	0.038	0.046
st_ph3	-447.8488	-420.2348	99.98%	99.98%	99.98%	99.98%	0.031	0.039	0.027	0.042
st_phex	-104.0000	-85.0000	99.96%	99.96%	99.71%	99.94%	0.088	0.088	0.166	0.059
st_qpc-m0	-6.0000	-5.0000	99.96%	99.96%	99.96%	99.96%	0.015	0.023	0.015	0.023
st_qpc-m1	-612.2714	-473.7778	99.99%	99.98%	99.95%	99.96%	0.223	0.233	0.162	0.276
st_qpc-m3a	-725.0518	-382.6950	98.10%	99.16%	99.64%	99.39%	3615.442	3727.123	776.578	291.805
st_qpc-m3b	-24.6757	0.0000	100.00%	100.00%	100.00%	100.00%	0.566	1.648	0.127	0.233
st_qpk1	-11.0000	-3.0000	99.98%	99.98%	99.97%	99.98%	0.110	0.053	0.076	0.057
st_qpk2	-21.0000	-12.2500	71.34%	83.33%	73.72%	84.68%	3599.788	3622.692	3612.000	3620.808
st_qpk3	-66.0000	-36.0000	33.53%	50.04%	32.91%	53.32%	3621.930	3778.200	3618.484	3655.638
st_rv1	-64.2359	-59.9439	96.19%	98.44%	98.48%	98.68%	3607.723	3602.339	829.402	2804.776
st_rv2	-73.0007	-64.4807	88.79%	81.85%	88.92%	96.25%	3601.528	44.550	3623.841	3641.686
st_rv3	-38.5155	-35.7607	40.40%	72.68%	58.70%	81.72%	112.028	3807.828	3599.892	3649.818
st_rv7	-148.9816	-138.1875	45.43%	62.28%	44.27%	44.49%	3640.861	3880.783	2314.720	314.910
st_rv8	-143.5829	-132.6616	29.90%	45.80%	37.15%	21.32%	3696.452	3874.801	3686.005	362.907
st_rv9	-134.9131	-120.1164	20.56%	31.64%	24.00%	27.68%	3920.213	3675.654	3643.930	3610.947
st_z	-0.9674	0.0000	99.96%	99.95%	99.93%	99.95%	2.749	0.790	1.262	1.253

Table 17 Marginal Value of Convex Quadratic Cuts (Part 3)