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Necessary Conditions for the Impulsive Optimal Control of Multibody Mechanical Systems

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In this work, necessary conditions for the impulsive optimal control of multibody mechanical systems are stated. The conditions are obtained by the application subdifferential calculus techniques to extended-valued lower semi-continuous generalized Bolza functional that is evaluated on multiple intervals. Contrary to the approach in literature so far, the instant of possibly impulsive transition is considered as a Lebesgue negligible instant. This approach is in comparison to other impulsive necessary conditions consistent with mainstream hybrid system modeling methods in which transitions happen instantaneously. The necessary conditions provide necessary criteria for the determination of optimal transition times and locations. The consideration of certain type of variations at the boundaries give birth to the concepts of internal boundary variations and discontinuous transversality conditions. The concepts are developed by the author and are presented and discussed in [20] and [22] with applications to optimal control. In this work, a characterization of these concepts in terms of upper and lower subderivatives to the extended-valued lower-semicontinuous value functional under several regularity assumptions is given. The properties of the transition sets are discussed.

1 Introduction

An impact in mechanics is defined as a discontinuity in the generalized velocities of a mechanical system which is induced by impulsive forces, therefore optimal control of such impulsive systems inevitably encompasses optimal control with discontinuous generalised velocities. In this work, necessary conditions for the impulsive optimal control of scleronomic rigidbody Lagrangian systems is studied.

The main issue in the optimal control of impactive mechanical systems has been the blending of impact mechanics with impulsive optimal control. The crux in the derivation of these necessary conditions is to handle joint discontinuity of the state and the dual state on a Lebesgue negligible interval. In the framework of integration theory, this has long been recognized as a problem if state and costate should become concurrently discontinuous as has been addressed in [10] and [15]. Rockafellar studied in [15] the discontinuity of the dual state in constrained convex optimal control problems but dispensed of attacking the problem concurrent discontinuity of state and costate. Moreau gave in [10] partial integration formulas for differential measures in general bilinear forms. In [27] Murray studies the extension and existence theorems of problems in calculus of variation to the setting when impulsive controls are applied and state discontinuity occurs. He bases his work on [15], and outlines in his motivation that jumps in the states may occur due to constraints on the dual dynamics which are reached by the costate, which has an application in economics. In [24] several classes of impulsive Lagrangian systems are studied. The main focus is impulses generated by sudden parameter changes such as inertial parameters that affect the momentum balance, or impulses arising due to structure of constraints of a mechanical system. A certain class of impulsive systems that resemble discontinuous diffusion processes. In this work, it is assumed that the instant of discontinuity is reduced to an instant with Lebesgue measure zero, instead of taking an interval opening approach, which is the approach considered in literature so far. In the approaches provided in reference such as [1], [8] and [21], the impulsive control problem is transformed into a problem of an ordinary differential inclusion problem, which requires to determine trajectories for the ”discontinuous” states during the ”impulsive” control action. In [7], impulses arising from unilateral constraints are considered but again in the framework of interval-opening approach and transformation technique. The instantaneous transition approach is in comparison to other impulsive necessary conditions consistent with different common hybrid system modeling methods in which transitions happen instantaneously such as in [2].

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In what follows next, the new concepts required to deal with this specific problem are introduced. In order to overcome the difficulties arising from joint discontinuity of state and costate, the instant of impulsive control action where discontinuity in the generalized velocities occur is considered as an internal boundary in the time domain. In [20], the concept of internal boundary variations are introduced literally, and as an application a theorem that states the necessary conditions for the impulsive time-optimal control of finite-dimensional Lagrangian systems is stated. In the framework of these concepts,philosophically, the instant of state discontinuity constitutes an internal boundary in the optimal control problem. The essential idea is thus to consider every point of the domain where continuity and differentiability ceases to exist, as a boundary of the problem. By introducing a boundary at an instant of a discontinuity, one has to notice that it has bilateral character, in the sense that the boundary constitutes an upper boundary for one segment of the interval whereas for the other segment a lower boundary in the time domain. The necessity that at a location of transition several conditions have to be fulfilled, gives rise to the idea of some sort of transversality conditions if one begins to consider an instant of discontinuity as a two-sided boundary where to arcs are "connected" discontinuously. This dependence is embedded in the concept of internal boundary variations. In order to obtain criteria for the optimality of the transition position, transition pre-, and post-transition generalised velocities, transition time and impulsive control, variations in these entities need to be considered, which represent in the setting of this work the internal boundary variations. At the boundaries of the time domain, the pre-transition state variations are considered separately from the post-transition variations. The absolute continuity of the generalized positions means that the total variation of the generalized positions at the pre-transition and post-transition instants are equal. The pre-transition and post-transition variations are interrelated by the transition conditions which can be seen as the bases of transversality conditions that join two trajectories discontinuously. The transition conditions are introduced symmetrically with respect to pre-, and post-transition states. The transition conditions are of two types, namely, the impact equation and the constitutive impact laws. The impact equations relate the discontinuity in the impulse of the Lagrangian system to the impulsive forces/controls. The impact law (.i.e. the moreau-newton impact law), however, is a constitutive law which is chosen depending on the modeling approach preferred. As a case study, in reference [19] the blocking of some DOF of an underactuated manipulator by tangential fully-inelastic impact is discussed and the necessary conditions are stated. In [20] the necessary conditions for the impulsive optimal control of Lagrangian systems in the Hamiltonian framework is investigated. By the application of subdifferential calculus techniques to extended-valued lower semi-continuous functionals, necessary conditions are obtained. In publications of R. T. Rockafellar such [13] and [14] a summary of the rules in subdifferential calculus are provided, which is one of the most flourishing branches of mathematics.

2 The Euler-Lagrange Equations in Impulsive Control Form

Let $\mathbf{q}$, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ represent the position, velocity and acceleration in the generalised coordinates of a scleronomic rigidbody mechanical system with n degrees of freedom (DOF), respectively. The equations of motion (EOM) can be obtained by using the Lagrange formalism for the smooth dynamics:

$$\frac{d}{dt}\left(\partial_{\dot{\mathbf{q}}}T(\mathbf{q},\dot{\mathbf{q}})\right)^{\mathrm{T}} - \left(\partial_{\mathbf{q}}T(\mathbf{q},\dot{\mathbf{q}})\right)^{\mathrm{T}} + \left(\partial_{\mathbf{q}}V(\mathbf{q})\right)^{\mathrm{T}} - \mathbf{f}_{\mathrm{T}} = \mathbf{0}. \tag{1}$$

Here $T(\mathbf{q},\dot{\mathbf{q}})$ denotes the total kinetic, and $V(\mathbf{q})$ the total smooth potential energy of the system. The controls is introduced into the equations of motion by means of the structure of $\mathbf{f}_{\mathrm{T}}$. In impact mechanics, the generalized accelerations and velocities are eligible to become discontinuous where as the generalized positions are of absolutely continuous character. The interaction of the mechanical system with the surroundings as well as the control actions imposed on the system necessitates to allow discontinuity events in the velocities and accelerations of the system. The equations have to be supplemented with some force laws that relate the external forces f_{T} and controls τ with the system's state $(\mathbf{q},\dot{\mathbf{q}})$. The coexistence of the generalized velocities $\dot{\mathbf{q}}$ and accelerations $\ddot{\mathbf{q}}$ is limited to the instants where $\dot{\mathbf{q}}$ and $\boldsymbol{\tau}$ are continuous. Because of the set of discontinuity points $\{t_{\mathrm{i}}\} \in \mathcal{I}_{\mathrm{T}}$ of $\dot{\mathbf{q}}$ and discontinuities in the controls $\boldsymbol{\tau}$, where $\ddot{\mathbf{q}}$ does not exist, the equations are stated in the following form:

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}} - \mathbf{h}(\mathbf{q},\dot{\mathbf{q}}) = \mathbf{f} + \mathbf{B}(\mathbf{q})\,\boldsymbol{\tau} = \mathbf{f}_{\mathrm{T}}, \quad \text{a.e.} \quad . \tag{2}$$

Here $\mathbf{M}$ is the symmetric and positive definite generalized mass matrix depending smoothly on the generalized positions $\mathbf{q}$, and $\mathbf{h}$ is a smooth function of $\mathbf{q}, \dot{\mathbf{q}}$ containing the gyroscopical, coriolis, centripedal accelerations of the Lagrangian system, as well as all smooth finite forces such as spring and damping forces. The linear operator $\mathbf{B}(\mathbf{q})$ includes the generalized directions of control forces. The linear operator $\mathbf{M}(\mathbf{q})$ and the vector $\mathbf{h}$ are related to the Lagrangian formalism by the following equations:

$$\mathbf{M}(\mathbf{q}) = \left(\partial^2_{\dot{\mathbf{q}}\,\dot{\mathbf{q}}}T(\mathbf{q},\dot{\mathbf{q}})\right)^{\mathrm{T}}, \quad \mathbf{h}(\mathbf{q},\dot{\mathbf{q}}) = \partial^2_{\dot{\mathbf{q}}\,\mathbf{q}}T(\mathbf{q},\dot{\mathbf{q}})^{\mathrm{T}}\dot{\mathbf{q}} - \left(\partial_{\mathbf{q}}(T(\mathbf{q},\dot{\mathbf{q}}) - V(\mathbf{q}))\right)^{\mathrm{T}}.$$

Definition 1 Transition Time A time instant of Lebesgue measure zero is considered as a transition time $t_\mathrm{i} \in \mathcal{I}_\mathrm{T}$ if one of the two events occur together or for itself:

- **Event 1** Some directions of motion of the system are opened or closed by the control strategy, which entails a change in the degrees of freedom (DOF) of the system.
- **Event 2** An impulsive control action is exerted on the system, which may be accompanied by a discontinuity of the generalized velocities of the Lagrangian system.

In order to investigate the discontinuity points of the velocities $\dot{\mathbf{q}}$ and accelerations $\ddot{\mathbf{q}}$ properly, equation (2) is replaced by the corresponding equality of measures as in [11]:

$$\mathbf{M}(\mathbf{q})\, d\dot{\mathbf{q}} - \mathbf{h}(\mathbf{q},\dot{\mathbf{q}})\, dt = d\mathbf{R} + \mathbf{B}(\mathbf{q})\, d\mathbf{\Gamma}\ , \tag{3}$$

where it has been introduced for uncontrolled rigidbody mechanical systems. The absolutely continuous part of the measure $d\dot{\mathbf{q}}$ is denoted by $\ddot{\mathbf{q}}\, dt$. The singular part of $d\dot{\mathbf{q}}$, can be represented as $(\frac{d\dot{\mathbf{q}}}{d\sigma})\, d\sigma$, where $d\sigma$ is some nonnegative singular measure (a regular Borel measure), and $\frac{d\dot{\mathbf{q}}}{d\sigma}$ is the Radon-Nikodym derivative of $d\dot{\mathbf{q}}$ with respect to $d\sigma$, which is also denoted as $\boldsymbol{\chi}'$. This form of representation of the evolution of the dynamics has a wider range of validity such that it is valid everywhere instead of almost everywhere on the time domain. For the force measure $d\mathbf{R}$ following decomposition is assumed:

$$d\mathbf{R} = \mathbf{f}\, dt + \mathbf{F}'\, d\sigma\ , \tag{4}$$

such that $\mathbf{f}$ and $\mathbf{F}'$ represent Lebesgue-measurable and Borel-measurable forces, respectively. The Radon-Nykodym derivative of $d\mathbf{R}$ with respect to $d\sigma$ is given by $\mathbf{F}'$. Similarly the differential measure of controls is decomposed as:

$$d\mathbf{\Gamma} = \boldsymbol{\tau}\, dt + \boldsymbol{\zeta}'\, d\sigma\ . \tag{5}$$

Here $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}'$ represent the Lebesgue-measurable and Borel-measurable controls, respectively. Here, the Radon-Nykodym derivative of $d\mathbf{\Gamma}$ with respect to $d\sigma$ is given by $\boldsymbol{\zeta}'$. The substitution of (4) into (3) along with $d\dot{\mathbf{q}} = \ddot{\mathbf{q}}\, dt + \boldsymbol{\chi}'\, d\sigma$ reveals:

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}}\, dt + \mathbf{M}(\mathbf{q})\,\boldsymbol{\chi}'\, d\sigma - \mathbf{h}(\mathbf{q},\dot{\mathbf{q}})\, dt = (\mathbf{f} + \mathbf{B}(\mathbf{q})\,\boldsymbol{\tau})\ dt + \left(\mathbf{F}' + \mathbf{B}(\mathbf{q})\,\boldsymbol{\zeta}'\right)\ d\sigma\ . \tag{6}$$

Equation (6) can be split into a Lebesgue and Borel part as given below:

$$\mathbf{M}(\mathbf{q})\,\boldsymbol{\chi}'\, d\sigma = \left(\mathbf{F}' + \mathbf{B}(\mathbf{q})\,\boldsymbol{\zeta}'\right) d\sigma\ , \tag{7}$$

$$\mathbf{M}(\mathbf{q})\,\ddot{\mathbf{q}}\, dt - \mathbf{h}(\mathbf{q},\dot{\mathbf{q}})\, dt = (\mathbf{f} + \mathbf{B}(\mathbf{q})\,\boldsymbol{\tau})\ dt\ . \tag{8}$$

The discontinuities of the generalized velocities $\dot{\mathbf{q}}$ are concentrated on countably many instants which are Lebesgue negligible. By evaluating the Lebesgue-Stieltjes Integral over an atomic time $t_\mathrm{i} \in \mathcal{I}_\mathrm{T}$ of equation (6) reveals the impact equation:

$$\int_{\{t_\mathrm{i}\}} \mathbf{M}(\mathbf{q})\,\boldsymbol{\chi}' - \left(\mathbf{F}' + \mathbf{B}(\mathbf{q})\,\boldsymbol{\zeta}'\right) d\sigma = \mathbf{M}(\mathbf{q}(t_\mathrm{i}))\,(\boldsymbol{\chi}_\mathrm{i}^+ - \boldsymbol{\chi}_\mathrm{i}^-) - (\mathbf{F}_\mathrm{i}^+ - \mathbf{F}_\mathrm{i}^-) - \mathbf{B}(\mathbf{q}(t_\mathrm{i}))\,(\boldsymbol{\zeta}_\mathrm{i}^+ - \boldsymbol{\zeta}_\mathrm{i}^-) = \mathbf{0}\,, \tag{9}$$

The Lebesgue part which remains unaffected by the points of discontinuity can be expressed in two forms as below:

$$\mathbf{M}(\mathbf{q}^+)\,\ddot{\mathbf{q}}^+\, dt - \mathbf{h}(\mathbf{q}^+,\dot{\mathbf{q}}^+)\, dt = \left(\mathbf{f}^+ + \mathbf{B}(\mathbf{q}^+)\,\boldsymbol{\tau}^+\right)\ dt\ , \tag{10}$$

$$\mathbf{M}(\mathbf{q}^-)\,\ddot{\mathbf{q}}^-\, dt - \mathbf{h}(\mathbf{q}^-,\dot{\mathbf{q}}^-)\, dt = \left(\mathbf{f}^- + \mathbf{B}(\mathbf{q}^-)\,\boldsymbol{\tau}^-\right)\ dt\ . \tag{11}$$

Here $\mathbf{f}^+$ and $\mathbf{f}^-$ are meant to be the right and left limits of $\mathbf{f}$ with respect to time, respectively. As a corollary, the directional Euler-Lagrange equations can be stated as follows:

$$\mathbf{M}(\mathbf{q}^+)\,\ddot{\mathbf{q}}^+ - \mathbf{h}(\mathbf{q}^+,\dot{\mathbf{q}}^+) = \mathbf{f}^+ + \mathbf{B}(\mathbf{q}^+)\,\boldsymbol{\tau}^+, \quad \text{a.e.}\ , \tag{12}$$

$$\mathbf{M}(\mathbf{q}^-)\,\ddot{\mathbf{q}}^- - \mathbf{h}(\mathbf{q}^-,\dot{\mathbf{q}}^-) = \mathbf{f}^- + \mathbf{B}(\mathbf{q}^-)\,\boldsymbol{\tau}^-, \quad \text{a.e.}\ . \tag{13}$$

An introduction to impacts in rigidbody mechanics can be found in [6], and a literature survey on Lagrangian impactive systems is provided in [3].

3 The Generalized Problem of Bolza in Impulsive Control Form for Rigidbody Lagrangian Systems

Consider a problem in Bolza form (GPB), in which the objective is to choose an absolutely continuous arcs $\mathbf{q} \in \mathcal{AC}$ and $\dot{\mathbf{q}} \in \mathcal{AC}$ in order to minimize problem P given by:

$$P: \quad J(\mathbf{q}, \dot{\mathbf{q}}) = l(\mathbf{q}(a), \dot{\mathbf{q}}(a), \mathbf{q}(b), \dot{\mathbf{q}}(b)) + \int_{\mathrm{a}}^{\mathrm{b}} L(t, \mathbf{q}(t), \dot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t))\, dt\ , \tag{14}$$

where the function $L : [a, b] \times \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}} \rightarrow \mathbb{R} \cup \{+\infty\}$ is $\mathcal{L} \times \mathcal{B}$ measurable. Here $\mathcal{L} \times \mathcal{B}$ denotes the σ-algebra of subsets of $[a, b] \times \mathbb{R}^{\mathrm{n}}$ generated by product sets $\mathcal{M} \times \mathcal{N}$, where $\mathcal{M}$ is a Lebesgue measurable subset of $[a, b]$ and $\mathcal{N}$ is a Borel subset of $\mathbb{R}^{3n}$. For each $t \in [a, b]$, the function l and L are lower semi-continuous on $\mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}$ and $\mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}$, respectively, with values in $\mathbb{R} \cup \{+\infty\}$. For each $(t, \mathbf{q}, \dot{\mathbf{q}})$ in $[a, b] \times \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}$, the function $L(t, \mathbf{q}, \dot{\mathbf{q}}, \cdot)$ is convex and l represents the endpoint cost. GPB concerns the minimization of a functional whose form is identical to that in the classical calculus of variations. The endpoint cost l and the integrand L are allowed to take the value $+\infty$, so that a variety of endpoint and differential constraints can be treated. An important class of optimal control problems constrain the derivative of an admissible arc and can be formulated in the form:

$$\min\{l(\mathbf{q}(a), \dot{\mathbf{q}}(a), \mathbf{q}(b), \dot{\mathbf{q}}(b)) : \mathbf{M}(\mathbf{q})\, \ddot{\mathbf{q}}(t) \in \mathcal{F}(t, \mathbf{q}, \dot{\mathbf{q}}) \quad \text{a.e.} \quad t \in [a, b]\}. \tag{15}$$

The problem in (15) can be seen as minimizing the Bolza functional J over all arcs $\mathbf{q}, \dot{\mathbf{q}}$. If one identifies the integrand in (14) with (16):

$$L(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{a}) = \Psi_{\mathcal{F}(t,\mathbf{q},\dot{\mathbf{q}})}(\mathbf{a}) = \begin{cases} 0, \ \mathbf{M}(\mathbf{q})\, \mathbf{a} \in \mathcal{F}(t, \mathbf{q}, \dot{\mathbf{q}})\,, \\ \qquad +\infty\ , \end{cases} \tag{16}$$

the general class of optimal control problems defined in (15) can be handled as a GPB. The function $\Psi_{\mathcal{C}}$ is the indicator function of the set $\mathcal{C}$. It is evident that for some arc $\mathbf{q}, \dot{\mathbf{q}}$, one then has

$$\int_{\mathrm{a}}^{\mathrm{b}} L(t, \mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})\, dt = \begin{cases} 0, \ \mathbf{M}(\mathbf{q})\, \ddot{\mathbf{q}}(t) \in \mathcal{F}(t, \mathbf{q}, \dot{\mathbf{q}}) \quad \text{a.e.} \quad t \in [a, b], \\ \qquad\qquad +\infty\ . \end{cases} \tag{17}$$

where the differential inclusion $\mathcal{F}$ is defined via the state-control triplet $(\boldsymbol{\tau}, \mathbf{q}, \dot{\mathbf{q}})$:

$$\mathbf{M}(\mathbf{q})\, \ddot{\mathbf{q}}(t) \quad \in \quad \mathcal{F}(t, \mathbf{q}, \dot{\mathbf{q}}) := \{\mathbf{a}\,|\, \mathbf{M}(\mathbf{q})\, \mathbf{a} = \mathbf{h}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + \mathbf{B}(\mathbf{q}(t))\, \boldsymbol{\tau}(t) \ \ \forall \boldsymbol{\tau}(t) \in \mathcal{C}_{\tau}\ \text{a.e.}\ t \in [a, b]\}. \tag{18}$$

In order to guarantee the well-behaving of $\mathcal{F}$ and l let following assumptions hold:

Assumptions 1. A pair of trajectories of generalised positions $\bar{\mathbf{q}} : [a, b] \rightarrow \mathbb{R}^{\mathrm{n}}$ and and velocities $\dot{\bar{\mathbf{q}}} : [a, b] \rightarrow \mathbb{R}^{\mathrm{n}}$ is given. On some relatively open subset $\Omega \subseteq [a, b] \times \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}}$ containing the graph of $(\bar{\mathbf{q}}, \dot{\bar{\mathbf{q}}})$, the following statements hold:

1. The multifunction $\mathcal{F}$ is $\mathcal{L} \times \mathcal{B}$ measurable on Ω. For each $(t, \mathbf{q}, \dot{\mathbf{q}})$ in Ω, the set $\mathcal{F}(t, \mathbf{q}, \dot{\mathbf{q}})$ is nonempty, compact and convex.
2. There are nonnegative integrable functions $k_{\mathbf{q}}(t)$, $k_{\dot{\mathbf{q}}}(t)$ and $\Phi(t)$ on $[a, b]$ such that
   (a) $\mathcal{F}(t, \mathbf{q}, \dot{\mathbf{q}}) \subseteq \Phi(t)\, \mathbb{B}$ for all $\mathbf{q}, \dot{\mathbf{q}} \in \Omega_{\mathrm{t}}$, almost everywhere, and
   (b) $\mathcal{F}(t, \mathbf{q}, \dot{\mathbf{q}}) \subseteq \mathcal{F}(t, \mathbf{q}, \dot{\mathbf{q}}) + k_{\mathbf{q}}(t)|\mathbf{p} - \mathbf{r}| + k_{\dot{\mathbf{q}}}(t)|\dot{\mathbf{p}} - \dot{\mathbf{r}}| \mathrm{cl}\, \mathbb{B}$ for all $(\mathbf{p}, \dot{\mathbf{p}}), (\mathbf{r}, \dot{\mathbf{r}}) \in \Omega_{\mathrm{t}}$, almost everywhere.
3. The endpoint cost function l is lower-semicontinuous on $\Omega_a \times \Omega_{\mathrm{b}}$.

Here Ω_{t} is given by $\Omega_{\mathrm{t}} = \{(\mathbf{q}, \dot{\mathbf{q}}) \in \mathbb{R}^{\mathrm{n}} \times \mathbb{R}^{\mathrm{n}} \,|\, (t, \mathbf{q}, \dot{\mathbf{q}}) \in \Omega, \quad \forall t \in [a, b] \backslash \{t_{\mathrm{i}}\} \in \mathcal{I}_{\mathrm{T}}\}$ and $\mathbb{B}$ is the unit open ball. It is assumed that conditions of assumptions (1) are fulfilled for the Lebesgue-measurable part of the Lagrangian dynamics in the "almost everywhere" sense. By the way Ω_{t} is defined, the instants of the discontinuity are excluded.

Impulsive optimal control of rigidbody mechanical systems requires to seek extremizing arcs in the space of locally bounded variation functions $\mathcal{LBV}$. Every generalized velocity $\dot{\mathbf{q}} : [t_0, t_1] \rightarrow \mathbb{R}^{\mathrm{n}}$ of bounded variation is associated with an $\mathbb{R}^{\mathrm{n}}$-valued regular Borel measure $d\dot{\mathbf{q}}$ on $[t_0, t_1]$. The atoms for $d\dot{\mathbf{q}}$ occur only at discontinuities of $\dot{\mathbf{q}}$, of which there are at most countably many. Trajectories of locally bounded variation in $\mathbb{R}^{\mathrm{n}}$ are defined to be an equivalence class, and the space of all arcs is denoted by $\mathcal{LBV}$. The space of absolutely continuous arcs $\mathcal{AC}$ is a subspace of $\mathcal{LBV}$. The generalized positions of a mechanical system are in the class $\mathcal{AC}$. There are uniquely determined functions $\dot{\mathbf{q}}^+(t)$ and $\dot{\mathbf{q}}^-(t)$ in $[t_0, t_1] \rightarrow \mathbb{R}^{\mathrm{n}}$, right and left continuous respectively, such that $\dot{\mathbf{q}}^+(t) = \dot{\mathbf{q}}^-(t) = \dot{\mathbf{q}}(t)$ at all the non-atomic points, and at the end points

$\dot{\mathbf{q}}^-(t_0) = \dot{\mathbf{q}}(t_0)$ and $\dot{\mathbf{q}}^+(t_\mathrm{f}) = \dot{\mathbf{q}}(t_\mathrm{f})$ are valid. Therefore, a further classification of $\dot{\mathbf{q}} \in \mathcal{LBV}$ is to subdivide these functions into left-continuous bounded variation ($\mathcal{LCBV}$) and right-continuous bounded variation ($\mathcal{RCBV}$) functions. A good overview on the topic of treatment of functions of bounded variation in time is provided in [10]. The right-continuous and left-continuous regularizations of a function f, which is a mapping of $\mathcal{I}$ to a Hausdorff topological space $\mathcal{E}$; becomes important if one considers that for every $t_\mathrm{i} \in \mathcal{I}_\mathrm{T}$ the right-side limit given by:

$$f^+(t_\mathrm{i}) = \lim_{s\to t_\mathrm{i},\, s>t} f(s) \tag{19}$$

may differ from $f(t_\mathrm{i})$, if it exists. Symmetrically, the left-side limit, if it exists, is denoted by $f^-(t_\mathrm{i})$. Following proposition is used often in this work:

Proposition 1 [10] Let $\mathcal{E}$ be regular and let $f : \mathcal{I} \to \mathcal{E}$ be such that for every $t \in \mathcal{I}$ different from the possible right end of $\mathcal{I}$, there exists $f^+(t)$; then

$$f^+(t) = \lim_{s\to t,\, s>t} f^+(s). \tag{20}$$

If, in addition, for every t different from the possible left end of $\mathcal{I}$, there exists ; then

$$f^-(t) = \lim_{s\to t,\, s>t} f^+(s). \tag{21}$$

As short hand notation one has:

$$(f^+)^+ = f^+, \quad (f^-)^+ = f^+, \quad (f^-)^- = f^-, \quad (f^+)^- = f^-. \tag{22}$$

Having set the stage, the necessary conditions for the impulsive optimal control problem of scleronomic multibody mechanical systems is formally derived by considering a problem in GPB, in which the objective is to choose arcs $\mathbf{q} \in \mathcal{AC}$ and $\dot{\mathbf{q}} \in \mathcal{LBV}$ on every interval $\left(t_\mathrm{i}^+,\, t_\mathrm{i+1}^-\right) \in \mathcal{I}_\mathrm{T}$ and transition location triplets $\left\{\mathbf{q}(t_\mathrm{i+1}), \dot{\mathbf{q}}(t_\mathrm{i+1}^-), \dot{\mathbf{q}}(\mathrm{i}^+)\right\}$, transition times t_i and final time t_f, $\forall\, t_\mathrm{i} \in \mathcal{I}_\mathrm{T}$ in order to minimize P_Tot:

$$\begin{aligned} &J(\mathbf{q}(t), \dot{\mathbf{q}}(t), \left\{\mathbf{q}(t_\mathrm{i}), \dot{\mathbf{q}}(t_\mathrm{i}^-), \dot{\mathbf{q}}(t_\mathrm{i}^+), t_\mathrm{i}^-, t_\mathrm{i}^+\right\}, t_\mathrm{f}) = \\ &\sum_{\mathrm{i}=1}^{N} l(\mathbf{q}(t_\mathrm{i+1}^-), \dot{\mathbf{q}}(t_\mathrm{i+1}^-), \mathbf{q}(t_\mathrm{i}^+), \dot{\mathbf{q}}(t_\mathrm{i}^+)) + \int_{t_\mathrm{i}^+}^{t_\mathrm{i+1}^-} L_\mathrm{i}(s_\mathrm{i}, \mathbf{q}(s_\mathrm{i}), \dot{\mathbf{q}}(s_\mathrm{i}), \ddot{\mathbf{q}}(s_\mathrm{i}))\, ds_\mathrm{i}\ . \end{aligned} \tag{23}$$

The theory at hand treats optimal solutions as solutions of multi-point boundary value problems (MBVP) with discontinuous transitions in the state. In this setting, the prespecification of the mode sequence and number of intervals must be given in advance. The overall problem as stated in (37) is seen as the union of several problems in the generalized Bolza form. Here it is assumed that the control horizon is composed of N different phases, which are separated from eachother by $N-1$ possibly discontinuous transitions in the generalized velocities. The importance of the transition process becomes clear if one considers the fact that at pre-transition and post-transition states the values of several functions may differ due to discontinuities. A transition process is common to the pre-transition configuration and post-transition configuration. Each problem P_i with a unique mechanical configuration is defined on a closed time domain $\mathrm{dom}(P_\mathrm{i})$ with variable boundary which is partitioned as follows:

$$\mathrm{dom}(P_\mathrm{i}) = \{t_\mathrm{i}^-,\, t_\mathrm{i}^+\} \bigcup \left(, t_\mathrm{i}^+, t_\mathrm{i+1}^-\right) \bigcup \{t_\mathrm{i+1}^-,\, t_\mathrm{i+1}^+\}\,. \tag{24}$$

The boundary of the domain $\mathrm{dom}(P_\mathrm{i})$ is given by:

$$\mathrm{bdy\ dom}(P_\mathrm{i}) = \{t_\mathrm{i}^-,\, t_\mathrm{i}^+\} \bigcup \{t_\mathrm{i+1}^-,\, t_\mathrm{i+1}^+\}\,. \tag{25}$$

The interior of the domain is given by:

$$\mathrm{int\ dom}(P_\mathrm{i}) = \left(t_\mathrm{i}^+, t_\mathrm{i+1}^-\right). \tag{26}$$

The domain of the overall problem P is given by the union:

$$\mathrm{dom}(P_\mathrm{Tot}) = \bigcup_{\forall\,\mathrm{i}} \mathrm{dom}(P_\mathrm{i}). \tag{27}$$

However, the domains of successive problems P_{i} and $P_{\mathrm{i+1}}$ are not disjoint:

$$\mathrm{dom}(P_{\mathrm{i}}) \bigcap \mathrm{dom}(P_{\mathrm{i+1}}) = \mathrm{bdy\ dom}(P_{\mathrm{i}}) \bigcap \mathrm{bdy\ dom}(P_{\mathrm{i+1}}) = \left\{t^{-}_{\mathrm{i+1}},\, t^{+}_{\mathrm{i+1}}\right\}, \tag{28}$$

The set $\mathrm{bdy\ dom}(P_{\mathrm{i}}) \bigcap \mathrm{bdy\ dom}(P_{\mathrm{i+1}}) = \left\{t^{-}_{\mathrm{i+1}},\, t^{+}_{\mathrm{i+1}}\right\}$ is the support of the transition process and is Lebesgue-negligible. The extended-valued integrand may differ on each interval based on the structure of the equations of motion given in (10) and (11). The difference in structure may arise due to change in parameters (i.e. mass, inertia) or degrees of freedom. In [22] a projection approach is presented in case, the mechanical configurations in successive intervals differ based on change in the number of degrees of freedom.

4 Statement of the Optimal Control Problem

The impulsive optimal control of multibody Lagrangian systems is considered, for which the transition times $t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}}$, final time t_{f} and transition locations characterized by triplets $\left\{\mathbf{q}(t_{\mathrm{i}}), \dot{\mathbf{q}}(t^{+}_{\mathrm{i}}), \dot{\mathbf{q}}(t^{-}_{\mathrm{i}})\right\}$ are free. The goal function is to minimize the functional $g(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\tau})$. The differential inclusion of $\{\mathbf{q}(t), \dot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t), \boldsymbol{\tau}(t)\}$ that fulfill the Lebesgue measurable part of the dynamics in every time-interval $(t^{+}_{\mathrm{i}}, t^{-}_{\mathrm{i+1}})$ is denoted by $\mathcal{F}_{\mathrm{i}}$:

$$\mathbf{M}(\mathbf{q}(t))\ddot{\mathbf{q}}(t) \in \mathcal{F}_{\mathrm{i}}\left(\mathbf{q}(t), \dot{\mathbf{q}}(t)\right), \quad t \in (t^{+}_{\mathrm{i}}, t^{-}_{\mathrm{i+1}}) \ . \tag{29}$$

The measurable controls $\boldsymbol{\tau}$ is constrained to a bounded closed polytopic convex set $\mathcal{C}_{\tau}$. The set $\mathcal{C}^{+}_{\mathrm{I_i}}$ denotes the set of $\{\mathbf{q}(t^{+}_{\mathrm{i}}), \dot{\mathbf{q}}(t^{+}_{\mathrm{i}}), \dot{\mathbf{q}}(t^{-}_{\mathrm{i}}), \boldsymbol{\zeta}^{+}_{\mathrm{i}}, \boldsymbol{\zeta}^{-}_{\mathrm{i}}\}$ that fulfill the post-transition impact equation:

$$\mathbf{M}\left(\mathbf{q}(t^{+}_{\mathrm{i}})\right)\left(\dot{\mathbf{q}}(t^{+}_{\mathrm{i}}) - \dot{\mathbf{q}}(t^{-}_{\mathrm{i}})\right) - \mathbf{B}_{\mathrm{i}}\left(\mathbf{q}(t^{+}_{\mathrm{i}})\right)\left(\boldsymbol{\zeta}^{+}_{\mathrm{i}} - \boldsymbol{\zeta}^{-}_{\mathrm{i}}\right) = \mathbf{0}\ , \quad \forall t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}}\ . \tag{30}$$

The set $\mathcal{C}^{-}_{\mathrm{I_i}}$ denotes the set of $\{\mathbf{q}(t^{-}_{\mathrm{i}}), \dot{\mathbf{q}}(t^{+}_{\mathrm{i}}), \mathbf{q}(t^{-}_{\mathrm{i}}), \boldsymbol{\zeta}^{+}_{\mathrm{i}}, \boldsymbol{\zeta}^{-}_{\mathrm{i}}\}$ that fulfill the pre-transition impact equation:

$$\mathbf{M}\left(\mathbf{q}(t^{-}_{\mathrm{i}})\right)\left(\dot{\mathbf{q}}(t^{+}_{\mathrm{i}}) - \dot{\mathbf{q}}(t^{-}_{\mathrm{i}})\right) - \mathbf{B}_{\mathrm{i}}\left(\mathbf{q}(t^{-}_{\mathrm{i}})\right)\left(\boldsymbol{\zeta}^{+}_{\mathrm{i}} - \boldsymbol{\zeta}^{-}_{\mathrm{i}}\right) = \mathbf{0}\ , \quad \forall t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}}\ . \tag{31}$$

The equations (30) and (31) represent smooth manifolds and are smoothly differentiable in their arguments. Analogously, let $\mathcal{C}^{+}_{\mathrm{T_i}}$ denote the set defined by the equality

$$\mathbf{p}^{+}_{\mathrm{i}}(\mathbf{q}(t^{+}_{\mathrm{i}}), \dot{\mathbf{q}}(t^{+}_{\mathrm{i}}), \dot{\mathbf{q}}(t^{-}_{\mathrm{i}})) = \mathbf{0}, \tag{32}$$

and $\mathcal{C}^{-}_{\mathrm{T_i}}$ denote the set defined by the equality

$$\mathbf{p}^{-}_{\mathrm{i}}(\mathbf{q}(t^{-}_{\mathrm{i}}), \dot{\mathbf{q}}(t^{+}_{\mathrm{i}}), \dot{\mathbf{q}}(t^{-}_{\mathrm{i}})) = \mathbf{0}, \tag{33}$$

that arise from the pre-transition and post-transition constitutive impact laws. Both $\mathbf{p}^{-}_{\mathrm{i}}$ and $\mathbf{p}^{+}_{\mathrm{i}}$ are at least C^2 in their arguments. The end state is to be in a convex set $\mathcal{C}_{\mathrm{f}}(\mathbf{q}(t_{\mathrm{f}}), \dot{\mathbf{q}}(t_{\mathrm{f}}))$. By the absolute continuity of the generalized positions, the relations:

$$\mathcal{C}_{\mathrm{T_i}} = \mathcal{C}^{+}_{\mathrm{T_i}} \cup \mathcal{C}^{-}_{\mathrm{T_i}} = \mathcal{C}^{+}_{\mathrm{T_i}} \cap \mathcal{C}^{-}_{\mathrm{T_i}}, \quad \forall t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}}\ , \tag{34}$$

$$\mathcal{C}_{\mathrm{I_i}} = \mathcal{C}^{+}_{\mathrm{I_i}} \cup \mathcal{C}^{-}_{\mathrm{I_i}} = \mathcal{C}^{+}_{\mathrm{I_i}} \cap \mathcal{C}^{-}_{\mathrm{I_i}}, \quad \forall t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}}\ , \tag{35}$$

are tractable.

The overall problem as stated in (37) is seen as the union of several problems in the generalized Bolza form. In its full glory the impulsive optimal control problem is stated as:

$$\min_{\{t_{\mathrm{i}}\}, t_{\mathrm{f}}, \boldsymbol{\tau}, \{\boldsymbol{\zeta}^{+}_{\mathrm{i}}, \boldsymbol{\zeta}^{-}_{\mathrm{i}}\}} J\ , \tag{36}$$

where J is given by:

$$J = l_0 + \sum_{\mathrm{i}=1}^{N} l_{\mathrm{i}} + \sum_{\mathrm{i}=1}^{N} \int_{t^{+}_{\mathrm{i}}}^{t^{-}_{\mathrm{i+1}}} L_{\mathrm{i}}(\mathbf{q}(s), \dot{\mathbf{q}}(s), \ddot{\mathbf{q}}(s))\, ds\ . \tag{37}$$

The costs associated with boundary terms and the integrand are composed in the following manner:

$$\begin{aligned} l_\mathrm{i} &= \Psi_{\mathcal{C}^+_{\mathrm{T_i}}}(\mathbf{q}(t^+_\mathrm{i}), \dot{\mathbf{q}}(t^+_\mathrm{i}), \dot{\mathbf{q}}(t^-_\mathrm{i})) + \Psi_{\mathcal{C}^-_{\mathrm{T_i}}}(\mathbf{q}(t^-_\mathrm{i}), \dot{\mathbf{q}}(t^+_\mathrm{i}), \dot{\mathbf{q}}(t^-_\mathrm{i})) \quad (38)\\ &+ \Psi_{\mathcal{C}^-_{\mathrm{I_i}}}(\mathbf{q}(t^-_\mathrm{i}), \dot{\mathbf{q}}(t^+_\mathrm{i}), \dot{\mathbf{q}}(t^-_\mathrm{i}), \boldsymbol{\zeta}^+_\mathrm{i}, \boldsymbol{\zeta}^-_\mathrm{i}) + \Psi_{\mathcal{C}^+_{\mathrm{I_i}}}(\mathbf{q}(t^+_\mathrm{i}), \dot{\mathbf{q}}(t^+_\mathrm{i}), \dot{\mathbf{q}}(t^-_\mathrm{i}), \boldsymbol{\zeta}^+_\mathrm{i}, \boldsymbol{\zeta}^-_\mathrm{i}),\\ l_0 &= \Psi_{\mathcal{C}_\mathrm{f}}(\mathbf{q}(t_\mathrm{f}), \dot{\mathbf{q}}(t_\mathrm{f})), \quad (39)\\ L_\mathrm{i} &= \lambda(t)\, g(\mathbf{q}(t), \dot{\mathbf{q}}(t), \boldsymbol{\tau}) + \Psi_{\mathrm{Gr}\,\mathcal{F}^+_\mathrm{i}(\mathbf{q}(t^+),\dot{\mathbf{q}}(t^+))}. \quad (40) \end{aligned}$$

The necessary conditions are derived by making use of following assumptions on the general problem:
Assumptions 2

1. the dual states $\boldsymbol{\nu}$ is assumed left-continuous locally bounded variation functions ($\mathcal{LCLBV}$), and the generalized velocities $\dot{\mathbf{q}}$ of the Lagrangian system is assumed right-continuous locally bounded variation functions ($\mathcal{RCLBV}$), whereas the generalized positions are in class $\mathcal{AC}$.
2. The mode sequence and number of intervals for the MBVP constitute a feasible hybrid trajectory.
3. The set $\mathcal{C}^+_{\mathrm{I_i}} \cap \mathcal{C}^+_{\mathrm{T_i}}$ is closed and nonempty.
4. The set $\mathcal{C}^-_{\mathrm{I_i}} \cap \mathcal{C}^-_{\mathrm{T_i}}$ is is closed and nonempty.
5. The goal functional $g(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\tau})$ is convex for all $t \in \Omega_\mathrm{t}$ and $t_\mathrm{i} \in \mathcal{I}_\mathrm{T}$.
6. The limiting partial subdifferential $\partial_{\dot{\mathbf{q}}} g(\mathbf{q}, \dot{\mathbf{q}}, \boldsymbol{\tau})$ is bounded for all $t \in \Omega_\mathrm{t}$ and $t_\mathrm{i} \in \mathcal{I}_\mathrm{T}$.
7. Each $L_\mathrm{i} : (t^+_\mathrm{i}, t^-_{\mathrm{i}+1}) \times \mathbb{R}^\mathrm{n} \times \mathbb{R}^\mathrm{n} \times \mathbb{R}^\mathrm{n} \to \mathbb{R} \cup \{+\infty\}$ is a Lebesgue normal integrand.
8. Each $L_\mathrm{i}(\mathbf{q}(s), \dot{\mathbf{q}}(s), \cdot)$ is convex for each $(\mathbf{q}(s), \dot{\mathbf{q}}(s))$.
9. Each $l_\mathrm{i} : \mathbb{R}^\mathrm{n} \times \mathbb{R}^\mathrm{n} \times \mathbb{R}^\mathrm{n} \times \mathbb{R}^\mathrm{n} \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous.

Assumption (2.8) is equivalent to the epigraph of each L_i

$$\mathrm{epi}\, L_\mathrm{i}(\cdot,\cdot,\cdot) = \{(\mathbf{q}(s), \dot{\mathbf{q}}(s), \ddot{\mathbf{q}}(s), u) \in \mathbb{R}^\mathrm{n} \times \mathbb{R}^\mathrm{n} \times \mathbb{R}^\mathrm{n} \,|\, u \geq L_\mathrm{i}(\mathbf{q}(s), \dot{\mathbf{q}}(s), \ddot{\mathbf{q}}(s))\} \quad (41)$$

being closed and depending Lebesgue measurably on t, in the sense that for each closed $\mathcal{V} \subset \mathbb{R}^\mathrm{n} \times \mathbb{R}^\mathrm{n} \times \mathbb{R}^\mathrm{n} \times \mathbb{R}$, the set

$$\left\{ t \in (t^+_\mathrm{i}, t^-_{\mathrm{i}+1}) \,|\, \mathcal{V} \bigcap \mathrm{epi}\, L_\mathrm{i}(\cdot,\cdot,\cdot) \neq \emptyset \right\} \quad (42)$$

is Lebesgue measurable. Normality implies that $L_\mathrm{i}(\cdot,\cdot,\cdot)$ is Lebesgue measurable in t whenever $(\mathbf{q}(t), \dot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t))$ is and that each $L_\mathrm{i}(\mathbf{q}(t), \dot{\mathbf{q}}(t), \ddot{\mathbf{q}}(t))$ is lower semicontinuous for each $t \in (t^+_\mathrm{i}, t^-_{\mathrm{i}+1})$. These results are given in [16].

The differentiability properties of the transition sets given by the impact equations and the constitutive impact laws are sufficient to render each l_i defined by (38) in the finite-dimensional case subdifferentially regular. As stated in [14] if an extended-valued function f is the indicator function of a set $\mathcal{C}$ that is tangentially regular then it is subdifferentially regular. In this case, the set of which l_i is the indicator function is given by:

$$\mathcal{C} = (\mathcal{C}^-_{\mathrm{I_i}} \cup \mathcal{C}^+_{\mathrm{I_i}}) \cap (\mathcal{C}^-_\mathrm{T} \cup \mathcal{C}^+_\mathrm{T}) = (\mathcal{C}^-_{\mathrm{I_i}} \cap \mathcal{C}^+_{\mathrm{I_i}}) \cap (\mathcal{C}^-_\mathrm{T} \cap \mathcal{C}^+_\mathrm{T}) \quad (43)$$

and is tangentially regular by the differentiability properties of the underlying equations. The set defined in (43) is nonempty and closed by assumptions (2.4) and (2.5) and properties (34) and (35). The closedness of this set is equivalent to the lower semicontinuity as required by assumption (2.10). Assumptions (2.4) and (2.5) are sufficient to render each l_i defined by (38) in the finite-dimensional case directionally Lipschitzian by theorem (1.e) in [14].

5 Internal Boundary Variations and Discontinuous Transversality Conditions

The upper subderivative and the lower subderivative of an extended-valued lower semicontinuous (l.s.c) function is defined as given [13].
Definition 2: Upper and Lower Subderivatives [13] Let f be any extended-real valued lower semi-continuous function

on a linear topological space $\mathcal{E}$, and let $\mathbf{x}$ be any point where f is finite. The upper subderivative of f at $\mathbf{x}$ with respect to $\mathbf{y}$ is defined by:

$$f^{\uparrow}(\mathbf{x};\mathbf{y}) = \limsup_{\substack{\mathbf{x}' \underset{f}{\to} \mathbf{x} \\ t \downarrow 0}} \inf_{\mathbf{y}' \to \mathbf{y}} \frac{f(\mathbf{x}' + t\,\mathbf{y}') - f(\mathbf{x}')}{t}. \tag{44}$$

The lower subderivative of f at $\mathbf{x}$ with respect to $\mathbf{y}$ is defined by:

$$f^{\downarrow}(\mathbf{x};\mathbf{y}) = \liminf_{\substack{\mathbf{x}' \underset{f}{\to} \mathbf{x} \\ t \downarrow 0}} \sup_{\mathbf{y}' \to \mathbf{y}} \frac{f(\mathbf{x}' + t\,\mathbf{y}') - f(\mathbf{x}')}{t}, \tag{45}$$

where

$$\mathbf{x}' \underset{f}{\to} \mathbf{x} \Leftrightarrow \quad \mathbf{x}' \Rightarrow \mathbf{x} \quad \wedge \quad f(\mathbf{x}') \Rightarrow f(\mathbf{x}).$$

Theorem [13](Theorem 4) Let f be any extended-real valued function on a linear topological space $\mathcal{E}$, and let $\mathbf{x}$ be any point where f is finite. Then the "upper" subdifferential $\partial f(\mathbf{x})$ is a weak*-closed convex subset of $\mathcal{E}^*$ and

$$\partial f(\mathbf{x}) = \left\{ \mathbf{z} \in \mathcal{E}^* \mid (\mathbf{z}, -1) \in \mathcal{N}_{\text{epi}\, f}(\mathbf{x},\, f(\mathbf{x})) \right\} \tag{46}$$

If $f^{\uparrow}(\mathbf{x}\,;\,\mathbf{0}) = -\infty$, then $\partial f(\mathbf{x})$ is empty, but otherwise $\partial f(\mathbf{x})$ is nonempty and

$$f^{\uparrow}(\mathbf{x}\,;\,\mathbf{y}) = \sup\,\{\, \langle \mathbf{y},\, \mathbf{z} \rangle \,|\, \mathbf{z} \in \partial f(\mathbf{x}), \quad \forall \mathbf{y} \in \mathcal{E}\} \tag{47}$$

Analogously, the "lower" subdifferential $\tilde{\partial} f(\mathbf{x})$ is a weak*-closed convex subset of $\mathcal{E}^*$ and

$$\tilde{\partial} f(\mathbf{x}) = \left\{ \mathbf{z} \in \mathcal{E}^* \mid (\mathbf{z}, -1) \in \mathcal{N}_{\text{hypo}\, f}(\mathbf{x},\, f(\mathbf{x})) \right\} \tag{48}$$

If $f^{\downarrow}(\mathbf{x}\,;\,\mathbf{0}) = \infty$, then $\tilde{\partial} f(\mathbf{x})$ is empty, but otherwise $\tilde{\partial} f(\mathbf{x})$ is nonempty and

$$f^{\downarrow}(\mathbf{x}\,;\,\mathbf{y}) = \inf \left\{ \langle \mathbf{y},\, \mathbf{z} \rangle \,|\, \mathbf{z} \in \tilde{\partial} f(\mathbf{x}), \quad \forall \mathbf{y} \in \mathcal{E} \right\}. \tag{49}$$

The evaluation of the total subderivative of the value function in the orthogonal boundary variations reveal the discontinuous transversality conditions for each transition time $t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}}$. The set of such orthogonal boundary variations is denoted by $\hat{\mathcal{V}}$. The optimality condition requires that the lower subderivatives of the value functional $J^{\downarrow}(\mathbf{q}, \dot{\mathbf{q}}; \hat{\psi})$ are all nonnegative with respect to the admissible boundary variations:

$$J^{\downarrow}(\mathbf{q}, \dot{\mathbf{q}}; \hat{\psi}) \geq 0, \quad \forall\, \hat{\psi} \in \hat{\mathcal{V}} \text{ and } \hat{\psi} \text{ admissible.} \tag{50}$$

The functional J is directionally Lipschitzian in all directions $\hat{\psi} \in \hat{\mathcal{V}}$. By theorem (6) in [13] this property of J implies the equivalence of the "lower" and "upper" subdifferentials of J:

$$\partial J = \tilde{\partial} J. \tag{51}$$

By reverting to the definition of the "upper" and "lower" subdifferential, one notices that the lower and upper subderivatives coincide in the directionally Lipschitzian case:

$$J^{\downarrow}(\mathbf{q}, \dot{\mathbf{q}}; \hat{\psi}) = J^{\uparrow}(\mathbf{q}, \dot{\mathbf{q}}; \hat{\psi}), \quad \forall\, \hat{\psi} \in \hat{\mathcal{V}}. \tag{52}$$

The upper subderivatives of the functional J have following structure in the evaluation of the internal boundary variations:

$$J^{\uparrow}(\cdot;\, \hat{\psi}_{\mathrm{i}}) = l_{\mathrm{i}}^{\uparrow}(\cdot; \hat{\psi}_{\mathrm{i}}) + \left(\int_{t_{\mathrm{i}}^{+}}^{t_{\mathrm{i}+1}^{-}} L_{\mathrm{i}}(\mathbf{q}(s), \dot{\mathbf{q}}(s), \ddot{\mathbf{q}}(s))\, ds \right)^{\uparrow} (\cdot; \hat{\psi}_{\mathrm{i}}), \quad \forall\, \hat{\psi}_{\mathrm{i}} \in \hat{\mathcal{V}}. \tag{53}$$

5.1 Internal Boundary Variations

In this subsection the internal boundary variations at the transition from subproblem $P_{\mathrm{i}-1}$ to P_{i} is investigated. In the classical calculus of variations where the final state and final time are free, the variations of the final state are composed of two parts, namely, the part that arises of the variations at a given time and the part arising from variations due to final time. Since the transitions times are assumed to be free, the two-part character of the variations at pre- and post-transition states is considered. By the Lebesgue-Stieltjes integration of the differential measure of the generalized velocities following relation:

$$\int_{\{t_{\mathrm{i}}\}} d\dot{\mathbf{q}} = \dot{\mathbf{q}}(t_{\mathrm{i}}^{+}) - \dot{\mathbf{q}}(t_{\mathrm{i}}^{-}) = \boldsymbol{\chi}_{\mathrm{i}}^{+} - \boldsymbol{\chi}_{\mathrm{i}}^{-} \tag{54}$$

is obtained, whereas for the generalized positions one has

$$\int_{\{t_{\mathrm{i}}\}} d\mathbf{q} = \mathbf{q}(t_{\mathrm{i}}^{+}) - \mathbf{q}(t_{\mathrm{i}}^{-}) = \mathbf{0}, \tag{55}$$

by assumption (3.2). Based on the relations (54) and (55), the variations of the pre-, and post-transition generalized positions and velocities at fixed time $\hat{\mathbf{q}}(t_{\mathrm{i}}^{+})$,$\hat{\dot{\mathbf{q}}}(t_{\mathrm{i}}^{+})$, $\hat{\mathbf{q}}(t_{\mathrm{i}}^{-})$, $\hat{\dot{\mathbf{q}}}(t_{\mathrm{i}}^{-})$ are brought in relation with the total variations in these entities $\hat{\mathbf{q}}_{\mathrm{i}}^{+}$,$\hat{\dot{\mathbf{q}}}_{\mathrm{i}}^{+}$, $\hat{\mathbf{q}}_{\mathrm{i}}^{-}$, $\hat{\dot{\mathbf{q}}}_{\mathrm{i}}^{-}$ at each $t_{\mathrm{i}} \in \mathcal{I}_{\mathrm{T}}$ by the following affine relations:

$$\hat{\mathbf{q}}(t_{\mathrm{i}}^{+}) = \hat{\mathbf{q}}_{\mathrm{i}}^{+} - \dot{\mathbf{q}}(t_{\mathrm{i}}^{+})\, \hat{t}_{\mathrm{i}}^{+}, \tag{56}$$

$$\hat{\mathbf{q}}(t_{\mathrm{i}}^{-}) = \hat{\mathbf{q}}_{\mathrm{i}}^{-} - \dot{\mathbf{q}}(t_{\mathrm{i}}^{-})\, \hat{t}_{\mathrm{i}}^{-}, \tag{57}$$

$$\hat{\dot{\mathbf{q}}}(t_{\mathrm{i}}^{+}) = \hat{\dot{\mathbf{q}}}_{\mathrm{i}}^{+} - \ddot{\mathbf{q}}(t_{\mathrm{i}}^{+})\, \hat{t}_{\mathrm{i}}^{+} - \hat{\boldsymbol{\chi}}_{\mathrm{i}}^{+}, \tag{58}$$

$$\hat{\dot{\mathbf{q}}}(t_{\mathrm{i}}^{-}) = \hat{\dot{\mathbf{q}}}_{\mathrm{i}}^{-} - \ddot{\mathbf{q}}(t_{\mathrm{i}}^{-})\, \hat{t}_{\mathrm{i}}^{-} - \hat{\boldsymbol{\chi}}_{\mathrm{i}}^{-}. \tag{59}$$

By making use of the affine relations given in equations (56) to (59) the boundary variations are decomposed into orthogonal independent variations in $\hat{t}_{\mathrm{i}}^{-}$, $\hat{t}_{\mathrm{i}}^{+}$, $\hat{\dot{\mathbf{q}}}_{\mathrm{i}}^{+}$, $\hat{\dot{\mathbf{q}}}_{\mathrm{i}}^{-}$, $\hat{\mathbf{q}}_{\mathrm{i}}^{+}$, $\hat{\mathbf{q}}_{\mathrm{i}}^{-}$, $\hat{\boldsymbol{\zeta}}_{\mathrm{i}}^{-}$, $\hat{\boldsymbol{\zeta}}_{\mathrm{i}}^{+}$ at each transition instant. Thus the internal boundary variations at each transition time are given in the finite-dimensional set $\hat{\mathcal{V}}$:

$$\hat{\mathcal{V}} = \left\{ \hat{t}_{\mathrm{i}}^{-}, \hat{t}_{\mathrm{i}}^{+}, \hat{\dot{\mathbf{q}}}_{\mathrm{i}}^{+}, \hat{\dot{\mathbf{q}}}_{\mathrm{i}}^{-}, \hat{\mathbf{q}}_{\mathrm{i}}^{+}, \hat{\mathbf{q}}_{\mathrm{i}}^{-}, \hat{\boldsymbol{\zeta}}_{\mathrm{i}}^{+}, \hat{\boldsymbol{\zeta}}_{\mathrm{i}}^{-}, \hat{\boldsymbol{\chi}}_{\mathrm{i}}^{-}, \hat{\boldsymbol{\chi}}_{\mathrm{i}}^{+} \right\}. \tag{60}$$

The assumptions during a possibly impactive transition are given as follows:
Assumptions 3

1. The transitions may be impactively.
2. The generalized position remain unchanged during transition.
3. The impulsive control action acts on the system at a time instant t_{i} which is Lebesgue-negligible.
4. At a possibly impactive transition, the pre-transition controller configuration is assumed to be effective.
5. There are no transitions at initial time t_0 and final time t_{f} without loss of generality.

The above stated assumptions are converted into requirements to the variations at the internal boundaries.

5.2 Discontinuous Transversality Conditions

The subderivatives of the integral part of condition (53) is considered first:

$$\left(\int_{t_{\mathrm{i}}^{+}}^{t_{\mathrm{i}+1}^{-}} L_{\mathrm{i}}(\mathbf{q}(s), \dot{\mathbf{q}}(s), \ddot{\mathbf{q}}(s))\, ds \right)^{\uparrow} (\cdot; \hat{\psi}_{\mathrm{i}}), \quad \forall\, \hat{\psi}_{\mathrm{i}} \in \hat{\mathcal{V}}. \tag{61}$$

In order to access the boundary variations and eliminate the variations with respect to $\hat{\dot{\mathbf{q}}}(t^{+})$ and $\hat{\dot{\mathbf{q}}}(t^{+})$ the Lemma of Du Bois-Reymond is used.

In the process of eliminating the variation with respect to $\hat{\dot{\mathbf{q}}}(t^+)$ on every open interval $(t_\mathrm{i}^+, t_{\mathrm{i}+1}^-)$ such that $\{t_\mathrm{i}, t_{\mathrm{i}+1}\} \in \mathcal{I}_\mathrm{T}$ the Lemma of Du Bois-Reymond is applied twice successively:

$$\int_{\mathrm{t_i}+}^{\mathrm{t}_{\mathrm{i}+1}^-} \boldsymbol{\nu}(t^-)\,\mathbf{M}(\mathbf{q}(t^+))\;\hat{\dot{\mathbf{q}}}(t^+)\,dt = \boldsymbol{\nu}(t^-)\,\mathbf{M}(\mathbf{q}(t^+))\;\hat{\mathbf{q}}(t^+)\,\Big|_{\mathrm{t_i}+}^{\mathrm{t}_{\mathrm{i}+1}^-} \tag{62}$$

$$- \int_{\mathrm{t_i}+}^{\mathrm{t}_{\mathrm{i}+1}^-} \left(\dot{\boldsymbol{\nu}}(t^-)\,\mathbf{M}(\mathbf{q}(t^+)) + \boldsymbol{\nu}(t^-)\,\dot{\mathbf{M}}(\mathbf{q}(t^+),\mathbf{q}(\dot{t}^+))\right)\;\hat{\mathbf{q}}(t^+)\,dt$$

$$= -\left(\dot{\boldsymbol{\nu}}(t^-)\,\mathbf{M}(\mathbf{q}(t^+)) + \boldsymbol{\nu}(t^-)\,\dot{\mathbf{M}}(\mathbf{q}(t^+),\dot{\mathbf{q}}(t^+))\right)\;\hat{\mathbf{q}}(t^+)\,\Big|_{\mathrm{t_i}+}^{\mathrm{t}_{\mathrm{i}+1}^-} \tag{63}$$

$$+\;\boldsymbol{\nu}(t^-)\,\mathbf{M}(\mathbf{q}(t^+))\;\hat{\mathbf{q}}(t^+)\Big|_{\mathrm{t_i}+}^{\mathrm{t}_{\mathrm{i}+1}^-} + \int_{\mathrm{t_i}+}^{\mathrm{t}_{\mathrm{i}+1}^-} \frac{d^2}{dt^2}\left(\boldsymbol{\nu}(t^-)\,\mathbf{M}(\mathbf{q}(t^+))\right)\;\hat{\mathbf{q}}(t^+)\,dt,$$

where the second-order time derivative of the integrand multiplier is given by:

$$\frac{d^2}{dt^2}\left(\boldsymbol{\nu}(t^-)\,\mathbf{M}(\mathbf{q}(t^+))\right) = \ddot{\boldsymbol{\nu}}(t^-)\,\mathbf{M}(\mathbf{q}(t^+)) + 2\,\dot{\boldsymbol{\nu}}(t^-)\,\dot{\mathbf{M}}(\mathbf{q}(t^+),\dot{\mathbf{q}}(t^+)) + \boldsymbol{\nu}(t^-)\,\ddot{\mathbf{M}}(\mathbf{q}(t^+),\dot{\mathbf{q}}(t^+),\ddot{\mathbf{q}}(t^+)).$$

In order to eliminate the variations with respect to $\hat{\dot{\mathbf{q}}}(t^+)$ on every open interval $(t_\mathrm{i}^+, t_{\mathrm{i}+1}^-)$ such that $\{t_\mathrm{i}, t_{\mathrm{i}+1}\} \in \mathcal{I}_\mathrm{T}$ the Lemma of Du Bois-Reymond is applied:

$$\int_{\mathrm{t_i}+}^{\mathrm{t}_{\mathrm{i}+1}^-} \lambda(t^+)\left(\partial_{\dot{\mathbf{q}}}\,g(\mathbf{q}(t^+),\dot{\mathbf{q}}(t^+),\boldsymbol{\tau}(t^+)) - \boldsymbol{\nu}(t^-)\,\nabla_{\dot{\mathbf{q}}}\mathbf{h}(\mathbf{q}(t^+),\dot{\mathbf{q}}(t^+))\right)\;\hat{\dot{\mathbf{q}}}(t^+)\,dt$$

$$= \left(\lambda(t^+)\partial_{\dot{\mathbf{q}}}\,g(\mathbf{q}(t^+),\dot{\mathbf{q}}(t^+),\boldsymbol{\tau}(t^+)) - \boldsymbol{\nu}(t^-)\,\nabla_{\dot{\mathbf{q}}}\,\mathbf{h}(\mathbf{q}(t^+),\dot{\mathbf{q}}(t^+))\,\right)\hat{\mathbf{q}}(t^+)\,\Big|_{\mathrm{t_i}+}^{\mathrm{t}_{\mathrm{i}+1}^-} \tag{64}$$

$$- \int_{\mathrm{t_i}+}^{\mathrm{t}_{\mathrm{i}+1}^-} \frac{d}{dt}\left(\lambda(t^+)\partial_{\dot{\mathbf{q}}}\,g(\mathbf{q}(t^+),\dot{\mathbf{q}}(t^+),\boldsymbol{\tau}(t^+)) - \boldsymbol{\nu}(t^-)\,\nabla_{\dot{\mathbf{q}}}\mathbf{h}(\mathbf{q}(t^+),\dot{\mathbf{q}}(t^+))\right)\;\hat{\mathbf{q}}(t^+)\,dt,$$

where

$$\frac{d}{dt}\left(\partial_{\dot{\mathbf{q}}}\,g(\mathbf{q}(t),\dot{\mathbf{q}}(t),\boldsymbol{\tau}(t))\right) = \partial^2_{\dot{\mathbf{q}}\dot{\mathbf{q}}}\,g(\mathbf{q}(t),\dot{\mathbf{q}}(t),\boldsymbol{\tau}(t))\ddot{\mathbf{q}}(t) + \partial^2_{\dot{\mathbf{q}}\mathbf{q}}\,g(\mathbf{q}(t),\dot{\mathbf{q}}(t),\boldsymbol{\tau}(t))\dot{\mathbf{q}}(t),$$

and

$$\frac{d}{dt}\nabla_{\dot{\mathbf{q}}}\,\mathbf{h}(\mathbf{q}(t),\dot{\mathbf{q}}(t)) = \nabla^2_{\dot{\mathbf{q}}\,\dot{\mathbf{q}}}\,\mathbf{h}(\mathbf{q}(t),\dot{\mathbf{q}}(t))\,\ddot{\mathbf{q}}(t) + \nabla^2_{\dot{\mathbf{q}}\,\mathbf{q}}\,\mathbf{h}(\mathbf{q}(t),\dot{\mathbf{q}}(t))\,\dot{\mathbf{q}}(t).$$

The boundary terms obtained by the application of the Lemma of Du Bois-Reymond in equations (63) and (64) can be combined in shorthand notation as given in (65):

$$\boldsymbol{\nu}(t)\,\mathbf{M}\;\hat{\dot{\mathbf{q}}}(t) + \left(\lambda(t)\partial_{\dot{\mathbf{q}}}\,g - \boldsymbol{\nu}(t)\,\nabla_{\dot{\mathbf{q}}}\,\mathbf{h} - \left(\dot{\boldsymbol{\nu}}(t)\,\mathbf{M} + \boldsymbol{\nu}(t)\,\dot{\mathbf{M}}\right)\right)\;\hat{\mathbf{q}}(t)\,\Big|_{\mathrm{t_i}+}^{\mathrm{t_i}^-}\;. \tag{65}$$

The resulting boundary terms as given in (65) obtained by the application of the Lemma of Du Bois-Reymond are rearranged by making use of the equations (56) to (59) in the following form:

$$\sum_{\forall\hat{\psi}_\mathrm{i}\in\hat{\mathcal{V}}}\left(\int_{t_\mathrm{i}^+}^{t_{\mathrm{i}+1}^-} L_\mathrm{i}(\mathbf{q}(s),\dot{\mathbf{q}}(s),\ddot{\mathbf{q}}(s))\,ds\right)^{\uparrow}(\cdot;\hat{\psi}_\mathrm{i}) = \boldsymbol{\Omega}_\mathrm{i}^-\;\hat{\mathbf{q}}_\mathrm{i}^- - \boldsymbol{\Omega}_\mathrm{i}^+\;\hat{\mathbf{q}}_\mathrm{i}^+ + \boldsymbol{\Upsilon}_\mathrm{i}^-\;\hat{t}_\mathrm{i}^- - \boldsymbol{\Upsilon}_\mathrm{i}^+\;\hat{t}_\mathrm{i}^+ \tag{66}$$

$$-\;\boldsymbol{\nu}(t_\mathrm{i}^-)\mathbf{M}(\mathbf{q}(t_\mathrm{i}^-))\,\hat{\chi}_\mathrm{i}^- - \boldsymbol{\nu}(t_\mathrm{i}^+)\mathbf{M}(\mathbf{q}(t_\mathrm{i}^+))\,\hat{\dot{\mathbf{q}}}_\mathrm{i}^+ + \boldsymbol{\nu}(t_\mathrm{i}^-)\mathbf{M}(\mathbf{q}(t_\mathrm{i}^-))\,\hat{\dot{\mathbf{q}}}_\mathrm{i}^- + \boldsymbol{\nu}(t_\mathrm{i}^+)\mathbf{M}(\mathbf{q}(t_\mathrm{i}^+))\;\hat{\chi}_\mathrm{i}^+. \tag{67}$$

Here $\boldsymbol{\Omega}_\mathrm{i}$ and $\boldsymbol{\Upsilon}_\mathrm{i}$ are defined by

$$\boldsymbol{\Omega}_\mathrm{i} = \lambda(t_\mathrm{i})\partial_{\dot{\mathbf{q}}(t_\mathrm{i})}\,g - (\dot{\boldsymbol{\nu}}(t_\mathrm{i})\mathbf{M} + \boldsymbol{\nu}(t_\mathrm{i})\dot{\mathbf{M}}) - \boldsymbol{\nu}(t_\mathrm{i})\nabla_{\dot{\mathbf{q}}(t_\mathrm{i})}\,\mathbf{h}_\mathrm{i} \tag{68}$$

$$\boldsymbol{\Upsilon}_\mathrm{i} = -\boldsymbol{\nu}(t_\mathrm{i})\,\mathbf{M}\;\ddot{\mathbf{q}}(t_\mathrm{i}) - \left(\lambda(t_\mathrm{i})\partial_{\dot{\mathbf{q}}(t_\mathrm{i})}\,g - \boldsymbol{\nu}(t_\mathrm{i})\,\nabla_{\dot{\mathbf{q}}(t_\mathrm{i})}\,\mathbf{h} - \left(\dot{\boldsymbol{\nu}}(t_\mathrm{i})\,\mathbf{M} + \boldsymbol{\nu}(t_\mathrm{i})\,\dot{\mathbf{M}}\right)\right)\;\dot{\mathbf{q}}(t_\mathrm{i}). \tag{69}$$

As can be seen in (66) the variations in $\hat{\chi}_\mathrm{i}^-$ and $\hat{\chi}_\mathrm{i}^+$ are linearly dependent on the variations $\hat{\dot{\mathbf{q}}}_\mathrm{i}^-$ and $\hat{\dot{\mathbf{q}}}_\mathrm{i}^+$, respectively. In passing from (65) to (66) proposition (1) is used where necessary. Since the time-derivatives and gradients of $\dot{\mathbf{M}}$, $\ddot{\mathbf{M}}$, $\nabla_{\dot{\mathbf{q}}}\mathbf{h}$, g, $\boldsymbol{\Omega}_\mathrm{i}$ and $\boldsymbol{\Upsilon}_\mathrm{i}$ involve the generalized velocities and accelerations of the system at pre-, and post-transition state, the right supscripted signs denote wether the pre-transition or post-transition values of the relevant entities are meant where these entities exist in the limit sense as stated in proposition (1). By the presence of transition sets that impose restrictions on the pre-, and post-transition generalised velocities and positions, the internal boundary variations are embedded into the discontinuous transversality conditions. By the tangential regularity of the transition sets, the sum of the indicator functions of the transition sets are rendered subdifferentially regular. By theorem (3) in [14] following is valid:

$$\left(\Psi_{\mathcal{C}^-_{\mathrm{I_i}}} + \Psi_{\mathcal{C}^-_{\mathrm{T_i}}} + \Psi_{\mathcal{C}^+_{\mathrm{I_i}}} + \Psi_{\mathcal{C}^+_{\mathrm{T_i}}}\right)^{\uparrow}(\cdot;\hat{\psi}) = \Psi^{\uparrow}_{\mathcal{C}^-_{\mathrm{I_i}}}(\cdot;\hat{\psi}) + \Psi^{\uparrow}_{\mathcal{C}^-_{\mathrm{T_i}}}(\cdot;\hat{\psi}) + \Psi^{\uparrow}_{\mathcal{C}^+_{\mathrm{I_i}}}(\cdot;\hat{\psi}) + \Psi^{\uparrow}_{\mathcal{C}^+_{\mathrm{T_i}}}(\cdot;\hat{\psi}), \tag{70}$$

$\forall\,\hat{\psi} \in \hat{\mathcal{V}}$. The equality in (70) arises from the subdifferential regularity of the tangential regularity of the sets given in (43). Considering together with (66), the upper subderivative of the value functional J in the direction $\hat{\dot{\mathbf{q}}}^+_{\mathrm{i}}$ is given by:

$$J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{\dot{\mathbf{q}}}^+_{\mathrm{i}}) = \left(\Psi_{\mathcal{C}^-_{\mathrm{I_i}}} + \Psi_{\mathcal{C}^-_{\mathrm{T_i}}} + \Psi_{\mathcal{C}^+_{\mathrm{I_i}}} + \Psi_{\mathcal{C}^+_{\mathrm{T_i}}}\right)^{\uparrow}(\cdot;\hat{\dot{\mathbf{q}}}^+_{\mathrm{i}}) - \boldsymbol{\nu}(t^+_{\mathrm{i}})\mathbf{M}\,\hat{\dot{\mathbf{q}}}^+_{\mathrm{i}}. \tag{71}$$

The condition of optimality stated in (50) becomes:

$$\left(\Psi_{\mathcal{C}^-_{\mathrm{I_i}}} + \Psi_{\mathcal{C}^-_{\mathrm{T_i}}} + \Psi_{\mathcal{C}^+_{\mathrm{I_i}}} + \Psi_{\mathcal{C}^+_{\mathrm{T_i}}}\right)^{\uparrow}(\cdot;\hat{\psi}) \geq \boldsymbol{\nu}(t^+_{\mathrm{i}})\mathbf{M}\,\hat{\dot{\mathbf{q}}}^+_{\mathrm{i}}, \quad \forall\,\hat{\dot{\mathbf{q}}}^+_{\mathrm{i}} \in \mathbb{R}^{\mathrm{n}}. \tag{72}$$

Under the given regularity assumptions on the sets $\mathcal{C}_{\mathrm{T}}$ and $\mathcal{C}_{\mathrm{I}}$, this equivalently means that the vector $\boldsymbol{\nu}(t^+_{\mathrm{i}})\mathbf{M}(\mathbf{q}(t^+_{\mathrm{i}}))$ is in the partial asymptotic limiting subdifferential of the indicator function of the set $(\mathcal{C}^-_{\mathrm{I_i}} \cup \mathcal{C}^+_{\mathrm{I_i}}) \cap (\mathcal{C}^-_{\mathrm{T}} \cup \mathcal{C}^+_{\mathrm{T}})$ with respect to $\dot{\mathbf{q}}(t^+_{\mathrm{i}})$:

$$\boldsymbol{\nu}(t^+_{\mathrm{i}})\mathbf{M}(\mathbf{q}(t_{\mathrm{i}})) \in \partial^{\infty}_{\dot{\mathbf{q}}(t^+_{\mathrm{i}})}\Psi_{(\mathcal{C}^-_{\mathrm{I_i}}\cup\mathcal{C}^+_{\mathrm{I_i}})\cap(\mathcal{C}^-_{\mathrm{T}}\cup\mathcal{C}^+_{\mathrm{T}})}, \tag{73}$$

which equivalently can be expressed as:

$$\boldsymbol{\nu}(t^+_{\mathrm{i}})\mathbf{M}(\mathbf{q}(t_{\mathrm{i}})) \in \partial^{\infty}_{\dot{\mathbf{q}}(t^+_{\mathrm{i}})}\Psi_{(\mathcal{C}^-_{\mathrm{I_i}}\cap\mathcal{C}^+_{\mathrm{I_i}})\cap(\mathcal{C}^-_{\mathrm{T}}\cap\mathcal{C}^+_{\mathrm{T}})}, \tag{74}$$

by theorem (4) in [13]. By the properties of indicator functions the optimality condition states that the vector $\boldsymbol{\nu}(t^+_{\mathrm{i}})\mathbf{M}(\mathbf{q}(t^+_{\mathrm{i}}))$ is in the partial singular limiting normal cone of the indicator function of the set $(\mathcal{C}^-_{\mathrm{I_i}} \cup \mathcal{C}^+_{\mathrm{I_i}}) \cap (\mathcal{C}^-_{\mathrm{T}} \cup \mathcal{C}^+_{\mathrm{T}})$ with respect to $\hat{\dot{\mathbf{q}}}(t^+_{\mathrm{i}})$:

$$\boldsymbol{\nu}(t^+_{\mathrm{i}})\mathbf{M}(\mathbf{q}(t_{\mathrm{i}})) \in {}^{\infty}\mathcal{N}_{(\mathcal{C}^-_{\mathrm{I_i}}\cup\mathcal{C}^+_{\mathrm{I_i}})\cap(\mathcal{C}^-_{\mathrm{T}}\cup\mathcal{C}^+_{\mathrm{T}})}(\cdot;\hat{\dot{\mathbf{q}}}(t^+_{\mathrm{i}})). \tag{75}$$

Considering together with (66), the upper subderivative of the value functional J in the direction $\hat{\dot{\mathbf{q}}}^-_{\mathrm{i}}$ is given by :

$$J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{\dot{\mathbf{q}}}^-_{\mathrm{i}}) = \left(\Psi_{\mathcal{C}^-_{\mathrm{I_i}}} + \Psi_{\mathcal{C}^-_{\mathrm{T_i}}} + \Psi_{\mathcal{C}^+_{\mathrm{I_i}}} + \Psi_{\mathcal{C}^+_{\mathrm{T_i}}}\right)^{\uparrow}(\cdot;\hat{\dot{\mathbf{q}}}^-_{\mathrm{i}}) + \boldsymbol{\nu}(t^-_{\mathrm{i}})\mathbf{M}\,\hat{\dot{\mathbf{q}}}^-_{\mathrm{i}}. \tag{76}$$

The condition of optimality stated in (50) becomes:

$$\left(\Psi_{\mathcal{C}^-_{\mathrm{I_i}}} + \Psi_{\mathcal{C}^-_{\mathrm{T_i}}} + \Psi_{\mathcal{C}^+_{\mathrm{I_i}}} + \Psi_{\mathcal{C}^+_{\mathrm{T_i}}}\right)^{\uparrow}(\cdot;\hat{\dot{\mathbf{q}}}^-_{\mathrm{i}}) \geq -\boldsymbol{\nu}(t^-_{\mathrm{i}})\mathbf{M}\,\hat{\dot{\mathbf{q}}}^-_{\mathrm{i}}, \quad \forall\,\hat{\dot{\mathbf{q}}}^-_{\mathrm{i}} \in \mathbb{R}^{\mathrm{n}}. \tag{77}$$

Under the given regularity assumptions on the sets $\mathcal{C}_{\mathrm{T}}$ and $\mathcal{C}_{\mathrm{I}}$, this equivalently means that the vector $-\boldsymbol{\nu}(t^-_{\mathrm{i}})\mathbf{M}(\mathbf{q}(t^-_{\mathrm{i}}))$ is in the partial asymptotic limiting subdifferential of the indicator function of the set $(\mathcal{C}^-_{\mathrm{I_i}} \cup \mathcal{C}^+_{\mathrm{I_i}}) \cap (\mathcal{C}^-_{\mathrm{T}} \cup \mathcal{C}^+_{\mathrm{T}})$ with respect to $\dot{\mathbf{q}}(t^+_{\mathrm{i}})$:

$$-\boldsymbol{\nu}(t^-_{\mathrm{i}})\mathbf{M}(\mathbf{q}(t_{\mathrm{i}})) \in \partial^{\infty}_{\dot{\mathbf{q}}(t^-_{\mathrm{i}})}\Psi_{(\mathcal{C}^-_{\mathrm{I_i}}\cup\mathcal{C}^+_{\mathrm{I_i}})\cap(\mathcal{C}^-_{\mathrm{T}}\cup\mathcal{C}^+_{\mathrm{T}})}, \tag{78}$$

which equivalently can be expressed as:

$$\boldsymbol{\nu}(t^-_{\mathrm{i}})\mathbf{M}(\mathbf{q}(t_{\mathrm{i}})) \in \partial^{\infty}_{\dot{\mathbf{q}}(t^-_{\mathrm{i}})}\Psi_{(\mathcal{C}^-_{\mathrm{I_i}}\cap\mathcal{C}^+_{\mathrm{I_i}})\cap(\mathcal{C}^-_{\mathrm{T}}\cap\mathcal{C}^+_{\mathrm{T}})}, \tag{79}$$

by theorem (4) in [13]. By the properties of indicator functions the optimality condition states that the vector $-\boldsymbol{\nu}(t^-_{\mathrm{i}})\mathbf{M}(\mathbf{q}(t^-_{\mathrm{i}}))$ is in the partial singular limiting normal cone of the indicator function of the set $(\mathcal{C}^-_{\mathrm{I_i}} \cup \mathcal{C}^+_{\mathrm{I_i}}) \cap (\mathcal{C}^-_{\mathrm{T}} \cup \mathcal{C}^+_{\mathrm{T}})$ with respect to $\dot{\mathbf{q}}(t^-_{\mathrm{i}})$:

$$\boldsymbol{\nu}(t^+_{\mathrm{i}})\mathbf{M}(\mathbf{q}(t_{\mathrm{i}})) \in {}^{\infty}\mathcal{N}_{(\mathcal{C}^-_{\mathrm{I_i}}\cup\mathcal{C}^+_{\mathrm{I_i}})\cap(\mathcal{C}^-_{\mathrm{T}}\cup\mathcal{C}^+_{\mathrm{T}})}(\cdot;\dot{\mathbf{q}}(t^-_{\mathrm{i}})). \tag{80}$$

Considering together with (66), the upper subderivative of the value functional J in direction $\hat{\mathbf{q}}_\mathrm{i}^-$ is given by:

$$\begin{aligned} J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{\mathbf{q}}_\mathrm{i}^+) &= \left(\Psi_{\mathcal{C}_{\mathrm{I_i}}^-}+\Psi_{\mathcal{C}_{\mathrm{T_i}}^-}+\Psi_{\mathcal{C}_{\mathrm{I_i}}^+}+\Psi_{\mathcal{C}_{\mathrm{T_i}}^+}\right)^{\uparrow}(\cdot;\hat{\mathbf{q}}_\mathrm{i}^+) \\ +\ & \left(-\lambda(t_\mathrm{i}^+)\partial_{\dot{\mathbf{q}}(t_\mathrm{i}^+)}\, g(\mathbf{q}(t_\mathrm{i}^+),\dot{\mathbf{q}}(t_\mathrm{i}^+),\boldsymbol{\tau}(t_\mathrm{i}^+))+(\dot{\boldsymbol{\nu}}(t_\mathrm{i}^+)\mathbf{M}+\boldsymbol{\nu}(t_\mathrm{i}^+)\dot{\mathbf{M}})+\boldsymbol{\nu}(t_\mathrm{i}^+)\nabla_{\dot{\mathbf{q}}}\,\mathbf{h}_\mathrm{i}\right)\hat{\mathbf{q}}_\mathrm{i}^+. \end{aligned} \tag{81}$$

Considering together with (66), the upper subderivative of the value functional J in direction $\hat{\mathbf{q}}_\mathrm{i}^+$ is given by:

$$\begin{aligned} J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{\mathbf{q}}_\mathrm{i}^-) &= \left(\Psi_{\mathcal{C}_{\mathrm{I_i}}^-}+\Psi_{\mathcal{C}_{\mathrm{T_i}}^-}+\Psi_{\mathcal{C}_{\mathrm{I_i}}^+}+\Psi_{\mathcal{C}_{\mathrm{T_i}}^+}\right)^{\uparrow}(\cdot;\hat{\mathbf{q}}_\mathrm{i}^-) \\ -\ & \left(-\lambda(t_\mathrm{i}^-)\partial_{\dot{\mathbf{q}}(t_\mathrm{i}^-)}\, g(\mathbf{q}(t_\mathrm{i}^-),\dot{\mathbf{q}}(t_\mathrm{i}^-),\boldsymbol{\tau}(t_\mathrm{i}^-))+(\dot{\boldsymbol{\nu}}(t_\mathrm{i}^-)\mathbf{M}+\boldsymbol{\nu}(t_\mathrm{i}^-)\dot{\mathbf{M}})+\boldsymbol{\nu}(t_\mathrm{i}^-)\nabla_{\dot{\mathbf{q}}}\,\mathbf{h}_\mathrm{i}\right)\hat{\mathbf{q}}_\mathrm{i}^-. \end{aligned} \tag{82}$$

As a corollary of assumption (3.2) the post and pre-transition variations of the generalized position are set equal:

$$\hat{\mathbf{q}}_\mathrm{i}^+=\hat{\mathbf{q}}_\mathrm{i}^-=\hat{\mathbf{q}}_\mathrm{i}. \tag{83}$$

The combination of the conditions given in (81) and (82) under assumption (3.2) reveals following optimality condition in variational inequality form:

$$\begin{aligned} J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{\mathbf{q}}_\mathrm{i}) \quad = \quad & J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{\mathbf{q}}_\mathrm{i}^-)+J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{\mathbf{q}}_\mathrm{i}^+)\geq 0, \quad \forall\left(\hat{\mathbf{q}}_\mathrm{i},\hat{\mathbf{q}}_\mathrm{i}^-,\hat{\mathbf{q}}_\mathrm{i}^+\right)\in\mathbb{R}^\mathrm{n}\times\mathbb{R}^\mathrm{n}\times\mathbb{R}^\mathrm{n} \\ & \text{such that } \hat{\mathbf{q}}_\mathrm{i}^+=\hat{\mathbf{q}}_\mathrm{i}^-=\hat{\mathbf{q}}_\mathrm{i}. \end{aligned} \tag{84}$$

which is equivalent to:

$$\left(\Psi_{\mathcal{C}_{\mathrm{I_i}}^-}+\Psi_{\mathcal{C}_{\mathrm{T_i}}^-}+\Psi_{\mathcal{C}_{\mathrm{I_i}}^+}+\Psi_{\mathcal{C}_{\mathrm{T_i}}^+}\right)^{\uparrow}(\cdot;\hat{\mathbf{q}}_\mathrm{i})\geq\left(\boldsymbol{\Omega}_\mathrm{i}^+-\boldsymbol{\Omega}_\mathrm{i}^-\right)\hat{\mathbf{q}}_\mathrm{i}, \quad \forall\,\hat{\mathbf{q}}_\mathrm{i}\in\mathbb{R}^\mathrm{n}. \tag{85}$$

Under the given regularity assumptions on the sets $\mathcal{C}_\mathrm{T}$ and $\mathcal{C}_\mathrm{I}$, this equivalently means that the vector $\boldsymbol{\Omega}_\mathrm{i}^+-\boldsymbol{\Omega}_\mathrm{i}^-$ is in the partial asymptotic limiting subdifferential of the set $(\mathcal{C}_{\mathrm{I_i}}^-\cup\mathcal{C}_{\mathrm{I_i}}^+)\cap(\mathcal{C}_\mathrm{T}^-\cup\mathcal{C}_\mathrm{T}^+)$ with respect to $\mathbf{q}(t_\mathrm{i})$ at the optimal solution:

$$\boldsymbol{\Omega}_\mathrm{i}^+-\boldsymbol{\Omega}_\mathrm{i}^-\in\partial^{\infty}_{\mathbf{q}(t_\mathrm{i})}\Psi_{(\mathcal{C}_{\mathrm{I_i}}^-\cup\mathcal{C}_{\mathrm{I_i}}^+)\cap(\mathcal{C}_\mathrm{T}^-\cup\mathcal{C}_\mathrm{T}^+)} \tag{86}$$

which equivalently can be expressed as:

$$\boldsymbol{\Omega}_\mathrm{i}^+-\boldsymbol{\Omega}_\mathrm{i}^-\in\partial^{\infty}_{\mathbf{q}(t_\mathrm{i})}\Psi_{(\mathcal{C}_{\mathrm{I_i}}^-\cap\mathcal{C}_{\mathrm{I_i}}^+)\cap(\mathcal{C}_\mathrm{T}^-\cap\mathcal{C}_\mathrm{T}^+)} \tag{87}$$

By the properties of indicator functions the optimality condition states that the vector $\boldsymbol{\Omega}_\mathrm{i}^+-\boldsymbol{\Omega}_\mathrm{i}^-$ is in the partial singular limiting normal cone of the indicator function of the set $(\mathcal{C}_{\mathrm{I_i}}^-\cup\mathcal{C}_{\mathrm{I_i}}^+)\cap(\mathcal{C}_\mathrm{T}^-\cup\mathcal{C}_\mathrm{T}^+)$ with respect to $\mathbf{q}(t_\mathrm{i})$:

$$\boldsymbol{\Omega}_\mathrm{i}^+-\boldsymbol{\Omega}_\mathrm{i}^-\in{}^{\infty}\mathcal{N}_{(\mathcal{C}_{\mathrm{I_i}}^-\cup\mathcal{C}_{\mathrm{I_i}}^+)\cap(\mathcal{C}_\mathrm{T}^-\cup\mathcal{C}_\mathrm{T}^+)}(\cdot;\mathbf{q}(t_\mathrm{i})). \tag{88}$$

The upper subderivatives of the value function with respect to impulsive controls $\hat{\zeta}_\mathrm{i}^+$ and $\hat{\zeta}_\mathrm{i}^-$ yields following variational inequalities as optimality conditions:

$$J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{\boldsymbol{\zeta}}_\mathrm{i}^+)=\left(\Psi_{\mathcal{C}_{\mathrm{I_i}}^-}+\Psi_{\mathcal{C}_{\mathrm{T_i}}^-}+\Psi_{\mathcal{C}_{\mathrm{I_i}}^+}+\Psi_{\mathcal{C}_{\mathrm{T_i}}^+}\right)^{\uparrow}(\cdot;\hat{\boldsymbol{\zeta}}_\mathrm{i}^+)\geq 0, \quad \forall\,\hat{\boldsymbol{\zeta}}_\mathrm{i}^+\in\mathbb{R}^\mathrm{n}, \tag{89}$$

and

$$J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{\boldsymbol{\zeta}}_\mathrm{i}^-)=\left(\Psi_{\mathcal{C}_{\mathrm{I_i}}^-}+\Psi_{\mathcal{C}_{\mathrm{T_i}}^-}+\Psi_{\mathcal{C}_{\mathrm{I_i}}^+}+\Psi_{\mathcal{C}_{\mathrm{T_i}}^+}\right)^{\uparrow}(\cdot;\hat{\boldsymbol{\zeta}}_\mathrm{i}^-)\geq 0, \quad \forall\,\hat{\boldsymbol{\zeta}}_\mathrm{i}^-\in\mathbb{R}^\mathrm{n}. \tag{90}$$

respectively. The optimality conditions given (95) and (96) translate by theorem by theorem (4) in [13] into following normal cone inclusions:

$$\left(\boldsymbol{\xi}_\mathrm{i}^++\boldsymbol{\xi}_\mathrm{i}^-\right)\mathbf{B}_\mathrm{i}(\mathbf{q}(t_\mathrm{i}))\in{}^{\infty}\mathcal{N}_{(\mathcal{C}_{\mathrm{I_i}}^-\cup\mathcal{C}_{\mathrm{I_i}}^+)\cap(\mathcal{C}_\mathrm{T}^-\cup\mathcal{C}_\mathrm{T}^+)}(\cdot;\boldsymbol{\zeta}_\mathrm{i}^-), \tag{91}$$

and

$$-\left(\boldsymbol{\xi}_\mathrm{i}^++\boldsymbol{\xi}_\mathrm{i}^-\right)\mathbf{B}_\mathrm{i}(\mathbf{q}(t_\mathrm{i}))\in{}^{\infty}\mathcal{N}_{(\mathcal{C}_{\mathrm{I_i}}^-\cup\mathcal{C}_{\mathrm{I_i}}^+)\cap(\mathcal{C}_\mathrm{T}^-\cup\mathcal{C}_\mathrm{T}^+)}(\cdot;\boldsymbol{\zeta}_\mathrm{i}^+), \tag{92}$$

for some $\boldsymbol{\xi}_{\rm i}^{+} \in \mathbb{R}^{1\times \rm n}$ and $\boldsymbol{\xi}_{\rm i}^{-} \in \mathbb{R}^{1\times \rm n}$.

By corollary (5) of theorem (3) in [14] the partial normal cones given in (75), (80), (88), (91) and (92) which are obtained from the variational inequalities (72), (77), (85), (95) and (96) via theorem (4) in [13] can be stated due to the tangential regularity of the sets given (43) as a normal cone inclusion of the form:

$$\begin{pmatrix} \boldsymbol{\nu}(t_{\rm i}^{+})\mathbf{M}(\mathbf{q}(t_{\rm i})) \\ -\boldsymbol{\nu}(t_{\rm i}^{-})\mathbf{M}(\mathbf{q}(t_{\rm i})) \\ \boldsymbol{\Omega}_{\rm i}^{+} - \boldsymbol{\Omega}_{\rm i}^{-} \\ \left(\boldsymbol{\xi}_{\rm i}^{+} + \boldsymbol{\xi}_{\rm i}^{-}\right) \mathbf{B}(\mathbf{q}(t_{\rm i})) \\ -\left(\boldsymbol{\xi}_{\rm i}^{+} + \boldsymbol{\xi}_{\rm i}^{-}\right) \mathbf{B}(\mathbf{q}(t_{\rm i})) \end{pmatrix} \in {}^{\infty}\mathcal{N}_{(\mathcal{C}_{\rm I_i}^{-}\cap\mathcal{C}_{\rm I_i}^{+})\cap(\mathcal{C}_{\rm T_i}^{-}\cap\mathcal{C}_{\rm T_i}^{+})}(\cdot;\mathbf{y}(t_{\rm i})) \tag{93}$$

and the vector $\mathbf{y}(t_{\rm i})$ is given by:

$$\mathbf{y}(t_{\rm i}) = \left(\ \dot{\mathbf{q}}(t_{\rm i}^{+}),\quad \dot{\mathbf{q}}(t_{\rm i}^{-}),\quad \mathbf{q}(t_{\rm i}),\quad \boldsymbol{\zeta}_{\rm i}^{+},\quad \boldsymbol{\zeta}_{\rm i}^{-}\ \right)^{\rm T}. \tag{94}$$

The variations due to the transition time are some more involved. The subderivatives with respect to pre-transitional time and post-transitional time instant yield (95) and (96) as variational inequalities as optimality condition:

$$J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{t}_{\rm i}^{-}) = \left(\Psi_{\mathcal{C}_{\rm I_i}^{-}} + \Psi_{\mathcal{C}_{\rm T_i}^{-}} + \Psi_{\mathcal{C}_{\rm I_i}^{+}} + \Psi_{\mathcal{C}_{\rm T_i}^{+}}\right)^{\uparrow}(\cdot;\hat{t}_{\rm i}^{-}) + \Upsilon^{-}\hat{t}_{\rm i}^{-} \geq 0,, \quad \forall \hat{t}_{\rm i}^{-} \in \mathbb{R}, \tag{95}$$

and

$$J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{t}_{\rm i}^{+}) = \left(\Psi_{\mathcal{C}_{\rm I_i}^{-}} + \Psi_{\mathcal{C}_{\rm T_i}^{-}} + \Psi_{\mathcal{C}_{\rm I_i}^{+}} + \Psi_{\mathcal{C}_{\rm T_i}^{+}}\right)^{\uparrow}(\cdot;\hat{t}_{\rm i}^{+}) + \Upsilon^{+}\hat{t}_{\rm i}^{+} \geq 0, \quad \forall \hat{t}_{\rm i}^{+} \in \mathbb{R}. \tag{96}$$

The entities Υ^{+} and Υ^{-} are defined as given in (66). As a corollary of assumption (3.3) the post and pre-transition variations of the transition instant are set equal:

$$\hat{t}_{\rm i}^{+} = \hat{t}_{\rm i}^{-} = \hat{t}_{\rm i}. \tag{97}$$

The condition

$$J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{t}_{\rm i}) = J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{t}_{\rm i}^{-}) + J^{\uparrow}(\mathbf{q},\dot{\mathbf{q}};\hat{t}_{\rm i}^{+}) \geq 0 \quad \forall \left(\hat{t}_{\rm i},\hat{t}_{\rm i}^{-},\hat{t}_{\rm i}^{+}\right) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R} \\ \text{such that } \hat{t}_{\rm i}^{+} = \hat{t}_{\rm i}^{-} = \hat{t}_{\rm i}. \tag{98}$$

reveals following variational inequality under consideration of affine relations from (56) to (59) and theorem (3) in [14]:

$$-\sum_{\forall\theta\in\Theta}\left\langle \left(\Psi_{\mathcal{C}_{\rm I_i}^{-}} + \Psi_{\mathcal{C}_{\rm T_i}^{-}} + \Psi_{\mathcal{C}_{\rm I_i}^{+}} + \Psi_{\mathcal{C}_{\rm T_i}^{+}}\right)^{\uparrow}(\cdot;\hat{\theta}),\dot{\theta}\right\rangle \hat{t}_{\rm i} + \left(\Upsilon^{+} + \Upsilon^{-}\right)\hat{t}_{\rm i} \geq 0, \quad \forall \hat{t}_{\rm i} \in \mathbb{R}. \tag{99}$$

where the index set Θ is given by:

$$\Theta = \left\{\dot{\mathbf{q}}(t_{\rm i}^{+}),\, \dot{\mathbf{q}}(t_{\rm i}^{-}),\, \mathbf{q}(t_{\rm i}^{-}),\, \mathbf{q}(t_{\rm i}^{+})\right\}. \tag{100}$$

6 Necessary Conditions

Under Assumptions (1), (2) and (3) the value function possesses regularity properties which enable the statement of "sharp" necessary conditions. The conditions under which "sharp" necessary conditions are obtained is discussed in [17].

Theorem 6.1 *Let assumptions (1), (2) and (3) be valid for the optimal control problem. If optimal trajectories of generalized positions $\mathbf{q}^{*}(t^{+}) \in \mathcal{AC}[\mathbb{R}^{\rm n}]$, velocities $\dot{\mathbf{q}}^{*}(t^{+}) \in \mathcal{RCLBV}[\mathbb{R}^{\rm n}]$ provide a minimum for the described optimal control problem, then there exist optimal controls $\boldsymbol{\tau}^{*}(t)$, optimal transition times $t_{\rm i}^{*} \in \mathcal{I}_{\rm T}$, dual multipliers $\boldsymbol{\xi}_{\rm i}^{+*}$, $\boldsymbol{\xi}_{\rm i}^{-*}$, $\boldsymbol{\alpha}_{\rm i}^{+*}$, $\boldsymbol{\alpha}_{\rm i}^{-*}$ in $\mathbb{R}^{1\times\rm n}$, $\forall t_{\rm i}^{*} \in \mathcal{I}_{\rm T}$, transition location triplets $\left\{\mathbf{q}^{*}(t_{\rm i}),\, \dot{\mathbf{q}}^{*}(t_{\rm i}^{+}),\, \dot{\mathbf{q}}^{*}(t_{\rm i}^{-})\right\}$, dual state $\boldsymbol{\nu}^{*}(t^{-}) \in \mathcal{LCLBV}^{\star}[\mathbb{R}^{1\times\rm n}]$ (where * denote dual space) and a scalar $\lambda(t^{+}) \in \{0,1\}$, such that $\lambda^{*}(t^{+}) + |\boldsymbol{\nu}^{*}(t^{-})| > 0$ forall $t \in \Omega_{\rm t} \cup \mathcal{I}_{\rm T}$, which fulfill:*

1. *the Lebesgue-measurable dynamics in every interval of motion $t \in (t_{\rm i}^{+*}, t_{\rm i+1}^{-*})$*

$$\mathbf{M}(\mathbf{q}^{*}(t^{+}))\ddot{\mathbf{q}}^{*}(t^{+}) - \mathbf{h}_{\rm i}(\mathbf{q}^{*}(t^{+}),\dot{\mathbf{q}}^{*}(t^{+})) - \mathbf{B}_{\rm i}(\mathbf{q}^{*}(t^{+}))\,\boldsymbol{\tau}^{+*} = \mathbf{0},\ a.e., \tag{101}$$

2. *the Lebesgue-measurable dual dynamics*

$$\ddot{\boldsymbol{\nu}}^*(t^-)\,\mathbf{D}_\mathrm{i} + \dot{\boldsymbol{\nu}}^*(t^-)\,\mathbf{E}_\mathrm{i} + \boldsymbol{\nu}^*(t^-)\,\mathbf{F}_\mathrm{i} + \mathbf{G}_\mathrm{i} = \mathbf{0}\ , \quad a.e., \quad t \in (t_\mathrm{i}^{+*}, t_{\mathrm{i}+1}^{-*}), \tag{102}$$

where the coefficients in the differential equation above are given by:

$$\begin{aligned}
\mathbf{D}_\mathrm{i} &= \mathbf{M}(\mathbf{q}^*(t^+)),\\
\mathbf{E}_\mathrm{i} &= 2\dot{\mathbf{M}}(\mathbf{q}^*(t^+)) + \nabla_{\dot{\mathbf{q}}}\,\mathbf{h}_\mathrm{i}(\mathbf{q}^*(t^+), \dot{\mathbf{q}}^*(t^+)),\\
\mathbf{F}_\mathrm{i} &= \nabla_{\mathbf{q}}[\mathbf{M}(\mathbf{q}^*(t^+))\,\ddot{\mathbf{q}}^*(t^+) - \mathbf{h}_\mathrm{i}(\mathbf{q}^*(t^+), \dot{\mathbf{q}}^*(t^+)) - \mathbf{B}_\mathrm{i}(\mathbf{q}^*(t^+))\,\boldsymbol{\tau}] + \ddot{\mathbf{M}}(\mathbf{q}^*(t^+))\\
&+ \frac{d}{dt}\left(\nabla_{\dot{\mathbf{q}}}[\mathbf{h}_\mathrm{i}(\mathbf{q}^*(t^+), \dot{\mathbf{q}}(t^+))]\right) \quad ,\\
\mathbf{G}_\mathrm{i} &= \lambda^*(t^+)\left(\partial_{\mathbf{q}}\, g(\mathbf{q}^*(t^+), \dot{\mathbf{q}}^*(t^+), \boldsymbol{\tau}^*(t^+)) - \frac{d}{dt}\,\partial_{\dot{\mathbf{q}}}\, g(\mathbf{q}^*(t^+), \dot{\mathbf{q}}^*(t^+), \boldsymbol{\tau}^*(t^+))\right)
\end{aligned}$$

3. *the optimal control law on every interval* $\left(t_\mathrm{i}^{+*}, t_{\mathrm{i}+1}^{-*}\right)$

$$-\lambda^*(t^+)\partial_\tau\, g(\mathbf{q}^*(t^+), \dot{\mathbf{q}}^*(t^+), \boldsymbol{\tau}^*(t^+)) - \boldsymbol{\nu}^*(t^-)\,\mathbf{B}_\mathrm{i}(\mathbf{q}^*(t^+)) \in \mathcal{N}_{\mathcal{C}_\tau}(\boldsymbol{\tau}^{+*}),\ a.e.\ , \tag{103}$$

4. *the condition*

$$\Upsilon^{+*} + \Upsilon^{-*} = \mathbf{r}_{\mathrm{i}1}\,\dot{\mathbf{q}}^*(t_\mathrm{i}^-) + \mathbf{r}_{\mathrm{i}2}\,\dot{\mathbf{q}}^*(t_\mathrm{i}^+) + \mathbf{r}_{\mathrm{i}3}\,\ddot{\mathbf{q}}^*(t_\mathrm{i}^-) + \mathbf{r}_{\mathrm{i}4}\,\ddot{\mathbf{q}}^*(t_\mathrm{i}^+), \quad \forall t_\mathrm{i}^* \in \mathcal{I}_\mathrm{T} \tag{104}$$

where the vectors $\mathbf{r}_{\mathrm{i}1}$, $\mathbf{r}_{\mathrm{i}2}$, $\mathbf{r}_{\mathrm{i}3}$ *and* $\mathbf{r}_{\mathrm{i}4}$ *are given by:*

$$\begin{aligned}
\mathbf{r}_{\mathrm{i}1} &= \boldsymbol{\alpha}_\mathrm{i}^{-*}\,\nabla_{\mathbf{q}(t_\mathrm{i})}\,\mathbf{p}_\mathrm{i}^-(\mathbf{q}^*(t_\mathrm{i}^-), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-))\\
&+ \boldsymbol{\xi}_\mathrm{i}^{-*}\,\nabla_{\mathbf{q}(t_\mathrm{i})}\left[\mathbf{M}\left(\mathbf{q}^*(t_\mathrm{i})\right)\left(\dot{\mathbf{q}}(t_\mathrm{i}^{+*}) - \dot{\mathbf{q}}(t_\mathrm{i}^{-*})\right) - \mathbf{B}_\mathrm{i}\left(\mathbf{q}^*(t_\mathrm{i})\right)\left(\boldsymbol{\zeta}_\mathrm{i}^{+*} - \boldsymbol{\zeta}_\mathrm{i}^{-*}\right)\right]\ ,\\
\mathbf{r}_{\mathrm{i}2} &= \boldsymbol{\alpha}_\mathrm{i}^{+*}\,\nabla_{\mathbf{q}(t_\mathrm{i})}\,\mathbf{p}_\mathrm{i}^+(\mathbf{q}^*(t_\mathrm{i}^+)\dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-))\\
&+ \boldsymbol{\xi}_\mathrm{i}^{+*}\,\nabla_{\mathbf{q}(t_\mathrm{i})}\left[\mathbf{M}\left(\mathbf{q}^*(t_\mathrm{i})\right)\left(\dot{\mathbf{q}}(t_\mathrm{i}^{+*}) - \dot{\mathbf{q}}(t_\mathrm{i}^{-*})\right) - \mathbf{B}_\mathrm{i}\left(\mathbf{q}^*(t_\mathrm{i})\right)\left(\boldsymbol{\zeta}_\mathrm{i}^{+*} - \boldsymbol{\zeta}_\mathrm{i}^{-*}\right)\right]\ ,\\
\mathbf{r}_{\mathrm{i}3} &= \boldsymbol{\alpha}_\mathrm{i}^{-*}\,\nabla_{\dot{\mathbf{q}}(t_\mathrm{i}^-)}\,\mathbf{p}_\mathrm{i}^-(\mathbf{q}^*(t_\mathrm{i}^-), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-))\\
&- \boldsymbol{\alpha}_\mathrm{i}^{+*}\,\nabla_{\dot{\mathbf{q}}(t_\mathrm{i}^-)}\,\mathbf{p}_\mathrm{i}^-(\mathbf{q}^*(t_\mathrm{i}^-), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-)) - \boldsymbol{\xi}_\mathrm{i}^{+*}\,\mathbf{M}(\mathbf{q}^*(t_\mathrm{i})) - \boldsymbol{\xi}_\mathrm{i}^{-*}\,\mathbf{M}(\mathbf{q}^*(t_\mathrm{i}))\ ,\\
\mathbf{r}_{\mathrm{i}4} &= \boldsymbol{\alpha}_\mathrm{i}^{-*}\,\nabla_{\dot{\mathbf{q}}(t_\mathrm{i}^+)}\,\mathbf{p}_\mathrm{i}^+(\mathbf{q}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-))\\
&- \boldsymbol{\alpha}_\mathrm{i}^{+*}\,\nabla_{\dot{\mathbf{q}}(t_\mathrm{i}^+)}\,\mathbf{p}_\mathrm{i}^+(\mathbf{q}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-)) + \boldsymbol{\xi}_\mathrm{i}^{+*}\,\mathbf{M}(\mathbf{q}^*(t_\mathrm{i})) + \boldsymbol{\xi}_\mathrm{i}^{-*}\,\mathbf{M}(\mathbf{q}^*(t_\mathrm{i}))\ ,
\end{aligned}$$

5. *the impact equation and transition conditions at a transition*

$$\mathcal{C}_{\mathrm{T_i}}^* = \mathcal{C}_{\mathrm{T_i}}^{*+} \cup \mathcal{C}_{\mathrm{T_i}}^{*-}, \quad \mathcal{C}_{\mathrm{I_i}}^* = \mathcal{C}_{\mathrm{I_i}}^{*+} \cup \mathcal{C}_{\mathrm{I_i}}^{*-}, \quad \forall t_\mathrm{i}^* \in \mathcal{I}_\mathrm{T}\ , \tag{105}$$

6. *the discontinuity conditions of the dual state* $\boldsymbol{\nu}$ *and of its time-derivative* $\dot{\boldsymbol{\nu}}$*:*

$$\begin{aligned}
&\left(\boldsymbol{\nu}^*(t_\mathrm{i}^+) - \boldsymbol{\nu}^*(t_\mathrm{i}^-)\right)\mathbf{M}(\mathbf{q}^*(t)) = \\
&\boldsymbol{\alpha}_\mathrm{i}^{+*}\,\nabla_{\dot{\mathbf{q}}(t_\mathrm{i}^+)}\left(\mathbf{p}_\mathrm{i}^+(\mathbf{q}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-)) + \mathbf{p}_\mathrm{i}^-(\mathbf{q}^*(t_\mathrm{i}^-), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-))\right)\\
+\ &\boldsymbol{\alpha}_\mathrm{i}^{-*}\,\nabla_{\dot{\mathbf{q}}(t_\mathrm{i}^-)}\left(\mathbf{p}_\mathrm{i}^+(\mathbf{q}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-)) + \mathbf{p}_\mathrm{i}^-(\mathbf{q}^*(t_\mathrm{i}^-), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-))\right) \quad \forall t_\mathrm{i}^* \in \mathcal{I}_\mathrm{T}\ ,
\end{aligned} \tag{106}$$

and

$$\begin{aligned}
&\boldsymbol{\Omega}_\mathrm{i}^{+*} - \boldsymbol{\Omega}_\mathrm{i}^{-*} = \\
-\ &\nabla_{\mathbf{q}}\left[\boldsymbol{\alpha}_\mathrm{i}^{+*}\mathbf{p}_\mathrm{i}^+(\mathbf{q}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-)) - \boldsymbol{\alpha}_\mathrm{i}^{-*}\,\mathbf{p}_\mathrm{i}^-(\mathbf{q}^*(t_\mathrm{i}^-), \dot{\mathbf{q}}^*(t_\mathrm{i}^+), \dot{\mathbf{q}}^*(t_\mathrm{i}^-))\right]\\
-\ &\left(\boldsymbol{\xi}_\mathrm{i}^{+*} + \boldsymbol{\xi}_\mathrm{i}^{-*}\right)\nabla_{\mathbf{q}}\left[\mathbf{M}\left(\dot{\mathbf{q}}^*(t_\mathrm{i}^+) - \dot{\mathbf{q}}^*(t_\mathrm{i}^-)\right) - \mathbf{B}_\mathrm{i}\left(\boldsymbol{\zeta}_\mathrm{i}^{+*} - \boldsymbol{\zeta}_\mathrm{i}^{-*}\right)\right]\ \forall t_\mathrm{i}^* \in \mathcal{I}_\mathrm{T}\ ,
\end{aligned} \tag{107}$$

7. *the impulsive optimal control law condition*

$$\left(\boldsymbol{\xi}_\mathrm{i}^{-*} + \boldsymbol{\xi}_\mathrm{i}^{+*}\right)^\mathrm{T} \in Ker\left(\mathbf{B}_\mathrm{i}(\mathbf{q}^*(t_\mathrm{i}))^\mathrm{T}\right), \quad \forall t_\mathrm{i}^* \in \mathcal{I}_\mathrm{T}\ , \tag{108}$$

8. *the boundary constraints $\mathcal{C}_{\mathrm{f}}$ at final state,*

9. *the variational inequality with respect to the variations at final time $\hat{t}_{\mathrm{f}}$:*

$$\left(\Upsilon_{\mathrm{f}}^{*} - \langle \Psi_{\mathcal{C}_{\mathrm{f}}}^{\uparrow}(\cdot;\hat{\mathbf{q}}(t_{\mathrm{f}})),\, \dot{\mathbf{q}}(t_{\mathrm{f}})\rangle - \langle \Psi_{\mathcal{C}_{\mathrm{f}}}^{\uparrow}(\cdot;\hat{\mathbf{q}}(t_{\mathrm{f}})),\, \ddot{\mathbf{q}}(t_{\mathrm{f}})\rangle\rangle\right) \hat{t}_{\mathrm{f}} \geq 0\,,\ \forall\, \hat{t}_{\mathrm{f}} \in \mathbb{R}, \tag{109}$$

10. *the transversality condition at final state:*

$$\begin{pmatrix} -\mathbf{\Omega}_{\mathrm{f}}^{*} \\ -\boldsymbol{\nu}^{*}(t_{\mathrm{f}})\,\mathbf{M}(\mathbf{q}^{*}(t_{\mathrm{f}})) \end{pmatrix} \in {}^{\infty}\mathcal{N}_{\mathcal{C}_{\mathrm{f}}}\left(\mathbf{q}^{*}(t_{\mathrm{f}}), \dot{\mathbf{q}}^{*}(t_{\mathrm{f}})\right)\ . \tag{110}$$

Since the derivatives and gradients of $\dot{\mathbf{M}}$, $\ddot{\mathbf{M}}$, $\nabla_{\dot{\mathbf{q}}}\mathbf{h}$ involve the generalized velocities and accelerations of the system at pre-, and post-transition state, the right supscripted signs denote wether the pre-transition or post-transition values of the relevant entities are meant. The gradients and time derivatives of several tensors in the Einstein notation convention are given as follows:

$$\begin{aligned} \dot{m}_{\mathrm{ij}} &= \nabla_{\mathbf{q}_{\mathrm{k}}} m_{\mathrm{ij}}\dot{\mathbf{q}}_{\mathrm{k}}(t), \\ \ddot{m}_{\mathrm{ij}} &= \nabla^{2}_{\mathbf{q}_{\mathrm{k}}\,\mathbf{q}_{\mathrm{l}}} m_{\mathrm{ij}}\dot{\mathbf{q}}_{\mathrm{k}}(t)\,\dot{\mathbf{q}}_{\mathrm{l}}(t) + \nabla_{\mathbf{q}_{\mathrm{k}}} m_{\mathrm{ij}}\ddot{\mathbf{q}}_{\mathrm{k}}(t)\,, \\ \frac{d}{dt}[\nabla_{\dot{\mathbf{q}}_{\mathrm{k}}}\mathbf{h}_{\mathrm{p}}] &= \nabla^{2}_{\dot{\mathbf{q}}_{\mathrm{k}}\,\dot{\mathbf{q}}_{\mathrm{l}}}\mathbf{h}_{\mathrm{p}}\ddot{\mathbf{q}}_{\mathrm{l}}(t) + \nabla^{2}_{\dot{\mathbf{q}}_{\mathrm{k}}\,\mathbf{q}_{\mathrm{l}}}\mathbf{h}_{\mathrm{p}}\dot{\mathbf{q}}_{\mathrm{l}}(t)\,, \end{aligned}$$

where a_{ij} denotes the relevant element of a second-order tensor $\mathbf{A}$.

7 Discussion and Conclusion

In this work, necessary conditions of strong local minimizers for the impulsive optimal control problem of finite-dimensional multibody Lagrangian systems is presented. The necessary conditions provide criteria for the determination of optimal transition times and locations in the presence of discontinuity of generalized velocities. In the proposed setting concurrent discontinuity on an Lebesgue-negligible atomic time instant of the generalized velocities $\dot{\mathbf{q}}$ and the dual state $\boldsymbol{\nu}$ is handled. The proposed discontinuous transversality conditions and the internal boundary variations by the author are capable, given the assumptions in the statement of the optimal control problem, to handle this problem properly. It is shown that the idea of internal boundary variations indeed a natural extension of the classical boundary variations, where the latter one is unilateral and the extension is a bilateral concept. In this work, a characterization of these concepts in terms of subderivatives to the extended-valued lower-semicontinuous directionally Lipschitzian value functional is given, showing that these variational principles are, given certain regularity assumptions, well founded in subdifferential calculus rather than being some ad-hoc assumptions. Transition sets for discontinuities in the generalized velocities of mechanical systems are first introduced in [20]. In this work, the properties of the transition sets are discussed, especially from the viewpoint of regularity.

The derivation of conditions benefit of the underlying Lagrangian structure. One of the advantages of the Lagrangian dynamics is the fact, that the generalized directions of control, which are the rows of the linear operator $\mathbf{B}$ are only dependent on the generalized positions $\mathbf{q}$. Since the generalized positions are of absolutely continuous character, the generalized directions of impulsive control remain unchanged during a transition. Another fact is that in the framework of finite-dimensional Lagrangian systems, impact equations and constitutive impact laws are provided, that are means to ”join” two optimal trajectories discontinuously.

The proposed necessary conditions are for strong local minimizers and are valid in singular intervals. The optimal control law as stated in equation (103) is valid in singular intervals, because the zero vector belongs to the normal cone.

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$\nabla_{\mathbf{x}}\, \mathbf{y}$	Gradient of $\mathbf{y}$ w.r.t $\mathbf{x}$
$\mathcal{A} \subset \mathcal{B}$	$\mathcal{A}$ is subset of $\mathcal{B}$
$\mathcal{A} \supset \mathcal{B}$	$\mathcal{A}$ is superset of $\mathcal{B}$
$\langle \cdot, \cdot \rangle$	dual pairing
$\mathbb{B}$	Open unit ball in Euclidean space
$\lvert\mathbf{x}\rvert$	Euclidean norm of $\mathbf{x}$
int$\mathcal{C}$	Interior of $\mathcal{C}$
bdy$\mathcal{C}$	Boundary of $\mathcal{C}$
$\bar{\mathcal{C}}$	Closure of $\mathcal{C}$
$\mathcal{N}_{\mathcal{C}}(\mathbf{x})$	Proximal normal cone to $\mathcal{C}$ at $\mathbf{x}$
$\hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x})$	Strict normal cone to $\mathcal{C}$ at $\mathbf{x}$
$\mathcal{N}_{\mathcal{C}}(\mathbf{x})$	Limiting normal cone to $\mathcal{C}$ at $\mathbf{x}$
$^{\infty}\mathcal{N}_{\mathcal{C}}(\mathbf{x})$	Singular Proximal normal cone to $\mathcal{C}$ at $\mathbf{x}$
$^{\infty}\hat{\mathcal{N}}_{\mathcal{C}}(\mathbf{x})$	Singular Strict normal cone to $\mathcal{C}$ at $\mathbf{x}$
$^{\infty}\mathcal{N}_{\mathcal{C}}(\mathbf{x})$	Singular Limiting normal cone to $\mathcal{C}$ at $\mathbf{x}$
$\mathcal{T}_{\mathcal{C}}(\mathbf{x})$	Bouligand tangent cone to $\mathcal{C}$ at $\mathbf{x}$
$\bar{\mathcal{T}}_{\mathcal{C}}(\mathbf{x})$	Clarke tangent cone to $\mathcal{C}$ at $\mathbf{x}$
epi f	Epigraph of f
$\partial\, f(\mathbf{x})$	Proximal subdifferential of f at $\mathbf{x}$
$\hat{\partial}\, f(\mathbf{x})$	Strict subdifferential of f at $\mathbf{x}$
$\partial\, f(\mathbf{x})$	Limiting subdifferential of f at $\mathbf{x}$
$\partial_P^{\infty}\, f(\mathbf{x})$	Asymptotic proximal subdifferential of f at $\mathbf{x}$
$\hat{\partial}_P^{\infty}\, f(\mathbf{x})$	Asymptotic strict subdifferential of f at $\mathbf{x}$
$\partial^{\infty}\, f(\mathbf{x})$	Asymptotic limiting subdifferential of f at $\mathbf{x}$

dom f	(Effective) domain of f
Gr $\mathcal{F}$	Graph of $\mathcal{F}$
epi f	Epigraph of f
$f^0(\mathbf{x}; \mathbf{v})$	Generalized subderivative of f at $\mathbf{x}$ in the direction of $\mathbf{v}$
$\Psi_{\mathcal{C}}(\mathbf{x})$	Indicator function of the set $\mathcal{C}$ at the point $\mathbf{x}$
$\nabla\, f(\mathbf{x})$	Gradient vector of f at $\mathbf{x}$
$\mathbf{x}_{\mathrm{i}} \underset{\mathcal{C}}{\rightarrow} \mathbf{x}$	$\mathbf{x}_{\mathrm{i}} \rightarrow \mathbf{x}$ and $\mathbf{x}_{\mathrm{i}} \in \mathcal{C}, \quad \forall i$
$\mathbf{x}_{\mathrm{i}} \underset{f}{\rightarrow} \mathbf{x}$	$\mathbf{x}_{\mathrm{i}} \rightarrow \mathbf{x}$ and $f(\mathbf{x}_{\mathrm{i}}) \rightarrow f(\mathbf{x}), \quad \forall i$
supp μ	Support of the measureμ
$\mathcal{AC}(\mathcal{I}; \mathbb{R}^n)$	Absolutely continuous functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$

List of Abbreviations

$\mathcal{BV}(\mathcal{I}; \mathbb{R}^n)$	Bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{LCBV}(\mathcal{I}; \mathbb{R}^n)$	Left-continuous bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{RCBV}(\mathcal{I}; \mathbb{R}^n)$	Right-continuous bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{LBV}(\mathcal{I}; \mathbb{R}^n)$	Locally bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{LCLBV}(\mathcal{I}; \mathbb{R}^n)$	Left-continuous locally bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{RCLBV}(\mathcal{I}; \mathbb{R}^n)$	Right-continuous locally bounded variation functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$
$\mathcal{AC}(\mathcal{I}; \mathbb{R}^n)$	Absolutely Continuous functions $f : \mathcal{I} \rightarrow \mathbb{R}^n$