A Note on Split Rank of Intersection Cuts

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September, 2008

Abstract

In this note, we present a simple geometric argument to determine a lower bound on the split rank of intersection cuts. As a first step of this argument, a polyhedral subset of the lattice-free convex set that is used to generate the intersection cut is constructed. We call this subset the *restricted lattice-free set*. It is then shown that $\lceil \log_2(l) \rceil$ is a lower bound on the split rank of the intersection cut, where l is the number of integer points lying on the boundary of the restricted lattice-free set satisfying the condition that no two points lie on the same facet of the restricted lattice-free set. The use of this result is illustrated to obtain a lower bound of $\lceil \log_2(n+1) \rceil$ on the split rank of n-row mixing inequalities.

Over the years, many classes of cuts have been proposed for solving unstructured mixed integer programs that can be used within the branch-and-cut framework; see Nemhauser and Wolsey [29], Marchand, Martin, Weismantel and Wolsey [27] and Johnson, Nemhauser and Savelsbergh [25]. Among the many classes of cutting planes proposed, *Split cuts* (Balas [5]) which are equivalent to the *Gomory Mixed Integer cuts* (Gomory [21]) and the *Mixed Integer Rounding inequalities* (Nemhauser and Wolsey [30]) form one of the most successful classes of cutting planes used to solve general mixed integer programs; see for example Balas *et al.* [7] and Bixby and Rothberg [11]. It is therefore natural to compare other classes of valid cutting planes with split cuts. One possible method of comparison is to ask if a recursive application of split cuts will generate the target class of inequalities. If the recursive application of split cuts does generate a target inequality, the minimum number of steps necessary to obtain the target inequality gives a measure of the efficacy of a cutting-plane-algorithm that uses only split cuts. These questions are related to the question of determining the *split rank* of the inequality.

While the exact split rank for a general class of inequalities may be difficult to obtain, bounds on the split rank are more easily obtainable. A finite upper bound on the split rank indicates that the inequality under study can be obtained by recursively applying split cuts. On the other hand, a lower bound on the split rank indicates how difficult it is to obtain an inequality using split cuts. Therefore, if the lower bound on the split rank of an inequality is high, it may be better to apply this inequality directly to the LP relaxation of the problem instead of generating it using a sequence of split cuts.

In this note, we study the split rank of valid inequalities for the following class of problems,

$$x \;=\; f + \sum_{i=1}^{n} r^i y_i \quad x \in \mathbb{Z}^m, y \in \mathbb{R}^n_+, \tag{1}$$

where $r^i \in \mathbb{Q}^m$ and $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$. Recently there have been a number of studies on the facet-defining inequalities of (1). Andersen *et al.* [2] and Cornuéjols and Margot [16] characterize the facet-defining inequalities of (1) when $m = 2$. Borozan and Cornuéjols [12] analyze non-dominated inequalities of an infinite version of (1) for general m. It can be verified that all non-dominated and facet-defining inequalities for (1) are *intersection cuts*, a concept introduced by Balas [3]. Since (1) is a natural relaxation of general mixed integer programs, valid inequalities for (1) can be used to generate cutting planes from multiple rows of any simplex tableau by first relaxing the non-basic integer variables to be continuous. (Andersen *et al.* [2], Cornuéjols and Margot [16]). Recent computational study by Espinoza [20] indicates that inequalities for (1) can generate effective cutting planes for solving general MIPs. By a procedure called *fill-in* (Gomory and Johnson [22], Johnson [24], Dey and Wolsey [19]) and by a *strengthening procedure* (Balas and Jeroslow [8]) valid inequalities for (1) can also be improved to take into account the integrality of non-basic variables in a simplex tableau.

In this note, we give a geometric argument to deduce a non-trivial lower bound on the split rank of intersection cuts for (1) under a mild condition on the columns of (1). This lower bound is related to the number and the orientation of integer feasible points of (1) that are satisfied at equality for the intersection cut. In particular, a polyhedral set, called the *restricted lattice-free set*, is constructed. A subset of integer feasible points of (1) satisfying the intersection cut at equality, lie on the boundary of this restricted lattice-free set. We prove that $\lceil \log_2(l) \rceil$ is a lower bound on the split rank of the intersection cut, where l is the maximum number of integer points such that no two points lie on the same facet of the restricted lattice-free set. We then show that a related lower bound result holds in more general scenarios, such as in the presence of non-negativity constraints on the x-variables in (1). We illustrate the use of this result by showing that the split rank of the n-row *mixing inequality* (Günlük and Pochet [23]) and of some related inequalities for the *constant capacity lot-sizing problems with n periods* is at least $\lceil \log_2(n+1) \rceil$.

The paper is organized as follows. We begin in Section 1 by presenting a formal definition of split rank and by providing a brief literature survey of known bounds on the split rank of cuts for mixed integer programs. In Section 2, we present the lower bound result on the split rank of intersection cuts. Then we illustrate the use of this result to obtain a lower bound on the split rank of mixing inequalities. We conclude in Section 3.

1 Split Rank: Definition and Literature Review

Consider a general mixed integer set $M^I := \{(x, y) \in \mathbb{Z}^p \times \mathbb{R}^q \,|\, Gx + Hy \leq b\}$ where $G \in \mathbb{Q}^{m \times p}$, $H \in \mathbb{Q}^{m \times q}$, and $b \in \mathbb{Q}^{m \times 1}$. Let $M^0 := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \,|\, Gx + Hy \leq b\}$ denote the linear programming relaxation of M^I. Given a vector $a \in \mathbb{Z}^p$ and $c \in \mathbb{Z}$, any vector $x \in \mathbb{Z}^p$ satisfies the *split disjunction* defined as $(a^T x \leq c) \vee (a^T x \geq c + 1)$. Define the sets $L^0_{a,c}, R^0_{a,c} \subseteq \mathbb{R}^p \times \mathbb{R}^q$, as $L^0_{a,c} := M^0 \cap \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \,|\, a^T x \leq c\}$ and $R^0_{a,c} := M^0 \cap \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \,|\, a^T x \geq c+1\}$. Any inequality that is valid for $M^0_{a,c} := \text{conv}(L^0_{a,c} \cup R^0_{a,c})$ is called a *split cut* (Balas [4]). Split cuts and other classes of general disjunctive cuts where introduced by Balas [5]. The term split cut is due to Cook *et al.* [13].

The concept of split rank follows from the concept of *split closure* of a mixed integer program, defined and analyzed in Cook *et al.* [13].

Definition 1 (Split closure) *Given the linear programming relaxation $M^0 := \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \,|\, Gx + Hy \leq b\}$ of $M^I = \{(x, y) \in \mathbb{Z}^p \times \mathbb{R}^q \,|\, Gx + Hy \leq b\}$, the first split closure M^1 is defined as $\cap_{a \in \mathbb{Z}^p, c \in \mathbb{Z}} M^0_{a,c}$.*

Cook *et al.* [13] prove that the first split closure of a mixed integer set is a polyhedron. Andersen *et al.* [1], Vielma [34] and Dash *et al.* [18] present alternative proofs of this key result. See Balas and Saxena [9] and Dash *et al.* [18] for empirical studies of the strength of the first split closure. Cornuéjols and Li [14] compare the closure with rest to 18 different classes of general purpose cuts. See Basu *et al.* [10] for a comparison of split closure with closure of inequalities based on two-variable disjunctions.

The split closure procedure applied to the polyhedron M^1 gives the second split closure M^2. In general, denote the k^{th} split closure by M^k.

Given two valid inequalities $(\alpha^1)^T x + (\beta^1)^T y \leq \gamma^1$ and $(\alpha^2)^T x + (\beta^2)^T y \leq \gamma^2$ for M^I, the inequality $(\alpha^1)^T x+(\beta^1)^T y \leq \gamma^1$ is said to dominate $(\alpha^2)^T x+(\beta^2)^T y \leq \gamma^2$ if $M^0 \cap \{(x, y) \,|\, (\alpha^1)^T x+ (\beta^1)^T y \leq \gamma^1\} \subseteq M^0 \cap \{(x, y) \,|\, (\alpha^2)^T x + (\beta^2)^T y \leq \gamma^2\}$.

Definition 2 (Split rank) *The split rank of a valid inequality $\alpha^T x + \beta^T y \leq \gamma$ for M^I is defined as the smallest integer k such that there exists a valid inequality for M^k that dominates $\alpha^T x + \beta^T y \leq \gamma$.* □

Balas [5] proves that for a *facial disjunctive set* S, a sequential convexification procedure generates the convex hull of feasible points in η steps, where η is the number of conjunctive clauses in the *conjunctive normal form* of S; see Balas [5] and Balas *et al.* [6] for details. Binary mixed integer sets form a sub class of facial disjunctive sets and this result implies that given a binary mixed integer set with n binary variables, the split rank is at most n. Nemhauser and Wolsey [30] prove that for a binary mixed integer program with n binary variables, any valid inequality can be obtained by using an enumeration tree whose dept is at most n, where the i^{th} level in the enumeration tree corresponds to the split disjunction $(x_i \leq 0) \vee (x_i \geq 1)$. Balas *et al.* [6] show a similar result using the *lift-and-project cuts.* (Lift-and-project cuts are split disjunctive cuts). Using a class of problems, Cornuéjols and Li [15] prove that the upper bound of n on the split rank of valid inequalities for a binary mixed integer set with n integer variables can be tight. For general mixed integer programs the split rank of an inequality may not be finite. Cook *et al.* [13] present an example of a problem where the facet-defining inequality cannot be obtained using split cuts. This result is generalized to a family of problems in Li and Richard [26]. Dash and Günlük [17] prove an upper bound of n on the split rank of a mixing inequality based on n rows. To the best of our knowledge there are no known lower bounds on the split rank of valid inequalities for general mixed integer programs.

2 Lower Bound on Split Rank

In this section, we present a lower bound on the split rank of intersection cuts for,

$$x = f + \sum_{i=1}^{n} r^i y_i \quad x \in \mathbb{Z}^m, y \in \mathbb{R}^n_+, \tag{2}$$

where $r^i \in \mathbb{Q}^m$ and $f \in \mathbb{Q}^m \setminus \mathbb{Z}^m$.

An intersection cut can be generated as follows: Let $P \subseteq \mathbb{R}^m$ be a convex set containing f in its interior that is lattice-free, i.e., interior$(P) \cap \mathbb{Z}^m = \emptyset$. Then a valid cutting plane for (2) is

$$\sum_{i=1}^{n} \pi^P(r^i) y_i \geq 1, \tag{3}$$

where

$$\pi^P(r^i) = \begin{cases} \lambda & \text{if } \exists \lambda \geq 0, \text{ s.t. } f + \frac{r^i}{\lambda} \in \text{boundary}(P) \\ 0 & \text{if } r^i \text{ is a ray for } P \end{cases}. \tag{4}$$

All non-dominated inequalities for (2) are intersection cuts derived using maximal lattice-free convex sets P such that $f \in \text{interior}(P)$; see Zambelli [35].

Throughout this section the following notation is used: Given a vector x the i^{th} component is represented as x_i, while x^i is used to represent the i^{th} vector in a list of vectors $\{x^1, x^2, ...\}$. Given a set $K = \{(x, y) \in \mathbb{R}^p \times \mathbb{R}^q \,|\, Gx + Hy \leq b\}$, $\text{proj}_x(K)$ represents the projection of K on the space of x- variables, i.e., $\text{proj}_x(K) = \{x \in \mathbb{R}^p \,|\, \exists y \in \mathbb{R}^q \text{ s.t. } (x, y) \in K\}$. We always use m to represent the number of integer variables of (2) and n to represent the number of continuous variables of (2).

2.1 General Lower Bound

The outline of the geometric scheme to determine a lower bound on the split rank of (3) is as follows:

- Step 1: Under a mild assumption on the columns of (2), we first construct a subset of P (P is the lattice-free convex set used to generate (3) as described in (4)) which we call the *restricted lattice-free set.*
- Step 2: Then we prove the following result in Theorem 8: $\lceil \log_2(l) \rceil$ is a lower bound on the split rank of (3) where l is the number of integer points that lie on different facets of the restricted lattice-free set.

We begin with a definition of a generalized simplex from Rockafellar [33].

Definition 3 (Generalized Simplex) *A generalized simplex is the convex hull of p points v^1, ..., $v^p \in \mathbb{R}^m$ (where $|p| \geq 1$) and q rays $s^1, s^2, ..., s^q \in \mathbb{R}^m$, i.e., conv$\left(\cup_{i=1}^p \{v^i\}\right) +$ cone$\left(\cup_{i=1}^q \{s^i\}\right)$, such that the resulting set has dimension $p + q - 1$.*

The lattice-free set $P \subseteq \mathbb{R}^m$ that is used to generate (3) may not be a bounded set. If P is unbounded, some of the columns r^i of (2) may be rays of P. Without loss of generality we assume that r^i is a ray for P if and only if $i \in \{n_1 + 1, ..., n\}$. Also note that since f lies in the interior of P, using (4) we obtain that if r^i is not a ray for P, i.e., $i \in \{1, ..., n_1\}$, then $\pi^P(r^i) > 0$.

We next present the assumption that we make on the columns of (2) that allows the construction of the restricted lattice-free set.

Assumption 1: There exist subsets $S_v \subseteq \{1, ..., n_1\}$ and $S_r \subseteq \{n_1 + 1, ..., n\}$ of the columns of (2) such that $\tilde{Q} := \text{conv}\left(\cup_{i \in S_v} \{f + \frac{r^i}{\pi^P(r^i)}\}\right) + \text{cone}\left(\cup_{i \in S_r} \{r^i\}\right)$ forms a generalized simplex and $f \in \text{affine.hull}(\tilde{Q})$. □

By the definition of π^P presented in (4), observe that the vertices of $\tilde{Q}$, i.e., $f + \frac{r^i}{\pi^P(r^i)}$ $\forall i \in S_v$ lie on the boundary of P. Also the rays of $\tilde{Q}$ are rays of P. Therefore, $\tilde{Q}$ is a subset of P. We next present an example illustrating Assumption 1 and the construction of $\tilde{Q}$.

Example 4 *Consider the set*

$$\begin{array}{rcrcrcrcrcrcr} x_1 & = & 0.5 & + & 3y_1 & + & 0y_2 & - & 3y_3 & + & 0y_4 & + & 0y_5 \\ x_2 & = & 0.5 & + & 0y_1 & + & 3y_2 & + & 0y_3 & - & 3y_4 & + & 0y_5 \\ x_3 & = & 0.5 & - & 1y_1 & - & 1y_2 & - & 1y_3 & - & 1y_4 & + & 1y_5 \end{array} \tag{5}$$

$$x_i \in \mathbb{Z} \, \forall i \in \{1, 2, 3\} \quad y_i \in \mathbb{R}_+ \, \forall j \in \{1, 2, 3, 4, 5\}. \tag{6}$$

Consider the lattice-free convex set $P \subseteq \mathbb{R}^3$ defined by the following system of inequalities,

$$\begin{aligned} -x_3 &\leq 0 \\ -2x_1 - 2x_2 + x_3 &\leq 1 \\ 2x_1 - 2x_2 + x_3 &\leq 3 \\ 2x_1 + 2x_2 + x_3 &\leq 5 \\ -2x_1 + 2x_2 + x_3 &\leq 3. \end{aligned}$$

P is illustrated in Figure 1. Observe that $f := (0.5, 0.5, 0.5)$ lies in the interior of P. The dashed rays represent points of the form $x = f + r^i y_i$, $y_i \geq 0$. Since in this example P is a polytope, we have that $n_1 = n$. Now observe that $\pi^P(r^1) = 2$, since $(0.5, 0.5, 0.5) + \frac{(3,0,-1)}{2} = (2, 0.5, 0) \in$ boundary(P). Similarly computing $\pi^P(r^2)$, $\pi^P(r^3)$, $\pi^P(r^4)$, and $\pi^P(r^5)$ we obtain the following valid inequality for (5),

$$2y_1 + 2y_2 + 2y_3 + 2y_4 + \frac{2}{5}y_5 \geq 1. \tag{7}$$

Now note that the points $f + \frac{r^1}{\pi^P(r^1)} = (2, 0.5, 0) =: t^1$, $f + \frac{r^2}{\pi^P(r^2)} = (0.5, 2, 0) =: t^2$, $f + \frac{r^3}{\pi^P(r^3)} = (-1, 0.5, 0) =: t^3$, and $f + \frac{r^5}{\pi^P(r^5)} = (0.5, 0.5, 3) =: t^5$ are affinely independent and therefore their convex hull is a simplex. Also note that f belongs to the affine hull of these points. Therefore Assumption 1 holds and it is possible to set $S_v = \{1, 2, 3, 5\}$, i.e., $\tilde{Q} = conv\{(2, 0.5, 0), (0.5, 2, 0), (-1, 0.5, 0), (0.5, 0.5, 3)\}$. Note that the choice of S_v is not unique as setting S_v as $\{1, 2, 4, 5\}$, $\{1, 3, 4, 5\}$, or $\{2, 3, 4, 5\}$ also satisfies the conditions of Assumption 1. □

While Assumption 1 is not trivial, observe that a sufficient condition for it to hold is that the affine hull of the points $f + \frac{r^i}{\pi^P(r^i)}$, $i \in \{1, ..., n_1\}$ and rays r^i, $i \in \{n_1 + 1, ..., n\}$ is $\mathbb{R}^m$. Also note that many recent studies of (2) with two integer variables, i.e., with $m = 2$ satisfy this sufficient condition. (Andersen *et al.* [2], Cornuéjols and Margot [16], Dey and Wolsey [19]).

Without loss of generality we assume that $S_v = \{1, ..., |S_v|\}$ and $S_r = \{n_1 + 1, ..., n_1 + |S_r|\}$. We next define the *restricted lattice-free set* that is used to obtain the lower bound result.

Definition 5 (restricted lattice-free set) *If Assumption 1 holds, we define the restricted lattice-free set, $Q^{S_v,S_r} \subseteq \mathbb{R}^m$ as $Q^{S_v,S_r} := conv(\tilde{Q} \cup \{f\})$.* □

Since $\tilde{Q}$ is a subset of P and $f \in$ interior(P), we have that Q^{S_v,S_r} is a subset of P.

Example 4 (contd.) *The set Q^{S_v,S_r}, which is the convex combination of $f = (0.5, 0.5, 0.5)$ and $\tilde{Q} = conv\{(2, 0.5, 0), (0.5, 2, 0), (-1, 0.5, 0), (0.5, 0.5, 3)\}$ is illustrated in Figure 1. Notice that in this example Q^{S_v,S_r} is a simplex, since in this example $f \in \tilde{Q}$ and therefore $Q^{S_v,S_r} = \tilde{Q}$. If $f \notin \tilde{Q}$, then Q^{S_v,S_r} may not be a generalized simplex.* □

Assumption 1 implies two properties that are used to arrive at a lower bound on the split rank of (3) in Theorem 8. These properties are presented in Propositions 6 and 7.

Proposition 6 *If Assumption 1 holds, then $\exists\ \lambda^{S_v,S_r} \in \mathbb{R}^n$ such that,*

1. $\sum_{i=1}^{n} \lambda_i^{S_v,S_r} r^i = \bar{0}$ *($\bar{0}$ is the origin in $\mathbb{R}^m$) and*

2. $\sum_{i=1}^{n} \pi^P(r^i) \lambda_i^{S_v,S_r} = 1.$

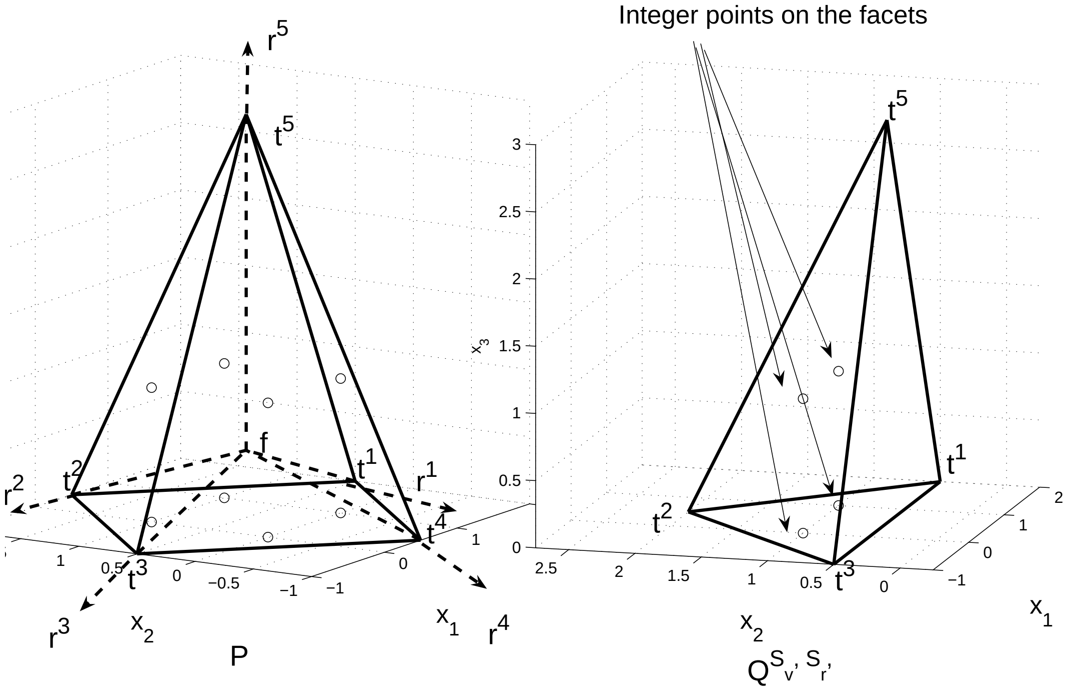

Figure 1: Example of P and Q^{S_v,S_r}

Proof: Since $\tilde{Q}$ is a generalized simplex and $f \in \text{affine.hull}(\tilde{Q})$, we obtain that $|S_v| \geq 1$ and $\exists \mu \in \mathbb{R}^{|S_v|+|S_r|}$ such that $\sum_{i\in(S_v\cup S_r)} \mu_i = 1$ and $\sum_{i\in S_v} \mu_i \left(f + \frac{r^i}{\pi^P(r^i)}\right) + \sum_{i\in S_r} \mu_i \left(f + \frac{r^1}{\pi^P(r^1)} + r^i\right) = f$. Therefore, $\left(\frac{\mu_1}{\pi^P(r^1)} + \sum_{i\in S_r} \frac{\mu_i}{\pi^P(r^1)}\right) r^1 + \sum_{i\in S_v\setminus\{1\}} \frac{\mu_i}{\pi^P(r^i)} r^i + \sum_{i\in S_r} \mu_i r^i = \bar{0}$. Define λ^{S_v,S_r} as follows:

$$\lambda_i^{S_v,S_r} = \begin{cases} \left(\frac{\mu_1}{\pi^P(r^1)} + \sum_{j\in S_r} \frac{\mu_j}{\pi^P(r^1)}\right) & \text{if } i = 1 \\ \frac{\mu_i}{\pi^P(r^i)} & \text{if } i \in S_v \setminus \{1\} \\ \mu_i & \text{if } i \in S_r \\ 0 & \text{if } i \in \{1,...,n\} \setminus (S_v \cup S_r) \end{cases} \tag{8}$$

Clearly by the construction of λ^{S_v,S_r} we obtain that $\sum_{i=1}^n \lambda_i^{S_v,S_r} r^i = \bar{0}$. Also observe that $\sum_{i=1}^n \pi^P(r^i)\lambda_i^{S_v,S_r} = \sum_{i\in(S_v\cup S_r)} \pi^P(r^i)\lambda_i^{S_v,S_r} = \sum_{i\in S_v} \pi^P(r^i)\lambda_i^{S_v,S_r} = \sum_{i\in(S_v\cup S_r)} \mu_i = 1$. The second equality follows from the fact that $\pi^P(r^i) = 0$ if r^i is a ray of P. □

Example 4 (contd.) *Using the proof of Proposition 6, we can obtain* $\lambda^{S_v,S_r} = (\frac{5}{24}, 0, \frac{5}{24}, 0, \frac{10}{24})$ *such that* $\sum_{i=1}^5 \lambda_i^{S_v,S_r} r^i = \bar{0}$ *and* $\sum_{i=1}^5 \lambda_i^{S_v,S_r} \pi^P(r^i) = 1$. □

In order to present the lower bound result we require some more notation that we present next.

- π^S: Define a function $\pi^S : \text{cone}\{r^1, ..., r^n\} \rightarrow \mathbb{R}_+$ as:

$$\pi^S(r^i) = \begin{cases} \pi^P(r^i) & \text{if } r^i \notin \text{cone}_{k\in(S_v\cup S_r)}\{r^k\} \\ \lambda & \text{if } \exists\lambda \geq 0, \text{ s.t. } f + \frac{r^i}{\lambda} \in \text{boundary}(Q^{S_v,S_r}) \\ 0 & \text{if } r^i \text{ is a ray of } Q^{S_v,S_r} \end{cases} . \tag{9}$$

  It is easily verified that $\pi^S(r^i) = \pi^P(r^i)\ \forall i \in (S_v \cup S_r)$ and in general $\pi^S(r^i) \geq \pi^P(r^i)$ since $Q^{S_v,S_r} \subseteq P$. Since $y_i \geq 0\ \forall i \in \{1, ..., n\}$ in (2), the inequality $\sum_{i=1}^n \pi^S(r^i)y_i \geq 1$ is a valid inequality for (2). The motivation for the definition of the function π^S is as follows: Theorem 8 will present a lower bound on the split rank of the inequality $\sum_{i=1}^n \pi^S(r^i)y_i \geq 1$, which is a valid lower bound on the split rank of (3) as $\pi^S(r^i) \geq \pi^P(r^i)\ \forall i \in \{1, ..., n\}$ implies that (3) dominates the inequality $\sum_{i=1}^n \pi^S(r^i)y_i \geq 1$. Note also that,

$$\sum_{i=1}^n \pi^S(r^i)\lambda_i^{S_v,S_r} = \sum_{i\in(S_v\cup S_r)} \pi^S(r^i)\lambda_i^{S_v,S_r} = \sum_{i\in(S_v\cup S_r)} \pi^P(r^i)\lambda_i^{S_v,S_r} = 1. \tag{10}$$

  The last equality is a consequence of Proposition 6.

- F^k, V^k, C^k: Let F^k be a facet of Q^{S_v,S_r} not containing f. Note that then F^k is a facet of $\tilde{Q}$. Therefore a non-empty subset of the vertices of $\tilde{Q}$ (denoted V^k) and a subset of extreme rays of $\tilde{Q}$ (denoted C^k) provide the internal description of F^k, i.e., $F^k = \text{conv}\left(\cup_{i\in V^k\subseteq\{1,...,|S_v|\}}(f + \frac{r^i}{\pi^Q(r^i)})\right) + \text{cone}\left(\cup_{i\in C^k\subseteq\{n_1+1,...,n_1+|S_r|\}}\{r^i\}\right)$.

- U^{S_v,S_r}, l: Let $U^{S_v,S_r} = \{x^1, x^2, x^3, ..., x^l\}$ where $x^j \in \mathbb{Z}^m\ \forall j \in \{1, ..., l\}$ *be points such that* x^j *belong to the boundary of* Q^{S_v,S_r} *and no two points* x^{j_1}, x^{j_2} $(j_1 \neq j_2)$ *lie on the same facet of* Q^{S_v,S_r}. The selection of the set U^{S_v,S_r} is not unique. However, in Theorem 8 we show that the split rank of (3) is at least $\lceil\log_2(|U^{S_v,S_r}|)\rceil$. Therefore, the set U^{S_v,S_r} with maximal cardinality provides the best bound. Henceforth, we use l to represent the cardinality of the set U^{S_v,S_r}.

- $y^j\ \forall j \in U^{S_v,S_r}$: By the definition of $x^j \in U^{S_v,S_r}$, we have that $x^j \in F^k$ where F^k is a facet of Q^{S_v,S_r}. Then $x^j = \sum_{i\in V^k} \mu_i(f + \frac{r^i}{\pi^P(r^i)}) + \sum_{i\in C^k} \mu_i r^i$, where $\mu_i \geq 0\ \forall i \in (V^k \cup C^k)$ and $\sum_{i\in V^k} \mu_i = 1$. Define $y^j \in \mathbb{R}^n_+$ as

$$y_i^j = \begin{cases} \frac{\mu_i}{\pi^P(r^i)} & \text{if } i \in V^K \\ \mu_i & \text{if } i \in C^k \\ 0 & \text{otherwise} \end{cases} . \tag{11}$$

  Since $\tilde{Q}$ is a generalized simplex, given any x^j the vector y^j in (11) is uniquely determined. Using (11) and the fact that $\pi^P(r^i) = \pi^S(r^i)\ \forall i \in (S_v \cup S_r)$, it is easily verified that $y^j \in \mathbb{R}^n_+$ satisfies,

$$f + \sum_{i=1}^n r^i y_i^j = x^j\ \forall j \in \{1, ..., l\}, \tag{12}$$

$$\sum_{i=1}^n \pi^S(r^i)y_i^j = 1\ \forall j \in \{1, ..., l\}. \tag{13}$$

  As $y^j \in \mathbb{R}^n_+$ and it satisfies (12) we obtain that $(x^j, y^j) \in M^I\ \forall j \in \{1, ..., l\}$. Note also that (12) and (13) together essentially imply that the points $(x^j, y^j)\ \forall j \in \{1, ..., l\}$ are satisfied at equality by (3).

- $x^{I,w}$, $y^{I,w}$: Let $w_j \in \mathbb{R}_{>0}$, i.e., $w_j > 0$ for all $j \in \{1, ..., l\}$. Let I be any subset of $\{1, ..., l\}$. We denote a point $x^{I,w} \in \mathbb{R}^m$ as $x^{I,w} = \frac{1}{\sum_{j\in I} w_j} \sum_{j\in I}(w_j x^j)$. The point $x^{I,w}$ is a strict convex combination of the points $x^j, j \in I$. This notation is used for convenience. Define $y^{I,w} \in \mathbb{R}^n_+$ similarly as $y^{I,w} = \frac{1}{\sum_{j\in I} w_j} \sum_{j\in I}(w_j y^j)$.

- $\gamma^{I,w,k}$: Finally define $\gamma^{I,w,k} = \max\{\delta \in \mathbb{R} \,|\, (x^{I,w}, y^{I,w} - \delta\lambda^{S_v,S_r}) \in M^k\}$. The motivation for the definition of $\gamma^{I,w,k}$ is as follows: By (13) we obtain that $\sum_{i=1}^n \pi^S(r^i) y^j_i = 1 \; \forall j \in \{1, ..., l\}$. Therefore $\sum_{i=1}^n \pi^S(r^i) y^{I,w}_i = 1$. By (10) we obtain that $\sum_{i=1}^n \pi^S(r^i)\lambda^{S_v,S_r}_i = 1$. Therefore observe that if $\gamma^{I,w,k} > 0$, then $(x^{I,w}, y^{I,w} - \gamma^{I,w,k}\lambda^{S_v,S_r}) \in M^k$ and $\sum_{i=1}^n \pi^S(r^i)(y^{I,w} - \gamma^{I,w,k}\lambda^{S_v,S_r})_i < 1$, i.e., we obtain a point $(x^{I,w}, y^{I,w} - \gamma^{I,w,k}\lambda^{S_v,S_r})$ which belongs to the k^{th} split closure but is not valid for the inequality $\sum_{i=1}^n \pi^S(r^i) y_i \geq 1$. Thus we obtain,

$$(\gamma^{I,w,k} > 0) \Rightarrow (\text{split rank of the cut (3)} \geq k + 1). \tag{14}$$

Observe that since $(x^j, y^j) \in M^I \; \forall j \in \{1, ..., l\}$, we obtain that $(x^{I,w}, y^{I,w})$ belongs to the convex hull of M^I. Since M^k is a relaxation of the convex hull of M^I, we obtain that $(x^{I,w}, y^{I,w}) \in M^k$. Therefore we obtain,

$$\gamma^{I,w,k} \geq 0 \, \forall w \in \mathbb{R}^l_{>0}, \forall k \in \mathbb{Z}_+, \text{ and } \forall I \subset \{1, ..., l\}. \tag{15}$$

Example 4 (contd.) *In the example observe that three of the facets of Q^{S_v,S_r} contain four integer points in total. The facet defined by the vertices $t_1 := (2, 0.5, 0)$, $t_2 := (0.5, 2, 0)$, $t_3 := (-1, 0.5, 0)$ contains two integer points $(1, 1, 0)$ and $(0, 1, 0)$. Therefore, one choice of the set U^{S_v,S_r} is $\{(1, 1, 0), (1, 1, 1), (0, 1, 1)\}$ as each of these integer points belong to different facets of Q^{S_v,S_r}. In this example it can be verified that $\pi^P(r^i) = \pi^S(r^i)$ for $i \in \{1, 2, 3, 4, 5\}$. However in general if $r^i \in cone(\cup_{j\in(S_v\cup S_r)}(r^j))$ and $r^i \neq r^j \; \forall j \in (S_v \cup S_r)$, then $\pi^S(r^i) > \pi^P(r^i)$. Next we compute y^j.*

1. *$x^1 = (1, 1, 0)$. Now observe that $x^1 = \frac{1}{2}\left(f + \frac{r^1}{\pi^P(r^1)}\right) + \frac{1}{3}\left(f + \frac{r^2}{\pi^P(r^2)}\right) + \frac{1}{6}\left(f + \frac{r^3}{\pi^P(r^3)}\right)$. Therefore, $y^1 = (\frac{1}{4}, \frac{1}{6}, \frac{1}{12}, 0, 0)$. Also note that $\sum_{i=1}^5 \pi^S(r^i) y^1_i = 1$.*

2. *$x^2 = (1, 1, 1)$. Now observe that $x^2 = \frac{1}{3}\left(f + \frac{r^1}{\pi^P(r^1)}\right) + \frac{1}{3}\left(f + \frac{r^2}{\pi^P(r^2)}\right) + \frac{1}{3}\left(f + \frac{r^5}{\pi^P(r^5)}\right)$. Therefore, $y^2 = (\frac{1}{6}, \frac{1}{6}, 0, 0, \frac{5}{6})$.*

3. *$x^3 = (0, 1, 1)$. Now observe that $x^3 = \frac{1}{3}\left(f + \frac{r^2}{\pi^P(r^2)}\right) + \frac{1}{3}\left(f + \frac{r^3}{\pi^P(r^3)}\right) + \frac{1}{3}\left(f + \frac{r^5}{\pi^P(r^5)}\right)$. Therefore, $y^3 = (0, \frac{1}{6}, \frac{1}{6}, 0, \frac{5}{6})$.*

Observe that $\forall i \in S_v = \{1, 2, 3, 5\}$ either $y^{j_1}_i > 0$ or $y^{j_2}_i > 0$ whenever $j_1 \neq j_2$. (This will be shown to be consequence of the fact that $\tilde{Q}$ is a simplex and that each of the points x^1, x^2, and x^3 belong to different facets of Q^{S_v,S_r}). This implies that $y^{I,w}_i > 0 \; \forall i \in S_v \; \forall w \in \mathbb{R}^5_{>0}$ whenever $|I| \geq 2$. Now note that since $\lambda^{S_v,S_r}_i = 0 \; \forall i \notin S_v$, and $\sum_{i=1}^5 \lambda^{S_v,S_r}_i r^i = \bar{0}$, we obtain that a point of the form $(x^{I,w}, y^{I,w} - \delta\lambda^{S_v,S_r})$, $\delta > 0$ belongs to the LP relaxation of (5). In other words, $\gamma^{I,w,0} > 0$ whenever $|I| \geq 2$. Proposition 7 verifies this result in general. □

We next present in Proposition 7 a property of $\gamma^{I,w,0}$ that is a key component in the proof of Theorem 8 which presents a lower bound on the split rank. Proposition 7 is a consequence of Assumption 1 and the construction of U^{S_v,S_r}.

Proposition 7 *If $I \subseteq \{1, ..., l\}$ and $|I| \geq 2$, then $\gamma^{I,w,0} > 0 \; \forall w \in \mathbb{R}^l_{>0}$.*

Proof: Note first that the dimension of Q^{S_v,S_r} is $|S_v|+|S_r|-1$. Therefore the dimension of a facet of Q^{S_v,S_r} is S_v+S_r-2. Hence $|C^k|+|V^k| = |S_v|+|S_r|-1$. Therefore, for convenience re-number the facets of Q^{S_v,S_r} so that $\{k\} = (S_v \cup S_r) \setminus (C^k \cup V^k)$.

Next note that since $\tilde{Q}$ is a generalized simplex, any face of Q^{S_v,S_r} not including f (which is therefore a face of $\tilde{Q}$) of dimension q, is described internally by exactly $q+1$ components (vertices and rays). Therefore we obtain that if $x^j \in$ relative.interior$(F^{k_1} \cap F^{k_2} \cap ... \cap F^{k_q})$ and x^j does not belong to any other facet of Q^{S_v,S_r}, then $x^j = \sum_{S_v \setminus \{k_1,k_2,...,k_q\}}(\mu_i(f+\frac{r^i}{\pi^Q(r^i)})) + \sum_{i \in S_r \setminus \{k_1,k_2,...,k_q\}} \mu_i r^i$ where $\mu_i > 0$ $\forall i \in (S_v \cup S_r) \setminus \{k_1,k_2,...,k_q\}$. Thus it follows from the construction of y^j (see (11)) that if $x^j \notin F^k$, then $y^j_k > 0$.

We prove this result for the case when $|I| = 2$. For $|I| > 2$, the proof is similar. By the construction of U^{S_v,S_r}, x^{j_1} and x^{j_2} ($j_1 \neq j_2$ and $j_1, j_2 \in I$) do not belong to the same facet. Therefore, since $x^j \notin F^k$ implies $y^j_k > 0$, we obtain that $\forall i \in (S_v \cup S_r)$ either $y^{j_1}_i > 0$ or $y^{j_2}_i > 0$. Thus $y^{I,w}_i > 0$ $\forall i \in (S_v \cup S_r)$. Now since $f + \sum_{i=1}^n y^{I,w}_i r^i = x^{I,w}$, $\sum_{i=1}^n \lambda^{S_v,S_r}_i r^i = \bar{0}$, and $\lambda^{S_v,S_r}_i = 0$ $\forall i \notin (S_v \cup S_r)$, $\exists \delta > 0$ such that $(x^{I,w}, y^{I,w} - \delta\lambda^{S_v,S_r}) \in M^0$. □

We now have all the tools necessary to prove the main result of this section. Before we present the proof, we illustrate the key ideas on Example 4.

Example 4 (contd.) *We prove that the split rank of the inequality (7) is at least 2: Consider any disjunction* $(a^Tx \leq c) \vee (a^Tx \geq c+1)$ *applied to the LP relaxation* M^0. *Let* $L^0_{a,c} := M^0 \cap \{(x,y)|a^Tx \leq c\}$ *and* $R^0_{a,c} := M^0 \cap \{(x,y)|a^Tx \geq c+1\}$. *Suppose* $proj_x(L^0_{a,c})$ *contains the points* $\{x^1,x^2\} := \{(1,1,1),(1,1,0)\}$ *and suppose* $proj_x(R^k_{a,c})$ *contains the point* $x^3 := \{(0,1,1)\}$. *Let* $w = (0.5,0.5,0.5)$. *Thus* $x^{\{1,2\},w} = (1,1,0.5)$ *and* $y^{I,w} = (\frac{5}{24},\frac{4}{24},\frac{1}{24},0,\frac{10}{24})$. *By Proposition 7, we know that if* $|I| \geq 2$, *then* $\gamma^{I,w,0} > 0$. *Therefore, there exists* $\bar{y} \in \mathbb{R}^5_+$ *such that* $\bar{y} = y^{\{1,2\},w} - \gamma^{\{1,2\},w,0}\lambda^{S_v,S_r}$ *and* $(x^{\{1,2\},w},\bar{y}) \in M^0$. *Here it is easily verified that* $\gamma^{\{1,2\},w,0} = \frac{1}{5}$. *Since by assumption* $proj_x(L^0_{a,c})$ *contains the points* $\{(1,1,1),(1,1,0)\}$, *we obtain that* $(x^{\{1,2\},w},\bar{y}) \in L^0_{a,c}$. *Also by assumption* $(x^3,y^3) \in R^0_{a,c}$. *Therefore any convex combination of the points* $(x^{\{1,2\},w},\bar{y})$ *and* (x^3,y^3) *belongs to* $conv(L^0_{a,c} \cup R^0_{a,c})$. *The convex combination* $\left(\frac{w_1+w_2}{w_1+w_2+w_3}(x^{\{1,2\},w},\bar{y}) + \frac{w_3}{w_1+w_2+w_3}(x^3,y^3)\right)$ *is a point of the form* $(x^{\{1,2,3\},w}, y^{\{1,2,3\},w} - \delta^*\lambda^{S_v,S_r})$ *where* $\delta^* > 0$. *Here* $\delta^* = \frac{2}{15}$. *Next note that for any disjunction such that* $proj_x(L^0_{a,c})$ *contains the points* $\{(1,1,1),(1,1,0)\}$ *and* $proj_x(R^k_{a,c})$ *contains the point* $\{(0,1,1)\}$, *the point* $(x^{\{1,2,3\},w}, y^{\{1,2,3\},w} - \delta^*\lambda^{S_v,S_r})$ *belongs to* $conv(L^0_{a,c} \cup R^0_{a,c})$. *It can be similarly shown that for any other partition of the integer points in* U^{S_v,S_r} *into the sets* $proj_x(L^0_{a,c})$ *and* $proj_x(R^0_{a,c})$, *a point of the form* $(x^{\{1,2,3\},w}, y^{\{1,2,3\},w} - \delta\lambda^{S_v,S_r})$, $\delta > 0$ *belongs to* $conv(L^0_{a,c} \cup R^0_{a,c})$. *Since there are only a finite number of partitions, this implies that a point of the form* $(x^{\{1,2,3\},w}, y^{\{1,2,3\},w} - \delta\lambda^{S_v,S_r})$, $\delta > 0$ *belongs to* M^1, *i.e.,* $\gamma^{\{1,2,3\},w,1} > 0$. *Therefore using (14) we obtain that the split rank of (7) is at least 2.* □

Theorem 8 *Let* $\{x^1,x^2,...,x^l\}$ *be a subset of integer points on the boundary of the restricted lattice-free set such that no two points lie on the same facet of* Q^{S_v,S_r}. *Then a lower bound on the split rank of (3) is* $\lceil log_2(l) \rceil$.

Proof: In order to prove that the split rank is at least $\lceil \log_2(l) \rceil$, by (14) it is sufficient to show that $\gamma^{I,w,\lceil \log_2(l) \rceil - 1} > 0$ for some $I \subseteq \{1,...,l\}$ and some $w \in \mathbb{R}^l_{>0}$. This is achieved by verifying that if $I \subseteq \{1,...,l\}$ and $|I| > 2^t$, then $\gamma^{I,w,t} > 0$ $\forall w \in \mathbb{R}^l_{>0}$. (Note that it is enough to verify this for a single $w \in \mathbb{R}^l_{>0}$).

The proof is by induction on t. By Proposition 7, the statement holds for $t = 0$. Assume that the statement is true for $t = 1, ..., k$. Choose any I such that $|I| > 2^{k+1}$. We need to prove that $\gamma^{I,w,k+1} > 0$.

For any disjunction $(a^T x \leq c) \vee (a^T x \geq c + 1)$ applied to the k^{th} split closure M^k, let $L^k_{a,c} := M^k \cap \{(x, y)|a^T x \leq c\}$ and $R^k_{a,c} := M^k \cap \{(x, y)|a^T x \geq c+1\}$. Let $(a^T x \leq c) \vee (a^T x \geq c+1)$ be any disjunction such that $L^k_{a,c}$ contains the points (x^j, y^j) where $j \in J$ $(J \subseteq I)$ and $R^k_{a,c}$ contains the points (x^j, y^j) where $j \in I \setminus J$. Observe now that the result is implied if we verify that a point of the form $(x^{I,w}, y^{I,w} - \delta^J \lambda^{S_v,S_r})$, $\delta^J > 0$ (where δ^J depends only on the sets I and J) is valid after applying this disjunction: Since $(x^{I,w}, y^{I,w}) \in M^I \subseteq \text{conv}(L^0_{a,c} \cup R^0_{a,c})$, by convexity of $\text{conv}(L^0_{a,c} \cup R^0_{a,c})$ all points of the form $(x^{I,w}, y^{I,w} - \delta\lambda^{S_v,S_r})$ where $0 \leq \delta \leq \delta^J$ belongs to $\text{conv}(L^0_{a,c} \cup R^0_{a,c})$. Since there is only a finite number of ways to partition I into J and $I \setminus J$, we therefore obtain that $\gamma^{I,w,k+1} > 0$.

Now we show that a point of the form $(x^{I,w}, y^{I,w} - \delta^J \lambda^{S_v,S_r})$, $\delta^J > 0$ is valid after applying this disjunction: Observe first that $x^{J,w} \in \text{proj}_x(L^k_{a,c})$, $x^{I\setminus J,w} \in \text{proj}_x(R^k_{a,c})$ and that

$$\frac{\sum_{j\in J} w_j}{\sum_{j\in I} w_j} x^{J,w} + \frac{\sum_{j\in I\setminus J} w_j}{\sum_{j\in I} w_j} x^{I\setminus J,w} = x^{I,w}. \tag{16}$$

Therefore when we apply the disjunction $(a^T x \leq c) \vee (a^T x \geq c+1)$, the point $(x^{I,w}, \frac{\sum_{j\in J} w_j}{\sum_{j\in I} w_j)} y^1 + \frac{\sum_{j\in I\setminus J} w_j}{\sum_{j\in I} w_j} y^2)$ belongs to $\text{conv}(L^k_{a,c} \cup R^k_{a,c})$, where $y^1 = y^{J,w} - \gamma^{J,w,k}\lambda^{S_v,S_r}$ and $y^2 = y^{I\setminus J,w} - \gamma^{I\setminus J,w,k}\lambda^{S_v,S_r}$. Note also that $\frac{\sum_{j\in J} w_j}{\sum_{j\in I} w_j} y^{J,w} + \frac{\sum_{j\in I\setminus J} w_j}{\sum_{j\in I} w_j} y^{I\setminus J,w} = y^{I,w}$. Therefore, to prove the result we need to verify that $\frac{\sum_{j\in J} w_j}{\sum_{j\in I} w_j}\gamma^{J,w,k} + \frac{\sum_{j\in I\setminus J} w_j}{\sum_{j\in I} w_j}\gamma^{I\setminus J,w,k} > 0$. Clearly, either $|J|$ or $|I \setminus J| > 2^k$. Without loss of generality assume that $|J| > 2^k$. Therefore by the induction argument $\gamma^{J,w,k} > 0$. Also by (15) we have that $\gamma^{I\setminus J,w,k} \geq 0$. □

Now we consider a more general problem than (2). From the definition of split rank, it is clear that given a problem with m integer variables, the split rank of the inequality $\sum_{i=1}^n \pi^P(r^i)y_i \geq 1$ cannot increase (and may decrease) if more constraints were present in the original system in addition to the constraints $x = f + \sum_{i=1}^n r^i y_i$, $x \in \mathbb{Z}^m, y \in \mathbb{R}^n_+$. Therefore to obtain a more general lower bound result we assume additional constraints added to (2) of the form,

$$Ax \quad \leq \quad b, \tag{17}$$

where $A \in \mathbb{Q}^{m_1\times m}$ and $b \in \mathbb{Q}^{m_1}$. At the very least, these constraints can be used to represent non-negativity of integer variables. The following result can be proven using an almost identical proof.

Corollary 9 *Let*

$$M^0 := \begin{cases} x = f + \sum_{i=1}^n r^i y_i, & x \in \mathbb{R}^m, y \in \mathbb{R}^n_+, \\ Ax \leq b. \end{cases}$$

and $M^I = M^0 \cap x \in \mathbb{Z}^m$. Let $P \subset \mathbb{R}^m$ be a lattice-free convex set containing f in its interior. Let the inequality

$$\sum_{i=1}^n \pi^P(r^i)y_i \geq 1 \tag{18}$$

be generated using (4). Let $\{x^1, x^2, ..., x^l\}$ be a subset of integer points on the boundary of the restricted lattice-free set (constructed as in Definition 5) such that no two points lie on the same facet of Q^{S_v,S_r} and $Ax^j \leq b\ \forall 1 \leq j \leq l$. Then a lower bound on the split rank of (18) is $\lceil log_2(l) \rceil$.
□

One implication of Corollary 9 is that if after addition of the inequalities (17) the integer points in U^{S_v,S_r} are still valid, then the split rank does not change.

Example 10 *Consider the set*

$$\begin{array}{rcrcrcrcrcrcr} x_1 & = & 0.5 & + & 3y_1 & + & 0y_2 & - & 3y_3 & + & 0y_4 & + & 0y_5 \\ x_2 & = & 0.5 & + & 0y_1 & + & 3y_2 & + & 0y_3 & - & 3y_4 & + & 0y_5 \\ x_3 & = & 0.5 & - & 1y_1 & - & 1y_2 & - & 1y_3 & - & 1y_4 & + & 1y_5 \end{array} \tag{19}$$

$$x_2 \geq 1 \tag{20}$$

$$x_i \in \mathbb{Z}\ \forall i \in \{1,2,3\} \quad y_i \in \mathbb{R}_+\ \forall j \in \{1,2,3,4,5\}. \tag{21}$$

Here (20) is an extra constraint when compared with the constraints present in Example 4. Consider again the same P and obtain again the inequality,

$$2y_1 + 2y_2 + 2y_3 + 2y_4 + \frac{2}{5}y_5 \geq 1. \tag{22}$$

Selecting the same set S_v, constructing Q^{S_v,S_r} and selecting the same U^{S_v,S_r} =$\{(1,1,0), (1,1,1), (0,1,1)\}$ we observe that every point in U^{S_v,S_r} satisfies (20). Therefore by Corollary 9, we obtain that 2 is a valid lower bound on the split rank of (22) when applying split inequalities to M^0 given by (19) - (21). □

2.2 Mixing Inequality

To illustrate the use of Theorem 8, we obtain a lower bound on the split rank of the mixing inequalities. The mixing set presented in Günlük and Pochet [23] is

$$x_i + y_0 \geq f_i \quad \forall 1 \leq i \leq n \tag{23}$$

$$y_0 \geq 0 \tag{24}$$

$$x_i \in \mathbb{Z}, \tag{25}$$

where we assume that $0 < f_1 \leq f_2 \leq f_3 \leq ... \leq f_n < 1$.

Theorem 11 ([23]) *The mixing inequality*

$$y_0 \geq f_n - f_1x_1 - \sum_{i=2}^{n}(f_i - f_{i-1})x_i, \tag{26}$$

is facet-defining for (23)-(25).

Next observe that by introducing slack variables y_i for each row of (23), the mixing set may be written as

$$\begin{array}{rcl} x_i & = & f_i - y_0 + y_i \quad \forall 1 \leq i \leq n \\ y_0 & \geq & 0,\ y_i \geq 0,\ x_i \in \mathbb{Z}, \end{array}$$

which is exactly in the form of (2). Also note that (26) can be rewritten as

$$\begin{aligned} & y_0 \geq f_n - f_1(f_1 - y_0 + y_1) - \sum_{i=2}^{n}(f_i - f_{i-1})(f_i - y_0 + y_i) \\ \Leftrightarrow \quad & (1-f_n)y_0 + f_1 y_1 + \sum_{i=2}^{n}(f_i - f_{i-1})y_i \geq \sum_{i=1}^{n-1} f_i(f_{i+1} - f_i) + f_n(1-f_n) \\ \Leftrightarrow \quad & \frac{(1-f_n)}{D}y_0 + \frac{f_1}{D}y_1 + \sum_{i=2}^{n}\frac{(f_i - f_{i-1})}{D}y_i \geq 1, \end{aligned} \tag{27}$$

where $D = \sum_{i=1}^{n-1} f_i(f_{i+1} - f_i) + f_n(1-f_n)$. Note that (27) is an inequality of the form (3).

Proposition 12 *If $0 < f_1 < f_2 < ... < f_n < 1$, then a lower bound on the split rank of the mixing inequality (26) is $\lceil log_2(n+1) \rceil$.*

Proof: Let $\bar{1}$ represent the n-dimensional vector with 1 in each component and e^i represent the unit vector in the direction of the i^{th} coordinate. Select $S_v = \{0, 1, 2, ..., n\}$ and $S_r = \emptyset$. Using (27) observe that $\tilde{Q}$ has $n+1$ vertices corresponding to the $n+1$ continuous variables:

1. $v^0 := f - \frac{D}{1-f_n}\bar{1}$,
2. $v^1 := f + \frac{D}{f_1}e^1$, and
3. $v^i := f + \frac{D}{f_i - f_{i-1}}e^i \; i \in \{2, ..., n\}$.

Since these points are affinely independent they form a n-dimensional lattice-free simplex.[1] Also note that in this case it is easily verified that $f \in \tilde{Q}$. Therefore $\tilde{Q} = Q^{S_v, \emptyset}$. The facets of $Q^{S_v,\emptyset}$ are the convex combination of the $n+1$ possible different selections of n vertices among the $n+1$ vertices of $Q^{S_v,\emptyset}$.

Next observe that $n+1$ integer points of the form $(1, 1, 1, ..., 1)$, $(0, 1, 1, 1, ..., 1)$, $(0, 0, 1, ..., 1)$, ..., $(0, 0, ..., 0)$ are satisfied at equality for the mixing inequality. Number the facets of $Q^{S_v,\emptyset}$ as $F^i = \text{conv}(\cup_{j \neq i} v^j)$. Now observe that:

1. $(1, 1, 1, ..., 1) = \frac{f_1(1-f_1)}{D}v^1 + \sum_{i=2}^{n}\frac{(1-f_i)(f_i - f_{i-1})}{D}v^i$. Note that $\frac{f_1(1-f_1)}{D} + \sum_{i=2}^{n}\frac{(1-f_i)(f_i - f_{i-1})}{D} = 1$ and that each of the multipliers for the vertices are positive. Therefore $(1, 1, 1, ..., 1)$ lies in the relative interior of the facet F^0.
2. $(0, 1, 1, ..., 1) = \frac{f_1(1-f_n)}{D}v^0 + \sum_{i=2}^{n}\frac{(1+f_1-f_i)(f_i - f_{i-1})}{D}v^i$. Note that $\frac{f_1(1-f_n)}{D} + \frac{(1+f_1-f_i)(f_i - f_{i-1})}{D} = 1$ and that each of the multipliers for the vertices are positive. Therefore $(0, 1, 1, ..., 1)$ lies in the relative interior of the facet F^1.
3. Consider the point $(0, 0, ..., 1, ..., 1)$, where the first j entries are zeros and the last $(n-j)$ entries are ones. In this case it can be verified that $(0, 0, ..., 1, ..., 1) = \frac{f_j(1-f_n)}{D}v^0 + \frac{(f_j - f_1)f_1}{D}v^1 + \sum_{i=2}^{j-1}\frac{(f_i - f_{i-1})(f_j - f_i)}{D}v^i + \sum_{i=j+1}^{n}\frac{(1+f_j-f_i)(f_i - f_{i-1})}{D}v^i$. It can be verified that $\frac{f_j(1-f_n)}{D} + \frac{(f_j - f_1)f_1}{D} + \sum_{i=2}^{j-1}\frac{(f_i - f_{i-1})(f_j - f_i)}{D} + \sum_{i=j+1}^{n}\frac{(1+f_j-f_i)(f_i - f_{i-1})}{D} = 1$ and that each of the multipliers for the vertices are positive. Therefore $(0, 0, ..., 1, ..., 1)$ lies in the relative interior of the facet F^j.

[1] The validity of mixing inequality implies that $\tilde{Q}$ is lattice-free. Also note that the generating lattice-free convex set P is implicit here and can be assumed to be the same at $\tilde{Q}$.

Therefore there are $n+1$ integer points that lie on different facets of $Q^{S_v,\emptyset}$. Thus, by application of Theorem 8, we obtain the result. □

We note here that if only k of the fractions in the list $\{f_1, ..., f_n\}$ are unique, then a lower bound of $\lceil \log_2(k+1) \rceil$ on the split rank of the mixing inequality can be proven. The proof is essentially similar to the proof of Proposition 12 with the following additional details: Let the list of unique fractions be $\{f_{i_1}, f_{i_2}, .., f_{i_k}\}$. Select this subset $\{i_1, i_2, ..., i_k\}$ of k inequalities from the mixing set and treat this as (2). Treat the rest of the constraints, i.e., $x_{i'} + y_0 \geq f_{i'}$ $i' \notin \{i_1, i_2, ..., i_k\}$ in the mixing set as the constraints (17). Note that $x_{i'} + y_0 \geq f_{i'}$ is not of the same form as (17), since it involves y_0. However, if $(x_{i_1}, x_{i_2}, ..x_{i_k}, y_0)$ satisfy the k mixing inequalities, then setting $x_{i'} = x_{i_j}$ whenever $f_{i'} = f_{i_j}$ gives a feasible solution to the mixing problem. Using this fact it can be verified that the result of Corollary 9 still holds. Now using the rest of the proof of Proposition 12, we obtain the result.

Dash and Günlük [17] prove an upper bound of n on the split rank of a mixing inequality based on n rows. We next present a simple example that shows that the lower bound obtained in Proposition 12 can be tight for certain instances.

Example 13 *Consider the* 3*-row mixing set with* $f_1 = \frac{1}{4}$, $f_2 = \frac{1}{2}$ *and* $f_3 = \frac{3}{4}$*. Observe that the inequalities* $y_0 + 0.5x_1 + 0.5x_3 \geq 0.75$ *and* $y_0 + 0.5x_2 \geq 0.5$ *are MIR inequalities based on the disjunctions* $(x_3 - x_1 \leq 0) \vee (x_3 - x_1 \geq 1)$ *and* $(x_2 \leq 0) \vee (x_2 \geq 1)$ *respectively. Now we show that the* 3*-row mixing inequality* $y_0 + \frac{1}{4}x_1 + \frac{1}{4}x_2 + \frac{1}{4}x_3 \geq \frac{3}{4}$ *may be obtainable as a split rank-2 cut. Consider the disjunction* $(x_1 - x_2 + x_3 \leq 0) \vee (x_1 - x_2 + x_3 \geq 1)$*. Then,*

1. $x_1 - x_2 + x_3 \leq 0$*: Then* $0.25[-x_1 + x_2 - x_3 \geq 0] + [y_0 + 0.5x_1 + 0.5x_3 \geq 0.75] \equiv y_0 + 0.25x_1 + 0.25x_2 + 0.25x_3 \geq 0.75$.
2. $x_1 - x_2 + x_3 \geq 1$*: Then* $0.25[x_1 - x_2 + x_3 \geq 1] + [y_0 + 0.5x_2 \geq 0.5] \equiv y_0 + 0.25x_1 + 0.25x_2 + 0.25x_3 \geq 0.75$. □

Next we illustrate more uses of Corollary 9, i.e., the fact that if the integer points that are on the boundary of Q^{S_v,S_r} are not cut off by the constraints (17) then the lower bound on the split rank does not change. A formulation for single item discrete lot-sizing problem with initial stock s_0, with binary variables v_u representing the decision to produce in the period u, and with constant capacity C is

$$\frac{s_0}{C} + \sum_{u=1}^{t} v_u \geq \frac{d_{1t}}{C} \quad \forall 1 \leq t \leq n \tag{28}$$
$$s_0 \geq 0, v_u \in \{0,1\} \, \forall u \in \{1, ..., n\},$$

where d_{1t} is the sum of the demands from periods 1 to t; see Pochet and Wolsey [32], Section 9.4 and Miller and Wolsey [28] for details. Setting $y_0 = \frac{s_0}{C}$ and $f_t = \frac{d_{1t}}{C}$, we can re-write (28) as

$$y_0 + x_t \geq f_t \quad \forall 1 \leq t \leq n \tag{29}$$
$$0 \leq x_t - x_{t-1} \leq 1 \, \forall 1 \leq t \leq n \tag{30}$$
$$x_t = \sum_{u=1}^{t} v_u \, \forall 1 \leq t \leq n \tag{31}$$
$$y_0 \geq 0, \;\; x_t \in \mathbb{Z}_+ \, \forall 1 \leq t \leq n, \;\; v_u \in \{0,1\} \, \forall 1 \leq u \leq n. \tag{32}$$

Observe that (29)-(32) is essentially a mixing set with non-negativity and additional constraints (30). The facet-defining inequality for (29)-(32) are the mixing inequalities. Note that any split cut obtained using a split disjunction on the v variables can be rewritten as a split disjunction on the x variables. Also note that the $n+1$ integer feasible points that are satisfied at equality for the mixing inequality are valid after the addition of (30). Therefore by use of Corollary 9, $\lceil \log_2(k+1) \rceil$ is a lower bound on the split rank of the facet-defining inequalities for (29) where k of the fractional parts of $\frac{d_{1t}}{C}$s are unique. We finally note that by using Corollary 9 and the approach used in this section, similar lower bound results can be obtained for other inequalities based on more involved variants of mixing sets, such as the lot-sizing problem with Wagner-Whitin costs and constant capacities. (Pochet and Wolsey [31]).

3 Discussion

A lower bound on the split rank of an inequality indicates how difficult it may be to obtain an inequality using split cuts. In this note, we have presented a non-trivial lower bound on the split rank of intersection cuts. The main insight from this result is the demonstration of the effect of the orientation of integer feasible points satisfied at equality by (3) on the split rank of the inequality. We also used this result to derive a lower bound on the split rank of mixing inequalities. We next discuss some avenues of possible strengthening and generalization of the results presented in this note.

First observe that Theorem 8 provides a lower bound on split rank based completely on the structure of the restricted lattice-free set. It is easily verified that there can exist two problems with different columns such that Q^{S_v,S_r} is the same. It is however not clear whether the split rank of an inequality is a function of the structure of Q^{S_v,S_r} (for the best choice of S_v and S_r) alone.

Second observe that typically it is very difficult to ascertain the split rank of a valid inequality for any given problem. As illustrated in Section 2.2, on some examples the lower bound on the split rank of intersection cuts presented in this note can be verified to be tight. However, this lower bound seems to be weak when the vertices of the lattice-free set Q^{S_v,S_r} are all integral and each facet contains an integer point in its relative interior. This is best illustrated by the following result from Li and Richard [26] (modified to the notation of this paper).

Theorem 14 ([26]) *If $Q^{S_v,S_r} \subset \mathbb{R}^m$ is the lattice-free simplex defined by the following vertices: $(0,0,0,...0)$, $(m,0,0,...0)$, $(0,m,0,...0)$, ..., $(0,0,0...,m)$, then a lower bound to the split rank of the inequality is infinite.* □

Applying Theorem 8, a lower bound of $\lceil \log_2(m+1) \rceil$ on the split rank can be obtained, which is clearly a weak bound. A more general result that combines the flexibility of Theorem 8 with results such as Theorem 14 is an important direction in understanding cutting-plane-algorithms based on split inequalities.

Acknowledgements I would like to thank Laurence Wolsey for many discussions and suggestions for improvement of the paper.

This text presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

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