MAXIMIZING NON-MONOTONE SUBMODULAR FUNCTIONS UNDER MATROID AND KNAPSACK CONSTRAINTS

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Abstract. Submodular function maximization is a central problem in combinatorial optimization, generalizing many important problems including Max Cut in directed/undirected graphs and in hypergraphs, certain constraint satisfaction problems, maximum entropy sampling, and maximum facility location problems. Unlike submodular minimization, submodular maximization is NP-hard. In this paper, we give the first constant-factor approximation algorithm for maximizing any non-negative submodular function subject to multiple matroid or knapsack constraints. We emphasize that our results are for *non-monotone* submodular functions. In particular, for any constant k, we present a $\left(\frac{1}{k+2+\frac{1}{k}+\epsilon}\right)$-approximation for the submodular maximization problem under k matroid constraints, and a $\left(\frac{1}{5}-\epsilon\right)$-approximation algorithm for this problem subject to k knapsack constraints ($\epsilon > 0$ is any constant). We improve the approximation guarantee of our algorithm to $\frac{1}{k+1+\frac{1}{k-1}+\epsilon}$ for $k \geq 2$ partition matroid constraints. This idea also gives a $\left(\frac{1}{k+\epsilon}\right)$-approximation for maximizing a *monotone* submodular function subject to $k \geq 2$ partition matroids, which improves over the previously best known guarantee of $\frac{1}{k+1}$.

1. Introduction. In this paper, we consider the problem of maximizing a *nonnegative submodular function f, defined on a ground set V, subject to matroid constraints or knapsack constraints.* A function $f : 2^V \to \mathbb{R}$ is *submodular* if for all $S, T \subseteq V$,

$$f(S \cup T) + f(S \cap T) \leq f(S) + f(T).$$

An alternative definition of submodularity is the property of *decreasing marginal values*: For all $A \subseteq B \subseteq V$ and $x \in V \setminus B$,

$$f(B \cup \{x\}) - f(B) \leq f(A \cup \{x\}) - f(A).$$

Throughout, we assume that our submodular function f is given by a *value oracle*; i.e., for a given set $S \subseteq V$, an algorithm can query an oracle to find its value $f(S)$. Furthermore, all submodular functions we deal with are assumed to be non-negative. We also denote the ground set $V = [n] = \{1, 2, \cdots, n\}$.

We emphasize that our focus is on submodular functions that are *not required to be monotone* (i.e., we do *not* require that $f(X) \leq f(Y)$ for $X \subseteq Y \subseteq V$). Non-monotone submodular functions appear in several places including cut functions in weighted directed or undirected graphs or even hypergraphs, maximum facility location problems, maximum entropy sampling, and certain constraint satisfaction problems.

Given a weight vector w for the ground set V, and a knapsack of capacity C, the associated *knapsack constraint* is that the sum of weights of elements in S should not exceed the capacity C, i.e, $\sum_{j \in S} w_j \leq C$. In our usage, we consider k knapsacks defined by weight vectors w^i and capacities C_i, for $i = 1, \ldots, k$.

We assume some familiarity with matroids [39] and associated algorithmics [44]. Briefly, for a matroid $\mathcal{M}$, we denote the ground set of $\mathcal{M}$ by $\mathcal{E}(\mathcal{M})$, its set of independent sets by $\mathcal{I}(\mathcal{M})$, and its set of bases by $\mathcal{B}(\mathcal{M})$. For a given matroid $\mathcal{M}$, the associated *matroid constraint* is $S \in \mathcal{I}(\mathcal{M})$ and the associated *matroid base constraint* is $S \in \mathcal{B}(\mathcal{M})$. As is standard, $\mathcal{M}$ is a *uniform matroid* of rank r if $\mathcal{I}(\mathcal{M}) := \{X \subseteq \mathcal{E}(\mathcal{M}) : |X| \leq r\}$, and a *partition matroid* is the direct sum of uniform matroids. Note that uniform matroid constraints are equivalent to cardinality constraints, i.e, $|S| \leq k$. In our usage, we mostly have k matroids $\mathcal{M}_1, \ldots, \mathcal{M}_k$ on the common ground set $V := \mathcal{E}(\mathcal{M}_1) = \cdots = \mathcal{E}(\mathcal{M}_k)$ (which is also the ground set of our submodular function f), and we let $\mathcal{I}_i := \mathcal{I}(\mathcal{M}_i)$ for $i = 1, \ldots, k$.

Background. Optimizing submodular functions is a central subject in operations research and combinatorial optimization [35]. This problem appears in many important optimization problems including cuts in graphs [19, 40, 26], rank function of matroids [12, 16], set covering problems [13], plant location problems [9, 10, 11, 2], and certain satisfiability problems [25, 14], and maximum entropy sampling [32, 33]. Other

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than many heuristics that have been developed for optimizing these functions [20, 21, 27, 42, 31], many exact and constant-factor approximation algorithms are also known for this problem [37, 38, 43, 26, 15, 47, 18]. In some settings such as set covering or matroid optimization, the relevant submodular functions are monotone. Here, we are more interested in the general case where $f(S)$ is not necessarily monotone.

Unlike submodular minimization [43, 26], submodular function maximization is NP-hard as it generalizes many NP-hard problems, like the Max-Cut problem [19, 14] and maximum facility location problem [9, 10, 2]. Other than generalizing combinatorial optimization problems like Max Cut [19], Max Directed Cut [4, 22], hypergraph cut problems, maximum facility location [2, 9, 10], and certain restricted satisfiability problems [25, 14], maximizing non-monotone submodular functions have applications in a variety of problems, e.g, computing the core value of supermodular games [45], and optimal marketing for revenue maximization over social networks [23]. As an example, we describe one important application in the statistical design of experiments. The *maximum entropy sampling problem* is as follows: Let A be the n-by-n covariance matrix of a set of Gaussian random variables indexed by $[n]$. For $S \subseteq [n]$, let $A[S]$ denote the principal submatrix of A indexed by S. It is well known that (up to constants depending on $|S|$), $\log\det A[S]$ is the entropy of the random variables indexed by S. Furthermore, $\log\det A[S]$ is submodular on $[n]$. In applications of locating environmental monitoring stations, it is desired to choose s locations from $[n]$ so as to maximize the entropy of the associated random variables, so that problem is precisely one of maximizing a non-monotone submodular function subject to a cardinality constraint. Of course a cardinality constraint is just a matroid base constraint for a uniform matroid. We note that the entropy function is not even approximately submodular (see [30]). The maximum entropy sampling problem has mostly been studied from a computational point of view, focusing on calculating optimal solutions for moderate-sized instances (say $n < 200$) using mathematical programming methodologies (e.g, see [32, 33, 34, 29, 6, 5]), and our results provide the first set of algorithms with provable constant-factor approximation guarantee.

Recently, a $\frac{2}{5}$-approximation was developed for maximizing non-negative non-monotone submodular functions without any side constraints [15]. This algorithm also provides a tight $\frac{1}{2}$-approximation algorithm for maximizing a symmetric[1] submodular function [15]. However, the algorithms developed in [15] for non-monotone submodular maximization do not handle any extra constraints.

For the problem of maximizing a monotone submodular function subject to a matroid or multiple knapsack constraints, tight $\left(1 - \frac{1}{e}\right)$-approximation are known [37, 7, 48, 46, 28]. Maximizing monotone submodular functions over k matroid constraints was considered in [38], where a $\left(\frac{1}{k+1}\right)$-approximation was obtained. This bound is currently the best known ratio, even in the special case of partition matroid constraints. However, none of these results generalize to non-monotone submodular functions.

Better results are known either for specific submodular functions or for special classes of matroids. The $\frac{1}{k}$-approximation algorithm of local search type was designed in [41] for the problem of maximizing a linear function subject to k matroid constraints. Constant factor approximation algorithms are known for the problem of maximizing directed cut [1] or hypergraph cut [3] subject to a uniform matroid (or cardinality) constraint.

Hardness of approximation results are known even for the special case of maximizing a linear function subject to k partition matroid constraints. The best known lower bound is an $\Omega(\frac{k}{\log k})$ hardness of approximation [24]. Moreover, for the unconstrained maximization of non-monotone submodular functions, it has been shown that achieving a factor better than $\frac{1}{2}$ cannot be done using a subexponential number of queries [15].

Our Results. In this paper, we give the first constant-factor approximation algorithms for maximizing a non-monotone submodular function subject to multiple matroid constraints, or multiple knapsack constraints. More specifically, we give the following new results:

1. For every constant positive integer k, we present a $\left(\frac{1}{k+2+\frac{1}{k}+\epsilon}\right)$-approximation algorithm for maximizing any non-negative submodular function subject to k matroid constraints. This implies a $\left(\frac{1}{4+\epsilon}\right)$-approximation algorithm for maximizing non-monotone submodular functions subject to a

[1]The function $f : 2^V \rightarrow \mathbb{R}$ is symmetric if for all $S \subseteq V$, $f(S) = f(V \setminus S)$. For example, cut functions in undirected graphs are well-known examples of symmetric (non-monotone) submodular functions

single matroid constraint. Moreover, this algorithm is a $\left(\frac{1}{k+2+\epsilon}\right)$-approximation in the case of *symmetric* submodular functions. This result is almost best possible because there is $\Omega(\frac{k}{\log k})$ hardness of approximation, even in the monotone case [24].

2. For every constant positive integer k, we present a $\left(\frac{1}{5}-\epsilon\right)$-approximation algorithm for maximizing any nonnegative submodular function subject to a k-dimensional knapsack constraint ($\epsilon > 0$ is any constant). To achieve the approximation guarantee, we first give a $\left(\frac{1}{4}-\epsilon\right)$-approximation algorithm for a fractional relaxation (similar to the fractional relaxation used in [48]). We then use a simple randomized rounding technique to convert a fractional solution to an integral one. A similar method was recently used in [28] for maximizing a monotone submodular function over knapsack constraints, but neither their algorithm for the fractional relaxation, nor their rounding method is directly applicable to non-monotone submodular functions.
3. For submodular maximization under $k \geq 2$ *partition matroid* constraints, we obtain improved approximation guarantees. We give a $\left(\frac{1}{k+1+\frac{1}{k-1}+\epsilon}\right)$-approximation algorithm for maximizing non-monotone submodular functions subject to k partition matroids, where $\epsilon > 0$ is any constant. Moreover, our idea gives a $\left(\frac{1}{k+\epsilon}\right)$-approximation algorithm for maximizing a monotone submodular function subject to $k \geq 2$ partition matroid constraints. This is an improvement over the previously best known bound of $\frac{1}{k+1}$ in [38].
4. Finally, we study submodular maximization subject to a matroid *base* constraint. We give a $\left(\frac{1}{3}-\epsilon\right)$-approximation in the case of symmetric submodular functions. Our result for general submodular functions only holds for special matroids: we obtain a $(\frac{1}{6}-\epsilon)$-approximation when the matroid contains two disjoint bases. In particular, this implies a $\left(\frac{1}{6}-\epsilon\right)$ approximation for the problem of maximizing any non-negative submodular function subject to an exact cardinality constraint. Previously, only special cases of directed cut [1] or hypergraph cut [3] subject to an exact cardinality constraint were considered.

Our main technique for the above results is local search. Our local search algorithms are different from the previously used variant of local search for unconstrained maximization of a non-negative submodular function [15], or the local search algorithms used for Max Directed Cut [4, 22]. In the design of our algorithms, we also use structural properties of matroids, a fractional relaxation of submodular functions, and a randomized rounding technique.

2. Submodular Maximization subject to k Matroid Constraints. In this section, we give an approximation for submodular maximization subject to k matroid constraints. The problem is as follows: Let f be a *non-negative* submodular function defined on ground set V. Let $\mathcal{M}_1, \cdots, \mathcal{M}_k$ be k arbitrary matroids on the common ground set V. For each matroid $\mathcal{M}_j$ (with $j \in [k]$) we denote the set of its independent sets by $\mathcal{I}_j$. We consider the following problem:

$$\max \left\{ f(S) \ : \ S \in \cap_{j=1}^{k} \mathcal{I}_j \right\}. \tag{2.1}$$

We give an approximation algorithm for this problem using value queries to f that runs in time $n^{O(k)}$. The starting point is the following local search algorithm. Starting with $S = \emptyset$, repeatedly perform one of the following local improvements:

- **Delete operation.** If $e \in S$ such that $f(S \setminus \{e\}) > f(S)$, then $S \leftarrow S \setminus \{e\}$.
- **Exchange operation.** If $d \in V \setminus S$ and $e_i \in S \cup \{\phi\}$ (for $1 \leq i \leq k$) are such that $(S \setminus \{e_i\}) \cup \{d\} \in \mathcal{I}_i$ for all $i \in [k]$ and $f((S \setminus \{e_1, \cdots, e_k\}) \cup \{d\}) > f(S)$, then $S \leftarrow (S \setminus \{e_1, \cdots, e_k\}) \cup \{d\}$.

When dealing with a single matroid constraint ($k = 1$), the local operations correspond to: *delete* an element, *add* an element (i.e. an exchange when no element is dropped), *swap* a pair of elements (i.e. an element from outside the current set is exchanged with an element from the set). With $k \geq 2$ matroid constraints, we permit more general exchange operations (not just swaps as in the case of a single matroid).

Note that the size of any local neighborhood is at most n^{k+1}, which implies that each local step can be performed in polynomial time for a constant k. Let S denote a locally optimal solution. Next we prove a key lemma for this local search algorithm, which is used in analyzing our algorithm. Before presenting the lemma, we state a useful exchange property of matroids (see [44]). Intuitively, this property states that for any two independent sets I and J, we can add any element of J to the set I, and kick out at most one element from I while keeping the set independent. Moreover, each element of I is allowed to be kicked out by at most one element of J.

THEOREM 2.1. *Let $\mathcal{M}$ be a matroid and $I, J \in \mathcal{I}(\mathcal{M})$ be two independent sets. Then there is a mapping $\pi : J \setminus I \to (I \setminus J) \cup \{\phi\}$ such that:*

1. *$(I \setminus \pi(b)) \cup \{b\} \in \mathcal{I}(\mathcal{M})$ for all $b \in J \setminus I$.*
2. *$|\pi^{-1}(e)| \leq 1$ for all $e \in I \setminus J$.*

Proof. We outline the proof for completeness. We proceed by induction on $t = |J \setminus I|$. If $t = 0$, there is nothing to prove. Suppose there is an element $b \in J \setminus I$ with $I \cup \{b\} \in \mathcal{I}(\mathcal{M})$. In this case we apply induction on I and $J' = J \setminus \{b\}$ (where $|J' \setminus I| = t - 1 < t$). Because $I \setminus J' = I \setminus J$, we obtain a map $\pi' : J' \setminus I \to (I \setminus J) \cup \{\phi\}$ satisfying the two conditions. The desired map π is then $\pi(b) = \phi$ and $\pi(b') = \pi'(b')$ for all $b' \in J \setminus I \setminus \{b\} = J' \setminus I$.

Now we may assume that I is a maximal independent set in $I \cup J$. Let $\mathcal{M}' \subseteq \mathcal{M}$ denote the matroid $\mathcal{M}$ truncated to $I \cup J$; so I is a base in $\mathcal{M}'$. We augment J to some base $\tilde{J} \supseteq J$ in $\mathcal{M}'$ (because any maximal independent set in $\mathcal{M}'$ is a base). Then we have two bases I and $\tilde{J}$ in $\mathcal{M}'$. Theorem 39.12 from [44] implies the existence of elements $b \in \tilde{J} \setminus I$ and $e \in I \setminus \tilde{J}$ such that both $(\tilde{J} \setminus b) \cup \{e\}$ and $(I \setminus e) \cup \{b\}$ are bases in $\mathcal{M}'$. Note that $J' = (J \setminus \{b\}) \cup \{e\} \subseteq (\tilde{J} \setminus \{b\}) \cup \{e\} \in \mathcal{M}$. By induction on I and J' (because $|J' \setminus I| = t - 1 < t$) we obtain map $\pi' : J' \setminus I \to I \setminus J'$ satisfying the two conditions. The map π is then $\pi(b) = e$ and $\pi(b') = \pi'(b')$ for all $b' \in J \setminus I \setminus \{b\} = J' \setminus I$. The first condition on π is satisfied by induction (for elements $J \setminus I \setminus \{b\}$) and because $(I \setminus e) \cup \{b\} \in \mathcal{M}$ (see above). The second condition on π is satisfied by induction and the fact that $e \notin I \setminus J'$. ∎

LEMMA 2.2. *For a local optimal solution S and any $C \in \cap_{j=1}^{k} \mathcal{I}_j$, $(k+1) \cdot f(S) \geq f(S \cup C) + k \cdot f(S \cap C)$. Additionally for $k = 1$, if $S \in \mathcal{I}_1$ is any locally optimal solution under only the swap operation, and $C \in \mathcal{I}_1$ with $|S| = |C|$, then $2 \cdot f(S) \geq f(S \cup C) + f(S \cap C)$.*

Proof. For each matroid $\mathcal{M}_j$ ($j \in [k]$), because both $C, S \in \mathcal{I}_j$ are independent sets, Theorem 2.1 implies a mapping $\pi_j : C \setminus S \to (S \setminus C) \cup \{\phi\}$ such that:

1. $(S \setminus \pi_j(b)) \cup b \in \mathcal{I}_j$ for all $b \in C \setminus S$.
2. $|\pi_j^{-1}(e)| \leq 1$ for all $e \in S \setminus C$.

When $k = 1$ and $|S| = |C|$, Corollary 39.12a from [44] implies the stronger condition that $\pi_1 : C \setminus S \to S \setminus C$ is in fact a *bijection*.

We add to the trivial inequality $f(S) \geq f(S)$, the following inequalities that are implied by the fact that S is a local optimum.

$$f(S) \geq f\left((S \setminus \{\pi_1(b), \cdots \pi_k(b)\}) \cup \{b\}\right) \qquad \forall b \in C \setminus S. \tag{2.2}$$

These follow from local optimality under exchange operations. In the case $k = 1$ with $|S| = |C|$, these are only *swap* operations (because π_1 is a bijection here).

Note that each element $a \in S \setminus C$ is missing in the right-hand side of at most k inequalities (as each π_j is a 'matching'). Letting $r = |C \setminus S|$ and $C \setminus S = \{b_1, \cdots, b_r\}$, we have:

$$(r+1) \cdot f(S) \geq f(S) + \sum_{i=1}^{r} f(S + b_i - \{\pi_1(b_i), \cdots \pi_k(b_i)\}).$$

The right-hand side has $r + 1$ terms and each element of S is missing in at most k of them. We will simplify the right hand side by applying only one rule based on submodularity of f, namely $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$ for any two sets A, B. We use the following simple fact about union and intersection.

FACT 1. *For any $A, B \subseteq V$ and element $e \in V$, the number of sets among $\{A, B\}$ containing e equals the number of sets among $\{A \cup B, A \cap B\}$ containing e.*

Hence throughout the simplification of the right hand side, each element of S is missing in at most k sets (out of $r + 1$). We successively combine the $r + 1$ terms to be left with an inequality of the form:

$$(r+1) \cdot f(S) \geq f(S \cup \{b_1, \cdots, b_r\}) + \sum_{i=1}^{r} f(S \setminus A_i) = f(S \cup C) + \sum_{i=1}^{r} f(S \setminus A_i), \tag{2.3}$$

where the sets $A_1, \cdots, A_r \subseteq S \setminus C$ are such that each element $a \in S \setminus C$ appears in $n_a \leq k$ of $\{A_i\}_{i=1}^r$. We add to (2.3), the following inequalities that are implied by the local optimality of S under deletions.

$$(k - n_a) \cdot f(S) \geq (k - n_a) \cdot f(S \setminus \{a\}) \qquad \forall a \in S \setminus C. \tag{2.4}$$

Note that these inequalities are not required when $k = 1$ and $|S| = |C|$, because then $n_a = 1 = k$ for all $a \in S \setminus C$. Let $\lambda = r + \sum_{a \in S \setminus C}(k - n_a)$; then we have:

$$(\lambda + 1) \cdot f(S) \geq f(S \cup C) + \sum_{i=1}^{\lambda} f(S \setminus A_i), \tag{2.5}$$

where sets $\{A_i \subseteq S \setminus C \ : \ 1 \leq i \leq \lambda\}$ are such that each element of $S \setminus C$ appears in *exactly* k of them. Elements of $S \cap C$ appear in none of $\{A_i\}_{i=1}^{\lambda}$.

We now give an iterative procedure (applying submodularity on a pair of sets in each iteration) to modify $\sum_{i=1}^{\lambda} f(S \setminus A_i)$ in a *non-increasing* manner.

1. Initialize $\mathcal{B} = \{A_1, \cdots, A_\lambda\}$.
2. While fewer than $\lambda - k$ sets in $\mathcal{B}$ are empty, do:
   (a) Let $\mathcal{B} = \{B_i\}_{i=1}^{\lambda}$, and $p = \min\{|B_i| : B_i \neq \emptyset, i \in [\lambda]\}$.
   (b) Choose a set $B_x \in \mathcal{B}$ with $|B_x| = p$ and any element $e \in B_x$.
   (c) Choose a *non-empty* set $B_y \in \mathcal{B}$ with $e \notin B_y$.
   (d) Combine sets B_x and B_y to obtain sets $B_x \cap B_y$ and $B_x \cup B_y$; i.e. $\mathcal{B} \leftarrow (\mathcal{B} \setminus \{B_x, B_y\}) \cup \{B_x \cup B_y, B_x \cap B_y\}$.

Claim 2.3. *The above procedure is well-defined and terminates in a finite number of steps.*

Proof. First observe that in each iteration, all the steps are well-defined. There are at least $k + 1 > 0$ non-empty sets in $\mathcal{B}$ in any iteration (from the termination condition). So the existence of set B_x is obvious. Furthermore each element of S is present in at most k of the sets in $\mathcal{B}$: this follows from the earlier observation on any modification of the form $f(A)+f(B) \geq f(A\cap B)+f(A\cup B)$. Hence any element $e \in B_x \subseteq S$ appears in at most k sets in $\mathcal{B}$; this guarantees the existence of a non-empty set $B_y \in \mathcal{B}$. Finally, the combination of B_x and B_y corresponds to an application of submodularity to the sets $S \setminus B_x$ and $S \setminus B_y$, i.e.

$$f(S\setminus B_x)+f(S\setminus B_y) \geq f((S\setminus B_x)\cup(S\setminus B_y))+f((S\setminus B_x)\cap(S\setminus B_y)) = f(S\setminus(B_x\cap B_y))+f(S\setminus(B_x\cup B_y)).$$

Therefore, $\sum_{T\in\mathcal{B}} f(S \setminus T)$ does not increase.

We now argue finite termination of this procedure. For this purpose we show that the quantity $\eta(\mathcal{B}) := \sum_{T\in\mathcal{B}} |T|^2$ increases strictly in each iteration. This suffices because $0 \leq \eta(\mathcal{B}) \leq \lambda \cdot |S|^2$ always holds. Let $\alpha = |B_x \cap B_y|$, $\beta = |B_x \setminus B_y|$ and $\gamma = |B_y \setminus B_x|$. Then the increase in η at this iteration is $(\alpha + \beta + \gamma)^2 + \alpha^2 - (\alpha + \beta)^2 - (\alpha + \gamma)^2 = 2\beta\gamma$. We will show $\beta, \gamma \geq 1$ that implies an increase in η. By definition of B_x and B_y, there is an element $e \in B_x \setminus B_y$: so $\beta \geq 1$. On the other hand $|B_y| \geq |B_x|$ because B_x was chosen to be a minimum cardinality non-empty set in $\mathcal{B}$: combined with the fact that $B_x \setminus B_y \neq \emptyset$, we have $\gamma = |B_y \setminus B_x| \geq 1$. ∎

At the end of this procedure, we have at least $\lambda - k$ empty sets in $\mathcal{B}$. However, each element of $S \setminus C$ is present in exactly k sets of $\mathcal{B}$: so we have exactly $\lambda - k$ empty sets in $\mathcal{B}$. Additionally, elements of $S \cap C$ are present in none of the $\mathcal{B}$ sets: so the remaining k sets in $\mathcal{B}$ are all $S \setminus C$. From (2.5), we now have:

$$(\lambda + 1) \cdot f(S) \geq f(S \cup C) + \sum_{T\in\mathcal{B}} f(S \setminus T) = f(S \cup C) + (\lambda - k) \cdot f(S) + k \cdot f(S \cap C). \tag{2.6}$$

Thus we obtain $(k + 1) \cdot f(S) \geq f(S \cup C) + k \cdot f(S \cap C)$, giving the claim. Observe that when $k = 1$ and $|S| = |C|$, we only used the inequalities (2.2) from the local search, which are only swap operations. Hence in this case, the statement also holds for any solution S that is locally optimal under only swap-operations. In the general case, we use both inequalities (2.2) (from exchange operations) and inequalities (2.4) (from deletion operations). ∎

A simple consequence of the Lemma 2.2 implies bounds analogous to known approximation factors [38, 41] in the cases when the set function f has additional structure.

Corollary 2.4. *For a locally optimal solution S and any $C \in \cap_{j=1}^{k}\mathcal{I}_j$ the following inequalities hold:*

Approximate Local Search Procedure B**:**

Input: Ground set X of elements and value oracle access to submodular function f.

1. Let $\{v\}$ be a singleton set with the maximum value $f(\{v\})$.
2. Set $S = \{v\}$.
3. While there exists the following delete or exchange local operation that increases the value of $f(S)$ by a factor of at least $(1+\frac{\epsilon}{n^4})$, then apply the local operation and update S accordingly.
   - **Delete operation on** S**.** If $e \in S$ such that $f(S \setminus \{e\}) \geq (1+\frac{\epsilon}{n^4})f(S)$, then $S \leftarrow S \setminus \{e\}$.
   - **Exchange operation on** S**.** If $d \in X \setminus S$ and $e_i \in S \cup \{\phi\}$ (for $1 \leq i \leq k$) are such that $(S \setminus \{e_i\}) \cup \{d\} \in \mathcal{I}_i$ for all $i \in [k]$ and $f((S \setminus \{e_1, \cdots, e_k\}) \cup \{d\}) > (1+\frac{\epsilon}{n^4})f(S)$, then $S \leftarrow (S \setminus \{e_1, \cdots, e_k\}) \cup \{d\}$.

Fig. 2.1. *The approximate local search procedure.*

Algorithm A**:**

1. Set $V_1 = V$.
2. For $i = 1, \cdots, k+1$, do:
   (a) Apply the approximate local search procedure B on the ground set V_i to obtain a solution $S_i \subseteq V_i$ corresponding to the problem:
   $$\max\{f(S) \ : \ S \in \cap_{j=1}^{k} \mathcal{I}_j, \ S \subseteq V_i\}. \tag{2.7}$$
   (b) Set $V_{i+1} = V_i \setminus S_i$.
3. Return the solution corresponding to $\max\{f(S_1), \cdots, f(S_{k+1})\}$.

Fig. 2.2. *Approximation algorithm for submodular maximization under k matroid constraints.*

1. $f(S) \geq f(C)/(k+1)$ if function f is monotone,
2. $f(S) \geq f(C)/k$ if function f is linear.

Unfortunately, the local search algorithm defined above could run an exponential amount of time until it reaches a locally optimal solution. The standard approach is to consider an approximate local search. We will show that any approximate local optimum satisfies an inequality analogous to the one in the Lemma 2.2.

Lemma 2.5. *For an approximately locallly optimal solution S and any $C \in \cap_{j=1}^{k} \mathcal{I}_j$, $(1+\epsilon)(k+1) \cdot f(S) \geq f(S \cup C) + k \cdot f(S \cap C)$ where $\epsilon > 0$ a parameter defined in the algorithm description. Additionally for $k = 1$, if $S \in \mathcal{I}_1$ is any locally optimal solution under only the swap operation, and $C \in \mathcal{I}_1$ with $|S| = |C|$, then $2(1+\epsilon) \cdot f(S) \geq f(S \cup C) + f(S \cap C)$.*

Proof.

The proof of this lemma is almost identical to the proof of the Lemma 2.2 the only difference is that left-hand sides of inequalities (2.2) and inequalities (2.4) are multiplied by $(1+\frac{\epsilon}{n^4})$. Therefore, after iteratively applying submodularity we obtain the inequality analogous to the inequality (2.6)

$$(k+1+\frac{\epsilon}{n^4}(\lambda+1)) \cdot f(S) \geq f(S \cup C) + k \cdot f(S \cap C).$$

Because $\lambda \leq (k+1)n$ and we may assume that $n^4 >> (k+1)n+1$, we obtain the statement of the lemma. ∎

Theorem 2.6. *Algorithm A in Figure 2.2 is a $(\frac{1}{(1+\epsilon)(k+2+\frac{1}{k})})$-approximation algorithm for maximizing a non-negative submodular function subject to any k matroid constraints, running in time $n^{O(k)}$.*

Proof. The parameter $\epsilon > 0$ in Procedure B is any value such that $\frac{1}{\epsilon}$ is at most a polynomial in n. Before stating the proof, we note that using approximate local operations in the local search procedure B (in Figure 2.1) makes the running time of the algorithm polynomial. The reason is as follows: one can easily show that for any ground set X of elements, the value of the initial set $S = \{v\}$ is at least $\mathsf{Opt}(X)/n$, where

$\mathsf{Opt}(X)$ is the optimal value of problem (2.1) restricted to X. Each local operation in procedure B increases the value of the function by a factor $1+\frac{\epsilon}{n^4}$. Therefore, the number of local operations for procedure B is at most $\log_{1+\frac{\epsilon}{n^4}} \frac{\mathsf{Opt}(X)}{\frac{\mathsf{Opt}(X)}{n}} = O(\frac{1}{\epsilon} n^4 \log n)$, and thus the running time of the whole procedure is $O(\frac{1}{\epsilon} n^{O(k)})$.

We now prove the performance guarantee of Algorithm A. Let C denote the optimal solution to the original problem $\max\{f(S) \ : \ S \in \cap_{j=1}^{k} \mathcal{I}_j, \ S \subseteq V\}$. Let $C_i = C \cap V_i$ for each $i \in [k+1]$; so $C_1 = C$. Observe that C_i is a feasible solution to the problem (2.7) solved in the ith iteration. Applying Lemma 2.5 to problem (2.7) using the local optimum S_i and solution C_i, we obtain:

$$(1+\epsilon)(k+1) \cdot f(S_i) \geq f(S_i \cup C_i) + k \cdot f(S_i \cap C_i) \qquad \forall 1 \leq i \leq k+1, \tag{2.8}$$

Using $f(S) \geq \max_{i=1}^{k+1} f(S_i)$, we add these $k+1$ inequalities and simplify inductively as follows.

Claim 2.7. *For any $1 \leq l \leq k+1$, we have:*

$$(1+\epsilon)(k+1)^2 \cdot f(S) \geq (l-1) \cdot f(C) + f(\cup_{p=1}^{l} S_p \cup C_1) + \sum_{i=l+1}^{k+1} f(S_i \cup C_i) + \sum_{p=1}^{l-1} (k-l+p) f(S_p \cap C_p) + k \cdot \sum_{i=l}^{k+1} f(S_i \cap C_i).$$

Proof. We argue by induction on l. The base case $l=1$ is trivial, by just considering the sum of the $(k+1)$ inequalities in statement (2.8) above. Assuming the statement for some value $1 \leq l < k+1$, we prove the corresponding statement for $l+1$.

$$\begin{aligned}
(1+\epsilon)(k+1)^2 \cdot f(S) &\geq (l-1) \cdot f(C) + f(\cup_{p=1}^{l} S_p \cup C_1) \\
&\quad + \sum_{i=l+1}^{k+1} f(S_i \cup C_i) + \sum_{p=1}^{l-1} (k-l+p) f(S_p \cap C_p) + k \cdot \sum_{i=l}^{k+1} f(S_i \cap C_i) \\
&= (l-1) \cdot f(C) + f(\cup_{p=1}^{l} S_p \cup C_1) + f(S_{l+1} \cup C_{l+1}) \\
&\quad + \sum_{i=l+2}^{k+1} f(S_i \cup C_i) + \sum_{p=1}^{l-1} (k-l+p) f(S_p \cap C_p) + k \cdot \sum_{i=l}^{k+1} f(S_i \cap C_i) \\
&\geq (l-1) \cdot f(C) + f(\cup_{p=1}^{l+1} S_p \cup C_1) + f(C_{l+1}) \\
&\quad + \sum_{i=l+2}^{k+1} f(S_i \cup C_i) + \sum_{p=1}^{l-1} (k-l+p) f(S_p \cap C_p) + k \cdot \sum_{i=l}^{k+1} f(S_i \cap C_i) \\
&= (l-1) \cdot f(C) + f(\cup_{p=1}^{l+1} S_p \cup C_1) + f(C_{l+1}) + \sum_{p=1}^{l} f(S_p \cap C_p) \\
&\quad + \sum_{i=l+2}^{k+1} f(S_i \cup C_i) + \sum_{p=1}^{l} (k-l-1+p) f(S_p \cap C_p) + k \cdot \sum_{i=l+1}^{k+1} f(S_i \cap C_i) \\
&\geq l \cdot f(C) + f(\cup_{p=1}^{l+1} S_p \cup C_1) \\
&\quad + \sum_{i=l+2}^{k+1} f(S_i \cup C_i) + \sum_{p=1}^{l} (k-l-1+p) f(S_p \cap C_p) + k \cdot \sum_{i=l+1}^{k+1} f(S_i \cap C_i).
\end{aligned}$$

The first inequality is the induction hypothesis, and the next two inequalities follow from submodularity using $\left(\cup_{p=1}^{l} S_p \cap C_p\right) \cup C_{l+1} = C$. ∎

Using the statement of Claim 2.7 when $l = k+1$, we obtain $(1+\epsilon)(k+1)^2 \cdot f(S) \geq k \cdot f(C)$ as claimed. ∎

Finally, we give an improved approximation algorithm for symmetric submodular functions for which $f(S) = f(\overline{S})$ for any subset $S \subset V$. Symmetric submodular functions have been considered widely in the literature [17, 40], and it appears that symmetry allows for better approximation results and thus deserves separate attention.

THEOREM 2.8. *There exists a* $(\frac{1}{(1+\epsilon)(k+2)})$*-approximation algorithm for maximizing non-negative symmetric submodular functions.*

Proof. The algorithm for symmetric submodular functions is simpler. In this case, we only need to perform *one* iteration of the approximate local search procedure B (as opposed to $k+1$ in Theorem 2.6). Let C denote the optimal solution, and S_1 the result of the local search (on V). Then Lemma 2.2 implies:

$$(1+\epsilon)(k+1)\cdot f(S_1) \geq f(S_1 \cup C) + k \cdot f(S_1 \cap C) \geq f(S_1 \cup C) + f(S_1 \cap C).$$

Because f is symmetric, we also have $f(S_1) = f(\overline{S_1})$. Adding these two,

$$(1+\epsilon)(k+2)\cdot f(S_1) \geq f(\overline{S_1}) + f(S_1 \cup C) + f(S_1 \cap C) \geq f(C \setminus S_1) + f(S_1 \cap C) \geq f(C)\ .$$

□

3. Knapsack constraints. In this section, we give an approximation algorithm for submodular maximization subject to multiple knapsack constraints. Let $f : 2^V \to \mathbb{R}_+$ be a submodular function. Let $w^1, \cdots, w^k$ denote k weight-vectors corresponding to k knapsacks having capacities $C_1, \cdots, C_k$ respectively. The problem we want to solve is:

$$\max\{f(S)\ :\ \sum_{j\in S} w^i_j \leq C_i,\ \forall 1 \leq i \leq k, S \subseteq V\}. \tag{3.1}$$

By scaling each knapsack, we assume that $C_i = 1$ for all $i \in [k]$. We denote $f_{max} = \max\{f(v)\ :\ v \in V\}$. We assume without loss of generality that for every $i \in V$, the singleton solution $\{i\}$ is feasible for the knapsacks (otherwise such elements can be dropped from consideration). To solve the above problem, we first define a fractional relaxation of the submodular function, and give an approximation algorithm for this fractional relaxation. Then, we show how to design an approximation algorithm for the original integral problem using the solution for the fractional relaxation. Let $F : [0,1]^n \to \mathbb{R}_+$, the *fractional relaxation of* f, be the 'extension-by-expectation' of f, i.e.,

$$F(x) = \sum_{S\subseteq V} f(S)\cdot \Pi_{i\in S} x_i \cdot \Pi_{j\notin S}(1-x_j).$$

Note that F is a multi-linear polynomial in variables $x_1, \cdots, x_n$, and has continuous derivatives of all orders. Furthermore, as shown in Vondrák [48], for all $i, j \in V$, $\frac{\partial^2}{\partial x_j \partial x_i} F \leq 0$ everywhere on $[0,1]^n$; we refer to this condition as *continuous submodularity.*

Extending function f on scaled ground sets. Let $s_i \in \mathbb{Z}_+$ be arbitrary values for each $i \in V$. Define a new ground-set U that contains s_i 'copies' of each element $i \in V$; so the total number of elements in U is $\sum_{i\in V} s_i$. We will denote any subset T of U as $T = \cup_{i\in V} T_i$ where each T_i consists of all copies of element $i \in V$ from T. Now define function $g : 2^U \to \mathbb{R}_+$ as $g(\cup_{i\in V} T_i) = F(\cdots, \frac{|T_i|}{s_i}, \cdots)$.

Our goal is to prove the useful lemma that g is submodular. In preparation for that, we first establish a couple of claims. The first claim is standard, but we give a proof for the sake of completeness.

CLAIM 3.1. *Suppose* $l : \mathcal{D} \to \mathbb{R}$ *has continuous partial derivatives everywhere on* $\mathcal{D}$ *with* $\frac{\partial l}{\partial x_i}(y) \leq 0$ *for all* $y \in \mathcal{D}$ *and* $i \in V$. *Then for any* $a_1, a_2 \in \mathcal{D}$ *with* $a_1 \leq a_2$ *coordinate-wise,* $l(a_1) \geq l(a_2)$.

Proof. Consider the line from a_1 to a_2 parameterized by $t \in [0,1]$ as $y(t) := a_1 + t(a_2 - a_1)$. Observe that all points on this line are in $\mathcal{D}$ (because $\mathcal{D}$ is a convex set). At any $t \in [0,1]$, we have:

$$\frac{\partial l(y(t))}{\partial t} = \sum_{j=1}^{n} \frac{\partial l(y(t))}{\partial x_j}\cdot\frac{\partial y_j(t)}{\partial t} = \sum_{j=1}^{n} \frac{\partial l(y(t))}{\partial x_j}\cdot(a_2(j) - a_1(j)) \leq 0.$$

Above, the first equality follows from the chain rule because l is differentiable, and the last inequality uses the fact that $a_2 - a_1 \geq 0$ coordinate-wise. This completes the proof of the claim.□

Next, we establish the following property of the fractional relaxation F.

CLAIM 3.2. *For any $a, q, d \in [0,1]^n$ with $q + d \in [0,1]^n$ and $a \le q$ coordinate-wise, we have $F(a+d) - F(a) \ge F(q+d) - F(q)$.*

Proof. Let $\mathcal{D} = \{y \in [0,1]^n \ : \ y + d \in [0,1]^n\}$. Define function $h : \mathcal{D} \to \mathbb{R}_+$ as $h(x) := F(x+d) - F(x)$, which is a multi-linear polynomial. We will show that $\frac{\partial h}{\partial x_i}(\alpha) \le 0$ for all $i \in V$, at every point $\alpha \in \mathcal{D}$. This combined with Claim 3.1 below would imply $h(a) \ge h(q)$ because $a \le q$ coordinate-wise, which gives the claim.

In the following, fix an $i \in V$ and denote $F'_i(y) = \frac{\partial F}{\partial x_i}(y)$ for any $y \in [0,1]^n$. To show $\frac{\partial h}{\partial x_i}(\alpha) \le 0$ for $\alpha \in \mathcal{D}$, it suffices to have $F'_i(\alpha+d) - F'_i(\alpha) \le 0$. From the continuous submodularity of F, for every $j \in V$ we have $\frac{\partial F'_i}{\partial x_j}(y) = \frac{\partial^2 F}{\partial x_j \partial x_i}(y) \le 0$ for all $y \in [0,1]^n$. Then applying Claim 3.1 to F'_i (a multi-linear polynomial) implies that $F'_i(\alpha + d) - F'_i(\alpha) \le 0$. This completes the proof of Claim 3.2. ∎

We are now ready to state and prove the lemma.

LEMMA 3.3. *Set function g is a submodular function on ground set U.*

Proof. To show submodularity of g, consider any two subsets $P = \cup_{i \in V} P_i$ and $Q = \cup_{i \in V} Q_i$ of U, where each P_i (resp., Q_i) are copies of element $i \in V$. We have $P \cap Q = \cup_{i \in V}(P_i \cap Q_i)$ and $P \cup Q = \cup_{i \in V}(P_i \cup Q_i)$. Define vectors $p, q, a, b \in [0,1]^n$ as follows:

$$p_i = \frac{|P_i|}{s_i}, \quad q_i = \frac{|Q_i|}{s_i}, \quad a_i = \frac{|P_i \cap Q_i|}{s_i}, \quad b_i = \frac{|P_i \cup Q_i|}{s_i} \qquad \forall i \in V.$$

It is clear that $p + q = a + b$ and $d := p - a \ge 0$. Submodularity condition on g at P, Q requires $g(P) + g(Q) \ge g(P \cap Q) + g(P \cup Q)$. But by the definition of g, this is equivalent to $F(a+d) - F(a) \ge F(q+d) - F(q)$, which is true by Claim 3.2. Thus we have established the lemma. ∎

Solving the fractional relaxation. We now argue how to obtain a near-optimal fractional feasible solution for maximizing a non-negative submodular function over k knapsack constraints. Let $w^1, \cdots, w^k$ denote the weight-vectors in each of the k knapsacks such that all knapsacks have capacity 1. The problem we consider here has additional upper bounds $\{u_i \in [0,1]\}_{i=1}^n$ on variables:

$$\max\{F(y) \ : \ w^s \cdot y \le 1 \ \ \forall s \in [k], \ \ 0 \le y_i \le u_i \ \ \forall i \in V\}. \tag{3.2}$$

Denote the region $\mathcal{U} := \{y \ : \ 0 \le y_i \le u_i \ \forall i \in V\}$. We define a local search procedure to solve problem (3.2). We only consider values for each variable from a discrete set of values in $[0,1]$, namely $\mathcal{G} = \{p \cdot \zeta \ : \ 0 \le p \le \frac{1}{\zeta}\}$ where $\zeta = \frac{1}{8n^4}$. Let $\epsilon > 0$ be a parameter to be fixed later. At any step with current solution $y \in [0,1]^n$, the following local moves are considered:

- Let $A, D \subseteq [n]$ with $|A|, |D| \le k$. Decrease the variables $y(D)$ to any values in $\mathcal{G}$ and increase variables $y(A)$ to any values in $\mathcal{G}$ such that the resulting solution y' still satisfies all knapsacks and $y' \in \mathcal{U}$. If $F(y') > (1+\epsilon) \cdot F(y)$ then set $y \leftarrow y'$.

Note that the size of each local neighborhood is $n^{O(k)}$. Let a be the index corresponding to $\max\{u_i \cdot f(\{i\}) \ : \ i \in V\}$. We start the local search procedure with the solution y_0 having $y_0(a) = u_a$ and zero otherwise. Observe that for any $x \in \mathcal{U}^n$, $F(x) \le \sum_{i=1}^n u_i \cdot f(\{i\}) \le n \cdot u_a \cdot f(\{a\}) = n \cdot F(y_0)$. Hence the number of iterations of local search is $O(\frac{1}{\epsilon} \log n)$, and the entire procedure terminates in polynomial time. Let y denote a local optimal solution. We first prove the following based on the discretization $\mathcal{G}$.

CLAIM 3.4. *Suppose $\alpha, \beta \in [0,1]^n$ are such that each has at most k positive coordinates, $y' = y - \alpha + \beta \in \mathcal{U}$, and y' satisfies all knapsacks. Then $F(y') \le (1+\epsilon) \cdot F(y) + \frac{1}{4n^2} f_{max}$.*

Proof. Let $z \le y'$ be the point in $\mathcal{U} \cap \mathcal{G}^n$ that minimizes $\sum_{i=1}^n (y'_i - z_i)$. Note that z is a feasible local move from y: it lies in $\mathcal{G}^n$, satisfies all knapsacks and the upper-bounds, and is obtainable from y by reducing k variables and increasing k others. Hence by local optimality $F(z) \le (1+\epsilon) \cdot F(y)$.

By the choice of z, it follows that $|z_i - y'_i| \le \zeta$ for all $i \in V$. Suppose B is an upper bound on all first partial derivatives of function F on $\mathcal{U}$: i.e. $\left| \frac{\partial F(x)}{\partial x_i} |_{\bar{x}} \right| \le B$ for all $i \in V$ and $\bar{x} \in \mathcal{U}$. Then because F has continuous derivatives, we obtain

$$|F(z) - F(y')| \le \sum_{i=1}^n B \cdot |z_i - y'_i| \le nB\zeta \le 2n^2 f_{max} \cdot \zeta \le \frac{f_{max}}{4n^2}.$$

Above $f_{max} = \max\{f(v) \ : \ v \in V\}$. The last inequality uses $\zeta = \frac{1}{8n^4}$, and the second to last inequality uses $B \leq 2n \cdot f_{max}$ which we show next. Consider any $\bar{x} \in [0,1]^n$ and $i \in V$. We have

$$\left|\frac{\partial F(x)}{\partial x_i}|_{\bar{x}}\right| = \left|\sum_{S\subseteq[n]\setminus\{i\}} [f(S\cup\{i\}) - f(S)] \cdot \Pi_{a\in S}\bar{x}_a \cdot \Pi_{b\in S^c\setminus i}(1-\bar{x}_b)\right|$$
$$\leq \max_{S\subseteq[n]\setminus\{i\}} [f(S\cup\{i\}) + f(S)] \ \leq \ 2n\cdot f_{max} \ .$$

Thus we have $F(y') \leq (1+\epsilon)\cdot F(y) + \frac{1}{4n^2} f_{max}$. ☐

For $x, y \in \mathbb{R}^n$, we define $x \vee y$ and $x \wedge y$ by $(x\vee y)_j := \max(x_j, y_j)$ and $(x\wedge y)_j := \min(x_j, y_j)$ for $j \in [n]$.

LEMMA 3.5. *For local optimal $y \in \mathcal{U}\cap\mathcal{G}^n$ and any $x \in \mathcal{U}$ satisfying the knapsack constraints, we have* $(2+2n\cdot\epsilon)\cdot F(y) \geq F(y\wedge x) + F(y\vee x) - \frac{1}{2n}\cdot f_{max}$.

Proof. For the sake of this proof, we assume that each knapsack $s \in [k]$ has a dummy element (which has no effect on function f) of weight 1 in knapsack s (and zero in all other knapsacks), and upper-bound of 1. So any fractional solution can be augmented to another of the same F-value, while satisfying all knapsacks at equality. We modify y and x using dummy elements so that both satisfy all knapsacks at equality: this does not change any of the values $F(y)$, $F(y\wedge x)$ and $F(y\vee x)$. Let $y' = y - (y\wedge x)$ and $x' = x - (y\wedge x)$. Note that for all $s\in[k]$, $w^s\cdot y' = w^s\cdot x'$ and let $c_s = w^s\cdot x'$. We will decompose y' and x' into an equal number of terms as $y' = \sum_t \alpha_t$ and $x' = \sum_t \beta_t$ with the additional property that $w^s\cdot\alpha_t = w^s\cdot\beta_t$ for all t.

1. Initialize $t \leftarrow 1$, $\gamma \leftarrow 1$, $x'' \leftarrow x'$, $y'' \leftarrow y'$.
2. While $\gamma > 0$, do:
   (a) Consider $LP_x := \{z \geq 0 \ : \ z\cdot w^s = c_s, \ \forall s\in[k]\}$ where the variables are restricted to indices $i\in[n]$ with $x''_i > 0$. Similarly $LP_y := \{z\geq 0 \ : \ z\cdot w^s = c_s, \ \forall s\in[k]\}$ where the variables are restricted to indices $i\in[n]$ with $y''_i > 0$. Let $u \in LP_x$ and $v\in LP_y$ be extreme points.
   (b) Set $\delta_1 = \max\{\chi \ : \ \chi u \leq x''\}$ $\delta_2 = \max\{\chi \ : \ \chi v\leq y''\}$, and $\delta = \min\{\delta_1, \delta_2\}$.
   (c) Set $\beta_t \leftarrow \delta\cdot u$, $\alpha_t \leftarrow \delta\cdot v$, $\gamma\leftarrow\gamma-\delta$, $x''\leftarrow x''-\beta_t$, and $y''\leftarrow y''-\alpha_t$.
   (d) Set $t\leftarrow t+1$.

We first show that this procedure is well-defined. In every iteration, $\gamma > 0$, and by induction $w^s\cdot x'' = w^s\cdot y'' = \gamma\cdot c_s$ for all $s\in[k]$. Thus in step 2a, LP_x (resp. LP_y) is non-empty: x''/γ (resp. y''/γ) is a feasible solution. From the definition of LP_x and LP_y it also follows that $\delta > 0$ in step 2b and at least one coordinate of x'' or y'' is zeroed out in step 2c. This implies that the decomposition procedure terminates in $r \leq 2n$ steps. At the end of the procedure, we have decompositions $x' = \sum_{t=1}^r \beta_t$ and $y' = \sum_{t=1}^r\alpha_t$. Furthermore, each α_t (resp. β_t) corresponds to an *extreme point* of LP_y (resp. LP_x) in some iteration: hence the number of positive components in any of $\{\alpha_t,\beta_t\}_{t=1}^r$ is at most k. Finally note that for all $t\in[r]$, $w^s\cdot\alpha_t = w^s\cdot\beta_t$ for all knapsacks $s\in[k]$.

Observe that Claim 3.4 applies to y, α_t and β_t (any $t\in[r]$) because each of α_t,β_t has support-size k, and $y-\alpha_t+\beta_t\in\mathcal{U}$ and satisfies all knapsacks with equality. Strictly speaking, Claim 3.4 requires the original local optimal $\tilde{y}$, which is not augmented with dummy elements. However even $\tilde{y}-\alpha_t+\beta_t\in\mathcal{U}$ and satisfies all knapsacks (possibly not at equality), and the claim does apply. This gives:

$$F(y-\alpha_t+\beta_t) \leq (1+\epsilon)\cdot F(y) + \frac{f_{max}}{4n^2} \qquad \forall t\in[r] \ . \tag{3.3}$$

Let $M\in\mathbb{Z}_+$ be large enough so that $M\alpha_t$ and $M\beta_t$ are integral for all $t\in[r]$. In the rest of the proof, we consider a scaled ground-set U containing M copies of each element in V. We define function $g: 2^U\to\mathbb{R}_+$ as $g(\cup_{i\in V}T_i) = F(\cdots,\frac{|T_i|}{M},\cdots)$ where each T_i consists of copies of element $i\in V$. Lemma 3.3 implies that g is submodular. Corresponding to y we have a set $P = \cup_{i\in V}P_i$ consisting of the first $|P_i| = M\cdot y_i$ copies of each element $i\in V$. Similarly, x corresponds to set $Q = \cup_{i\in V}Q_i$ consisting of the first $|Q_i| = M\cdot x_i$ copies of each element $i\in V$. Hence $P\cap Q$ (resp. $P\cup Q$) corresponds to $x\wedge y$ (resp. $x\vee y$) scaled by M. Again, $P\setminus Q$ (resp. $Q\setminus P$) corresponds to y' (resp. x') scaled by M. The decomposition of y' from above suggests *disjoint* sets $\{A_t\}_{t=1}^r$ such that $\cup_t A_t = P\setminus Q$; i.e. each A_t corresponds to scaled α_t. Similarly there are *disjoint* sets $\{B_t\}_{t=1}^r$ such that $\cup_t B_t = Q\setminus P$. Observe also that $g((P\setminus A_t)\cup B_t) = F(y-\alpha_t+\beta_t)$, so (3.3) corresponds to:

$$g((P \setminus A_t) \cup B_t) \le (1+\epsilon) \cdot g(P) + \frac{f_{max}}{4n^2} \qquad \forall t \in [r] \ . \tag{3.4}$$

Adding all these r inequalities to $g(P) = g(P)$, we obtain $(r + \epsilon \cdot r + 1)g(P) + \frac{r}{4n^2} f_{max} \ge g(P) + \sum_{t=1}^r g((P \setminus A_t) \cup B_t)$. Using submodularity of g and the disjointness of families $\{A_t\}_{t=1}^r$ and $\{B_t\}_{t=1}^r$, we obtain $(r + \epsilon \cdot r + 1) \cdot g(P) + \frac{r}{4n^2} f_{max} \ge (r-1) \cdot g(P) + g(P \cup Q) + g(P \cap Q)$. Hence $(2 + \epsilon \cdot r) \cdot g(P) \ge g(P \cup Q) + g(P \cap Q) - \frac{r}{4n^2} f_{max}$. This implies the lemma because $r \le 2n$. ∎

THEOREM 3.6. *There exists a* $\frac{1}{4} - \delta$*-approximation algorithm for problem* (3.2) *with all upper bounds equal to* 1*, for any constant* $\delta > 0$*.*

Proof. Because each singleton solution $\{i\}$ is feasible for the knapsacks and upper bounds are 1, we have a feasible solution of value f_{max}. Choose $\epsilon = \frac{\delta}{n^2}$. The algorithm runs the fractional local search algorithm (with all upper bounds 1) to get locally optimal solution $y_1 \in [0,1]^n$. Then we run another fractional local search, this time with each variable $i \in V$ having upper bound $u_i = 1 - y_1(i)$; let y_2 denote the local optimum obtained here. The algorithm outputs the better of the solutions y_1, y_2, and value f_{max}.

Let x denote the globally optimal fractional solution to (3.2), where upper bounds are 1. We will show $(2+\delta) \cdot (F(y_1) + F(y_2)) \ge F(x) - f_{max}/n$, which would prove the theorem. Observe that $x' = x - (x \wedge y_1)$ is a feasible solution to the second local search. Lemma 3.5 implies the following for the two local optima:

$$(2+\delta) \cdot F(y_1) \ge F(x \wedge y_1) + F(x \vee y_1) - \frac{f_{max}}{2n} \ ,$$
$$(2+\delta) \cdot F(y_2) \ge F(x' \wedge y_2) + F(x' \vee y_2) - \frac{f_{max}}{2n} \ .$$

We show that $F(x \wedge y_1) + F(x \vee y_1) + F(x' \vee y_2) \ge F(x)$, which suffices to prove the theorem. For this inequality, we again consider a scaled ground-set U having M copies of each element in V (where $M \in \mathbb{Z}_+$ is large enough so that Mx, My_1, My_2 are all integral). Define function $g : 2^U \to \mathbb{R}_+$ as $g(\cup_{i \in V} T_i) = F(\cdots, \frac{|T_i|}{M}, \cdots)$ where each T_i consists of copies of element $i \in V$. Lemma 3.3 implies that g is submodular. Also define the following subsets of U: A (representing y_1) consists of the first $My_1(i)$ copies of each element $i \in V$, C (representing x) consists of the first $Mx(i)$ copies of each element $i \in V$, and B (representing y_2) consists of $My_2(i)$ copies of each element $i \in V$ (namely the copies numbered $My_1(i) + 1$ through $My_1(i) + My_2(i)$) so that $A \cap B = \phi$. Note that we can indeed pick such sets because $y_1 + y_2 \le 1$ coordinate-wise. Also we have the following correspondences via scaling:

$$A \cap C \equiv x \wedge y_1, \quad A \cup C \equiv x \vee y_1, \quad (C \setminus A) \cup B \equiv x' \vee y_2 \ .$$

Thus it suffices to show $g(A \cap C) + g(A \cup C) + g((C \setminus A) \cup B) \ge g(C)$. But this follows from submodularity and non-negativity of g:

$$g(A \cap C) + g(A \cup C) + g((C \setminus A) \cup B) \ge g(A \cap C) + g(C \setminus A) + g(C \cup A \cup B) \ge g(C).$$

Hence we have the desired approximation for the fractional problem (3.2). ∎

3.1. Rounding the fractional solution. Fix a constant $\eta > 0$ and let $c = \frac{16}{\eta}$. We give a $(\frac{1}{5} - \eta)$-approximation for submodular maximization over k knapsack constraints, which is problem (3.1). Define parameter $\delta = \frac{1}{4c^3k^4}$. We call an element $e \in V$ *heavy* if $w^i(e) \ge \delta$ for some knapsack $i \in [k]$. All other elements are called *light*. Let H and L denote the heavy and light elements in an optimal integral solution. Note that $|H| \le k/\delta$. Hence enumerating over all possible sets of heavy elements, we can obtain profit at least $f(H)$ in $n^{O(k/\delta)}$ time, which is polynomial for fixed k. We now focus only on light elements and show how to obtain profit at least $\frac{1}{4} \cdot f(L)$. Later we show how these can be combined into an approximation algorithm for problem (3.1). Let $\mathsf{Opt} \ge f(L)$ denote the optimal value of the knapsack constrained problem, restricted to only light elements.

Algorithm for light elements. Restricted to light elements, the algorithm first solves the fractional relaxation (3.2) with all upper bounds 1, to obtain solution x with $F(x) \ge (\frac{1}{4} - \frac{\eta}{2}) \cdot \mathsf{Opt}$, as described in the previous subsection (see Theorem 3.6). Again by adding dummy light elements for each knapsack, we assume that fractional solution x satisfies all knapsacks with equality. Fix a parameter $\epsilon = \frac{1}{ck}$, and pick each element e into solution S independently with probability $(1-\epsilon) \cdot x_e$. We declare failure if S violates any knapsack and claim zero profit in this case (output the empty set as solution). Clearly this algorithm always outputs a feasible solution. In the following we lower bound the expected profit. Let $\alpha(S) := \max\{w^i(S) \ : \ i \in [k]\}$.

CLAIM 3.7. *For any $a \geq 1$, $Pr[\alpha(S) \geq a] \leq k \cdot e^{-cak^2}$.*

Proof. Fixing a knapsack $i \in [k]$, we will bound $Pr[w^i(S) \geq a]$. Let X_e denote the binary random variable which is set to 1 iff $e \in S$, and let $Y_e = \frac{w^i(e)}{\delta} X_e$. Because we only deal with light elements, each Y_e is a $[0,1]$ random variable. Let $Z_i := \sum_e Y_e$, then $E[Z_i] = \frac{1-\epsilon}{\delta}$. By scaling, it suffices to upper bound $Pr\left[Z_i \geq a(1+\epsilon)E[Z_i]\right]$. Because the Y_e are independent $[0,1]$ random variables, Chernoff bounds [36] imply:

$$Pr\left[Z_i \geq a(1+\epsilon)E[Z_i]\right] \leq e^{-E[Z_i]\cdot a\epsilon^2/2} \leq e^{-a\epsilon^2/4\delta} = e^{-cak^2}.$$

Finally by a union bound, we obtain $Pr[\alpha(S) \geq a] \leq \sum_{i=1}^{k} Pr[w^i(S) \geq a] \leq k \cdot e^{-cak^2}$. ∎

CLAIM 3.8. *For any $a \geq 1$, $\max\{f(S) \ : \ \alpha(S) \leq a+1\} \leq 2(1+\delta)k(a+1) \cdot \mathsf{Opt}$.*

Proof. We will show that for any set S with $\alpha(S) \leq a+1$, $f(S) \leq 2(1+\delta)k(a+1) \cdot \mathsf{Opt}$, which implies the claim. Consider partitioning set S into a number of smaller parts each of which satisfies all knapsacks as follows. As long as there are remaining elements in S, form a group by greedily adding S-elements until no more addition is possible, then continue to form the next group. Except for the last group formed, every other group must have filled up some knapsack to extent $1-\delta$ (otherwise another light element can be added). Thus the number of groups partitioning S is at most $\frac{k(a+1)}{1-\delta} + 1 \leq 2k(a+1)(1+\delta)$. Because each of these groups is a feasible solution, the claim follows by the subadditivity of the function f. ∎

LEMMA 3.9. *The algorithm for light elements obtains expected value at least $(\frac{1}{4} - \eta) \cdot \mathsf{Opt}$.*

Proof. Define the following disjoint events: $A_0 := \{\alpha(S) \leq 1\}$, and $A_l := \{l < \alpha(S) \leq 1 + l\}$ for any $l \in \mathbb{N}$. Note that the expected value of the algorithm is $\mathsf{ALG} = E[f(S) \mid A_0] \cdot Pr[A_0]$. We can write:

$$F(x) = E[f(S)] = E[f(S) \mid A_0] \cdot Pr[A_0] + \sum_{l \geq 1} E[f(S) \mid A_l] \cdot Pr[A_l] = \mathsf{ALG} + \sum_{l \geq 1} E[f(S) \mid A_l] \cdot Pr[A_l].$$

For any $l \geq 1$, from Claim 3.7 we have $Pr[A_l] \leq Pr[\alpha(S) > l] \leq k \cdot e^{-clk^2}$. From Claim 3.8 we have $E[f(S) \mid A_l] \leq 2(1+\delta)k(l+1) \cdot \mathsf{Opt}$. So,

$$E[f(S) \mid A_l] \cdot Pr[A_l] \leq k \cdot e^{-clk^2} \cdot 2(1+\delta)k(l+1) \cdot \mathsf{Opt} \leq 8 \cdot \mathsf{Opt} \cdot lk^2 \cdot e^{-clk^2}.$$

Consider the expression $\sum_{l \geq 1} lk^2 \cdot e^{-clk^2} \leq \sum_{t \geq 1} t \cdot e^{-ct} \leq \frac{1}{c}$, for large enough constant c. Thus:

$$\mathsf{ALG} = F(x) - \sum_{l \geq 1} E[f(S) \mid A_l] \cdot Pr[A_l] \geq F(x) - 8 \cdot \mathsf{Opt} \sum_{l \geq 1} lk \cdot e^{-clk} \geq F(x) - \frac{8}{c}\mathsf{Opt}.$$

Because $\eta = \frac{16}{c}$ and $F(x) \geq (\frac{1}{4} - \frac{\eta}{2}) \cdot \mathsf{Opt}$ from Theorem 3.6, we obtain the lemma. ∎

THEOREM 3.10. *For any constant $\eta > 0$, there is a $(\frac{1}{5} - \eta)$-approximation algorithm for maximizing a non-negative submodular function over k knapsack constraints.*

Proof. As mentioned in the beginning of this subsection, let H and L denote the heavy and light elements in an optimal integral solution. The enumeration algorithm for heavy elements produces a solution of value at least $f(H)$. Lemma 3.9 implies that the rounding algorithm for light elements produces a solution of expected value at least $(\frac{1}{4} - \eta) \cdot f(L)$. By subadditivity, the optimal value $f(H \cup L) \leq f(H) + f(L)$. The better of the two solutions (over heavy and light elements respectively) found by our algorithm has value:

$$\max\{f(H), (\frac{1}{4} - \eta) \cdot f(L)\} \geq \frac{1}{5} \cdot f(H) + \frac{4}{5} \cdot (\frac{1}{4} - \eta) \cdot f(L) \geq (\frac{1}{5} - \eta) \cdot f(H \cup L).$$

This implies the desired approximation guarantee. ∎

4. Improved Bounds under Partition Matroids. In this section, we consider a special case of maximizing a submodular function over $k \geq 2$ *partition* matroids. In this case, we obtain an algorithm with a better approximation ratio: $\frac{1}{k+1+\frac{1}{k-1}+\epsilon}$ for general submodular functions, and $\frac{1}{k+\epsilon}$ for monotone submodular functions ($\epsilon > 0$ is any fixed constant).

The algorithm is again based on local search. In the *exchange* local move of the general case (Section 2), the algorithm only attempts to include one new element at a time (while dropping upto k elements). Here we generalize that step to allow including p new elements while dropping up to $(k-1)\cdot p$ elements, for some fixed constant $p \geq 1$. We show that this yields an improvement under partition matroid constraints. Given a current solution $S \in \cap_{j=1}^{k} \mathcal{I}_j$, the local moves we consider are:

- **Delete operation.** If $e \in S$ such that $f(S \setminus \{e\}) > f(S)$, then $S \leftarrow S \setminus \{e\}$.
- **Exchange operation.** For some $q \leq p$, if $d_1, \cdots, d_q \in V \setminus S$ and $e_i \in S \cup \{\phi\}$ (for $1 \leq i \leq (k-1)\cdot q$) are such that: **(i)** $S' = (S \setminus \{e_i \;:\; 1 \leq i \leq (k-1)q\}) \cup \{d_1, \cdots, d_q\} \in \mathcal{I}_j$ for all $j \in [k]$, and **(ii)** $f(S') > f(S)$, then $S \leftarrow S'$.

We prove the following strengthening of Lemma 2.2.

Lemma 4.1. *For a local optimal solution S and any $C \in \cap_{j=1}^{k} \mathcal{I}_j$, we have $k \cdot f(S) \geq (1 - \frac{1}{p}) \cdot f(S \cup C) + (k-1) \cdot f(S \cap C)$.*

Proof. We use an exchange property (see Schrijver [44]), which implies for any *partition matroid* $\mathcal{M}$ and $C, S \in \mathcal{I}(\mathcal{M})$ the existence of a map $\pi : C \setminus S \to (S \setminus C) \cup \{\phi\}$ such that

1. $(S \setminus \{\pi(b) \;:\; b \in T\}) \cup T \in \mathcal{I}(\mathcal{M})$ for all $T \subseteq C \setminus S$.
2. $|\pi^{-1}(e)| \leq 1$ for all $e \in S \setminus C$.

Let π_j denote the mapping under partition matroid $\mathcal{M}_j$ (for $1 \leq j \leq k$).

Combining partition matroids $\mathcal{M}_1$ and $\mathcal{M}_2$. We use π_1 and π_2 to construct a multigraph G on vertex set $C \setminus S$ and edge-set labeled by $E = \pi_1(C \setminus S) \cap \pi_2(C \setminus S) \subseteq S \setminus C$ with an edge labeled $a \in E$ between $e, f \in C \setminus S$ iff $\pi_1(e) = \pi_2(f) = a$ or $\pi_2(e) = \pi_1(f) = a$. Each edge in G has a unique label because there is exactly one edge (e, f) corresponding to any $a \in E$. Note that the maximum degree in G is at most 2. Hence G is a union of disjoint cycles and paths. We index elements of $C \setminus S$ in such a way that elements along any path or cycle in G are consecutive. For any $q \in \{0, \cdots, p-1\}$, let R_q denote the elements of $C \setminus S$ having an index that is *not* q modulo p. It is clear that the induced graph $G[R_q]$ for any $q \in [p]$ consists of disjoint paths/cycles, each of length at most p. Furthermore each element of $C \setminus S$ appears in exactly $p-1$ sets among $\{R_q\}_{q=0}^{p-1}$.

Claim 4.2. *For any $q \in \{0, \cdots, p-1\}$, $k \cdot f(S) \geq f(S \cup R_q) + (k-1) \cdot f(S \cap C)$.*

Proof. The following arguments hold for any $q \in [p]$, and for notational simplicity we denote $R = R_q \subseteq C \setminus S$. Let $\{D_l\}_{l=1}^{t}$ denote the vertices in connected components of $G[R]$, which form a partition of R. As mentioned above, $|D_l| \leq p$ for all $l \in [t]$. For any $l \in [t]$, let E_l denote the labels of edges in G incident to vertices D_l. Because $\{D_l\}_{l=1}^{t}$ are distinct connected components in $G[R]$, $\{E_l\}_{l=1}^{t}$ are disjoint subsets of $E \subseteq S \setminus C$. Consider any $l \in [t]$: we claim $S_l = (S \setminus E_l) \cup D_l \in \mathcal{I}_1 \cap \mathcal{I}_2$. Note that $E_l \supseteq \{\pi_1(b) \;:\; b \in D_l\}$ and $E_l \supseteq \{\pi_2(b) \;:\; b \in D_l\}$. Hence $S_l \subseteq (S \setminus \{\pi_i(b) \;:\; b \in D_l\}) \cup D_l$ for $i = 1, 2$. But from the property of mapping π_i (where $i = 1, 2$), $(S \setminus \{\pi_i(D_l)) \cup D_l \in \mathcal{I}_i$. This proves that $S_l \in \mathcal{I}_1 \cap \mathcal{I}_2$ for all $l \in [t]$.

From the properties of the maps π_j for each partition matroid $\mathcal{M}_j$, we have $(S \setminus \pi_j(D_l)) \cup D_l \in \mathcal{I}_j$ for each $3 \leq j \leq k$. Thus the following sets are independent in all matroids $\mathcal{M}_1, \cdots, \mathcal{M}_k$:

$$\big(S \setminus (\cup_{j=3}^{k} \pi_j(D_l) \cup E_l)\big) \cup D_l \qquad \forall\, l \in [t].$$

Additionally, because $|D_l| \leq p$ and $|(\cup_{j=3}^{k} \pi_j(D_l) \cup E_l| \leq (k-1) \cdot p$, each of the above sets are in the local neighborhood of S. But local optimality of S implies:

$$f(S) \geq f\big(\big(S \setminus (\cup_{j=3}^{k} \pi_j(D_l) \cup E_l)\big) \cup D_l\big) \qquad \forall l \in [t].$$

Recall that $\{E_l\}$ are disjoint subsets of $S \setminus C$. Also each element of $S \setminus C$ is missing in the right-hand side of at most $k-1$ terms (the π_js are 'matchings' onto $S \setminus C$). Note that elements of $S \cap C$ are present in all of them. Adding these t inequalities to $f(S) = f(S)$ and successively combining terms $f(X), f(Y)$ using $f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y)$, we have

$$(t+1) \cdot f(S) \geq f(S) + \sum_{l=1}^{t} f\big(\big(S \setminus (\cup_{j=3}^{k} \pi_j(D_l) \cup E_l)\big) \cup D_l\big) \geq f(S \cup R) + \sum_{l=1}^{t} f(S \setminus A_l). \tag{4.1}$$

where $\{A_l \subseteq S \setminus C\}_{l=1}^t$ are such that each element $a \in S \setminus C$ appears in $n_a \leq k-1$ sets from $\{A_l\}_{l=1}^t$ (using Fact 1). Furthermore, elements of $S \cap C$ do not appear in any of $\{A_l\}_{l=1}^t$. The second inequality also uses the fact that $\cup_{l=1}^t D_l = R$. Using local optimality under deletions, we have the inequalities:

$$(k-1-n_a) \cdot f(S) \geq f(S \setminus \{a\}) \qquad \forall a \in S \setminus C. \tag{4.2}$$

Letting $\lambda = t + \sum_{a \in S \setminus C}(k-1-n_a)$, adding inequalities from (4.1) and (4.2), we obtain

$$(\lambda+1) \cdot f(S) \geq f(S \cup R) + \sum_{i=1}^{\lambda} f(S \setminus A_i).$$

where $\{A_i\}_{i=1}^{\lambda}$ are such that each element of $S \setminus C$ appears in exactly $k-1$ of them (and elements of $S \cap C$ appear in none of them). An identical simplification procedure as Claim 2.3 of Lemma 2.2 gives the following which implies the claim.

$$(\lambda+1) \cdot f(S) \geq f(S \cup R) + (\lambda - k + 1) \cdot f(S) + (k-1) \cdot f(S \cap C).$$

□

Adding the p inequalities given by Claim 4.2, we get $pk \cdot f(S) \geq \sum_{q=0}^{p-1} f(S \cup R_q) + p(k-1) \cdot f(S \cap C)$. Note that each element of $C \setminus S$ is missing in exactly 1 set $\{S \cup R_q\}_{q=0}^{p-1}$. Hence a simplification procedure as in Claim 2.3 gives $\sum_{q=0}^{p-1} f(S \cup R_q) \geq (p-1) \cdot f(S \cup C) + f(S)$. Thus,

$$(pk-1) \cdot f(S) \geq (p-1) \cdot f(S \cup C) + p(k-1) \cdot f(S \cap C),$$

which implies $k \cdot f(S) \geq (1 - \frac{1}{p}) \cdot f(S \cup C) + (k-1) \cdot f(S \cap C)$. This completes the proof of the lemma. □

THEOREM 4.3. *For any $k \geq 2$ and fixed constant $\epsilon > 0$, there exists a $(\frac{1}{k+1+\frac{1}{k-1}+\epsilon})$-approximation algorithm for maximizing a non-negative submodular function over k partition matroids. This bound improves to $\frac{1}{k+\epsilon}$ for monotone submodular functions.*

Proof. We set $p = 1 + \lceil \frac{2k}{\epsilon} \rceil$. The algorithm for the monotone case is just the local search procedure with p-exchanges. Lemma 4.1 applied to local optimal S and the global optimal C implies $f(S) \geq (\frac{1}{k} - \frac{1}{pk}) \cdot f(S \cup C) \geq (\frac{1}{k} - \frac{1}{pk}) \cdot f(C)$ (by non-negativity and monotonicity). From the setting of p, S is a $k+\epsilon$ approximate solution.

For the non-monotone case, the algorithm repeats the p-exchange local search k times as in Theorem 2.6. If C denotes a global optimum, an identical analysis yields $(1 + \frac{1}{p-1})k^2 \cdot f(S) \geq (k-1) \cdot f(C)$. This uses the inequalities

$$(\frac{p}{p-1})k \cdot f(S_i) \geq f(S_i \cup C_i) + (k-1) \cdot f(S_i \cap C_i) \qquad \forall 1 \leq i \leq k,$$

where S_i denotes the local optimal solution in iteration $i \in \{1, \cdots, k\}$ and $C_i = C \setminus \cup_{j=1}^{i-1} S_j$. Using the value of p, S is a $k + 1 + \frac{1}{k-1} + \epsilon$ approximate solution. Observe that the algorithm has running time $n^{O(k/\epsilon)}$. □

We note that the result for monotone submodular functions is the first improvement over the greedy $\frac{1}{k+1}$-approximation algorithm [38], even for the special case of partition matroids. It is easy to see that the greedy algorithm is a $\frac{1}{k}$-approximation for *modular* functions. But it is only a $\frac{1}{k+1}$-approximation for monotone submodular functions. The following example shows that this bound is tight for every $k \geq 1$. The submodular function f is the coverage function defined on a family $\mathcal{F}$ of sets. Consider a ground set $E = \{e : 0 \leq e \leq p(k+1)+1\}$ of natural numbers (for $p \geq 2$ arbitrarily large); we define a family $\mathcal{F} = \{S_i : 0 \leq i \leq k\} \cup \{T_1, T_2\}$ of $k+3$ subsets of E. We have $S_0 = \{e : 0 \leq e \leq p\}$, $T_1 = \{e : 0 \leq e \leq p-1\}$, $T_2 = \{p\}$, and for each $1 \leq i \leq k$, $S_i = \{e : p \cdot i + 1 \leq e \leq p \cdot (i+1)\}$. For any subset $S \subseteq \mathcal{F}$, $f(S)$ equals the number of elements in E covered by S; f is clearly monotone submodular. We now define k partition matroids over $\mathcal{F}$: for $1 \leq j \leq k$, the j^{th} partition has $\{S_0, S_j\}$ in one group and all other sets in singleton groups. In other words, the partition constraints require that for every $1 \leq j \leq k$, at most one of S_0 and S_j be chosen. Observe that $\{S_i : 1 \leq i \leq k\} \cup \{T_1, T_2\}$ is a feasible solution of value $|E| = p(k+1)+1$. However the greedy algorithm picks S_0 first (because it has maximum size), and gets only value $p+1$.

5. Matroid Base Constraints. A base in a matroid is any maximal independent set. In this section, we consider the problem of maximizing a non-negative submodular function over *bases* of some matroid $\mathcal{M}$.

$$\max\{f(S) \ : \ S \in \mathcal{B}(\mathcal{M})\}. \tag{5.1}$$

We first consider the case of symmetric submodular functions.

THEOREM 5.1. *There is a* $(\frac{1+\epsilon}{3})$*-approximation algorithm for maximizing a non-negative symmetric submodular function over bases of any matroid.*

Proof. We use the natural local search algorithm based only on swap operations. The algorithm starts with any maximal independent set and performs improving *swaps* until none is possible. From the second statement of Lemma 2.2, if S is a local optimum and C is the optimal base, we have $2 \cdot f(S) \geq f(S \cup C) + f(S \cap C)$. Adding to this inequality, the fact $f(S) = f(\overline{S})$ using symmetry, we obtain $3 \cdot f(S) \geq f(S \cup C) + f(\overline{S}) + f(S \cap C) \geq f(C \setminus S) + f(S \cap C) \geq f(C)$. Using an approximate local search procedure to make the running time polynomial, we obtain the theorem. ☐

However, the approximation guarantee of this algorithm can be arbitrarily bad if the function f is not symmetric. An example is the directed-cut function in a digraph with a vertex bipartition (U, V) with $|U| = |V| = n$, having $t \gg 1$ edges from each U-vertex to V and 1 edge from each V-vertex to U. The matroid in this example is just the uniform matroid with rank n. It is clear that the optimal base is U; on the other hand V is a local optimum under swaps.

We are not aware of a constant approximation for the problem of maximizing a submodular function subject to an arbitrary matroid base constraint. For a special class of matroids we obtain the following.

THEOREM 5.2. *There is a* $(\frac{1+\epsilon}{6})$*-approximation algorithm for maximizing any non-negative submodular function over bases of matroid* $\mathcal{M}$*, when* $\mathcal{M}$ *contains at least two disjoint bases.*

Proof. Let C denote the optimal base. The algorithm here first runs the local search algorithm using only swaps to obtain a base S_1 that satisfies $2 \cdot f(S_1) \geq f(S_1 \cup C) + f(S_1 \cap C)$, from Lemma 2.2. Then the algorithm runs a local search on $V \setminus S_1$ using both exchanges and deletions to obtain an independent set $S_2 \subseteq V \setminus S_1$ satisfying $2 \cdot f(S_2) \geq f(S_2 \cup (C \setminus S_1)) + f(S_2 \cap (C \setminus S_1))$. Consider the matroid $\mathcal{M}'$ obtained by contracting S_2 in $\mathcal{M}$. Our assumption implies that $\mathcal{M}'$ also has two disjoint bases, say B_1 and B_2 (which can also be computed in polynomial time). Note that $S_2 \cup B_1$ and $S_2 \cup B_2$ are bases in the original matroid $\mathcal{M}$. The algorithm outputs solution S which is the better of the three bases: S_1, $S_2 \cup B_1$ and $S_2 \cup B_2$. We have

$$\begin{aligned} 6f(S) &\geq 2f(S_1) + 2\left(f(S_2 \cup B_1) + f(S_2 \cup B_2)\right) \;\geq\; 2f(S_1) + 2f(S_2) \\ &\geq f(S_1 \cup C) + f(S_1 \cap C) + f(S_2 \cup (C \setminus S_1)) \;\geq\; f(C). \end{aligned}$$

The second inequality uses the disjointness of B_1 and B_2.☐

A consequence of this result is the following.

COROLLARY 5.3. *Given any non-negative submodular function* $f : 2^V \to \mathbb{R}_+$ *and an integer* $0 \leq c \leq |V|$*, there is a* $(\frac{1+\epsilon}{6})$*-approximation algorithm for the problem* $\max\{f(S) \ : \ S \subseteq V, \ |S| = c\}$.

Proof. If $c \leq |V|/2$ then the assumption in Theorem (5.2) holds for the rank c uniform matroid, and the theorem follows. We show that $c \leq |V|/2$ can be ensured without loss of generality. Define function $g : 2^V \to \mathbb{R}_+$ as $g(T) = f(V \setminus T)$ for all $T \subseteq V$. Because f is non-negative and submodular, so is g. Furthermore, $\max\{f(S) \ : \ S \subseteq V, \ |S| = c\} = \max\{g(T) \ : \ T \subseteq V, \ |T| = |V| - c\}$. Clearly one of c and $|V| - c$ is at most $|V|/2$, and we can apply Theorem 5.2 to the corresponding problem. ☐

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