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# The Rotational Dimension of a Graph

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**Abstract**

Given a connected graph $G = (N, E)$ with node weights $s \in \mathbb{R}^N_+$ and nonnegative edge lengths, we study the following embedding problem related to an eigenvalue optimization problem over the second smallest eigenvalue of the (scaled) Laplacian of $G$: Find $v_i \in \mathbb{R}^{|N|}$, $i \in N$ so that distances between adjacent nodes do not exceed prescribed edge lengths, the weighted barycenter of all points is at the origin, and $\sum_{i\in N} s_i \|v_i\|^2$ is maximized. In the case of a two dimensional optimal solution this corresponds to the equilibrium position of a quickly rotating net consisting of weighted mass points that are linked by massless cables of given lengths. We define the rotational dimension of $G$ to be the minimal dimension $k$ so that for all choices of lengths and weights an optimal solution can be found in $\mathbb{R}^k$ and show that this is a minor monotone graph parameter. We give forbidden minor characterizations up to rotational dimension 2 and prove that the rotational dimension is always bounded above by the tree-width of $G$ plus one.
**Keywords:** spectral graph theory, semidefinite programming, eigenvalue optimization, embedding, graph partitioning, tree-width
**MSC 2000:** 05C50; 90C22, 90C35, 05C10, 05C78

## 1 Introduction and Main Results

Given an undirected simple graph $G = (N, E)$ on node set $N = \{1, \ldots, n\}$ and edge set $E \subseteq \{\{i, j\} : i, j \in N, i \neq j\}$ with edge weights $w_e \in \mathbb{R}_+$, $e \in E$, the *weighted Laplacian of* $G$ is the matrix $L_w(G) := \sum_{e\in E} w_e E_e$, where $E_{\{i,j\}}$ ($i, j \in N$ with $i \neq j$) is a real symmetric $N \times N$ matrix having value 1 in diagonal elements $(i, i)$ and $(j, j)$, value $-1$ in offdiagonal elements $(i, j)$ and $(j, i)$ and value 0 otherwise. For brevity, we will write $ij$ instead of $\{i, j\}$ whenever there is no danger of confusion. For each $ij \in E$ the matrices $E_{ij}$ are positive semidefinite ($E_{ij} \succeq 0$) with smallest eigenvalue 0 and they possess an associated eigenvector of all ones, so these properties also hold for $L_w(G)$. Spectral properties of the Laplacian and their connections to structural properties of the graph are a prominent research topic in spectral graph theory [4, 9, 10, 2]. Our starting point is the *absolute algebraic connectivity*

$$\hat{a}(G) := \max\{\lambda_2(L_w) : w \in \mathbb{R}^E_+, \sum_{e\in E} w_e = |E|\}, \tag{1}$$

introduced by Fiedler [6, 7] who motivated the name by exhibiting several connections between $\hat{a}(G)$ and the node and edge connectivity of the graph, the most direct being that $\hat{a}(G) > 0$ if and only if $G$ is connected. Interest in fastest mixing Markov chains and graph conductivity led Boyd, Diaconis and Xiao [11] to investigate the same object (up to a trivial scaling). There and in [8] it was observed for connected $G$, that via semidefinite duality $\hat{a}(G)$ may also be expressed as the

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embedding problem

$$
\begin{array}{rll}
\dfrac{|E|}{\hat{a}(G)} = & \text{maximize} & \displaystyle\sum_{i\in N} \|v_i\|^2 \\
& \text{subject to} & \displaystyle\sum_{i\in N} v_i = 0, \\
& & \|v_i - v_j\| \le 1 \qquad \text{for } ij \in E, \\
& & v_i \in \mathbb{R}^n \text{ for } i \in N.
\end{array} \tag{2}
$$

It asks for an embedding of the nodes of the graph in $n$-space so that their barycenter is at the origin (we will call this the *equilibrium constraint*), the distances of adjacent nodes are bounded by one (the *distance constraints*), and the sum of their squared norms is maximized. For optimal solutions whose points $v_i$ span a two dimensional subspace, a direct physical analog exists. For this, view the graph as a net consisting of nodes of mass 1 connected by massless cables of length 1 (the edges) that is being spun around its barycenter at high speed. The optimal $v_i$ then give the relative positions of the nodes within their rotational equilibrium and the (scaled) optimal weights $w_{ij}$ of (1) describe the stress or forces acting along the cables. This interpretation also provides a link to rigidity theory, see, e.g., [3].

A central question is the existence of low dimensional optimal embeddings of (2) in dependence of structural properties of $G$ (high dimensional optimal embeddings may even exist for very simple structures like stars and are therefore less interesting). It was proven in [8] that for connected graphs $G$ there always exist optimal solutions of (2) having dimension at most tree-width of $G$ plus one (see Section 4 for the definition of tree-width). Intuitively, it is clear that the complexity of the structure of the optimal embedding mainly depends on some kind of central separator enclosing the barycenter in the optimal embedding. In consequence, even graphs with highly complex separator structures are likely to have optimal embeddings of very small dimension, if these structures only appear on the periphery. In this paper we therefore allow to shift the barycenter to the more interesting parts of the graph by introducing node weights $s_i \in \mathbb{R}_+$, $i \in N$, as well as edge lengths $l_e \in \mathbb{R}_+$, $e \in E$, and consider the generalized embedding problem

$$
\begin{array}{rrl}
\text{maximize} & \sum_{i\in N} s_i\|v_i\|^2 & \\
\text{subject to} & \sum_{i\in N} s_i v_i = 0, & \\
& \|v_i - v_j\| \le l_{ij} & \text{for } ij \in E, \\
& v_i \in \mathbb{R}^n & \text{for } i \in N.
\end{array} \tag{3}
$$

For appropriate choices of the data $s$ and $l$ (nonnegative integer values will turn out to suffice as a ground set) the dimension of optimal embeddings of minimal dimension should be tightly linked to the structural complexity of the entire graph. These considerations motivate the following definitions.

**Definition 1** *For a connected graph $G = (N, E)$, the* rotational dimension of $G$ with respect to node weights $s \in \mathbb{R}_+^N$ and edge lengths $l \in \mathbb{R}_+^E$ *is*

$$\text{rotdim}_G(s,l) := \min\{\dim \text{span}\,\{v_i,\, i \in N\} : v_i,\, i \in N, \text{ is an optimal solution of } (3)\}.$$

*(by convention,* $\dim \emptyset = -1$*) and the* rotational dimension of a connected graph $G$ *is*

$$\text{rotdim}(G) := \max\{\text{rotdim}_G(s,l) : s \in \mathbb{N}_0^N,\, l \in \mathbb{N}_0^E\}.$$

*For a graph $G$ consisting of several connected components the* rotational dimension of $G$ *is*

$$\text{rotdim}(G) := \max\{\text{rotdim}(C) : C \text{ is a connected component of } G\}$$

*A graph $G$ is called* $d$-embeddable *if* $\text{rotdim}(G) \le d$.

A rather straight forward but important consequence of these definitions is minor monotonicity of the rotational dimension.

**Theorem 2** *The rotational dimension is a minor monotone graph parameter and d-embeddability is a minor monotone graph property.*

While we chose to define the rotational dimension using $s \in \mathbb{N}_0^N$ and $l \in \mathbb{N}_0^E$ in order to emphasize its discrete nature, the same object would be obtained by allowing for $s \in \mathbb{R}_+^N$ and $l \in \mathbb{R}_+^E$ or even $s > 0$ and $l > 0$. This last equivalence is of major importance in one of our proofs.

**Theorem 3** *Given a connected graph $G = (N, E)$, then for any dense subset $S$ of $\mathbb{R}_+^N$ and any dense subset $L$ of $\mathbb{R}_+^E$ there holds* $\operatorname{rotdim}(G) = \max\{\operatorname{rotdim}_G(s, l) : s \in S, l \in L\}$.

The main results of [8] for (2) can be transferred, without too much difficulty, to the more general problem (3). To begin with, there is also a dual problem resembling (1). With $D := \operatorname{Diag}(s_1^{-1/2}, \ldots, s_n^{-1/2})$ it reads

$$\begin{array}{ll} \text{maximize} & \lambda_2(DL_wD) \\ \text{subject to} & \sum_{ij\in E} l_{ij}^2 w_{ij} \leq 1, \\ & w \geq 0. \end{array} \tag{4}$$

Its optimal value is the reciprocal of the optimal value of (3). Regarding the properties of optimal embeddings of (3) we will need a slightly sharpened version of the separator-shadow theorem of [8]. It describes a characteristic structural property of optimal solutions of (3) irrespective of dimensional considerations.

**Theorem 4 (Separator-Shadow)** *Given optimal $v_i \in \mathbb{R}^n$, $i \in N$, of (3) and an optimal $w \in \mathbb{R}_+^E$ of (4) for a connected graph $G = (N, E)$ with node weights $s > 0$ and edge lengths $l > 0$, define* the strictly active subgraph $G_w = (N, E_w = \{ij \in E : w_{ij} > 0\})$ *and let $S$ be a separator in $G_w$ giving rise to a partition $N = S \cup C_1 \cup C_2$ where there is no edge in $E_w$ between $C_1$ and $C_2$. For at least one $C_j$ with $j \in \{1, 2\}$*

$$\operatorname{conv}\{0, v_i\} \cap \operatorname{conv}\{v_s : s \in S\} \neq \emptyset \qquad \forall i \in C_j. \tag{5}$$

*In words, the straight line segments* $\operatorname{conv}\{0, v_i\}$ *of all nodes $i \in C_j$ intersect the convex hull of the points in $S$.*

For an explanation of the theorem's name imagine the origin as a light source and the convex hull of the points of the separator as solid body, then at least one of the components has all its points embedded in the shadow of the separator body (note, any separator in $G$ is also a separator in $G_w$). A second important result of [8] is the tree-width bound on the minimal dimension of optimal embeddings of (2). Its generalization to (3) yields an immediate bound on the rotational dimension of graphs.

**Theorem 5** *For any connected graph $G = (N, E)$ and any data $s \in \mathbb{R}^N$, $l \in \mathbb{R}^E$ there exists an optimal embedding of (3) of dimension at most the tree-width of $G$ plus one.*

The same examples as in [8] may serve to show that this bound is tight for an infinite family of graphs. In general, however, we expect this bound to be rather weak. Because $d$-embeddability is a minor monotone graph property, the Graph-Minor-Theorem of Robertson and Seymore (cf. [5]) allows to characterize all graphs, that are $d$-embeddable by a finite list of forbidden minors. Denoting by $\operatorname{Forb}_{\preceq}(G_1, \ldots, G_k)$ the class of graphs that do not have the graphs $G_1, \ldots, G_k$ as minors, we prove the following.

**Theorem 6** *For a graph $G = (N, E)$ holds:*

(*i*) $\operatorname{rotdim}(G) = 0 \Leftrightarrow$ *G has no edges, i.e.,* $G \in \operatorname{Forb}_{\preceq}(K_2)$.

(*ii*) $\operatorname{rotdim}(G) \leq 1 \Leftrightarrow$ *G is a disjoint union of paths, i.e.,* $G \in \operatorname{Forb}_{\preceq}(K_3, K_{1,3})$.

(*iii*) $\operatorname{rotdim}(G) \leq 2 \Leftrightarrow$ *G is outerplanar, i.e.,* $G \in \operatorname{Forb}_{\preceq}(K_4, K_{2,3})$.

There are some obvious similarities of the rotational dimension to the Colin de Verdière number $\mu(G)$; see the excellent survey [12]. Unfortunately, our efforts to exhibit any clear relation between these two numbers were in vain so far. Even though the forbidden minor characterizations are the same for values 0, 1, and 2 we currently suspect that they differ already for value 3. Seeing the connection of the rotational dimension to tensegrities one might wonder whether the rotational dimension is related to graph $d$-realizability, see [1]. Graph $d$-realizability, however, works with bars (instead of cables) and without a potential. As, in addition, the forbidden minor characterizations differ significantly even for values 1 and 2 we have refrained from searching for other possible relations.

The paper is structured as follows. Section 2 is devoted to the proofs of theorems 2 and 3. In Section 3 we explore the duality relation between (3) and (4) and give optimality conditions. In Section 4 we explain how to transfer results for (2) of [8] to the setting in (3). In particular, we will give the proof of Theorem 4 in detail, but only list a few important steps that lead to Theorem 5 that will be needed later. Finally, Section 5 contains the proof of Theorem 6 giving the forbidden minor characterizations of rotational dimensions 0, 1, and 2.

We use basic notions and notation from graph theory and convex optimization. In particular, all vectors are column vectors and for symmetric $H \in \mathbb{R}^{n\times n}$, $H \succeq 0$ is used to denote positive semidefiniteness. For matrices $A, B \in \mathbb{R}^{m\times n}$, $\langle A, B\rangle := \sum_{ij} A_{ij}B_{ij}$ is the canonical inner product; in the case of vectors $a, b \in \mathbb{R}^n$ we will simply use $a^T b$. $\|\cdot\|$ refers to the usual Euclidean norm. A bold face $\mathbf{1}^N$ denotes the vector of all ones indexed by $N$, there may be no superscript set if the index set is clear from the context. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, $\operatorname{conv}\mathcal{S}$ refers to the convex hull of $\mathcal{S}$ and $\operatorname{cone}\mathcal{S}$ to its convex conic hull. The projection on a closed convex set $C$ is denoted by $p_C(\cdot)$. For a graph $G = (N, E)$ and a node subset $S \subseteq N$, $G[S]$ is the subgraph induced by $S$ and $G - S := G[N \setminus S]$ is the subgraph obtained by deleting the nodes in $S$. Likewise, for $E' \subset E$, $G - E'$ is the graph $(N, E \setminus E')$.

# 2 Basic Properties of the Rotational Dimension

We start by showing that the rotational dimension and $d$-embeddability are minor monotone.

**Proof (of Theorem 2).** By definition, every minor of a graph can be obtained by consecutive application of the following three operations: contraction of an edge, deletion of an edge, deletion of an isolated node. Loops or multiple edges are removed whenever they appear. It suffices to show that these operations preserve $d$-embeddability. This is clear for the deletion of an isolated node, which just removes the corresponding component. So, let us consider the edge operations. Given graphs $G = (N, E)$ and $\hat{G} = (\hat{N}, \hat{E})$ with $\hat{G}$ arising from $G$ by edge deletion or contraction, we show $\operatorname{rotdim}(\hat{G}) \le \operatorname{rotdim}(G)$ by expressing the embedding problem (3) for $\hat{G}$ and any given data $\hat{s} \in \mathbb{N}_0^{\hat{N}}$, $\hat{l} \in \mathbb{N}_0^{\hat{E}}$ as an embedding problem (3) for $G$ and appropriately chosen $s \in \mathbb{N}_0^N$, $l \in \mathbb{N}_0^E$. It suffices to do this for the component of $G$ where the operation takes place. So, w.l.o.g., let $G$ be connected and let the considered edge of $G$ be $\{1, 2\}$.

- Edge deletion. The node sets $N$ and $\hat{N}$ are identical and $\hat{E} = E \setminus \{\{1, 2\}\}$. We set $l_e := \hat{l}_e$ for all $e \in \hat{E}$, and $l_{12} := \sum_{e\in\hat{E}} l_e + 1$. First suppose $\hat{G}$ is connected. In this case we set $s := \hat{s}$. Because the edge length of $\{1, 2\}$ exceeds every possible length with respect to the other edges, its length restriction is never active in feasible solutions of (3) for $G$. Therefore every optimal solution of (3) for $G$ is also an optimal solution of (3) for $\hat{G}$ and vice versa. Thus, $\operatorname{rotdim}_G(s, l) = \operatorname{rotdim}_{\hat{G}}(\hat{s}, \hat{l})$. If $\hat{G}$ consists of two components $C_1$, $C_2$ with $\operatorname{rotdim}(\hat{G}) = \operatorname{rotdim}(C_1) \ge \operatorname{rotdim}(C_2)$, then we set $s_i := \hat{s}_i$, $i \in C_1$, and $s_i := 0$, $i \in C_2$. Now, every optimal solution of (3) for $G$ is also an optimal solution of (3) for $C_1$ and vice versa. Hence $\operatorname{rotdim}_{C_1}(\hat{s}, \hat{l}) = \operatorname{rotdim}_{\hat{G}}(\hat{s}, \hat{l}) = \operatorname{rotdim}_G(s, l)$.

- Edge contraction. Calling the newly arising node in $\hat{G}$ node 0 we have $\hat{N} = N \cup \{0\} \setminus \{1, 2\}$. Set the node weights to $s_1 := \hat{s}_0$, $s_2 := 0$, $s_i := \hat{s}_i$, $i \in \hat{N} \setminus \{0\}$ and the edge lengths to $l_{12} := 0$, $l_{1i} := \hat{l}_{0i}$, $1i \in E$, $l_{2i} := \hat{l}_{0i}$, $2i \in E$. Thus, the edge $\{1, 2\}$ in $G$ behaves like node 0 in $\hat{G}$. Again, every optimal solution of (3) for $G$ is also an optimal solution of (3) for $\hat{G}$ and vice versa. Therefore, $\mathrm{rotdim}_G(s,l) = \mathrm{rotdim}_{\hat{G}}(\hat{s},\hat{l})$.

This holds for all $\hat{s} \in \mathbb{N}_0^{\hat{N}}$ and $\hat{l} \in \mathbb{N}_0^{\hat{E}}$, thus $\mathrm{rotdim}(\hat{G}) \le \mathrm{rotdim}(G)$. ■

Note that this proof does not make use of the integrality of the data. Indeed, we will show below that replacing $\mathbb{N}_0^N$ and $\mathbb{N}_0^E$ by any dense subsets of $\mathbb{R}_+^N$ and $\mathbb{R}_+^E$ gives rise to the same notion of rotational dimension. Before this we prove the geometrically evident fact that embedding problem (3) is invariant under orthogonal transformations.

**Observation 7** *Let $G = (N, E)$ be a connected graph, $s \in \mathbb{R}_+^N$, $l \in \mathbb{R}_+^E$ data for the embedding problem* (3)*, and $Q \in \mathbb{R}^{n\times n}$ an orthogonal matrix. For any feasible solution $v_i$, $i \in N$, of the embedding problem* (3) *the vectors $Qv_i$, $i \in N$, are also a feasible solution of* (3) *with the same objective value.*

**Proof.** This holds because $\|Qv\| = \|v\|$ for all $v \in \mathbb{R}^n$ and $Q\sum_{i\in N} s_i v_i = 0 \Leftrightarrow \sum_{i\in N} s_i v_i = 0$. ■

Next, we show that one can replace $\mathbb{N}_0^N$ and $\mathbb{N}_0^E$ by $\mathbb{Q}_+^N$ and $\mathbb{Q}_+^E$ in the definition of the rotational dimension.

**Observation 8** *For any connected graph $G = (N, E)$,*

$$\mathrm{rotdim}(G) = \max\{\mathrm{rotdim}_G(s,l) : s \in \mathbb{Q}_+^N,\ l \in \mathbb{Q}_+^E\}.$$

**Proof.** It suffices to prove that for given $s \in \mathbb{Q}_+^N$, $l \in \mathbb{Q}_+^E$ we have $\mathrm{rotdim}(G) \ge \mathrm{rotdim}_G(s,l)$. Let $\alpha \in \mathbb{N}$ be a common denominator of the entries of $s$ and $\beta \in \mathbb{N}$ be a common denominator of the entries of $l$. Then the embedding problem (3) with data $\hat{s} := \alpha s \in \mathbb{N}_0^N$, $\hat{l} := \beta l \in \mathbb{N}_0^E$ has an optimal solution $\hat{v}_i$, $i \in N$, with $\mathrm{span}\,\{\hat{v}_i : i \in N\} \le \mathrm{rotdim}(G)$. It is now straight forward to show, that $v_i = \hat{v}_i/\beta$ is an optimal solution to the embedding problem (3) with data $s$ and $l$. ■

Now we are set for the decisive step.

**Lemma 9** *Given a connected graph $G = (N, E)$ and data $\hat{s} \in \mathbb{R}_+^N$, $\hat{l} \in \mathbb{R}_+^E$,*

$$\mathrm{rotdim}_G(\hat{s},\hat{l}) \le \max\{\mathrm{rotdim}_G(s,l) : s \in \mathbb{Q}_+^N, s > 0,\ l \in \mathbb{Q}_-^E, l > 0\} = \mathrm{rotdim}(G).$$

**Proof.** Once the left hand inequality is proved, the right hand equality follows from Obs. 8, so it suffices to prove the inequality. Let $(s^k)_{k\ge1}$ and $(l^k)_{k\ge1}$ be sequences of rational data with $s^k > 0$, $l^k > 0$, $k \ge 1$, and $s^k \to \hat{s}$, $l^k \to \hat{l}$, $k \to \infty$. Let $\mathrm{rotdim}_G(s^k, l^k) \le d$ for all $k \ge 1$. We will show that also $\mathrm{rotdim}_G(s,l) \le d$, which eventually will complete the proof. Denote by $(P^k)$ the embedding problems (3) for $G$ with data $s^k$, $l^k$ and by $(\hat{P})$ the embedding problem (3) for $G$ with data $\hat{s}$, $\hat{l}$. For $k \ge 1$ let $v_i^k$, $i \in N$ be an optimal solution of dimension at most $d$. By Obs. 7 we may assume that all $v_i^k$, $i \in N$, $k \ge 1$, lie in the same subspace $\mathcal{L} \subset \mathbb{R}^n$ with $\dim \mathcal{L} = d$. Since the sequence $(l^k)_{k\ge1}$ converges, it is certainly bounded and therefore the $(v_i^k)_{k\ge1}$ remain in a compact subset of $\mathcal{L}$ for all $i \in N$. Therefore we may assume (by selecting an appropriate subsequence if needed) that there exist $v_i \in \mathcal{L}$, $i \in N$, with $v_i^k \to v_i$ for $k \to \infty$ and $i \in N$. It remains to show that the $v_i$, $i \in N$ are an optimal solution of $(\hat{P})$. Let $\hat{v}_i$, $i \in N$, be an arbitrary optimal solution of $(\hat{P})$. Fix some $k \ge 1$. For $i \in N$ put $\alpha_i^k := s_i/s_i^k$ and for $ij \in E$ put

$$\beta_{ij}^k := \begin{cases} l_{ij}^k/(\hat{l}_{ij} + |1-\alpha_i^k|\|v_i\| + |1-\alpha_j^k|\|v_j\|) \text{ if } \hat{l}_{ij} > 0,\\ 1 \text{ if } \hat{l}_{ij} = 0. \end{cases}$$

With $\beta^k := \min_{ij\in E} \beta_{ij}^k$ define $\hat{v}_i^k := \alpha_i^k \beta^k \hat{v}_i$, $i \in N$. This is a feasible solution of $(P^k)$:

$$\sum_{i\in N} s_i^k \hat{v}_i^k = \beta^k (\sum_{i\in N} s_i \hat{v}_i) = 0,$$

$$\begin{aligned}
\|\hat{v}_i^k - \hat{v}_j^k\| &\le \beta^k(\|\hat{v}_i - \hat{v}_j\| + \|\alpha_i^k \hat{v}_i - \hat{v}_i\| + \|\hat{v}_j - \alpha_j^k \hat{v}_j\|) \\
&\le l_{ij}^k \beta^k (\hat{l}_{ij} + |1-\alpha_i^k| \|\hat{v}_i\| + |1-\alpha_j^k| \|\hat{v}_j\|)/l_{ij}^k \\
&\le l_{ij}^k.
\end{aligned}$$

Because of the optimality of $v_i^k$, $i \in N$ for $(P_k)$ we obtain

$$\sum_{i\in N} s_i^k \|v_i^k\|^2 \ge \sum_{i\in N} s_i^k \|\hat{v}_i^k\|^2 = \sum_{i\in N} \alpha_i^k (\beta^k)^2 s_i \|\hat{v}_i\|^2.$$

Using the continuity of the objective function and $\alpha_i^k \to 1$, $k \to \infty$ for all $i \in N$, $\beta^k \to 1$, $k \to \infty$ we get for $k \to \infty$ :

$$\sum_{i\in N} s_i \|v_i\|^2 \ge \sum_{i\in N} s_i \|\hat{v}_i\|^2.$$

Therefore the $v_i$, $i \in N$ form an optimal solution of $(\hat{P})$. ■

Theorem 3 now follows directly from the limit consideration of the previous proof. Consequently, it does not matter whether we consider $\mathrm{rotdim}(G)$ for data $s \in \mathbb{N}_0^N$, $l \in \mathbb{N}_0^E$, or for data $s \ge 0$, $l \ge 0$, or for data $s > 0$, $l > 0$.

# 3 A primal-dual pair

In the remainder of the paper we will assume that $G = (N, E)$ is connected. As mentioned in the introduction, for data $s = \mathbf{1}^N$, $l = \mathbf{1}^E$ the embedding problem (3) simplifies to problem (2) which is the (scaled) semidefinite dual of the absolute algebraic connectivity (1). The purpose of this section is to show that (4) is the corresponding dual of (3) for general data $s > 0$ and $l > 0$ as well as to derive explicit optimality conditions, that we will need in the proof of Theorem 6 in Section 6.

Let $s > 0$, $l > 0$ be data for the embedding problem (3), and $D := \mathrm{Diag}(s_1^{-1/2}, \ldots, s_n^{-1/2})$. We begin with Fiedler's basic result on the second smallest eigenvalue of the Laplacian.

**Lemma 10** *Given a connected graph $G = (N, E)$ and data $s > 0$, $w \ge 0$, the following statements are equivalent:*

*(i) The graph $G_w = (N, E_w := \{ij \in E : w_{ij} > 0\})$ is connected.*

*(ii) $\lambda_2(L_w) > 0$.*

*(iii) $\lambda_2(DL_wD) > 0$.*

**Proof.** The equivalence of (i) and (ii) was shown in Theorem 6.1 of [6]. Since $D$ is nonsingular we have $\dim\ker DL_wD = \dim\ker L_wD = \dim\ker L_w$. This implies the equivalence of (ii) and (iii). ■

In the optimization problem (4) the objective function $\lambda_2(DL_wD)$ as well as the constraint function $\sum_{ij\in E} c_{ij}w_{ij}$ are positive homogeneous, therefore we may also minimize the constraint function subject to a lower bound on the objective:

$$\begin{array}{rl}
\omega := \text{minimize} & \sum_{ij\in E} l_{ij}^2 w_{ij} \\
\text{subject to} & \lambda_2(DL_wD) \ge 1 \\
& w \ge 0.
\end{array} \qquad (6)$$

Lemma 10 yields that this program has a feasible solution and that the objective value is attained with $\omega > 0$ (actually, given optimal $w$ of (6), the optimal solution of (4) is attained in $w/\omega$ and $1/\omega$ is its optimal value).

**Observation 11** *Let $G = (N, E)$ be a connected graph and $s \in \mathbb{R}^N$, $l \in \mathbb{R}^E$ with $s > 0$, $l > 0$. Then we have:*

*(i) The programs* (6) *and* (3) *are a primal-dual pair satisfying strong duality.*

*(ii) The Complementary Slackness Conditions are equivalent to*

$$\begin{aligned} s_j v_j + \sum_{ij \in E} w_{ij}(v_i - v_j) = 0, \forall j \in N, \\ w_{ij}(l_{ij} - \|v_i - v_j\|) = 0, \forall ij \in E. \end{aligned} \tag{7}$$

*Together with primal and dual feasibility they form the KKT conditions. These are necessary and sufficient for $w \geq 0$ and $v_i \in \mathbb{R}^n$, $i \in N$ to be optimal.*

**Proof.** (i) The idea is to use semidefinite programming formulations of (6) and (3) together with the duality theory of semidefinite programming, v. [13]. We start from (6). Let $\mathbf{1}$ denote the vector of all ones of appropriate dimension. As $E_{ij}\mathbf{1} = 0$ for all $ij \in E$, the matrix $DL_wD$ has a single eigenvalue zero with eigenvector $D^{-1}\mathbf{1}$. Adding at least $(1/\|D^{-1}\mathbf{1}\|^2)D^{-1}\mathbf{1}(D^{-1}\mathbf{1})^T$ shifts this eigenvalue of $DL_wD$ to a value at least one. Having done this, replace the constraint $\lambda_2(DL_wD) \geq 1$ by $DL_wD + \mu D^{-1}\mathbf{1}\mathbf{1}^T D^{-1} \succeq I$ to arrive at

$$\begin{array}{ll} \min & \sum_{ij \in E} l_{ij}^2 w_{ij} \\ \text{s.t.} & \sum_{ij \in E} w_{ij} DE_{ij}D + \mu D^{-1}\mathbf{1}\mathbf{1}^T D^{-1} \succeq I \\ & w \geq 0,\ \mu \text{ free.} \end{array} \tag{8}$$

This program has a strictly feasible solution, because by Lemma 10 sufficiently large $w$ and $\mu$ ensure the positive definiteness of $DL_wD + \mu D^{-1}\mathbf{1}\mathbf{1}^T D^{-1} - I$. Semidefinite duality theory now asserts that strong duality holds for its dual, which reads

$$\begin{array}{lll} \text{maximize} & \langle I, X\rangle & \\ \text{subject to} & \left\langle D^{-1}\mathbf{1}\mathbf{1}^T D^{-1}, X\right\rangle = 0 & \\ & \langle DE_{ij}D, X\rangle \leq l_{ij}^2 & \text{for } ij \in E \\ & X \succeq 0. & \end{array} \tag{9}$$

It remains to recover (3) from this problem. We have $X \succeq 0$ if and only if $DXD \succeq 0$ if and only if there is a Gram representation $DXD = V^TV$ with $V = [v_1, \ldots, v_n] \in \mathbb{R}^{n \times n}$. Using $[DXD]_{ij} = v_i^T v_j$ we obtain

$$\begin{aligned} &\langle I, X\rangle = \left\langle D^{-2}, DXD\right\rangle = \sum_{i \in N} s_i \|v_i\|^2, \\ &\left\langle D^{-1}\mathbf{1}\mathbf{1}^T D^{-1}, X\right\rangle = (VD^{-2}\mathbf{1})^T(VD^{-2}\mathbf{1}) = \|\sum_{i \in N} s_i v_i\|^2, \\ &\langle DE_{ij}D, X\rangle = \langle E_{ij}, DXD\rangle = v_i^T v_i - 2v_i^T v_j + v_j^T v_j = \|v_i - v_j\|^2. \end{aligned}$$

Substituting this into (9) and taking square roots where appropriate yields (3) and proves (i).

(ii) The KKT conditions for (8) and (9) consist of primal and dual feasibility and the complementary slackness conditions. The latter read

$$\begin{aligned} w_{ij}(l_{ij}^2 - \langle E_{ij}, DXD\rangle) = 0 \qquad \text{for } ij \in E \\ (DL_wD + \mu D^{-1}\mathbf{1}\mathbf{1}^T D^{-1} - I)X = 0. \end{aligned} \tag{10}$$

For $DXD = V^TV$ the first line readily transforms to the second line of (7). In order to derive the first line of (7), put

$$\Xi := [\xi_1, \ldots, \xi_n] := D^{-1}V^T$$

so that $X = \Xi\Xi^T = \sum \xi_i \xi_i^T$. Recall that the product of two positive semidefinite matrices is zero if and only if their trace inner product is zero if and only if the range space of one matrix is in the null space of the other and vice versa. Therefore $\langle D^{-1}\mathbf{1}\mathbf{1}^T D^{-1}, X\rangle = 0$ and the second line of (10) imply $DL_w D\xi_i = \xi_i$ for $i \in N$. Thus, for $i \in N$,

$$\begin{aligned}(s_j v_j)_i = (D^{-1}\xi_i)_j &= (L_w D\xi_i)_j \\ &= (L_w D)_{j,\cdot}\xi_i \\ &= \sum_{k:jk\in E} -w_{jk}(v_k)_i + (\sum_{k:jk\in E} w_{jk})(v_j)_i \\ &= \sum_{k:jk\in E} w_{jk}(v_j - v_k)_i.\end{aligned}$$

This yields $s_j v_j + \sum_{ij\in E} w_{ij}(v_i - v_j) = 0$, $\forall j \in N$. ■

**Remark 12** *Complementarity can also be used to show that for optimal solutions $w \geq 0$ of (6) and $v_i \in \mathbb{R}^n$, $i \in N$ of (3) the projection of the $v_i$ onto an arbitrary one-dimensional subspace spanned by some $u \in \mathbb{R}^n$, yields an eigenvector to the eigenvalue 1 of $DL_w D$ via $\xi := (\sqrt{s_1}(u^T v_1), \dots, \sqrt{s_n}(u^T v_n))^T$. So one may view an optimal embedding as a map of (a subset of) the eigenvectors of $DL_w D$.*

In the following we will often restrict ourselves to the edges whose length constraints are tight.

**Definition 13** *Let $G = (N, E)$ be a connected graph, $w \geq 0$ be a feasible solution of (6), $v_i \in \mathbb{R}^n$, $i \in N$ be a feasible solution of (3) for data $s > 0$, $l > 0$ and put $V := [v_1, \dots, v_n]$. We call the graph $G_V = (N, E_V := \{ij \in E : \|v_i - v_j\| = l_{ij}\})$* active subgraph of $G$ and data $s, l$ *and the graph $G_w(N, E_w := \{ij \in E : w_{ij} > 0\})$* strictly active subgraph of $G$ to $w$.

Lemma 10 and complementarity (the second line in (7) of Observation 11(ii)) imply:

**Lemma 14** *Let $G = (N, E)$ be a connected graph, let $w \geq 0$ and $v_i \in \mathbb{R}^n$, $i \in N$ be optimal solutions of (6) and (3), respectively, for data $s > 0$, $l > 0$. The strictly active subgraph $G_w$ and the active subgraph $G_V$ are connected and $E_w \subseteq E_V \subseteq E$.*

# 4 Structural properties of optimal embeddings

Throughout, we consider connected graphs $G = (N, E)$ with given data $s > 0$ and $l > 0$ and study properties of optimal solutions $v_i$, $i \in N$, of the associated embedding problem (3). The results and techniques of this section are straight forward generalizations of those in [8] for the special case $s = \mathbf{1}^N$, $l = \mathbf{1}^E$ corresponding to problem (2). Mostly this involves nothing more than replacing $\sum v_i$ by $\sum s_i v_i$ or $\sum \|v_i\|^2$ by $\sum s_i \|v_i\|^2$ and there is no need to repeat this here. Some items, however, are essential for an understanding of the proof of Theorem 6 in Section 6. Therefore we give a rough sketch of the main ideas in [8] and present a detailed proof only for the slightly sharpened separator-shadow theorem 4.

Most proofs in [8] involve manipulating the current embedding by rotating or folding part of the points $v_i$ around some affine subspace into new feasible positions. There is always room for these operations, because by the equilibrium constraint $\sum s_i v_i = 0$, the $n$ vectors $v_i$ are linearly dependent. So there exists some $h \in \mathbb{R}^n$, $\|h\| = 1$, with $h^T v_i = 0$, $i \in N$. The subspace orthogonal to $h$ will be denoted by $\mathcal{H} := \{x \in \mathbb{R}^n : h^T x = 0\} \supseteq \operatorname{span}\{v_i : i \in N\}$. The points to be manipulated mostly lie in one halfspace of $\mathcal{H}$ with respect to a hyperplane $b^T x = \beta$ for some fixed $b \in \mathcal{H}$, $\|b\| = 1$ and $\beta \in \mathbb{R}$. The corresponding affine subspace will be referred to by $\mathcal{B} := \{x \in \mathcal{H} : b^T x = \beta\}$. The function $\varphi : \mathcal{H} \times [-\pi, \pi] \to \mathbb{R}^n$,

$$\varphi(x, \gamma) := \begin{cases} p_{\mathcal{B}}(x) - (\beta - b^T x)[b\cos\gamma + h\sin\gamma] & \text{if } b^T x < \beta, \\ x & \text{if } b^T x \geq \beta, \end{cases}$$

rotates a point $x \in \mathcal{H}$ around the affine subspace $\mathcal{B}$ by an angle $\gamma$. The effect of such an operation on the cost function can be obtained by direct calculation as in [8] and is stated here without proof.

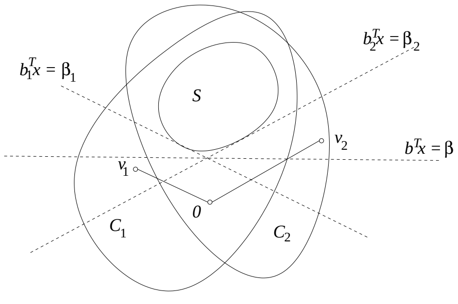


Figure 1: Initial setting in the separator-shadow proof.

**Observation 15 (The Cost of Folding)** *For $h$, $b$, $\beta > 0$, $\mathcal{H}$, $\mathcal{B}$, and $\varphi$ as as above and given $v_i \in \{x \in \mathcal{H} : b^T x < \beta\}$ ($i \in C \subseteq N$), let $\bar{v} := (\sum_{i \in C} s_i)^{-1} \sum_{i \in C} s_i v_i$, $\gamma \in [-\pi, \pi]$, and set for $i \in C$*

$$v_i' := \varphi(v_i, \gamma).$$

*Then,*

$$\sum_{i \in C} s_i \|v_i'\|^2 = \sum_{i \in C} s_i \|v_i\|^2 + 2 \sum_{i \in C} s_i r(1 - \cos\gamma)\beta \qquad \text{where } r := \beta - b^T \bar{v} > 0.$$

The separator-shadow theorem 4 gives necessary optimality conditions for feasible solutions of (3). Its proof is, up to the details of dealing with data $s$ and $l$ and separators of the strictly active subgraph, identical to the one given in [8] and works by contradiction, constructing a folding that improves a solution not satisfying the requirements.

**Proof of the separator-shadow theorem 4.** We first prove the result only for a separator $S$ in the original graph $G$. So let $S$ be a separator in $G_w$ giving rise to a partition $N = S \cup C_1 \cup C_2$ where there is no edge in $E$ between $C_1$ and $C_2$. Let $h \in \mathbb{R}^n$ with $\|h\| = 1$ satisfy $h^T v_i = 0$ for all $i \in N$ and let $\mathcal{S} := \text{conv}\{v_s : s \in S\}$. Assume, for contradiction, that the theorem does not hold for $S$. Then there is a node in $C_1$, call it node 1, and a node in $C_2$, call it node 2, embedded in $v_1$ and $v_2$ respectively, that satisfy $\text{conv}\{0, v_1\} \cap \mathcal{S} = \text{conv}\{0, v_2\} \cap \mathcal{S} = \emptyset$. By convex separation each set $\text{conv}\{0, v_j\}$ can be separated from $\mathcal{S}$ by a separating hyperplane within the subspace $\text{span}\{v_i : i \in N\}$. So for $j \in \{1, 2\}$ there are vectors $b_j \in \text{span}\{v_i : i \in N\}$ (these satisfy $b_j^T h = 0$) and scalars $\beta_j > 0$ so that $b_j^T x \geq \beta_j$ for all $x \in \mathcal{S}$ and $b_j^T x < \beta_j$ for all $x \in \text{conv}\{0, v_j\}$.

Next we show that we can find a convex combination of these two inequalities by choosing an appropriate $\alpha \in [0, 1]$ so that for $b(\alpha) := (1 - \alpha)b_1 + \alpha b_2$, $\beta(\alpha) := (1 - \alpha)\beta_1 + \alpha\beta_2$ the open halfspace $\{x : b(\alpha)^T x < \beta(\alpha)\}$ contains points of both $C_1$ and $C_2$ (illustrated in Fig. 1). Indeed, for $\alpha = 0$ the halfspace contains $v_1$ and so a point of $C_1$, for $\alpha = 1$ it contains $v_2$ which belongs to $C_2$, and it contains the origin for all $\alpha \in [0, 1]$. Suppose, for contradiction, that in sweeping $\alpha$ through $[0, 1]$ the halfspace looses the last point of $C_1$ before it encounters the first point of $C_2$ at some particular $\bar{\alpha}$. Then the corresponding hyperplane defined by $b(\bar{\alpha})^T x = \beta(\bar{\alpha}) > 0$ would separate 0 strictly from $\text{conv}\{v_i : i \in N\}$; but this contradicts the feasibility of the $v_i$ as the origin is a convex combination of the $v_i$ by the equilibrium constraint $(\sum_{i \in N} s_i)^{-1} \sum_{i \in N} s_i v_i = 0$.

Thus we have found $b := b(\alpha)$ and $\beta := \beta(\alpha) > 0$ such that the open halfspace $\{x : b^T x < \beta\}$ contains points from $C_1$ and $C_2$. Note that $b^T h = 0$ holds and by scaling $b$ and $\beta$ we may assume w.l.o.g. $\|b\| = 1$. Let, for $j \in \{1, 2\}$, $M_j := \{i \in C_j : b^T v_i < \beta\}$, $\bar{s}_j := \sum_{i \in M_j} s_i > 0$, and $\bar{v}_j := \frac{1}{\bar{s}_j} \sum_{i \in M_j} s_i v_i$. Next, observe that we may rotate all the points of nodes in $M_j$ around the affine subspace $\mathcal{B} = \{x \in \mathbb{R}^n : h^T x = 0, b^T x = \beta\}$ by $\varphi(\cdot, \gamma_j)$ without violating the distance constraints. Indeed, for $i_1 \in M_1$ and $i_2 \in M_2$ there holds $\{i_1, i_2\} \notin E$. For all nodes $i \in N \setminus (M_1 \cup M_2)$ we have $b^T v_i \geq \beta$, so they are not rotated and their distance to points of $M_1 \cup M_2$ can only be reduced by this rotation.

We show that rotating the points in $M_1$ in direction $h$ and the points in $M_2$ against direction $h$ by certain small angles $\gamma_1$ and $\gamma_2$ improves the solution (see Fig. 2). Denote, for rotation

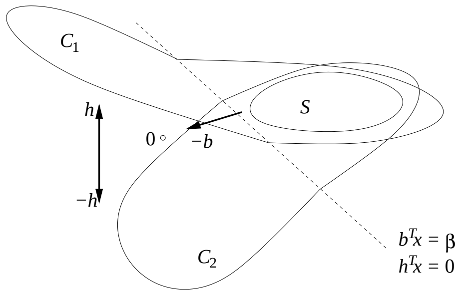


Figure 2: Improving movement in the separator-shadow proof.

$j \in \{1,2\}$, the radius of $\bar{v}_j$ by $r_j := \beta - b^T\bar{v}_j > 0$ and the resulting displacement of $\bar{v}_j$ by $d_j := r_j[(\sin\gamma_j)h + (1-\cos\gamma_j)b]$. By Obs. 15, the rotation of $M_j$ yields an improvement of $2\bar{s}_j r_j(1-\cos\gamma_j)\beta$. At the same time it adds $\bar{s}_j d_j$ to the barycenter of all points and has to be compensated in order to maintain feasibility with respect to the equilibrium constraint. Shifts of the global barycenter in the direction of $h$ can be avoided by requiring $\bar{s}_1 d_1^T h = -\bar{s}_2 d_2^T h$, i.e., given $\gamma_1$ choose $\gamma_2$ in dependence of $\gamma_1$ so that $\bar{s}_1 r_1 \sin\gamma_1 = -\bar{s}_2 r_2 \sin\gamma_2$. After carrying out these rotations it therefore remains to shift all points by

$$d := -(\bar{s}_1 d_1^T b + \bar{s}_2 d_2^T b)b/(\sum_{i\in N} s_i) = -[\bar{s}_1 r_1(1-\cos\gamma_1) + \bar{s}_2 r_2(1-\cos\gamma_2)]b/(\sum_{i\in N} s_i)$$

for feasibility in (3), which results in a further objective change of $-d^T d \sum_{i\in N} s_i$. The total objective improvement is

$$\sum_{j\in\{1,2\}} 2\bar{s}_j r_j(1-\cos\gamma_j)\beta - d^T d\sum_{i\in N} s_i =$$
$$= \sum_{j\in\{1,2\}} 2\bar{s}_j r_j(1-\cos\gamma_j)\beta - \frac{1}{\sum_{i\in N} s_i}[\bar{s}_1 r_1(1-\cos\gamma_1) + \bar{s}_2 r_2(1-\cos\gamma_2)]^2.$$

This is positive for $\gamma_1$ and $\gamma_2(\gamma_1)$ close enough to zero, yielding a contradiction to the optimality of the embedding.

It remains to extend the result to separators of the strictly active subgraph $G_w$ where, as explained in the paragraph following (6), we may assume that $w$ is an optimal solution of (6). Note that a solution $v_i$, $i \in N$ of (3) for $G$ and data $s > 0$, $l > 0$ is optimal if it is feasible and its objective value equals that of (4) for some feasible $w \geq 0$. Now any optimal embedding $v_i$, $i \in N$ of (3) for $G$ and data $s > 0$ and $l > 0$ is trivially feasible for the graph $G_w$, because for $G_w$ we only work with a subset of the constraints required for $G$. Furthermore the value of this solution is still the same as the value of the dual problem (6) of $G$ for $w$ . But the restriction of $w$ to $E_w$ is trivially feasible for the dual for $G_w$ and yields the same objective value, because dropping indices with $w_{ij} = 0$ neither affects the constraint nor the objective value of (6). Therefore any optimal embedding for $G$ is also an optimal embedding for the graph $G_w$ and thus the structural results also hold for separators of $G_w$. ■

All further results aim at proving the existence of low dimensional optimal embeddings. In this regard the separator shadow theorem only helps for those parts of the graph that are embedded in the shadow of a separator that does not contain the origin in the convex hull of its points. Indeed, the points of such parts have to lie in the span of the point of the separator and are therefore bounded in dimension by the size of the separator. If, however, the origin is contained in the convex hull of the points of the separator, then the separator shadow theorem holds trivially and yields no further information.

The key to the tree-width bound of theorem 5 is to get a hold on separators $S$ with $0 \in \mathcal{S} = \operatorname{conv}\{v_i : i \in S\}$. Note that rotating a point in a plane orthogonal to the subspace $\mathcal{L} = \operatorname{span}\mathcal{S}$

neither changes its contribution to the objective value nor does it change the distance to points in $S$. If operations are restricted to such rotations then in order to preserve optimality one only has to ensure the overall equilibrium condition with respect to the subspace $\mathcal{L}^\perp$ (no weights are shifted along $\mathcal{L}$) and, separately for each connected component of $G-S$, the validity of the distance constraints.

Let us ignore the equilibrium constraint for a moment and concentrate on reducing the dimension of just one of the $m \in N$ connected components $C_j$, $j \in M := \{1,\dots,m\}$ of $G-S$. Fix some $j \in M$, put $\bar{s}_j = \sum_{i\in C_j} s_i$, get rid of the subspace $\mathcal{L}$ by projecting it to 0, and imagine the remaining part of $C_j$ as being embedded in a rather flat subspace $\mathcal{H}$ with the handle $h$ sticking out orthogonally from the origin like the handle of an opened flat umbrella. Collapsing this umbrella corresponds to folding all of $\mathcal{H}$ towards $h$. For each single point of $C_j$ this operation can be implemented as an admissible rotation and, if executed in a coordinated manner, this only decreases the distances between points in $C_j$. In the end, all points of $C_j$ lie on the flat halfspace $\mathcal{L} + \{\alpha h : \alpha \geq 0\}$. In fact, the resulting continuous transformations $v_i(t)$, $i \in C_j$, parameterized jointly in $t \in [0,1]$ have the property that for the weighted barycenter $\bar{v}_j(t) := \sum_{i\in C_j} s_i v_i(t)/\bar{s}_j$ of $C_j$ the weighted norm $\delta_j(t) := \bar{s}_j \|p_{\mathcal{L}^\perp}\bar{v}_j(t)\|$ is monotonically increasing, reaching its maximum in $\tilde{\delta}_j := \bar{s}_j\|p_{\mathcal{L}^\perp}\bar{v}_j(1)\| = \sum_{i\in C_j} s_i\|p_{\mathcal{L}^\perp}(v_i)\|$. Note that, once the folding is completed, we can exchange $h$ against any $b_j \in \mathcal{L}^\perp$ with $\|b_j\| = 1$ and obtain an embedding that satisfies all distance constraints involving nodes of $C_j$.

Returning to the equilibrium constraint it remains to check whether it is possible to find normalized $b_j \in \mathcal{L}^\perp$, $j \in M$, so that $\sum_{j\in M}\tilde{\delta}_j b_j = 0$. It is not hard to prove that at most three linearly dependent directions $d_1, d_2, d_3 \in \mathcal{L}^\perp$ with $\dim\operatorname{span}\{d_1,d_2,d_3\} \leq 2$ suffice for the $b_j$ whenever $\tilde{\delta}_{\hat{\jmath}} \leq \sum_{j\in M\setminus\{\hat{\jmath}\}}\tilde{\delta}_j$ holds for all $\hat{\jmath} \in M$. In this case one may even require that $d_1$ is only assigned to a single $b_j$ for some $j \in M$ while $d_2$ and $d_3$ are shared by the others. This friendly case is the content of the next lemma.

**Lemma 16** *Let $v_i \in \mathbb{R}^n$ for $i \in N$ be an optimal solution of* (3) *for a connected graph $G = (N,E)$ and data $s > 0$, $l > 0$, and let $S \subset N$ with $0 \in \mathcal{S} := \operatorname{conv}\{v_s : s \in S\}$ be a separator in $G$ giving rise to separated sets $C_j \subset N$, $j \in M := \{1,\dots,m\}$. Put $\mathcal{L} := \operatorname{span}\mathcal{S}$ and, for $j \in M$, $\tilde{\delta}_j := \sum_{i\in C_j} s_i\|p_{\mathcal{L}^\perp}(v_i)\|$.*

*If $\tilde{\delta}_{\hat{\jmath}} \leq \sum_{j\in M\setminus\{\hat{\jmath}\}}\tilde{\delta}_j$ for all $\hat{\jmath} \in M$ then there exist vectors $d_1, d_2, d_3 \in \mathcal{L}^\perp$, $\|d_1\| = \|d_2\| = \|d_3\| = 1$ with $\dim\operatorname{span}\{d_1,d_2,d_3\} \leq 2$, $b_j \in \{d_1,d_2,d_3\}$, $j \in M$, so that the embedding $v_i'$, $i \in N$, with*

$$v_i' := \begin{cases} v_i & \text{for } i \in S, \\ p_{\mathcal{L}}(v_i) + \|p_{\mathcal{L}^\perp}(v_i)\|b_j & \text{for } i \in C_j. \end{cases}$$

*is also an optimal embedding of* (3)*. Furthermore, such an embedding exists with $b_j = d_1$ for at most one $j \in M$ and satisfies $\dim\operatorname{span}\{v_i' : i \in N\} \leq \dim\mathcal{L} + 2 \leq |S| + 1$.*

If one component $C_{\hat{\jmath}}$, $\hat{\jmath} \in M$ is 'heavier' than the other sets, namely $\tilde{\delta}_{\hat{\jmath}} > \sum_{j\in M\setminus\{\hat{\jmath}\}}\tilde{\delta}_j$, the need to maintain the equilibrium constraint will not allow to fold respectively collapse $C_{\hat{\jmath}}$ in full. It is possible, however, to fold and collapse all the other components and stop the transformation in $C_{\hat{\jmath}}$ at the point when $\delta_{\hat{\jmath}}(t) = \sum_{j\in M\setminus\{\hat{\jmath}\}}\tilde{\delta}_j$.

**Lemma 17** *Given the setting of Lemma 16 assume that there is a $\hat{\jmath} \in M$ with $\tilde{\delta}_{\hat{\jmath}} > \sum_{j\in M\setminus\{\hat{\jmath}\}}\tilde{\delta}_j$. There exists an $h \in \operatorname{span}\{v_i : i \in N\}^\perp$ and an optimal embedding $v_i'$ $(i \in N)$ of* (3) *with*

$$\begin{aligned} &v_i' \in \operatorname{span}\{h, v_i : i \in C_{\hat{\jmath}}\} && \text{for } i \in C_{\hat{\jmath}}, \\ &v_i' = v_i && \text{for } i \in S, \\ &v_i' = p_{\mathcal{L}}(v_i) + \|p_{\mathcal{L}^\perp}(v_i)\|\bar{b} && \text{for } i \in C_j \text{ with } j \in M\setminus\{\hat{\jmath}\}, \end{aligned}$$

*where $\bar{b} := -\frac{p_{\mathcal{L}^\perp}(\bar{v}'_{\hat{\jmath}})}{\|p_{\mathcal{L}^\perp}(\bar{v}'_{\hat{\jmath}})\|}$ if $\bar{v}'_{\hat{\jmath}} := \sum_{i\in C_{\hat{\jmath}}} s_i v_i'/\bar{s}_{\hat{\jmath}} \notin \mathcal{L}$ and $\bar{b} := 0$ otherwise.*

*Furthermore, if there is some direction $\hat{b} \in \operatorname{span}\{v_i : i \in C_{\hat{\jmath}}\} \cap \mathcal{L}^\perp \setminus \{0\}$ with $\hat{b}^T v_i \geq 0$ for $i \in C_{\hat{\jmath}}$, then such an embedding exists with $v_i' \in \operatorname{span}\{v_i : i \in C_{\hat{\jmath}}\}$ for $i \in C_{\hat{\jmath}}$.*

In fact, a further refinement of Lemma 16 is needed for the tree width bound.

**Lemma 18** *Given the setting of Lemma 16 assume that $\tilde{\delta}_{\hat{\jmath}} \leq \sum_{j \in M \setminus \{\hat{\jmath}\}} \tilde{\delta}_j$ holds for all $\hat{\jmath} \in M$ and let $\bar{\jmath} \in M$ be the only index with $b_{\bar{\jmath}} = d_1$ within the new embedding of Lemma 16. If at most $|S| - 1$ nodes of $S$ are adjacent to nodes in $C_{\bar{\jmath}}$, then there is an optimal embedding of dimension at most $|S|$.*

Let us now recall the definition of the tree-width of a graph.

**Definition 19** *For a graph $G = (N, E)$ a* tree-decomposition *of $G$ is a tree $(\mathcal{N}, \mathcal{E}) =: T$ with $\mathcal{N} \subseteq 2^N$ and $\mathcal{E} \subseteq \binom{\mathcal{N}}{2}$ satisfying the following requirements:*
*(i)* $N = \bigcup_{U \in \mathcal{N}} U$.
*(ii) For every $e \in E$ there is a $U \in \mathcal{N}$ with $e \subseteq U$.*
*(iii) If $U_1, U_2, U_3 \in \mathcal{N}$ with $U_2$ on the $T-$path from $U_1$ to $U_3$, then $U_1 \cap U_3 \subseteq U_2$.*
*The width of $T$ is the number* $\max\{|U| - 1 : U \in \mathcal{N}\}$. *The* tree-width *$tw(G)$ is the least width of any tree-decomposition of $G$.*

In general, it is $NP$-complete to determine the tree-width, but any valid tree-decomposition gives an upper bound. An important property of tree-decompositions is that each node $U \in \mathcal{N}$ as well as $U \cap V$ for $\{U, V\} \in \mathcal{E}$ is a separator in $G$. If, for a given optimal embedding $v_i$, $i \in N$ of (3) a node $U \in \mathcal{N}$ satisfies $0 \in \operatorname{conv}\{v_i : i \in U\}$, we call $U$ a *zero-node* of the tree-decomposition (with respect to this embedding). Likewise, we call an edge $\{U, V\} \in \mathcal{E}$ a *zero-edge* if $0 \in \operatorname{conv}\{v_i : i \in U \cap V\}$. Every tree-decomposition has zero-nodes.

**Lemma 20** *Consider a tree-decomposition $T = (\mathcal{N}, \mathcal{E})$ of a connected graph $G = (N, E)$ and an optimal embedding $v_i \in \mathbb{R}^n$ $(i \in N)$ of* (3). *There is a $U \in \mathcal{N}$ with $0 \in \operatorname{conv}\{v_u : u \in U\}$.*

By the separator-shadow theorem the zero-nodes form a subtree of the tree decomposition. The proof of the tree-width theorem 5 is algorithmic and may be sketched as follows. Given a tree-decomposition, start at a zero-node and check for the balancedness condition that allows to invoke Lemma 16 or Lemma 18. If this condition holds, one of these two lemmas yields an embedding of the desired dimension. If the condition does not hold, there is a unique zero-edge in the tree-decomposition that leads into the heavy component. Use Lemma 17 to modify the embedding so that the lighter side is straightened out and try to move over the zero-edge to the next zero-node of the tree-decomposition. If the heavy side turns out to flip back, one can prove that using the separator associated with this zero-edge yields the desired low dimensional embedding. Otherwise continue with the next zero-node. The process stops with the desired low-dimensional embedding because no zero-node is visited twice. This also ends the sketch of the proof of the tree-width theorem 5. For more details, see [8].

# 5 Characterizing $d$-embeddable graphs for $d = 0, 1, 2$

In the proof of Theorem 6 we may restrict considerations to the rotational dimension of data instances satisfying $s > 0$ and $l > 0$ by Theorem 3. The proof itself splits into two parts and will be given by the next observations. First we show that the rotational dimension of the forbidden minors specified in Theorem 6 is consistent with the claim.

**Observation 21**

*(i)* $\operatorname{rotdim}(K_n) = n - 1$ *for* $n \geq 1$,

*(ii)* $\operatorname{rotdim}(K_{1,3}) = 2$,

*(iii)* $\operatorname{rotdim}(K_{2,3}) = 3$.

**Proof.** Feasible embeddings in $d$-space will be described by specifying appropriate $v_i \in \mathbb{R}^d$, $i \in N$.

(i) Example 21 of [8] proves $\operatorname{rotdim}_{K_n}(\mathbf{1}, \mathbf{1}) = n - 1$. Furthermore $\operatorname{rotdim}_{K_n}(s, l) \leq n - 1$ for arbitrary data $s > 0$, $l > 0$ because of the equilibrium constraint $\sum_{i \in N} s_i v_i = 0$.

(ii) Suppose, w.l.o.g., node 1 is the central node of the $K_{1,3}$ and consider the data $s = \mathbf{1}^N$, $l = \mathbf{1}^E$. A best 1-embedding is $v_1 = \frac{1}{4}$, $v_2 = \frac{5}{4}$, $v_3 = v_4 = -\frac{3}{4}$. Its objective value is $\frac{11}{4}$. But this is smaller than 3, which is the objective value of the following feasible 2-embedding: $v_1 = (0,0)$, $v_2 = (\cos 0, \sin 0)$, $v_3 = (\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3})$, $v_4 = (\cos\frac{4\pi}{3}, \sin\frac{4\pi}{3})$ (this is optimal). We have shown that $\operatorname{rotdim}_{K_{1,3}} \geq 2$.

Now, given arbitrary $s > 0$, $l > 0$, any optimal embedding $v_i$, $i \in N$ is at most 2-dimensional. Indeed, if the center of the star is not embedded in zero ($v_1 \neq 0$) the embedding is 1-dimensional by Theorem 4 with $S = \{1\}$. If $v_1 = 0$, the equilibrium constraint $\sum_{i\in N} s_i v_i = 0$ requires the remaining three points to be linearly dependent.

(iii) Let $G$ be the complete bipartite graph on the node partition $\{1,2\} \cup \{3,4,5\}$ and put $s := \mathbf{1}^N$, $l_{1i} := 1$, $l_{2i} := 2$, $i = 3,4,5$. We will show, that an arbitrary optimal solution of the embedding problem (3) with these data requires at least three dimensions. Let $w \geq 0$ be an optimal solution of the dual problem (6) for this $s$ and $l$. According to Lemma 14 the graph $G_w$ is connected. Therefore at least one edge $e = 1k$, $k \in \{3,4,5\}$ and one edge $\tilde{e} = 2\tilde{k}$, $\tilde{k} \in \{3,4,5\}$ have positive weights $w_e > 0$ and $w_{\tilde{e}} > 0$. By symmetry and convexity of (6) we may take convex combinations of appropriate permutations to construct an optimal solution $\tilde{w} > 0$ of (6). Hence, complementary slackness in (7) implies that all distance constraints are tight in any optimal embedding of (3), in particular $v_1 \neq v_2$. Furthermore $0 \in \operatorname{conv}\{v_1, v_2\}$, otherwise Theorem 4 with $S = \{1,2\}$ would yield a contradiction to the equilibrium constraint $\sum_{i\in N} s_i v_i = 0$. The only feasible solution of dimension at most two that is not in contradiction to equilibrium constraint or Theorem 4 is, in fact, one dimensional, $v_1 = \frac{6}{5}b$, $v_2 = -\frac{9}{5}b$, $v_3 = v_4 = v_5 = \frac{1}{5}b$ for a vector $b \in \mathbb{R}^2$, $\|b\| = 1$ and has objective value $\frac{24}{5}$. The value of the following feasible 3-dimensional embedding, however, is $\frac{27}{5}$: $v_1 = (\frac{\sqrt{3}}{5}, 0, 0)$, $v_2 = (-\frac{4\sqrt{3}}{5}, 0, 0)$, $v_3 = (\frac{\sqrt{3}}{5}, \cos 0, \sin 0)$, $v_4 = (\frac{\sqrt{3}}{5}, \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3})$, $v_5 = (\frac{\sqrt{3}}{5}, \cos\frac{4\pi}{3}, \sin\frac{4\pi}{3})$.

It remains to prove $\operatorname{rotdim}(K_{2,3}) \leq 3$. Let $s > 0$, $l > 0$ be arbitrary data, and $v_i$, $i \in N$ an optimal embedding. If $0 \notin \operatorname{conv}\{v_1, v_2\}$ the embedding is two dimension due to Theorem 4 with $S = \{1,2\}$. If $0 \in \operatorname{conv}\{v_1, v_2\}$ we apply Lemma 16 with $S = \{1,2\}$ which yields an optimal 3-embedding. ■

We proceed to show that an arbitrary graph of the given classes is embeddable in the respective rotational dimension and start with the easiest case.

**Observation 22** *For $G = (\{1\}, \emptyset)$ and data $s > 0$ the optimal embedding of* (3) *is $v_1 = 0$.*

**Proof.** The equilibrium constraint reads $s_1 v_1 = 0$. ■

The next result handles graphs that are disjoint unions of paths.

**Observation 23** *Let $G = (N, E)$ be a nontrivial path. Every optimal solution of the embedding problem* (3) *with data $s > 0$, $l > 0$ has dimension* 1.

**Proof.** Let $w \geq 0$ be an optimal solution of (6), and $v_i \in \mathbb{R}^n$, $i \in N$ be an optimal embedding of (3). Because of Lemma 10 the strictly active subgraph $G_w$ is $G$ itself and thus $w > 0$. By complementarity in (7), all distance constraints are tight in the optimal embedding. Consider the tree-decomposition $\mathcal{T} = (\mathcal{N}, \mathcal{E})$ of $G$ with $\mathcal{N} = \{\{i,j\} \subseteq N : ij \in E\}$ and $\mathcal{E} = \{N_1N_2 \in \binom{\mathcal{N}}{2} : |N_1 \cap N_2| = 1\}$. By Lemma 20 there is a $U \in \mathcal{N}$ with $0 \in \operatorname{conv}\{v_u : u \in U\}$. Let $U = ij \in E$ be this edge of $G$. If 0 is contained in the relative interior of the convex hull of $v_i$ and $v_j$, then applying Theorem 4 to $i$ and to $j$ as separators yields the claim. Otherwise there is one node, say node $i$, that is embedded to 0. As its neighbors cannot be embedded in zero, because all edges are tight, we may apply Theorem 4 to its neighbors as separators. The equilibrium condition $\sum_{i\in N} s_i v_i = 0$ then asserts that the neighbors lie on a straight line. ■

Finally, we turn to outerplanar graphs. The next result shows that it suffices to consider outerplanar graphs of maximum degree at most three.

**Lemma 24** *A minor monotone graph property holds for all outerplanar graphs if and only if it holds for outerplanar graphs of maximum degree three.*

**Proof.** Necessity is immediate. The following construction shows sufficiency. Each outerplanar graph $G$ is a minor of an outerplanar graph $G'$ obtained as follows: Since $G$ is outerplanar, it has a plane embedding such that each vertex is situated on the boundary of the outer (infinite) face of this embedding. Let $r$ be a positive real number small enough, so that for each node the circle of radius $r$ around the node intersects each edge incident to this node (an only those) exactly once and all circles are disjont. The edges subdivide the circles into arcs. Because the embedding is outerplanar, each of the circles contains an arc belonging to the outer face. For each of the circles delete it's whole interior together with exactly one such arc from the drawing. Interpret the intersection points of the circles with the edges of $G$ as vertices of $G'$ and the remaining arcs and parts of the edges of $G$ as edges of $G'$. Clearly, the maximum degree of $G'$ is three, the outer face of $G'$ contains the interior of all circles and hence we constructed an outerplanar embedding of $G'$. Finally, contracting all arcs of the drawing results in $G$. ■

By Theorem 3 we only need to consider algebraically independent edge lengths and for these, optimal embeddings have favorable properties.

**Observation 25** *Given an outerplanar graph $G = (N, E)$ with maximum degree 3 and data $s > 0$, $l > 0$, suppose the entries of $l$ are algebraically independent and let $v_i \in \mathbb{R}^n$, $i \in N$ be an optimal solution of* (3). *Suppose $C$ is the set of nodes of a chordless cycle in the active subgraph $G_V$, then* $\dim \operatorname{span} \{v_i : i \in C\} = 2$.

**Proof.** Let $C \subseteq N$ be a set of nodes so that $G_V[C]$ is a cycle. Note that $\dim \operatorname{span} \{v_i : i \in C\}$ cannot be 0 or 1, because all edge lengths are tight in the embedding and by algebraic independence of the entries in $l$ any sum of signed lengths of cycle edges cannot cancel out. It remains to show that $\dim \operatorname{span} \{v_i : i \in C\} \leq 2$. Assume, for contradiction, that there are three nodes in $C$, say 1, 2, and 3 with $\dim \operatorname{span} \{v_1, v_2, v_3\} = 3$. Then $0 \notin \operatorname{conv}\{v_1, v_2, v_3\}$ and in $G_V$ any node $i \in N \setminus \{1, 2, 3\}$ is separated from one of the three nodes by the other two, because otherwise $G_V$ would contain a forbidden minor or $G_V[C]$ would have a chord. So, if $i$ is separated, say, from 1 in $G_V$ by 2 and 3 then Theorem 4 implies $\operatorname{conv}\{v_i, 0\} \cap \operatorname{conv}\{v_2, v_3\} \neq \emptyset$, because $\operatorname{conv}\{v_1, 0\} \cap \operatorname{conv}\{v_2, v_3\} = \emptyset$. Thus, for $i \in N$ we have $\operatorname{conv}\{v_i, 0\} \cap \operatorname{conv}\{v_1, v_2, v_3\} \neq \emptyset$. Therefore zero is strictly separated from the $v_i$, $i \in N$ by the affine plane containing the points $v_1$, $v_2$ and $v_3$. This contradicts the equilibrium constraint. ■

We are now ready for the decisive step.

**Lemma 26** *Given an outerplanar graph $G = (N, E)$ with maximum degree 3 and data $s > 0$, $l > 0$, suppose the entries of $l$ are algebraically independent and let $v_i \in \mathbb{R}^n$, $i \in N$ be an optimal solution of* (3). *The dimension of the embedding is at most two,* $\dim \operatorname{span} \{v_i : i \in N\} \leq 2$.

**Proof.** Because the active subgraph $G_V = (N, E_V)$ is outerplanar, the sets of nodes spanning chordless cycles in $G_V$ together with the sets of endnodes of bridges of $G_V$ form the set $\mathcal{N}$ of a tree decomposition $T = (\mathcal{N}, \mathcal{E})$ of $G_V$. By Lemma 20 there is at least one set $U \in \mathcal{N}$ with $0 \in \operatorname{conv}\{v_i, i \in U\}$.

Suppose first that each such $U \in \mathcal{N}$ with $0 \in \operatorname{conv}\{v_i : i \in U\}$ is the node set of a bridge in $G_V$. If there is such a $U = \{i, j\} \in \mathcal{N}$ with $0 \in \operatorname{conv}\{v_i, v_j\} \setminus \{v_i, v_j\}$ then Theorem 4 implies that the embedding is in fact one dimensional. Otherwise there is a node $\hat{\imath} \in N$ with $v_{\hat{\imath}} = 0$. By the current assumption on the zero-nodes of $\mathcal{N}$, each edge $\{\hat{\imath}, j\} \in E_V$ is a bridge in $G_V$ with $v_j \neq 0$ because $l_{\hat{\imath}j} > 0$. Furthermore, there are at most three edges incident to $\hat{\imath}$. Thus each $i \in N$ is separated from $\hat{\imath}$ in $G_V$ by some neighbor $j$ of $\hat{\imath}$, and Theorem 4 implies $v_j \in \operatorname{conv}\{v_i, 0\}$. In consequence, all points are embedded in at most three halfrays emanating from the origin. By the equilibrium constraint, these have to lie in a common two dimensional subspace.

So we may assume in the following that there is a $C \in \mathcal{N}$ with $G_V[C]$ a cycle and $0 \in \operatorname{conv}\{v_i : i \in C\}$. Put $E_C := \{\{i, j\} \in E_V : i, j \in C\}$ and call $e \in E_C$ an *inner edge* if it is contained in another chordless cycle of $G_V$ and an *outer edge* otherwise. Because $G_V$ is outerplanar, for each node $k \in N \setminus C$ there is an edge $ij \in E_C$ such that $k$ and $C \setminus \{i, j\}$ are in different components of $G_V - \{i, j\}$. If $0 \notin \operatorname{conv}\{v_i, v_j\}$ for all $ij \in E_C$, then by Theorem 4 all nodes of $G_V$ are embedded in the subspace span $\{v_c : c \in C\}$ which is two-dimensional by Obs. 25. The same argument still works if $v_h \neq 0$ for all $h \in C$ and $0 \notin \operatorname{conv}\{v_{i'}, v_{j'}\}$ for all inner edges $i'j' \in E_C$. It remains to consider the case of $0 \in \operatorname{conv}\{v_i, v_j\}$ for some edge $ij \in E_C$ where $ij$ is either an inner edge or $v_i = 0$ for some $i \in C$ and both edges incident to $i$ in $G_V[C]$ are outer edges. By $l_{ij} > 0$ we may assume $v_j \neq 0$. Let $N' \subset N$ be the set of nodes in the connected component of $G_V - \{i, j\}$ that contains $C - \{i, j\}$ and put $G_1 := G_V[N' \cup \{i, j\}]$ and $G_2 := G_V[N \setminus N']$.

We claim that $G_1$ is embedded in a halfplane bounded by span $\{v_j\}$.

Indeed, let $k \in C$ with $k \neq j$ be the other neighbour of $i$ in $G_V[C]$. By the properties of $ij$ and because the degree of $i$ is at most 3, the edge $ki$ is an outer edge. Hence, if $v_k$ and $v_j$ are linearly independent, all vertices $h$ of $G - \{i, j\}$ in the component of $k$ are separated from $i$ by the set $\{j, k\}$ and so Theorem 4 asserts $\operatorname{conv}\{v_h, 0\} \cap \operatorname{conv}\{v_j, v_k\} \neq \emptyset$ proving the claim in this case.

If $v_k$ and $v_j$ are linearly dependent, Obs. 25 ensures the existence of a vertex $k' \in C$ such that $v_{k'}$ is linearly independent to $v_j$. Because $l_{ik} > 0$, there is a $i' \in \{i, k\}$ such that $0 \neq v_{i'}$. Now Theorem 4 applied to the separators $\{i', k'\}$ and $\{j, k'\}$ of $G_V$ completes the proof of the claim.

It remains to show that the embedding of $G_2$ aligns nicely with that of $G_1$. If the edge $ij$ is an inner edge, the same argument shows that $G_2$ is embedded in a halfplane bounded by span $\{v_j\}$. The equilibrium constraint then ensures that the halfplanes of $G_1$ and $G_2$ lie in a common two dimensional subspace.

If the edge $ij$ is an outer edge we have $v_i = 0$, $ij$ is a bridge in $G_2$ and $i$ has at most one other neighbor $k \neq j$ in $G_2$. Consider some node $h \in N \setminus (N' \cup \{i, j\})$. In $G_2$ node $h$ is separated from $i$ by either $j$ or $k$. If $h$ is separated from $i$ by $j$ then Theorem 4 implies $v_j \in \operatorname{conv}\{0, v_h\}$, otherwise we get in the same way $v_k \in \operatorname{conv}\{0, v_h\}$. So nodes of $G_2$ lie in span $\{v_j\}$ or are embedded in a halfray emanating from the origin through $v_k$. Again the equilibrium constraint forces the halfplane of $G_1$ and the halfray of $G_2$ to lie in a common two dimensional subspace. ■

It remains to put everything together.

**Proof of Theorem 6.** Necessity of (i), (ii), and (iii) follows from Obs. 21(i),(ii), and (iii). For proving sufficiency we only need to consider connected graphs, because rotdim is defined as the maximum over rotational dimension of the graphs connected components. Sufficiency of (i) and (ii) is proved by observations 22 and 23, while sufficiency of (iii) is implied by Lemma 24, Lemma 26, and Theorem 3 together with the fact that the vectors $l$ with algebraically independent entries form a dense subset of $\mathbb{R}^E_+$. ■

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