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Abstract: This paper provides an analysis of Pontryagine mimina satisfying a quadratic growth condition, for optimal control problems of ordinary differential equations with constraints on initial-final state, as well as control constraints satisfying the uniform positive linear independence condition.

Key-words: Optimal control, Pontryagine's principle, second-order optimality conditions, quadratic growth condition, weak and strong minima.

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Analyse au second ordre de problèmes de commande optimale avec contraintes sur la commande et sur l'état initial-final

Résumé : Cet article présente une analyse des minima de Pontryagine satisfaisant une condition de croissance quadratique, pour des problèmes de commande optimale d'équations différentielles avec contraintes sur l'état initial-final et avec des contraintes sur la commande satisfaisant la condition d'indépendance linéaire positive uniforme.

Mots-clés : Commande optimale, principe de Pontryagine, conditions d'optimalité du second ordre, condition de croissance quadratique, minima faibles et forts.

1 Introduction

In this paper we discuss necessary or sufficient conditions for weak or bounded strong minima of optimal control problems with control constraints and constraint on the initial-final state. There is already an important literature on this subject.

Osmolovskii [10, 11, 12] analyzed second-order optimality conditions for such problems assuming linear independence of gradients of active constraints (LIG); see also Levitin, Milyutin and Osmolovskii [6, p. 155-156] where these conditions were first formulated.

Malanowski [7] obtained stability and sensitivity results in the case of convex cost and constraints (including state constraints) assuming the LIG hypothesis. More recently, Hermant and the first author [1] studied similar problems, without convexity assumption except for the (local) dependence of the Hamiltonian w.r.t. the control variable, and again with the LIG hypothesis.

The main novelty is that we do not assume any more the LIG hypothesis but instead a qualification condition that implies the uniform positive linear independence of gradients of active inequality constraints. Also, we do not assume the (local) convex dependence of the Hamiltonian w.r.t. the control variable, which makes the discussion of sufficient conditions more complex (since the Hessian of the Lagrangian of the problem is not a Legendre form, and so it is no more possible to pass to the limit in weakly convergent directions).

The paper is organized as follows. Section 2 sets the problem, recalls some basic concepts, and gives a decomposition principle, in the setting of abstract control constraints. Section 3 analyzes the multipliers associated with control constraints parameterized by finitely many inequalities. Section 4 proves, under a restoration property (of initial final state constraints), for which a verifiable sufficient condition is provided, that Pontryagines'principle makes the link between weak and bouded strong minima. Finally in section 5 we characterize the quadratic growth condition for weak minima.

2 Pontryagine minima

2.1 Pontryagine's principle

Let $\mathcal{U} := L^\infty(0,T;\mathbb{R}^m)$ and $\mathcal{Y} := W^{1,\infty}(0,T;\mathbb{R}^n)$ denote the control and state space. Set $\mathcal{W} := \mathcal{U}\times\mathcal{Y}$. When needed we denote $w=(u,y)$, $\bar{w}=(\bar{u},\bar{y})$, etc. the elements of $\mathcal{W}$. Similarly we denote when needed $\eta=(y(0),y(T))$, $\bar{\eta}=(\bar{y}(0),\bar{y}(T))$, etc. the pair of initial-final states. The cost function is defined by

$$J(w) := \int_0^T \ell(u(t),y(t))\mathrm{d}t + \phi(\eta), \tag{1}$$

where $\ell:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}$ (*running cost*) and $\phi:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ (*initial-final cost*) are twice continuously differentiable (C^2) mappings. Consider the state equation

$$\dot{y}(t) = f(u(t),y(t)) \quad \text{for a.a. } t\in[0,T]; \tag{2}$$

where $f:\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^n$ is a Lipschitz and C^2 mapping. We know that the state equation (2) has for any $u\in\mathcal{U}$ and given initial condition $y(0)=y_0$ a unique solution denoted $y_{u,y_0} \in \mathcal{Y}$. We consider problems having both control constraints

$$u(t) \in U, \quad \text{for a.a. } t \in (0,T), \tag{3}$$

where U is a closed subset of $\mathbb{R}^m$, and initial-final state constraints of the form

$$\Phi(\eta) \in K, \quad \text{with } K := \{0\}_{\mathbb{R}^{r_1}} \times \mathbb{R}^{r_2}_-, \tag{4}$$

and $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^r$, $r = r_1 + r_2$, r_1 and r_2 are nonnegative integers. In other words, there are a finite number of equality and inequality constraints on the initial-final state:

$$\Phi_i(\eta) = 0, \quad i = 1, \dots, r_1, \qquad \Phi_i(\eta) \leq 0, \quad i = r_1 + 1, \dots, r. \tag{5}$$

Consider the following optimal control problem:

$$\operatorname*{Min}_{w \in \mathcal{W}} J(w) \quad \text{subject to (2)-(4).} \tag{P}$$

We call *trajectory* an element w of $\mathcal{W}$ that satisfies the state equation (2). If in addition the constraints (3)-(4) hold, we say that w is a *feasible point* of problem (P); the set of feasible points is denoted by $F(P)$.

We briefly recall several concepts of solution. Denote by $\|\cdot\|_s$ the norm of the space $L^s(0,T,\mathbb{R}^q)$ for any q, and $s \in [1, +\infty]$. A *weak (resp. strong) solution* of (P) (or *weak, strong minimum*) is an element $\bar{w} \in F(P)$ such that $J(\bar{w}) \leq J(w)$ for all $w \in F(P)$ such that $\|w - \bar{w}\|_{\mathcal{W}}$ (resp. $\|y - \bar{y}\|_\infty$) is small enough. Equivalently, $\bar{w} \in F(P)$ is a weak (resp. strong) solution of (P) if, for any sequence $w_k \in F(P)$, such that $w_k \to \bar{w}$ in $\mathcal{W}$ (resp. $y_k \to \bar{y}$ uniformly), we have that $J(\bar{w}) \leq J(w_k)$ for large enough k.

Following [6, p.156] and [9, p. 291], we say that $\bar{w} \in F(P)$ is a *bounded strong solution (minimum)* if for any *bounded* sequence $w_k \in F(P)$, such that $y_k \to \bar{y}$ uniformly, we have that $J(\bar{w}) \leq J(w_k)$ when k is large enough. An element $\bar{w}$ of $F(P)$ is called *Pontryagine solution (minimum)* see [6, p.156] and [9, p. 2-3], if for any sequence $w_k \in F(P)$, bounded in $\mathcal{W}$, such that $y_k \to \bar{y}$ uniformly and $\|u_k - \bar{u}\|_1 \to 0$, we have that $J(\bar{w}) \leq J(w_k)$ when k is large enough.

Equivalently, $\bar{w}$ is a bounded strong (Pontryagine) solution if for any $M > 0$, there exists $\varepsilon_M > 0$ such that if $w \in F(P)$ is such that $\|u\|_\infty \leq M$, $\|y - \bar{y}\|_\infty \leq \varepsilon_M$ (and in addition $\|u - \bar{u}\|_1 \leq \varepsilon_M$ in the case of a Pontryagine solution) we have that $J(\bar{w}) \leq J(w)$.

Obviously any of the following concepts is implied by the previous one: strong, bounded strong, Pontryagine, weak solution. We define a strong, bounded strong, Pontryagine, weak *perturbation* of $\bar{w} \in F(P)$ as a sequence w_k of trajectories in $\mathcal{W}$ associated with the corresponding optimality concept, i.e., such that $y_k \to \bar{y}$ uniformly, and in addition u_k is bounded in L^∞ for the other types of perturbations, $u_k \to u$ in L^1 (uniformly) for a Pontryagine (weak) perturbation. We say that the perturbation is *feasible* if the elements of the sequence belong to $F(P)$. We call $\delta w_k := w_k - \bar{w}$ a strong, bounded strong, Pontryagine, weak *variation*.

We say that a (strong, bounded strong, Pontryagine, weak) solution $\bar{w}$ satisfies the *quadratic growth condition* if there exists $\alpha > 0$ (depending on M in the case of a bounded strong or Pontryagine solution) such that $\bar{w}$ is a solution of the same kind for the cost function

$$J_\alpha(u,y) := \int_0^T \ell_\alpha(t,u(t),y(t))\mathrm{d}t + \phi(\eta), \tag{6}$$

where

$$\ell_\alpha(t,u,y) := \ell(u,y) - \tfrac{1}{2}\alpha[|u-\bar{u}(t)|^2 + |y-\bar{y}(t)|^2]. \tag{7}$$

Then we say that the (strong, bounded strong, Pontryagine, weak) quadratic growth condition is satisfied. So for instance the quadratic growth condition for a weak solution $\bar{w}$ (we speak then of *weak quadratic growth*) means that

$$\begin{cases} \text{There exist } \ \alpha > 0,\, \varepsilon > 0: \ J(w) \geq J(\bar{w}) + \frac{1}{2}\alpha\|w-\bar{w}\|_2^2, \\ \text{for all } \ w \in F(P), \quad \|w-\bar{w}\|_\infty < \varepsilon, \end{cases} \tag{8}$$

and the *bounded strong quadratic growth* condition means that

$$\begin{cases} \text{For any } M>0, \text{ there exist } \ \alpha_M > 0, \varepsilon_M > 0: \ J(w) \geq J(\bar{w}) + \frac{1}{2}\alpha_M\|w-\bar{w}\|_2^2 \\ \text{for all } \ w \in F(P), \ \|y-\bar{y}\|_\infty < \varepsilon_M, \|u\|_\infty \leq M. \end{cases} \tag{9}$$

We now recall the formulation of Pontryagine's principle at the point $\bar{w} \in F(P)$. Let us denote by $\mathbb{R}^{q*}$ the dual of $\mathbb{R}^q$ (identified with the set of q dimensional horizontal vectors). We remind that K was defined in (4). The negative dual cone to K (set of vectors of $\mathbb{R}^{r*}$ having a nonpositive duality product with each elements of K) is $K^- = \mathbb{R}^{r_1 *} \times \mathbb{R}_+^{r_2 *}$. We say that $(\theta,\mu) \in K \times K^-$ is a *complementary pair* if $\mu_i\theta_i = 0$, for $i = 1,\dots,r$. The normal cone to K at the point $\theta \in K$ is the set of elements of the negative dual cone that are complementary to θ. In particular, the expression of the normal cone to K at $\Phi(\bar{\eta})$ is

$$N_K(\Phi(\bar{\eta})) := \{\mu \in \mathbb{R}^{r*};\ \mu_i \geq 0,\ \mu_i\Phi_i(\bar{\eta}) = 0,\ i > r_1\}. \tag{10}$$

Let the *end-point Lagrangian* be defined by

$$\Phi^\mu(y_0,y_T) := \phi(y_0,y_T) + \sum_{i=1}^r \mu_i\Phi_i(y_0,y_T). \tag{11}$$

Consider the *Hamiltonian function* $H: \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n*} \to \mathbb{R}$ defined by

$$H(u,y,p) := \ell(u,y) + pf(u,y). \tag{12}$$

Set $\mathcal{P} := W^{1,\infty}(0,T;\mathbb{R}^{n*})$. For any $\mu \in \mathbb{R}^{r*}$ and $p \in \mathcal{P}$, consider the following set of relations (reminding that $\bar{\eta} = (\bar{y}(0),\bar{y}(T))$):

$$\begin{array}{lrcl} \text{(i)} & -\dot{p}(t) & = & H_y(\bar{w}(t),p(t)), \ \text{ for a.a. } t \in (0,T); \\ \text{(ii)} & p(T) & = & \Phi^\mu_{y_T}(\bar{\eta}); \\ \text{(iii)} & p(0) & = & -\Phi^\mu_{y_0}(\bar{\eta}). \end{array} \tag{13}$$

We call *costate* associated with μ at the point $\bar{w} \in F(P)$, and denote by p^μ, the unique solution in $\mathcal{P}$ of the backward equation (13)(i-ii). Existence and uniqueness of the costate follow from the fact that this is a Cauchy problem for a linear o.d.e. with measurable and bounded coefficients. Obviously the mapping $\mu \mapsto p^\mu$ is affine. We will obtain (13)(iii) as a necessary optimality condition.

Definition 2.1. Let $\bar{w} \in F(P)$. We say that $\mu \in N_K(\Phi(\bar{\eta}))$ is a *(regular) Pontryagine multiplier* associated with $\bar{w}$ if the associated costate p^μ satisfies (13)(iii), and is such that the following Hamiltonian inequality holds:

$$H(\bar{u}(t), \bar{y}(t), p^\mu(t)) \le H(v, \bar{y}(t), p^\mu(t)), \quad \text{for all } v \in U, \text{ a.a. } t \in (0,T). \tag{14}$$

We denote by $M^P(\bar{w})$ the set of Pontryagine multipliers associated with $\bar{w}$; if this closed convex set is non empty, we say that $\bar{w}$ satisfies Pontryagine's principle (in qualified form), or that $\bar{w}$ is a *Pontryagine extremal.*

Remark 2.2. Let $\bar{w}$ be a Pontryagine extremal, and let $\mu \in M^P(\bar{w})$. We know (see e.g. [9, p. 24-25]) that there exists a constant $c_\mu \in \mathbb{R}$ such that

$$c_\mu := \inf_{v \in U} H(v, \bar{p}(t), p^\mu(t)), \quad \text{for all } t \in [0,T]. \tag{15}$$

By (13)(i-ii), the function $\mu \mapsto c_\mu$ is affine. Set

$$h(v, \mu, t) := H(v, \bar{y}(t), p^\mu(t)), \quad t \in (0,T). \tag{16}$$

By (14), we have that $h(\bar{u}(t), \mu, t) = c_\mu$ for a.a. $t \in (0,T)$. Define

$$U_M := U \cap B(0, M), \quad \text{where } M > \|\bar{u}\|_\infty. \tag{17}$$

Let us show that there exists a representative $\tilde{u}$ of $\bar{u}$ such that

$$\tilde{u}(t) \in U_M \quad \text{and} \quad h(\tilde{u}(t), \mu, t) = c_\mu, \quad \text{for all } t \in [0,T]. \tag{18}$$

Let $\hat{t} \in [0,T]$. If $h(\bar{u}(\hat{t}), \mu, \hat{t}) = c_\mu$ and $\bar{u}(\hat{t}) \in U_M$, let $\tilde{u}(\hat{t}) := \bar{u}(\hat{t})$. Otherwise, since $h(\bar{u}(t), \mu, t) = c_\mu$ a.e., there exists a sequence $t_k \in [0,T]$, $t_k \to \hat{t}$ such that $\bar{u}(t_k) \in U_M$ and $h(\bar{u}(t_k), \mu, t_k) = c_\mu$. Extracting if necessary a subsequence, we may assume that $\bar{u}(t_k)$ converges to some limit whose value will be $\tilde{u}(\hat{t})$. Passing to the limit in the relation $h(\bar{u}(t_k), \mu, t_k) = c_\mu$, we deduce that $h(\tilde{u}(\hat{t}), \mu, \hat{t}) = c_\mu$ so that $h(\tilde{u}(t), \mu, t) = c_\mu$ for all time. Clearly $\tilde{u}(\hat{t}) \in U_M$, and hence, (18) holds.

It is known that Pontryagine's principle, in a non qualified form, is satisfied by Pontryagine solutions of (P), see [9, p. 24]. The qualified form is satisfied under some qualifications conditions to be presented in the next section.

2.2 Hamiltonian functions with a unique minimum

If A is a convex subset of a finite-dimensional space, we denote by $\text{ri}(A)$ its relative interior, in the sense of convex analysis (the interior of A, in the topology induced by its affine hull). We check below that a relatively interior Pontryagine multiplier (i.e., some $\mu \in \text{ri}(M^P(\bar{w}))$) obtains an increase of Hamiltonian of the same growth rate as the maximum over bounded sets of Pontryagine multipliers:

Lemma 2.3. *Let $\bar{w} \in F(P)$ satisfy Pontryagine's principle, and let $M^C(\bar{w})$ be a nonempty, convex and compact subset of $M^P(\bar{w})$. Then for any $\bar{\mu} \in \text{ri}(M^C(\bar{w}))$, there exists $\beta > 0$ such that, for a.a. t, and any $v \in U$:*

$$\begin{aligned} H(v, \bar{y}(t), p^{\bar{\mu}}(t)) - H(\bar{u}(t), \bar{y}(t), p^{\bar{\mu}}(t)) \ge \\ \beta \max_{\mu \in M^C(\bar{w})} \left(H(v, \bar{y}(t), p^\mu(t)) - H(\bar{u}(t), \bar{y}(t), p^\mu(t)) \right); \end{aligned} \tag{19}$$

$$\beta \mu_i \le \bar{\mu}_i, \ \textit{for all } i > r_1. \tag{20}$$

Proof. Since $\bar\mu \in \mathrm{ri}(M^C(\bar w))$, and $M^C(\bar w)$ is compact, there exists $\varepsilon_0 > 0$ such that $\bar\mu + \varepsilon_0(\bar\mu - \mu) \in M^C(\bar w)$, for any $\mu \in M^C(\bar w)$. The function $\mu \mapsto a(\mu) := H(v, \bar y(t), p^\mu(t)) - H(\bar u(t), \bar y(t), p^\mu(t))$ is affine. For given $\mu \in M^C(\bar w)$, we have that $\mu' := \bar\mu + \varepsilon_0(\bar\mu - \mu)$ belongs to $M^C(\bar w)$. Since $\bar\mu = \frac{1}{1+\varepsilon_0}\mu' + \frac{\varepsilon_0}{1+\varepsilon_0}\mu$, $\mu'_i \geq 0$ for $i > r_1$, and $a(\cdot)$ is affine and nonnegative when $v \in U$, it follows that $\bar\mu_i \geq \frac{\varepsilon_0}{1+\varepsilon_0}\mu_i$ for $i > r_1$, and $a(\bar\mu) \geq \frac{\varepsilon_0}{1+\varepsilon_0} a(\mu)$. The conclusion follows with $\beta = \varepsilon_0/(1+\varepsilon_0)$. □

We next relate the convergence of cost, for bounded strong perturbations, to some integral of difference of Hamiltonian functions. For $\mu \in \mathbb{R}^{r*}$, we define

$$J^\mu(w) := J(w) + \mu\Phi(\eta) = \int_0^T \ell(w(t))\mathrm{d}t + \Phi^\mu(\eta). \tag{21}$$

If w is a trajectory, then for any $p \in \mathcal{P}$, we have that

$$J^\mu(w) = \int_0^T [H(w(t), p(t)) - p(t)\dot y(t)]\,\mathrm{d}t + \Phi^\mu(\eta). \tag{22}$$

Lemma 2.4. *Let $\bar w \in \mathcal{W}$ be a trajectory, let $\mu \in \mathbb{R}^{r*}$, with associated costate p^μ solution of* (13)*(i-ii), and let w be any trajectory. Denote $\eta := (y(0), y(T))$ and $\delta\eta := \eta - \bar\eta$. Then*
(i) *The following expansion holds:*

$$\begin{aligned} J^\mu(w) - J^\mu(\bar w) = &\int_0^T [H(w, p^\mu) - H(\bar w, p^\mu) - H_y(\bar w, p^\mu)\delta y]\mathrm{d}t \\ &+ (p^\mu(0) + \Phi^\mu_{y_0}(\bar\eta))\delta y(0) + \tfrac{1}{2}(\Phi^\mu)''(\bar\eta)(\delta\eta)^2 + o(|\delta\eta|^2). \end{aligned} \tag{23}$$

(ii) *Let w_k be a bounded strong perturbation of $\bar w$, and $\mu \in \mathbb{R}^{r*}$. Then*

$$J(w_k) - J(\bar w) = \int_0^T [H(u_k, \bar y, p^\mu) - H(\bar u, \bar y, p^\mu)]\mathrm{d}t + o(1). \tag{24}$$

(iii) *If $\bar w$ is a Pontryagine extremal, $\mu \in M^P(\bar w)$, and w_k is a feasible bounded strong perturbation of $\bar w$, then $\liminf_k J(w_k) \geq J(\bar w)$, with equality iff the following holds:*

$$\liminf_k \int_0^T [H(u_k, \bar y, p^\mu) - H(\bar u, \bar y, p^\mu)]\mathrm{d}t = 0. \tag{25}$$

In particular, $J(w_k) \to J(\bar w)$ iff the integral in (25) *converges to 0.*

Proof. Denoting $\delta y := y - \bar y$, observe that for any $\mu \in \mathbb{R}^{r*}$, using (22) with $p = p^\mu$, we have that

$$J^\mu(w) - J^\mu(\bar w) = \int_0^T [H(w, p^\mu) - H(\bar w, p^\mu) - p^\mu\delta\dot y]\mathrm{d}t + \Phi^\mu(\eta) - \Phi^\mu(\bar\eta). \tag{26}$$

Since by the integration by parts formula, we get

$$-\int_0^T p^\mu\delta\dot y\mathrm{d}t = p^\mu(0)\delta y(0) - p^\mu(T)\delta y(T) + \int_0^T \dot p^\mu\delta y\mathrm{d}t, \tag{27}$$

using the costate equation (13) and a second-order expansion of $\Phi^\mu(\eta)$, obtain (i). Since for a bounded strong perturbation we have that $\|y_k - \bar{y}\|_\infty \to 0$, we deduce from (23) that $J^\mu(w_k) - J^\mu(\bar{w})$ is equal to the r.h.s. of (24). Since $\eta_k \to \bar{\eta}$, $J^\mu(w_k) - J^\mu(\bar{w}) = J(w_k) - J(\bar{w}) + o(1)$. Relation (ii) follows. Combining with (14), we deduce (iii). □

We next show that the uniqueness of the minimum of the Hamiltonian function for all times t implies that the control is continuous.

Given a Pontryagine extremal $\bar{w}$, and $M > \|\bar{u}\|_\infty$, we say that $\mu \in M^P(\bar{w})$ satisfies the hypothesis of unique minimum of the Hamiltonian over U_M if the associated costate p^μ is such that, for all $t \in [0,T]$, the function $h(\cdot,\mu,t) = H(\cdot,\bar{y}(t),p^\mu(t))$ has a unique minimum over U_M.

Remark 2.5. If this hypothesis holds, then (i) by lemma 2.3, it holds for any element of $\mathrm{ri}(M^P(\bar{w}))$, and (ii) for given $\varepsilon > 0$ and $M > 0$, there exists $\varepsilon_M > 0$ such that

$$\begin{cases} \text{For a.a. } t \in (0,T), \text{ whenever } v \in U_M \text{ and } |v - \bar{u}(t)| \geq \varepsilon : \\ H(v,\bar{y}(t),p^\mu(t)) \geq H(\bar{u}(t),\bar{y}(t),p^\mu(t)) + \varepsilon_M. \end{cases} \tag{28}$$

Lemma 2.6. *Let $\bar{w}$ be a Pontryagine extremal, and $\mu \in M^P(\bar{w})$ satisfy the hypothesis of unique minimum of the Hamiltonian over U_M, with $M > \|\bar{u}\|_\infty$. Then (one representative of) $\bar{u}(t)$ is a continuous function of time, equal to this unique minimum.*

Proof. Let $t \in [0,T]$; then for $t_k \to t$ in $[0,T]$ the function $\tilde{u}$ constructed in remark 1.2 is such that $h(\tilde{u}(t),\mu,t) = h(\tilde{u}(t_k),\mu,t_k) = c_\mu$. Passing to the limit obtain $h(\tilde{u}(t),\mu,t) = h(v,\mu,t) = c_\mu$ for all limit point v of $\tilde{u}(t_k)$ (they exist since $\tilde{u}(t) \in U_M$ for all t$\in [0,T]$). Since $h(\cdot,\mu,t)$ has a unique minimum over U_M we see that $\tilde{u}(t) = v$, which proves that $\tilde{u}$ is continuous. The conclusion follows. □

Consider the condition similar to (25), but with a limit instead of a lower limit:

$$\lim_k \int_0^T [H(u_k,\bar{y},p^\mu) - H(\bar{u},\bar{y},p^\mu)]\mathrm{d}t = 0. \tag{29}$$

Lemma 2.7. *Let $\bar{w}$ be a Pontryagine extremal, and let $\mu \in M^P(\bar{w})$ be such that the Hamiltonian satisfies the hypothesis of unique minimum over U. If w_k is a feasible bounded strong perturbation of $\bar{w}$, then the four conditions below are equivalent:* (i) $\limsup_k J(w_k) \leq J(\bar{w})$, (ii) $\lim_k J(w_k) = J(\bar{w})$, (iii) (29) *holds,* (iv) *Any subsequence of u_k has itself a subsequence converging to $\bar{u}$ a.e.*

Proof. The equivalence of (i),(ii) and (iii) follows from lemma 2.4. If (iii) holds, then by (28) $u_k \to \bar{u}$ in measure (i.e., for all $\varepsilon > 0$, $\mathrm{meas}(\{t \in (0,T); |u_k(t) - \bar{u}(t)| > \varepsilon\}) \to 0$), and hence (since this holds also for an arbitrary subsequence of u_k) condition (iv) holds. Finally assume that (iv) holds, but not (iii). Taking if necessary a subsequence we may assume that there exists $\varepsilon > 0$ such that $\int_0^T [H(u_k,\bar{y},p^\mu) - H(\bar{u},\bar{y},p^\mu)]\mathrm{d}t > \varepsilon$. But then, taking again a subsequence for which $u_k \to \bar{u}$ a.e., we obtain a contradiction to Lebesgue's dominated convergence theorem. □

Corollary 2.8. *Let $\bar{w}$ be a Pontryagine extremal. Assume that, for some $\mu \in M^P(\bar{w})$, the Hamiltonian satisfies the hypothesis of unique minimum over U. Then (i) any feasible bounded strong perturbation of $\bar{w}$ such that* $\limsup_k J(w_k) \le J(\bar{w})$ *is a Pontryagine perturbation, and (ii) $\bar{w}$ is a bounded strong minimum iff it is a Pontryagine minimum.*

Proof. (i) Let $\bar{w}$ satisfy the hypotheses of the corollary. If w_k is a feasible bounded strong perturbation of $\bar{w}$ such that $\limsup_k J(w_k) \le J(\bar{w})$, then, by the above lemma, we deduce that $u_k \to \bar{u}$ a.e., and hence, by the dominated convergence theorem, $\|u_k - \bar{u}\|_1 \to 0$, so that w_k is a Pontryagine perturbation. (ii) As already observed, a bounded strong minimum is a Pontryagine solution. So it remains to prove that if $\bar{w}$ is a Pontryagine solution, then it is a bounded strong solution. This follows from point (i). □

Definition 2.9. Let $\bar{w}$ be a Pontryagine extremal. We say that the Hamiltonian function satisfies a *local quadratic growth condition* for $\mu \in M^P(\bar{w})$ if there exist $\alpha > 0$ and $\varepsilon > 0$ such that

$$\begin{array}{l} \text{For a.a. } t \in (0,T), \text{ whenever } v \in U,\ |v - \bar{u}(t)| \le \varepsilon : \\ H(v, \bar{y}(t), p^\mu(t)) \ge H(\bar{u}(t), \bar{y}(t), p^\mu(t)) + \alpha |v - \bar{u}(t)|^2. \end{array} \tag{30}$$

Remark 2.10. In view of lemma 2.3, we have that, if $M^C(\bar{w})$ is a nonempty, convex and compact subset of $M^P(\bar{w})$, and there exist $\alpha > 0$ and $\varepsilon > 0$ such that

$$\begin{array}{l} \text{For a.a. } t \in (0,T), \text{ whenever } v \in U,\ |v - \bar{u}(t)| \le \varepsilon : \\ \max\limits_{\mu \in M^C(\bar{w})} [H(v, \bar{y}(t), p^\mu(t)) - H(\bar{u}(t), \bar{y}(t), p^\mu(t))] \ge \alpha |v - \bar{u}(t)|^2. \end{array} \tag{31}$$

then the Hamiltonian function satisfies the local quadratic growth condition, for any $\mu \in \mathrm{ri}(M^C(\bar{w}))$.

If $\mu \in M^P(\bar{w})$ is such that (28) and (30) hold, for $M > \|\bar{u}\|_\infty$, then $\bar{u}$ has a representative $\tilde{u}$ such that, for all $t \in [0,T]$, $v \mapsto h(v, \mu, t)$ has a unique minimum at $\tilde{u}(t)$ over U_M, and we have that

$$\left\{ \begin{array}{l} H(v, \bar{y}(t), p^\mu(t)) \ge H(\tilde{u}(t), \bar{y}(t), p^\mu(t)) + \min(\alpha |v - \tilde{u}(t)|^2, \varepsilon_M), \\ \text{whenever } \ v \in U_M, \quad \text{for all } \ t \in [0,T]. \end{array} \right. \tag{32}$$

2.3 Second-order expansion of the (weighted) cost function

Having in mind Pontryagine perturbations, we now introduce the natural notion of linearization of the state equation in the framework of the study of Pontryagine minima. Given a Pontryagine perturbation w_k of $\bar{w} \in \mathcal{W}$, satisfying the state equation (2), reminding that $\delta y_k := y_k - \bar{y}$ is the variation of states, denote by $\delta_L y_k$ the solution of the *Pontryagine linearization* of the state equation at the point $\bar{w} = (\bar{u}, \bar{y})$:

$$\delta_L \dot{y}_k = f_y(\bar{u}, \bar{y}) \delta_L y_k + f(u_k, \bar{y}) - f(\bar{u}, \bar{y}); \quad \delta_L y_k(0) = y_k(0) - \bar{y}(0). \tag{33}$$

As shows the following lemma, $\delta_L y_k$ gives a good approximation of δy_k. We denote the remaining terms in a second-order Taylor expansion in L^1 as

$$R_{1k} := O(\|u_k - \bar{u}\|_1^2 + |y_k(0) - \bar{y}(0)|^2); \quad r_{1k} = o(\|u_k - \bar{u}\|_1^2 + |y_k(0) - \bar{y}(0)|^2).$$

A lemma similar to the one below can be found in Milyutin and Osmolovskiĭ [9, p. 40-42, prop.8.1 to 8.3]. We give the (short) proof in order to make the paper self-contained.

Lemma 2.11. *Let $\bar{w} \in \mathcal{W}$ satisfy the state equation* (2)*, and w_k be a Pontryagine perturbation of $\bar{w}$. Then*

$$\|\delta_L y - y_k - \bar{y}\|_\infty = R_{1k}. \tag{34}$$

Proof. We may write

$$\delta \dot{y}_k = f_y(\bar{u}, \bar{y})\delta y_k + f(u_k, \bar{y}) - f(\bar{u}, \bar{y}) + \Delta_k \tag{35}$$

where Δ_k satisfies, by the mean value theorem, for some $\theta : [0,T] \to (0,1)$:

$$\begin{aligned} \Delta_k \quad &:= f(u_k, y_k) - f(u_k, \bar{y}) - f_y(\bar{u}, \bar{y})\delta y_k \\ &= [f_y(u_k, \bar{y} + \theta \delta y_k) - f_y(\bar{u}, \bar{y})]\delta y_k = O((|u_k - \bar{u}| + |\delta y_k|)|\delta y_k|), \end{aligned} \tag{36}$$

so that $\|\Delta_k\|_1 = R_{1k}$. The conclusion follows then from Gronwall's lemma. □

Lemma 2.12. *Let $\bar{w} \in \mathcal{W}$ be a trajectory, and w_k be a Pontryagine perturbation of $\bar{w}$. Then for any $\mu \in \mathbb{R}^{r*}$, denoting by p^μ the costate associated with μ at the point $\bar{w}$ (solution of* (13)*(i-ii)), we have that:*

$$\begin{aligned} J^\mu(w_k) - J^\mu(\bar{w}) = \quad & \int_0^T [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu) + \tfrac{1}{2} H_{yy}(\bar{u}, \bar{y}, p^\mu)(\delta_L y_k)^2]\mathrm{d}t \\ & + \int_0^T [H_y(u_k, \bar{y}, p^\mu) - H_y(\bar{u}, \bar{y}, p^\mu)]\delta_L y_k \,\mathrm{d}t \\ & + (p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}))\delta y_k(0) \\ & + \tfrac{1}{2}(\Phi^\mu)''(\bar{\eta})(\delta_L y_k(0), \delta_L y_k(T))^2 + r_{1k}. \end{aligned} \tag{37}$$

Proof. Expanding w.r.t. $\delta y_k := y_k - \bar{y}$ the r.h.s. of the expression below:

$$\begin{aligned} H(u_k, y_k, p^\mu) - H(\bar{u}, \bar{y}, p^\mu) = \quad & [H(u_k, y_k, p^\mu) - H(u_k, \bar{y}, p^\mu)] \\ & + [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu)], \end{aligned} \tag{38}$$

and since (all derivatives of f and ℓ being uniformly bounded and continuous over bounded sets)

$$H(u_k, y_k, p^\mu) - H(u_k, \bar{y}, p^\mu) = H_y(u_k, \bar{y}, p^\mu)\delta_k y + \tfrac{1}{2} H_{yy}(u_k, \bar{y}, p^\mu)\delta y_k^2 + r_{1k}, \tag{39}$$

and using the relation $\int_0^T H_{yy}(u_k, \bar{y}, p^\mu)\delta y_k^2 \mathrm{d}t = \int_0^T H_{yy}(\bar{u}, \bar{y}, p^\mu)\delta y_k^2 \mathrm{d}t + r_{1k}$, obtain with (23):

$$\begin{aligned} & J^\mu(w_k) - J^\mu(\bar{w}) = \int_0^T [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu) + \tfrac{1}{2} H_{yy}(\bar{u}, \bar{y}, p^\mu)\delta y_k^2]\mathrm{d}t \\ & + \int_0^T [H_y(u_k, \bar{y}, p^\mu) - H_y(\bar{u}, \bar{y}, p^\mu)]\delta y_k \mathrm{d}t \\ & + (p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}))\delta y_k(0) + \tfrac{1}{2}(\Phi^\mu)''(\bar{\eta})(\delta y_k(0), \delta y_k(T))^2 + r_{1k}. \end{aligned} \tag{40}$$

The conclusion follows with lemma 2.11. □

2.4 A decomposition principle

Now given $\bar{w} \in \mathcal{W}$ and a Pontryagine perturbation w_k, both satisfying the state equation (2), we start to use second-order expansions of the cost function with the remainder

$$r_{2k} := o(\|\delta u_k\|_2^2 + |y_k(0) - \bar{y}(0)|^2).$$

Note that the norm $(\|u\|_2^2 + |y(0)|^2)^{1/2}$ is equivalent to $\|w\|_2$ over the set of trajectories. Consider a sequence of "measurable partitions" of $[0,T]$, i.e.,

$$[0,T] = A_k \cup B_k; \quad \text{meas}(A_k \cap B_k) = 0, \tag{41}$$

such that $\text{meas}(B_k) \to 0$, and a Pontryagine perturbation w_k of $\bar{w} \in \mathcal{W}$. Denote the restriction of variations of control variables to the set A_k by

$$\delta_A u_k = \mathbf{1}_{A_k}(u_k - \bar{u}); \quad u_{A,k} := \bar{u} + \delta_A u_k, \tag{42}$$

and similarly for $\delta_B u_k$ and $u_{B,k}$. The associated states are defined as solution of the Cauchy problems

$$\dot{y}_{A,k} = f(u_{A,k}, y_{A,k}); \quad y_{A,k}(0) = y_k(0), \tag{43}$$

$$\dot{y}_{B,k} = f(u_{B,k}, y_{B,k}); \quad y_{B,k}(0) = \bar{y}(0), \tag{44}$$

that is, the variation in the initial condition is "absorbed" by the "A" part. We set

$$\delta_A y_k = y_{A,k} - \bar{y}; \quad \delta_B y_k = y_{B,k} - \bar{y}. \tag{45}$$

Theorem 2.13 (Decomposition principle)**.** *Let $\bar{w}$ be a trajectory in $\mathcal{W}$, w_k be a Pontryagine perturbation, and (A_k, B_k) be a measurable partition of $(0,T)$, such that* $\text{meas}(B_k) \to 0$*. Then, for all $\mu \in \mathbb{R}^{r*}$, we have a decomposition principle*

$$J^\mu(w_k) = J^\mu(w_{A,k}) + (J^\mu(w_{B,k}) - J^\mu(\bar{w})) + r_{2k}, \tag{46}$$

and, p^μ being the costate associated with μ at the point $\bar{w}$, $J^\mu(w_{B,k})$ has the following expansion:

$$J^\mu(w_{B,k}) = J^\mu(\bar{w}) + \int_{B_k} [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu)]\mathrm{d}t + O(\|\delta_B u_k\|_1^2). \tag{47}$$

If in addition $\|\delta_A u_k\|_\infty \to 0$, then setting $r_{A2k} := o(\|\delta_A u_k\|_2^2 + |y_k(0) - \bar{y}(0)|^2)$, we have that

$$J^\mu(w_{A,k}) = J^\mu(\bar{w}) + \delta_A J_k^\mu + r_{A2k}, \tag{48}$$

with

$$\begin{aligned} \delta_A J_k^\mu := \quad & \int_0^T [\tfrac{1}{2} H_{ww}(\bar{w}, p^\mu)(\delta_A w_k)^2 + H_u(\bar{w}, p^\mu)\delta_A u_k]\mathrm{d}t \\ & + (p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}))\delta y_k(0) + \tfrac{1}{2}(\Phi^\mu)''(\bar{\eta})(\delta_A y_k(0), \delta_A y_k(T))^2. \end{aligned} \tag{49}$$

Proof. The Pontryagine linearization (33) being the sum of the ones for perturbations $u_{A,k}$ and $u_{B,k}$, it follows from lemma 2.11 that $\|\delta y_k - \delta_A y_k - \delta_B y_k\|_\infty = R_{1,k}$, and we have

$$\text{(i)} \ \ \|\delta_A y_k\|_\infty = O(\|\delta_A u_k\|_1 + |y_k(0) - \bar{y}(0)|); \quad \text{(ii)} \ \ \|\delta_B y_k\|_\infty = O(\|\delta_B u_k\|_1). \tag{50}$$

Relation (47) follows then from (37) and (50)(ii). Now by the Cauchy-Schwarz inequality:

$$\|\delta_B u_k\|_1 \leq \text{meas}(B_k)^{1/2}\|\delta_B u_k\|_2 = o(\|\delta_B u_k\|_2), \tag{51}$$

so that $\|\delta_B y_k\|_\infty = o(\|\delta_B u_k\|_2)$, and using $r_{1k} = O(r_{2k})$, obtain with (37):

$$\begin{array}{rl} J^\mu(w_k) - J^\mu(\bar{w}) = & \int_0^T [H(u_k, \bar{y}, p^\mu) - H(\bar{w}, p^\mu) + \frac{1}{2} H_{yy}(\bar{w}, p^\mu)\delta_A y_k^2 \\ & +(H_y(u_k, \bar{y}, p^\mu) - H_y(\bar{w}, p^\mu))\delta_A y_k]\text{d}t \\ & +(p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}))\delta y_k(0) \\ & +\frac{1}{2}(\Phi^\mu)''(\bar{\eta})(\delta_A y_k(0), \delta_A y_k(T))^2 + r_{2k}. \end{array} \tag{52}$$

Using (50) and(51), get

$$\int_{B_k} [H_y(u_k, \bar{y}, p^\mu) - H_y(\bar{u}, \bar{y}, p^\mu)]\delta_A y_k \text{d}t = O(\|\delta_B u_k\|_1 \|\delta_A y_k\|_\infty) = r_{2k}. \tag{53}$$

We deduce with (52)-(53) that

$$J^\mu(w_k) - J^\mu(\bar{w}) = \hat{\delta}_A J^\mu_k + \int_{B_k} [H(u_k, \bar{y}, p^\mu) - H(\bar{u}, \bar{y}, p^\mu)]\text{d}t + r_{2k}, \tag{54}$$

where

$$\begin{array}{rl} \hat{\delta}_A J^\mu_k := & \int_{A_k} [H(u_k, \bar{y}, p^\mu) - H(\bar{w}, p^\mu)]\text{d}t + \frac{1}{2}\int_0^T H_{yy}(\bar{w}, p^\mu)\delta_A y_k^2 \text{d}t \\ & + \int_{A_k} (H_y(u_k, \bar{y}, p^\mu) - H_y(\bar{w}, p^\mu))\delta_A y_k \text{d}t \\ & +(p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}))\delta y_k(0) \\ & +\frac{1}{2}(\Phi^\mu)''(\bar{\eta})(\delta_A y_k(0), \delta_A y_k(T))^2. \end{array} \tag{55}$$

Specializing to the case when $\delta_B u_k = 0$, we obtain

$$J^\mu(w_{A,k}) = J^\mu(\bar{w}) + \hat{\delta}_A J^\mu_k + r_{A2k}, \tag{56}$$

which combined with (47), (51) and (54), proves (46). Finally when $\|\delta_A u_k\|_\infty \to 0$, it follows from a Taylor expansion that $\hat{\delta}_A J^\mu_k = \delta_A J^\mu_k + r_{A2k}$, proving (48). The conclusion follows. □

3 Inequality control constraints

We assume in this section that the control constraints are parameterized by finitely many inequalities:

$$U := \{u \in \mathbb{R}^m; \;\; g(u) \leq 0\}, \tag{57}$$

where $g : \mathbb{R}^m \to \mathbb{R}^q$ is a C^2 mapping. In other words, the control constraints are defined by

$$g_i(u(t)) \leq 0, \quad \text{for a.a. } t \in (0,T), \quad i = 1, \dots, q. \tag{58}$$

We consider the "abstract" formulation where the state is a function of initial state and control. So we may write the state as $y_{u,y_0}(t)$, and define $G : \mathcal{U} \times \mathbb{R}^n \to \mathbb{R}^r$ by

$$G_i(u, y_0) := \Phi_i(y_0, y_{u,y_0}(T)), \quad i = 1, \dots, r. \tag{59}$$

By $G_{1:r_1}(u, y_0)$ we denote the (vertical) vector of components 1 to r_1 of $G(u, y_0)$. We say that the following *qualification condition* [13] (a natural infinite dimension generalization of the Mangasarian-Fromovitz condition [8]; see also [2, Section 2.3.4])) holds at $\bar{w} \in F(P)$ if

$$\begin{array}{lll} G'_{1:r_1}(\bar{u}, \bar{y}_0) \text{ is onto,} & & \\ \text{There exists } \beta > 0 \text{ and } & (\bar{v}, \bar{z}_0) \in \operatorname{Ker} G'_{1:r_1}(\bar{u}, \bar{y}_0); & \\ \quad g(\bar{u}(t)) + g'(\bar{u}(t))\bar{v}(t) & \leq -\beta, & \text{for a.a. } t \in [0,T] \\ \quad G'_i(\bar{u}, \bar{y}_0)(\bar{v}, \bar{z}_0) & \leq -\beta, & \text{for all } i > r_1 \text{ such that } G_i(\bar{u}, \bar{y}_0) = 0. \end{array} \tag{60}$$

We also define for future reference

$$\hat{J}(u, y_0) := J(u, y_{u,y_0}); \quad \hat{J}^\mu(u, y_0) := \hat{J}(u, y_0) + \mu G(u, y_0), \tag{61}$$

as well as the *augmented Hamiltonian function* by

$$H^a(u, y, p, \lambda) := H(u, y, p) + \lambda g(u) = \ell(u, y) + pf(u, y) + \lambda g(u), \tag{62}$$

where $u \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $p \in \mathbb{R}^{n*}$, and $\lambda \in \mathbb{R}^{q*}$. Given $\bar{w} = (\bar{u}, \bar{y}) \in F(P)$, we recall that the set of normal directions to K at the point $\Phi(\bar{\eta})$ was defined in (10). The costate $p^\mu \in \mathcal{P}$ associated with $\mu \in N_K(\Phi(\bar{\eta}))$ was defined as the solution of (13)(i-ii). For $\bar{w} \in F(P)$, $\mu \in N_K(\Phi(\bar{\eta}))$ such that (13)(iii) holds and $t \in [0,T]$, define

$$\begin{array}{rcl} \Lambda_t(\bar{w}, \mu) & := & \{\lambda \in \mathbb{R}^{q*}_+ \cap g(\bar{u}(t))^\perp ;\ H^a_u(\bar{w}(t), p^\mu(t), \lambda) = 0\}, \\ LM^L(\bar{w}) & := & \{(\lambda, \mu);\ \mu \in N_K(\Phi(\bar{\eta}));\ (13) \text{ holds };\ \lambda \in L^\infty(0, T, \Lambda_t(\bar{w}, \mu))\}. \end{array}$$

We call $LM^L(\bar{w})$ (the superscript L is reminiscent of Lagrange) the set of first-order multipliers, denote by $M^L(\bar{w})$ its projection on the second component, and say that $(\bar{w}, p, \lambda, \mu)$ is a *first-order* extremal if $\bar{w} \in F(P)$, $(\lambda, \mu) \in LM^L(\bar{w})$, and p is the associated costate. Remember that $J^\mu(w)$ was defined in (21). The Lagrangian for the abstract formulation is

$$\mathcal{L}(u, y_0, \lambda, \mu) := \hat{J}^\mu(u, y_0) + \langle \lambda, g(u) \rangle, \tag{63}$$

the last duality product being in the space $L^\infty(0, T, \mathbb{R}^q)$.

Theorem 3.1. *Let $\bar{w}$ be a weak solution of (P), satisfying the qualification hypothesis* (60). *Then* (i) *the sets $LM^L(\bar{w})$ and $M^L(\bar{w})$ are nonempty and bounded, and* (ii) *$M^P(\bar{w})$ is a (possibly empty) subset of $M^L(\bar{w})$.*

Proof. (i) An abstract formulation of problem (P) is

$$\operatorname*{Min}_{u,y_0} \hat{J}(u, y_0); \quad g(u) \leq 0; \quad G(u, y_0) \in K. \tag{64}$$

Let $K_g := L^\infty(0, T, \mathbb{R}^q_-)$, with associated normal cone at the point $g(\bar{u})$ denoted $N_{K_g}(g(\bar{u}))$. The corresponding set of Lagrange multipliers at point $(\bar{u}, \bar{y}_0)$ is defined as

$$(LM^L)^\sharp(\bar{w}) := \{(\lambda, \mu) \in N_{K_g}(g(\bar{u})) \times N_K(G(\bar{u}, \bar{y}_0)); D_{(u,y_0)}\mathcal{L}(\bar{u}, \bar{y}_0, \lambda, \mu) = 0\}. \tag{65}$$

The qualification hypothesis (60) being a particular case of Robinson's qualification condition [13], we know that $(LM^L)^\sharp(\bar{w})$ is nonempty and bounded in $L^\infty(0,T,\mathbb{R}^q)^* \times \mathbb{R}^{r*}$. It remains to prove that any $(\lambda,\mu) \in (LM^L)^\sharp(\bar w)$ is such that λ can be identified with some $\tilde\lambda$ in $L^\infty(0,T,\mathbb{R}^{q*})$, the norm in $L^\infty(0,T,\mathbb{R}^{q*})$ of $\tilde\lambda$ being uniform over all $(\lambda,\mu) \in (LM^L)^\sharp(\bar w)$. More precisely, we have to check the existence of $c>0$ such that, $|\langle\lambda,a\rangle| \le c\|a\|_{L^1(0,T,\mathbb{R}^q)}$, for all $a \in L^\infty(0,T,\mathbb{R}^q)$. If this holds then, as $L^\infty(0,T,\mathbb{R}^q)$ is a dense subset of $L^1(0,T,\mathbb{R}^q)$, λ has a unique extension $\tilde\lambda$ in the dual space of $L^1(0,T,\mathbb{R}^q)$, i.e., $L^\infty(0,T,\mathbb{R}^{q*})$.

Since the norm of $a \in L^\infty(0,T,\mathbb{R}^q)$ is the sum of the norms of its positive and negative parts, it suffices to check this inequality when $a \ge 0$, i.e., since $\lambda \ge 0$, $\langle\lambda,a\rangle \le c\|a\|_{L^1(0,T,\mathbb{R}^q)}$. We can write $a(t) = \alpha(t)\bar a(t)$, with $\alpha(t) = |a(t)|$ and $|\bar a(t)| = 1$. Set $h := -(g(\bar u) + g'(\bar u)\bar v)$. By (60), $\beta \le h_i(t)$, $i = 1,\dots,q$, for a.a. t. Since $a_i(t) \le \alpha(t)$, $i=1,\dots,q$, for a.a. t, we have that $\beta a(t) \le \alpha(t)h(t)$, and so

$$\beta\langle\lambda,a\rangle = \langle\lambda,\beta a\rangle \le \langle\lambda,\alpha h\rangle. \tag{66}$$

Since $\lambda \ge 0$, $a \ge 0$ and $g(\bar u) \le 0$, we have that:

$$0 \ge \langle\lambda,\alpha g(\bar u)\rangle \ge \langle\lambda,\|a\|_\infty g(\bar u)\rangle = \|a\|_\infty\langle\lambda,g(\bar u)\rangle = 0, \tag{67}$$

the last equality being the complementarity condition between elements of a convex cone and elements of the corresponding normal cone. It follows that $\langle\lambda,\alpha g(\bar u)\rangle = 0$. Combining with (66), and using (65), obtain

$$\begin{array}{ll}\beta\langle\lambda,a\rangle & \le -\langle\lambda,\alpha g'(\bar u)\bar v\rangle = -\langle\lambda,g'(\bar u)\alpha\bar v\rangle = D_u\hat J^\mu(\bar u,\bar y_0)(\alpha\bar v) \\ & \le \|D_u\hat J^\mu(\bar u,\bar y_0)\|_\infty\|\alpha\|_1\|\bar v\|_\infty = \|D_u\hat J^\mu(\bar u,\bar y_0)\|_\infty\|\bar v\|_\infty\|a\|_1,\end{array} \tag{68}$$

which, since μ remains in a bounded set, gives the desied estimate. Point (i) follows.
(ii) Let $\mu \in M^P(\bar w)$. Then $\bar u$ is solution of the problem

$$\operatorname*{Min}_{u\in L^\infty(0,T,\mathbb{R}^m)} \int_0^T H(u(t),\bar y(t),p^\mu(t))\mathrm{d}t; \quad g(u(t)) \le 0, \quad \text{for a.a. } t \in (0,T). \tag{69}$$

In view of the qualification condition, there exists some λ such that $(\lambda,\mu) \in (LM^L)(\bar w)$. The conclusion follows. □

4 Quadratic growth with initial-final state constraints

When dealing with initial-final state constraints we need to combine the previous decomposition principle with a certain restoration hypothesis, for which we will give sufficient conditions.

Given a Lagrange extremal $\bar w$, and an arbitrary $\bar\mu \in \mathrm{ri}(M^L(\bar w))$ (the relative interior of $M^L(\bar w)$), denote the set of strictly (non strictly) complementary active constraints by

$$\begin{array}{ll} I_+ & := \{1,\dots,r_1\} \cup \{r_1 < i \le r;\ \bar\mu_i > 0\}, \\ I_0 & := \{r_1 < i \le r;\ \Phi_i(\bar\eta) = 0\} \setminus I_+. \end{array} \tag{70}$$

Similarly to (20), all $\mu \in \mathrm{ri}(M^L(\bar w))$ have the same set of positive components, as can be easily checked, so that the definition does not depend on the choice of the particular $\bar\mu$. Define

$$K_+ := \{\theta \in I\!R^r;\ \theta_i = 0,\, i \in I_+,\ \theta_i \le 0,\, i \in I_0\}. \tag{71}$$

Note that, if $\bar\mu \in \mathrm{ri}(M^L(\bar w))$, then $K_+ = K \cap \bar\mu^\perp$. The function

$$d(\eta) := \sum_{i \in I_0} \Phi_i(\eta)_+ + \sum_{i \in I_+} |\Phi_i(\eta)| \tag{72}$$

is the distance of the initial-final state constraint to the set K_+, in the $L^1(\mathbb{R}^r)$ norm (the unique projection of $\theta \in \mathbb{R}^r$ in this norm being θ' defined by $\theta'_i = 0$ if $i \in I_+$, and $\theta'_i = \min(\theta_i, 0)$ otherwise). We have that

$$d(\eta) = O(-\bar\mu\Phi(\eta)) \quad \text{whenever } w \in F(P), \tag{73}$$

since then $\Phi_{1:r_1}(\eta) = 0$ and $\Phi_{r_1+1:r}(\eta) \le 0$. Call *Pontryagine norm* the following one:

$$\|w\|_P := \|u\|_1 + \|y\|_\infty. \tag{74}$$

For given $\varepsilon_D > 0$, and $u \in \mathcal{U}$, define the set times of ε_D-deviation of $\bar u$ as

$$B_{\varepsilon_D}(u) := \left\{ t \in (0,T);\ \ |u(t) - \bar u(t)| \ge \frac{\|u - \bar u\|_1}{\varepsilon_D} \right\}. \tag{75}$$

We have the following relation.

Lemma 4.1. *For any $u \in \mathcal{U}$, we have that* $\mathrm{meas}(B_{\varepsilon_D}(u)) \le \varepsilon_D$.

Proof. The conclusion follows from the relations below:

$$\mathrm{meas}(B_{\varepsilon_D}(u)) = \int_0^T \mathbf{1}_{\{|u(t)-\bar u(t)| \ge \frac{\|u-\bar u\|_1}{\varepsilon_D}\}} \mathrm{d}t \le \frac{\varepsilon_D}{\|u - \bar u\|_1} \int_0^T |u(t) - \bar u(t)| \mathrm{d}t = \varepsilon_D.$$

□

Definition 4.2. Let $\bar w$ be a Pontryagine extremal. We say that the *restoration property* (for the initial-final state constraints) is satisfied at $\bar w \in F(P)$ for $\bar\mu \in M^P(\bar w)$ in the Pontryagine sense if there exists $\varepsilon_P > 0$ and $\varepsilon_B > 0$ such that, for any trajectory w such that $\|w - \bar w\|_P \le \varepsilon_P$ and $u(t) \in U$ a.e., and measurable set $B \subset (0,T)$ such that $\mathrm{meas}(B) \le \varepsilon_B$ over which u and $\bar u$ coincide, there exists $w' \in F(P)$ such that $u' = \bar u$ on B and

$$\|w' - w\|_\infty = O(d(\eta)); \quad J(w') = J^{\bar\mu}(w) + O(\|w - \bar w\|_P d(\eta)). \tag{76}$$

Let us give a sufficient condition for the restoration property. Denote the kernel of derivatives of almost active control constraints (relative to $\bar w \in \mathcal{W}$), parameterized by $\varepsilon_R > 0$ (this notation is a reminder of "restoration");

$$\mathcal{U}_{\varepsilon_R} := \left\{ \begin{array}{r} v \in \mathcal{U};\ g'_i(\bar u(t))v(t) = 0 \ \text{ whenever } \ g_i(\bar u(t)) \ge -\varepsilon_R, \\ i = 1, \dots, q, \ \text{ for a.a. } \ t \in (0,T) \end{array} \right\}. \tag{77}$$

We will call *special qualification condition* the Mangasarian-Fromovitz qualification condition [8] for constraints on initial-final state in K_+, over the Banach space $E_{\varepsilon_R} := \mathcal{U}_{\varepsilon_R} \times \mathbb{R}^n$. Setting $\bar e := (\bar u, \bar y(0))$, it can be formulated as:

$$\left\{ \begin{array}{ll} \text{(i)} & \text{There exists } e^j \subset E_{\varepsilon_R},\ j = 1, \dots, |I_+|, \text{ such that} \\ & \{G'_{I_+}(\bar e)e^j\}_{j=1,\dots,|I_+|} \text{ is an independent family,} \\ \text{(ii)} & \text{There exists } \ e^0 \in E_{\varepsilon_R} \text{ such that } \ G'_{I_+}(\bar e)e^0 = 0;\ G'_{I_0}(\bar e)e^0 < 0. \end{array} \right. \tag{78}$$

We first note that this condition implies the uniqueness of the "μ part" of the multiplier.

Lemma 4.3. *Let $\bar{w} \in F(P)$ satisfy the special qualification condition* (78). *Let $(\lambda^i, \mu^i) \in LM^L(\bar{w})$, for $i = 1, 2$. Then $\mu^1 = \mu^2$.*

Proof. Let $(\lambda^i, \mu^i) \in LM^L(\bar{w})$, for $i = 1, 2$. Set $\lambda := \lambda^2 - \lambda^1$ and $\mu := \mu^2 - \mu^1$. Let $e = (v, z_0) \in E_{\varepsilon_R}$. Then $\lambda(t)g'(\bar{u}(t))v(t) = 0$ for a.a. t, and therefore, taking the difference of equations of stationarity of Lagrangians (last relation in (65)), $\mu G'(\bar{u}, \bar{y}_0)(v, z_0) = 0$. Since by (78) $G'_{I_+}(\bar{e})$ is onto from E_{ε_R} onto $\mathbb{R}^{|I_+|}$, it follows that $\mu = 0$, as was to be proved. □

When dealing with a Pontryagine perturbation $w_k = (u_k, y_k)$, in the process of restoration, it is useful to freeze control variations set of small measure where $|u_k - \bar{u}|$ is large. So, given an arbitrary measurable subset B (of small measure) of $[0, T]$, we need the following notation:

$$\mathcal{U}_{\varepsilon_R}(B) := \{v \in \mathcal{U}_{\varepsilon_R};\ v(t) = 0 \ \text{ for all } \ t \in B\}; \quad E_{\varepsilon_R}(B) := \mathcal{U}_{\varepsilon_R}(B) \times \mathbb{R}^n. \tag{79}$$

Denote by $e^j = (u^j, z_0^j)$ the components of vectors e^j in (78), $j = 0, \ldots, |I_+|$, and define $e_B^j = (u_B^j, z_{0B}^j)$ in $E_{\varepsilon_R}(B)$, for $j = 1, \ldots, |I_+|$, by

$$z_{0B}^j = z_0^j; \quad u_B^j(t) = u^j(t) \quad \text{ if } t \notin B,\ u_B^j(t) = 0 \text{ otherwise.} \tag{80}$$

Since the functions u^j, $j = 1, \ldots, |I_+|$, are essentially bounded, we have that

$$|G'(\bar{e})(e_B^j - e^j)| = O(\text{meas}(B)), \quad j = 1, \ldots, |I_+|. \tag{81}$$

For $j = 0$, we proceed in two steps. First define $\hat{e}_B^0 = (\hat{u}_B^0, z_{0B}^0)$ in the same way, i.e.,

$$z_{0B}^0 = z_0^0; \quad \hat{u}_B^0(t) = u^0(t) \quad \text{ if } t \notin B,\ \hat{u}_B^0(t) = 0 \text{ otherwise.} \tag{82}$$

Then

$$|G'(\bar{e})(\hat{e}_B^0 - e^0)| = O(\text{meas}(B)). \tag{83}$$

Now define

$$e_B^0 := \hat{e}_B^0 + \sum_{j=1}^{|I_+|} \bar{\alpha}_j e_B^j, \tag{84}$$

where the coefficient $\bar{\alpha}$ is solution of

$$G'_{I_+}(\bar{e}) \left(\hat{e}_B^0 + \sum_{j=1}^{|I_+|} \alpha_j e_B^j \right) = 0. \tag{85}$$

Lemma 4.4. *Let $\bar{w} \in F(P)$ satisfy the special qualification condition* (78). *Then there exists $\varepsilon_B > 0$ such that, for any measurable subset B of $[0, T]$, if* $\text{meas}(B) \leq \varepsilon_B$, *then* (85) *has a unique solution, and $\{e_B^0, \ldots, e_B^{|I_+|}\}$ are such that*

$$\begin{cases} \text{(i)} \ |\bar{\alpha}| = O(\text{meas}(B)); \quad \text{(ii)} \ \|e_B^0 - e^0\|_1 = O(\text{meas}(B)), \\ \text{(iii)} \ G'_{I_0}(\bar{e})e_B^0 < \frac{1}{2} G'_{I_0}(\bar{e})e^0 < 0. \end{cases} \tag{86}$$

Proof. It suffices to prove (86)(i). We see that this is just the analysis of the square linear system (85). We know that the matrix inversion is locally Lipschitz over the set of invertible square matrices of a given dimension. When $\text{meas}(B) = 0$, the solution is $\bar{\alpha} = 0$; otherwise the perturbation of the matrix and of the r.h.s. is, by (81) and (83), of order $O(\text{meas}(B))$. The result follows. □

Lemma 4.5. *Let $\bar{w} \in F(P)$ satisfy the special qualification condition* (78)*, for some $\varepsilon_R > 0$. Let $M > \|\bar{u}\|_\infty$. Then, if $\varepsilon_B > 0$ and $\varepsilon_P > 0$ are small enough, for any measurable $B \subset (0,T)$ such that* $\text{meas}(B) \leq \varepsilon_B$*, any trajectory w such that $\|u\|_\infty \leq M$ and $\|w - \bar{w}\|_P \leq \varepsilon_P$, there exists a trajectory $w'' \in \mathcal{W}$ (that does not in general satisfy the control constraints) such that*

$$\|w'' - w\|_\infty = O(d(\eta)); \quad G(u'', y_0'') \in K_+; \quad u'' - u \in E_{\varepsilon_R}(B). \tag{87}$$

Proof. Denote by F_B the finite dimensional space spanned by $e_B^0, \dots, e_B^{|I_+|}$, endowed with the norm of $\mathcal{U} \times \mathbb{R}^n$. Consider the mapping $\mathcal{T}$ from $\mathcal{U} \times \mathbb{R}^n$ into itself, that with a given $e := (u, y_0) \in \mathcal{U} \times \mathbb{R}^n$ associates $e + \delta e$, where $\delta e = (\delta u, \delta y_0)$ is a vector of F_B satisfying the following conditions:

$$\text{(i)} \;\; G_{I_+}(e) + G'_{I_+}(\bar{e})\delta e = 0; \quad \text{(ii)} \;\; G_{I_0}(e) + G'_{I_0}(\bar{e})\delta e \leq 0, \tag{88}$$

so we can write $\delta e = \sum_{j=0}^{|I_+|} \theta_j e_B^j$. Coefficients θ_j, for $j = 1$ to $|I_+|$, are uniquely determined by (88)(i), and for θ_0 we choose the smallest possible nonnegative value. In view of lemma 4.4, we have that, for some $c_1 > 0$, if ε_B is small enough:

$$\|\delta e\|_\infty \leq c_1 \left(|G_{I_+}(e)| + |G_{I_0}(e)_+| \right). \tag{89}$$

Over the set $\mathcal{V}_{1,M} := \{w \in \mathcal{W};\ \|w - \bar{w}\|_P \leq 1;\ \|u\|_\infty \leq M\}$, the mapping $e \mapsto G'(e)$ is, when E_{ε_R} is endowed with the L^1 norm, Lipschitz from E_{ε_R} into $L(E_{\varepsilon_R}, \mathbb{R}^r)$, so that for some $c_2 > 0$:

$$|G(e + \delta e) - G(e) - G'(e)\delta e| \leq c_2 \|\delta e\|_P^2. \tag{90}$$

In view of (88) and (89), and since (making the abuse of notation of denoting also by P the induced norm for $G'(e)$) $\|G'(\bar{e}) - G'(e)\|_P \to 0$ when $\|u\|_\infty \leq M$ and $\|w - \bar{w}\|_P \to 0$, it follows that when $\|w - \bar{w}\|_P$ is small enough, say $\|w - \bar{w}\|_P \leq \varepsilon_{P1}$ we have that

$$|G_{I_+}(e + \delta e)| + |G_{I_0}(e + \delta e)_+| \leq \tfrac{1}{2}(|G_{I_+}(e)| + |G_{I_0}(e)_+|). \tag{91}$$

Consider the sequence defined by $e^{k+1} := \mathcal{T}(e^k)$, for $k \in I\!N$. It follows from (89) and (91) that, if the trajectory w corresponding to e^0 satisfies the hypotheses of the lemma for small enough ε_P, then the sequence w^k is well-defined, remains in $\mathcal{V}_{1,M}$, and converges to some e'' such that $\|e'' - e^0\| \leq 2c_1 \left(|G_{I_+}(e)| + |G_{I_0}(e)_+| \right)$. The corresponding associated state y'' is such that $w'' := (u'', y'')$ satisfies (87). □

Given $w \in \mathcal{W}$ satisfying the state equation (2), we consider the following measure of constraint defect:

$$D(w) := \|g(u)_+\|_\infty + \sum_{i=1}^{r_1} |\Phi_i(\eta)| + \sum_{i=r_1+1}^{r} \Phi_i(\eta)_+. \tag{92}$$

It is well-known that Robinson's qualification condition (60) implies the following local Hoffman bound: there exists $c > 0$ such that, if $w \in \mathcal{W}$ satisfies the state equation (2), and is close enough to $\bar{w}$, then there exists $\hat{w} \in F(P)$ such that

$$\|\hat{w} - w\|_{\mathcal{W}} \leq cD(w). \tag{93}$$

We now check that this inequality still holds in some cases if $w \in \mathcal{W}$ is close enough to $\bar{w}$ in the Pontryagine norm.

Lemma 4.6. *Robinson's qualification condition* (60) *implies a local Hoffman bound, in the following sense. For any $M > \|\bar{u}\|_\infty$, there exists $c > 0$ and $\varepsilon_M > 0$ such that, for any measurable subset B of $[0,T]$ and any trajectory w, such that $\|u\|_\infty \leq M$, $\|w - \bar{w}\|_P \leq \varepsilon_M$, $\text{meas}(B) \leq \varepsilon_M$, and $g(u(t)) \leq 0$, a.e. on B, there exists $\hat{w} \in F(P)$ such that* (93) *holds, and $\hat{u}(t) = u(t)$ a.e. on B.*

Proof. The proof is somewhat in the spirit of the one of lemma 4.5, but including the control constraints. The first step is similar to lemma 4.4. Denote $\bar{e} := (\bar{u}, \bar{y}(0))$. By Robinson's condition (60) there exist $\{e^1, \ldots, e^{r_1}\}$ in $\mathcal{U} \times \mathbb{R}^n$ such that $\{G'_{1:r_1}(\bar{e})e^i\}_{1 \leq i \leq r_1}$ is of rank r_1. Denote by e^i_B the vector obtained by setting to zero the components of the control over the measurable subset B of $[0,T]$. Then

$$|G'(\bar{e})(e^j_B - e^j)| = O(\text{meas}(B)), \quad j = 1, \ldots, r_1. \tag{94}$$

Let $e^0 := (z\bar{v}, \bar{z}_0)$ and $\beta > 0$ be the direction and constant stated in (60). By arguments similar to those in the proof of lemma 4.4 we obtain the existence of a direction $e^0_B = (v^0_B, z^0_B) \in \mathcal{U} \times \mathbb{R}$ such that

$$\left\{\begin{array}{ll} \text{(i)} & \|e^0_B - e^0\|_P = O(\text{meas}(B)); \\ \text{(ii)} & g(\bar{u}(t)) + g'(\bar{u}(t))v^0_B(t) \leq -\frac{1}{2}\beta, \quad \text{for a.a. } t \in [0,T] \setminus B, \\ \text{(iii)} & v^0_B(t) = 0, \quad \text{for a.a. } t \in B, \\ \text{(iv)} & G'_i(\bar{e})e^0_B = 0, \quad i = 1, \ldots, r_1, \\ \text{(v)} & G'_i(\bar{e})e^0_B \leq -\frac{1}{2}\beta, \quad \text{for all } i > r_1 \text{ such that } G_i(\bar{u}, \bar{y}_0) = 0. \end{array}\right. \tag{95}$$

Let $\mathcal{T}$ be the mapping from $\mathcal{U} \times \mathbb{R}^n$ into itself, that with a given $(u, y_0) \in \mathcal{U} \times \mathbb{R}^n$ associates $(u, y_0) + \delta e$, where $\delta e = (\delta u, \delta y_0)$ is a vector of $\text{Span}\{e^0_B, \ldots, e^{r_1}_B\}$ of minimum norm satisfying the following conditions:

$$\left\{\begin{array}{ll} \text{(i)} & g(u(t)) + g'(\bar{u}(t))\delta u(t) \leq 0, \quad \text{for a.a. } t \in [0,T], \\ \text{(ii)} & G_i(e) + G'_i(\bar{e})\delta e = 0 \quad i = 1, \ldots, r_1, \\ \text{(iii)} & G_i(e) + G'_i(\bar{e})\delta e \leq 0, \quad \text{for all } i > r_1 \text{ such that } G_i(\bar{u}, \bar{y}_0) = 0. \end{array}\right. \tag{96}$$

We obtain by arguments similar to those at the end of the proof of lemma 4.5 that the sequence computed by the mapping $\mathcal{T}$, with initial point (u, y_0) such that $\|u\|_\infty \leq M$ and $\|w - \bar{w}\|_P$ is small enough converges to a point $(\hat{u}, \hat{y}_0)$ such that the corresponding associated state $\hat{y}$ satisfies the conclusion of the lemma. □

Lemma 4.7. *Let $\bar{w} \in F(P)$ satisfy* (60) *and* (78)*. Then the restoration property (definition 4.2) is satisfied in the Pontryagine sense.*

Proof. Let the trajectory w satisfy $g(u(t)) \leq 0$ a.e. on $[0,T]$. Given a constant $M > \max\{\|u\|_\infty, \|\bar{u}\|_\infty\}$, choose $\varepsilon_B > 0$ as in lemma 4.5, and set $\varepsilon_D = \varepsilon_B$,

$B = B_{\varepsilon_D}$, where B_{ε_D} is the deviation set introduced in (75). Then, by lemma 4.1, $\text{meas}\, B_{\varepsilon_D} \leq \varepsilon_D$ and by definition of the deviation set we have that

$$|u(t) - \bar{u}(t)| \leq {\varepsilon_D}^{-1} \|w - \bar{w}\|_P, \quad \text{for a.a. } t \in (0,T) \setminus B_{\varepsilon_D}. \tag{97}$$

By lemma 4.5, there exists a trajectory $w'' \in W$ satisfying (87), and then using (87) and (97) we get that, for some $c_1 > 0$ and for a.a. $t \in (0,T) \setminus B_{\varepsilon_D}$:

$$\begin{aligned} g_i(u''(t)) &= g_i(u(t)) + g_i'(u(t))(u''(t) - u(t)) + O(d(\eta)^2), \\ &\leq g_i(u(t)) + g_i'(\bar{u}(t))(u''(t) - u(t)) + c_1 \left(\|w - \bar{w}\|_P d(\eta) + d(\eta)^2 \right). \end{aligned} \tag{98}$$

If $t \in B_{\varepsilon_D}$ then $g_i(u''(t)) = g_i(u(t)) \leq 0$. Otherwise, if $g_i(\bar{u}(t)) \geq -\varepsilon_R$ (we remind that ε_R was introduced in (77)), since $g_i(u(t)) \leq 0$ and $g_i(\bar{u}(t))(u''(t) - u(t)) = 0$, we get

$$g_i(u''(t)) \leq c_1 \left(\|w - \bar{w}\|_P d(\eta) + d(\eta)^2 \right). \tag{99}$$

If $g_i(\bar{u}(t)) < -\varepsilon_R$ and $t \notin B_{\varepsilon_D}$, then by (97), whenever ε_P is small enough, we have $g_i(u(t)) < -\frac{1}{2}\varepsilon_R$. Using lemma 4.4, (98) and the estimate $d(\eta) = O(\|w - \bar{w}\|_P)$, we obtain:

$$g_i(u''(t)) \leq -\tfrac{1}{2}\varepsilon_R + O(d(\eta)) \leq -\tfrac{1}{2}\varepsilon_R + O(\varepsilon_P) \leq 0, \tag{100}$$

so that finally with (99), and using again $d(\eta) = O(\|w - \bar{w}\|_P)$:

$$g(u''(t)) \leq c_2 \|w - \bar{w}\|_P d(\eta), \qquad \text{a.e. on } (0,T). \tag{101}$$

Let μ be the element of the singleton $M^P(\bar{w})$. We next apply lemma 2.12 at point w, denoting therefore $\hat{p}^\mu$ the costate evaluated at the point w (and not $\bar{w}$). Note that $o(\cdot)$ and $O(\cdot)$ in the statement of this theorem come from Taylor expansions on bounded sets, and hence, are uniform over the reference point. Since $\|w'' - w\|_\infty = O(d(\eta))$, we deduce that, denoting by p^μ the costate evaluated at the point $\bar{w}$ and associated with μ:

$$\begin{aligned} J^\mu(w'') - J^\mu(w) = \; & \int_0^T H_u(w, \hat{p}^\mu)(u'' - u)\mathrm{d}t \\ & + (\hat{p}^\mu(0) + (\Phi^\mu_{y_0})(\eta))(y''(0) - y(0)) + O(d(\eta)^2). \end{aligned} \tag{102}$$

Since $\|w - \bar{w}\|_P \leq \varepsilon_P$, using

$$\begin{aligned} \left| \left(\hat{p}^\mu(0) + \Phi^\mu_{y_0}(\eta) \right) - \left(p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}) \right) \right| &= O(\|w - \bar{w}\|_P), \\ \|H_u(w, \hat{p}^\mu) - H_u(\bar{w}, p^\mu)\|_1 &= O(\|w - \bar{w}\|_P) \end{aligned} \tag{103}$$

as well as the relations $p^\mu(0) + \Phi^\mu_{y_0}(\bar{\eta}) = 0$ and $\|w'' - w\|_\infty = O(d(\eta))$, we can write

$$J^\mu(w'') - J^\mu(w) = \int_0^T H_u(\bar{w}, p^\mu)(u'' - u)\mathrm{d}t + O(\|w - \bar{w}\|_P d(\eta)). \tag{104}$$

Since $u'' - u \in \mathcal{U}_{\varepsilon_R}$, we deduce from the first-order optimality conditions that

$$H_u(\bar{w}(t), p^\mu(t))(u''(t) - u(t)) = -\lambda_t g'(\bar{u}(t))(u''(t) - u(t)) = 0. \tag{105}$$

With (104)-(105), we get $J^\mu(w'') - J^\mu(w) = O(\|w - \bar{w}\|_P d(\eta))$. Now by the definition (87) of w'', we have that $J(w'') = J^\mu(w'')$. By lemma 4.6, (87) and (101), there exists $w' \in F(P)$, such that $\|w' - w''\|_{\mathcal{W}} = O(\|w - \bar{w}\|_P d(\eta))$ and $u'(t) = u''(t) = \bar{u}(t)$ a.e. on B_{ε_D}. Consequently, $J(w') - J(w'') = O(\|w - \bar{w}\|_P d(\eta))$; therefore (76) holds. The conclusion follows. □

As shows the following theorem, the quadratic growth condition for the Hamiltonian makes a bridge between the notions of weak and bounded strong quadratic growth of the cost functional J.

Theorem 4.8. a) *Let $\bar{w} \in F(P)$ satisfy the qualification condition* (60). *Then the bounded strong quadratic growth condition* (9) *at $\bar{w}$ implies the three following conditions:* (i) *the weak minimum quadratic growth condition* (8)*,* (ii) *Pontryagine's principle with hypothesis of unique minimum of the Hamiltonian over U for some $\mu \in M^P(\bar{w})$, and* (iii) *the local quadratic growth condition for Hamiltonian* (30)*, for some $\mu \in M^P(\bar{w})$.*
b) *Conversely, if* (i)-(iii) *holds as well as the restoration property* (76)*, then the bounded strong quadratic growth condition holds at $\bar{w}$.*

Proof. a) Let $\bar{w}$ satisfy the bounded strong quadratic growth condition. Then of course the condition of weak quadratic growth holds, and in addition, for any $M > \|\bar{u}\|_\infty$, Pontryagine's principle holds for the problem of minimizing $J(w) - \alpha_M \|w - \bar{w}\|_2^2$ over the set

$$\mathcal{W}_M := \{w \in F(P);\ |u(t)| \leq M \text{ a.e.}\}. \tag{106}$$

from which the local quadratic growth condition for Hamiltonian function (30) follows, as well as the hypothesis of unique minimum of the Hamiltonian over U for some $\mu \in M^P(\bar{w})$.
b) Assume that $\bar{w}$ satisfies conditions (i)-(iii) of the theorem, as well as the restoration property (76). If it does not satisfy the bounded strong quadratic growth condition, then there exists a sequence of bounded strong perturbation w_k such that

$$J(w_k) \leq J(\bar{w}) + o(\|w_k - \bar{w}\|_2^2). \tag{107}$$

By Corollary 2.8, w_k is a Pontryagine perturbation of $\bar{w}$. Let $M > \sup_k \|u_k\|_\infty$. We know that (32) holds for some constants $\alpha > 0$ and $\varepsilon_M > 0$. Let $\varepsilon > 0$ be such that $\alpha\varepsilon^2 < \varepsilon_M$, and set

$$B_k := \{t \in (0,T);\ |u_k(t) - \bar{u}(t)| > \varepsilon\}; \quad A_k := (0,T) \setminus B_k. \tag{108}$$

In view of lemma 2.3 we may assume that conditions (ii) and (iii) hold for the same $\mu \in \mathrm{ri}(M^P(\bar{w}))$. By (30), since $\mathrm{meas}(B_k) \to 0$ by lemma 4.1, we can write, ←
by theorem 2.13 and using (32):

$$\begin{aligned} J^\mu(w_k) &= J^\mu(w_{A,k}) + \int_{B_k} [H(u_k, \bar{y}, p^\mu) - H(\bar{w}, p^\mu)]\mathrm{d}t + r_{2k} \\ &\geq J^\mu(w_{A,k}) + \ \alpha\varepsilon^2 \,\mathrm{meas}(B_k) + r_{2k}. \end{aligned} \tag{109}$$

Applying the restoration property (76) to $w_{A,k}$, get the existence of $w'_{A,k} \in F(P)$ such that

$$\left\{ \begin{array}{l} \text{(i) } \|w'_{A,k} - w_{A,k}\| = O(d(\eta_{A,k})); \\ \text{(ii) } J(w'_{A,k}) = J^\mu(w_{A,k}) + O(\|w_{A,k} - \bar{w}\|_P d(\eta_{A,k})). \\ \text{(iii) } u'_{A,k}(t) = u_{A,k}(t) = \bar{u}(t) \ \text{ a.e. on } \ B_k. \end{array} \right. \tag{110}$$

Combining (110)(ii) with (109), we obtain

$$J^\mu(w_k) \geq J(w'_{A,k}) + \alpha\varepsilon^2 \,\mathrm{meas}(B_k) + O(\|w_{A,k} - \bar{w}\|_P d(\eta_{A,k})) + r_{2k}. \tag{111}$$

On the other hand, since $\|y_{B,k} - \bar{y}\|_\infty = O(\text{meas}(B_k))$, we have that

$$\Phi(\eta_k) = \Phi(\eta_{A,k}) + O(\text{meas}(B_k)). \tag{112}$$

We deduce that

$$d(\eta_{A,k}) = d(\eta_k) + O(\text{meas}(B_k)). \tag{113}$$

Also, since $w_k \in F(P)$, and hence, $\Phi_{1:r_1}(\eta_k) = 0$ and $\Phi_{r_1+1:r}(\eta_k) \leq 0$, combining (111) and (113), and finally with (73), get

$$\begin{array}{rl} J(w_k) & \geq J(w'_{A,k}) + \alpha\varepsilon^2 \,\text{meas}(B_k) - \mu\Phi(\eta_k) + O(\|w_{A,k} - \bar{w}\|_P d(\eta_{A,k})) + r_{2k} \\ & \geq J(w'_{A,k}) + \frac{1}{2}\alpha\varepsilon^2 \,\text{meas}(B_k) - \mu\Phi(\eta_k) + O(\|w_{A,k} - \bar{w}\|_P d(\eta_k)) + r_{2k} \\ & \geq J(w'_{A,k}) + \frac{1}{2}\alpha\varepsilon^2 \,\text{meas}(B_k) - \frac{1}{2}\mu\Phi(\eta_k) + r_{2k}. \end{array} \tag{114}$$

Since $w'_{A,k}$ is feasible, by (110)(i), we have that

$$\begin{array}{rl} \|w'_{A,k} - \bar{w}\| & = O\left(\|u'_{A,k} - \bar{u}\|_\infty + |y'_{A,k,0} - \bar{y}_0|\right) \\ & = O\left(\|u_{A,k} - \bar{u}\|_\infty + |y_{A,k,0} - \bar{y}_0| + d(\eta_{A,k})\right) = O(\varepsilon) + o(1), \end{array} \tag{115}$$

then for small enough $\varepsilon > 0$, since $\bar{w}$ is a weak minimum, we have that $J(w'_{A,k}) \geq J(\bar{w})$, and so, using $\frac{1}{2}\mu\Phi(\eta_k) \leq 0$, (114) implies

$$J(w_k) \geq J(\bar{w}) + \tfrac{1}{2}\alpha\varepsilon^2 \,\text{meas}(B_k) + r_{2k}. \tag{116}$$

In view of (107), we must have $\text{meas}(B_k) = r_{2k}$. By lemma 4.6 and (113), there exists $w'_k \in F(P)$ such that $\|w'_k - w_{A,k}\|_\infty = r_{2k}$, and therefore $\|w'_k - \bar{w}\|_2^2 = \|w_{A,k} - \bar{w}\|_2^2 + r_{2k}$ so that, using theorem 2.13 for the first equality, when $\varepsilon > 0$ is small enough,

$$\begin{array}{rl} J(w_k) = J(w_{A,k}) + r_{2k} = J(w'_k) + r_{2k} & \geq J(\bar{w}) + \frac{1}{2}\alpha\|w_{A,k} - \bar{w}\|_2^2 + r_{2k}, \\ & = J(\bar{w}) + \frac{1}{2}\alpha\|w_k - \bar{w}\|_2^2 + r_{2k}, \end{array} \tag{117}$$

contradicting (107). □

5 Second-order necessary or sufficient conditions

5.1 Critical directions

Since the qualification hypothesis (60) is a particular case of Robinson's qualification condition, the second-order optimality condition due to Cominetti [3] (see also [2, Thm. 3.45]) holds. We denote $\bar{y}(0)$ as $\bar{y}_0$, and recall that $\hat{J}(u, y_0) := J(u, y_{u,y_0})$, where y_{u,y_0} is the solution of the state equation (2), with initial condition $y(0) = y_0$, and that G was defined in (59). In order to state it, reminding the notations in (61) and (63), define the set of active inequalities at time t and for the initial-final state constraints:

$$I_t := \{1 \leq i \leq q;\ g_i(\bar{u}(t)) = 0\}; \quad I_F := \{r_1 + 1 \leq j \leq r;\ G_j(\bar{u}, \bar{y}_0) = 0\}. \tag{118}$$

Set $\mathcal{U}_2 := L^2(0, T, \mathbb{R}^m)$. The linear mappings $\hat{J}'(\bar{u}, \bar{y}_0)$ and $G'(\bar{u}, \bar{y}_0)$ have a unique extension over $\mathcal{U}_2 \times \mathbb{R}^n$ that will be denoted in the same way. We define the set of *extended* tangent directions to the control and initial-final state constraints (they are extended in the sense that we take L^2 spaces instead of L^∞):

$$T_g(\bar{u}) := \{v \in \mathcal{U}_2;\ g'_{I_t}(\bar{u}(t))v(t) \le 0,\ \text{a.a. } t \in (0,T)\}, \tag{119}$$

$$T_\Phi(\bar{u},\bar{y}_0) := \{(v,z_0) \in \mathcal{U}_2 \times \mathbb{R}^n;\ G'_{1:r_1}(\bar{u},\bar{y}_0)(v,z_0) = 0;\ G'_{I_F}(\bar{u},\bar{y}_0)(v,z_0) \le 0\}, \tag{120}$$

the set of *extended* critical directions:

$$C_2(\bar{u},\bar{y}_0) := \{(v,z_0) \in T_\Phi(\bar{u},\bar{y}_0);\ v \in T_g(\bar{u});\ \hat{J}'(\bar{u},\bar{y}_0)(v,z_0) \le 0\}. \tag{121}$$

The set of critical directions (in the original space) is

$$C_\infty(\bar{u},\bar{y}_0) := \{(v,z_0) \in C_2(\bar{u},\bar{y}_0);\ \ v \in L^\infty(0,T,\mathbb{R}^m)\}. \tag{122}$$

Finally $T^2_-(g(\bar{u}), g'(\bar{u})v)$ stands for the second-order tangent set to $L^\infty(0,T,\mathbb{R}^q_-)$ at the point $g(\bar{u})$, in the direction $g'(\bar{u})v$, i.e., for $s > 0$:

$$T^2_-(g(\bar{u}), g'(\bar{u})v) = \{\hat{v} \in L^\infty(0,T,\mathbb{R}^q);\ \operatorname*{supess}_t(g(\bar{u}) + sg'(\bar{u})v + \tfrac{1}{2}s^2\hat{v}) \le o(s^2)\}. \tag{123}$$

By Cominetti [3], we have that (as usual, $\sigma_K(\cdot)$ denotes the support function to a set K, i.e. the supremum of duality products with elements of K, and the corresponding term in (124) is called "sigma-term"):

Theorem 5.1. *Let $\bar{w} = (\bar{u},\bar{y})$ be a weak solution of (P), satisfying the qualification hypothesis* (60). *Then*

$$\max_{(\lambda,\mu)\in LM^L(\bar{w})} \left(\mathcal{L}_{(u,y_0)^2}(\bar{u},\bar{y}_0,\lambda,\mu)(v,z_0)^2 - \sigma(\lambda, T^2_-(g(\bar{u}), g'(\bar{u})v)\right) \ge 0, \quad \textit{for all } (v,z_0) \in C_\infty(\bar{u},\bar{y}_0). \tag{124}$$

Note that the second-order tangent set in L^∞ has no practical characterization (see however [4]). We do not detail the proof of theorem 5.1 since it is a standard application of [3] (note that since the constraint on the initial-final state consist in a finite number of inequalities they have no contribution to the "sigma term") and since we next prove a stronger result. Let us denote

$$\Omega(v,z_0) := \max_{(\lambda,\mu)\in LM^L(\bar{w})} \mathcal{L}_{(u,y_0)^2}(\bar{u},\bar{y}_0,\lambda,\mu)(v,z_0)^2. \tag{125}$$

Note that the maximum is indeed attained since the function to be maximized is continuous for the weak∗ topology, the set of multipliers is bounded in the L^∞ norm, and any bounded sequence in an L^∞ space has a weak∗ converging subsequence. Since the sigma term is nonpositive (e.g. [2, Equation (3.110)]), and $C_\infty(\bar{u},\bar{y}_0) \subset C_2(\bar{u},\bar{y}_0)$, a sufficient condition for (124) is

$$\Omega(v,z_0) \ge 0, \quad \text{for all } (v,z_0) \in C_2(\bar{u},\bar{y}_0). \tag{126}$$

Note that the above condition makes sense since, for any $(\lambda,\mu) \in LM^L(\bar{w})$, the quadratic form $(v,z_0) \mapsto \mathcal{L}_{(u,y_0)^2}(\bar{u},\bar{y}_0,\lambda,\mu)(v,z_0)^2$, defined over $\mathcal{U} \times \mathbb{R}^n$, has a unique extension over $\mathcal{U}_2 \times \mathbb{R}^n$.

Theorem 5.2. *Let $\bar{w} = (\bar{u},\bar{y})$ be a weak solution of (P), satisfying the qualification hypotheses* (60) *and* (78). *Then condition* (126) *holds.*

Proof. Define the set of times with small negative constraints as

$$I_\varepsilon = \{t \in (0,T);\ -\varepsilon \leq g_i(\bar{u}(t)) < 0, \text{ for some } 1 \leq i \leq q\}. \tag{127}$$

Let $(v, z_0) \in C_2(\bar{u}, \bar{y}_0)$ be an extended critical direction. For given $\varepsilon > 0$, consider the perturbed direction $(v'_\varepsilon, z_0) \in C_\infty(\bar{u}, \bar{y}_0)$, defined by

$$v'_\varepsilon(t) = 0 \text{ if either } t \in I_\varepsilon, \text{ or } |v(t)| > \varepsilon^{-1};\ v'_\varepsilon(t) = v(t) \text{ otherwise.} \tag{128}$$

By (78), where vectors e^k were defined, there exists $(v_\varepsilon, z_{0\varepsilon}) \in C_\infty(\bar{u}, \bar{y}_0)$ of the form

$$(v_\varepsilon, z_{0\varepsilon}) = (v'_\varepsilon, z_0) + \sum_{k=0}^{|I_+|} \alpha_{\varepsilon,k} e^k \tag{129}$$

such that $v_\varepsilon - v'_\varepsilon \in \mathcal{U}_{\varepsilon_R}$, $|\alpha_\varepsilon| \to 0$ when $\varepsilon \downarrow 0$, and so $\|(v_\varepsilon, z_{0\varepsilon}) - (v'_\varepsilon, z_0)\|_\infty \to 0$. When $\varepsilon < \varepsilon_R$, if $-\varepsilon \leq g_i(\bar{u}(t))$, we have that $v'_\varepsilon(t) = 0$ and $g'_i(\bar{u}(t))e^k(t) = 0$. It follows that $g(\bar{u}) + \rho g'(\bar{u})v_\varepsilon \leq 0$ when $\rho > 0$ is small enough. In that case we know that $\sigma(\lambda, T^2_-(g(\bar{u}), g'(\bar{u})v_\varepsilon) = 0$. Therefore, by theorem 5.1, $\Omega(v_\varepsilon, z_{0\varepsilon}) \geq 0$, so that there exists $\lambda_\varepsilon \in L^\infty(0, T, \mathbb{R}^q)$ such that $(\lambda_\varepsilon, \mu) \in LM^L(\bar{w})$ and

$$\mathcal{L}_{(u,y_0)^2}(\bar{u}, \bar{y}_0, \lambda_\varepsilon, \mu)(v_\varepsilon, z_{0\varepsilon})^2 \geq 0. \tag{130}$$

Since the set of Lagrange multipliers is bounded, λ_ε has weak$*$ limit points when $\varepsilon \downarrow 0$. As $(v_\varepsilon, z_{0\varepsilon}) \to (v, z_0)$ in the L^2 norm (so that the term in product of λ_ε strongly converges in the L^1 norm), we may pass to the limit in this inequality when $\varepsilon \to 0$. The conclusion follows. □

Remark 5.3. The method of proof is a variant of the one used for "extended polyedricity", see [2, Section 3.2.3]. The basic concept there is the one of *radial* critical directions, i.e., critical directions v for which there exists $\kappa > 0$ such that (in our notations) $g(u) + \kappa g'(u)v \leq 0$. Here the L^∞ smoothness of the multiplier compensates the fact that extended critical directions belong to L^2 spaces.

We next discuss relations with weak quadratic growth, defined in (8).

Corollary 5.4. *Let $\bar{w}$ be a weak solution of (P), satisfying the qualification hypotheses* (60) *and* (78) *and the weak quadratic growth condition* (8) *with parameter α. Then*

$$\Omega(v, z_0) \geq \alpha(\|v\|_2^2 + |z_0|^2), \textit{ for all } (v, z_0) \in C_2(\bar{u}, \bar{y}_0). \tag{131}$$

This is a consequence of the second-order necessary condition for the problem of minimizing over $F(P)$ the perturbed cost function (the proof follows the one of theorem 5.2)

$$\hat{J}_\alpha(u, y_0) := \hat{J}(u, y_0) - \tfrac{1}{2}\alpha\left(\|u - \bar{u}\|_2^2 + |y_0 - \bar{y}_0|^2\right). \tag{132}$$

5.2 Characterization of the weak quadratic growth condition

Let $\bar{w} \in F(P)$ satisfy the special qualification hypothesis (78). We know that the projection $M^L(\bar{w})$ of the set $LM^L(\bar{w})$ on the second component is a singleton $\{\mu\}$. Let $\bar{p}$ denote the associated costate. We consider the following two conditions: *uniform local quadratic growth of Hamiltonian functions along the trajectory* (analogous to (30)) i.e.,

$$\left\{\begin{array}{l}\text{There exists } c_H > 0,\ \varepsilon_\infty > 0; \quad \text{for a.a. } t \in (0,T),\\ H(u,\bar{y}(t),\bar{p}(t)) \geq H(\bar{u}(t),\bar{y}(t),\bar{p}(t)) + c_H|u-\bar{u}(t)|^2,\\ \text{for all } u \in \mathbb{R}^m;\ g(u) \leq 0;\ |u-\bar{u}(t)| \leq \varepsilon_\infty,\\ \text{where } \bar{p} \text{ is the costate associated with } \bar{\mu},\end{array}\right. \tag{133}$$

and *uniform quadratic growth along critical directions* (131).

Theorem 5.5. *Let $\bar{w} \in F(P)$ satisfy the qualification hypotheses* (60) *and* (78)*. Then $\bar{w}$ satisfies the weak quadratic growth condition iff* (131) *and* (133) *hold.*

Proof. Since the weak quadratic growth condition means that $\bar{w}$ is, for some $\alpha > 0$, a weak solution of the perturbed problem $\text{Min}_{w\in\mathcal{W}}\, J(w) - \frac{1}{2}\alpha\|w-\bar{w}\|_2^2$, and the latter has the same first-order multipliers, conditions (133) and (131) (for the latter, in view of corollary 5.4) are necessary conditions for weak quadratic growth.

Let us show the converse by contradiction. So let us assume that, while $\bar{w}$ satisfies (131) and (133), there exist a sequence $w_k \in F(P)$ such that $w_k \to \bar{w}$ in $\mathcal{W}$, $w_k \neq \bar{w}$ for all k, and

$$J(w_k) \leq J(\bar{w}) + o(\|w_k - \bar{w}\|_2^2). \tag{134}$$

Define the *local critical cone* C_t at time $t \in [0,T]$ as (I_t was defined in (118)):

$$C_t \quad := \{v \in \mathbb{R}^m;\ H_u(\bar{u}(t),\bar{y}(t),\bar{p}(t))v \leq 0;\ \ g_i'(\bar{u}(t))v \leq 0, i \in I_t\}. \tag{135}$$

Note that, by (121):

$$C_2(\bar{u},\bar{y}_0) = \{(v,z_0) \in T_\Phi(\bar{u},\bar{y}_0);\ v(t) \in C_t \ \text{ for a.a. } t;\, \bar{\mu}G'(\bar{u},\bar{y}_0)(v,z_0) = 0\}. \tag{136}$$

Since C_t is a polyhedron, by Hoffman's lemma [5], there exists a constant κ_t such that

$$\text{dist}(v, C_t) \leq \kappa_t \left[[H_u(\bar{w}(t),\bar{p}(t))v]_+ + \sum_{i\in I_t} [g_i'(\bar{u}(t))v]_+ \right]. \tag{137}$$

This constant κ_t can be estimated as a finite maximum of those for projections over vector subspaces corresponding to a subset of active inequalities (for the problem of projection over the local critical cone), and therefore is a measurable function of time. Take a sequence of positive numbers $\varepsilon_k \to 0$ (we will be more precise later) and consider the measurable partition

$$B_k := \{t \in (0,T);\ \kappa_t \geq 1/\varepsilon_k\}; \quad A_k := (0,T) \setminus B_k. \tag{138}$$

Denote

$$u_k' := \bar{u} + (u_k - \bar{u})\mathbf{1}_{A_k}; \quad u_k'' := \bar{u} + (u_k - \bar{u})\mathbf{1}_{B_k}. \tag{139}$$

Denote also

$$\gamma_k' := \|u_k' - \bar{u}\|_2^2 + |y_{k0} - \bar{y}_0|^2; \quad \gamma_k'' := \|u_k'' - \bar{u}\|_2^2; \quad \gamma_k := \gamma_k' + \gamma_k''. \tag{140}$$

Note that $\gamma_k = \|u_k - \bar{u}\|_2^2 + |y_{k0} - \bar{y}_0|^2$. Let y'_k be the state associated with control u'_k and initial state y_{k0}. Since $\varepsilon_k \to 0$, we have that $\text{meas}(B_k) \to 0$. The decomposition principle (theorem 2.13) implies

$$J^{\bar{\mu}}(w_k) = J^{\bar{\mu}}(w'_k) + \int_{B_k} [H(u''_k, \bar{y}, \bar{p}) - H(\bar{u}, \bar{y}, \bar{p})]\text{d}t + o(\gamma_k). \tag{141}$$

In view of the uniform local quadratic growth condition for Hamiltonian functions (133), this implies

$$J^{\bar{\mu}}(w_k) \geq J^{\bar{\mu}}(w'_k) + c_H \gamma''_k + o(\gamma_k). \tag{142}$$

Let $(\lambda, \bar{\mu}) \in LM^L(\bar{w})$. Adding $0 \geq \int_0^T \lambda(t) g(u'_k(t))\text{d}t$ to (37), obtain by a second-order expansion that

$$J^{\bar{\mu}}(w'_k) \geq J^{\bar{\mu}}(w'_k) + \langle \lambda, g(u'_k) \rangle = J(\bar{w}) + O(\gamma'_k). \tag{143}$$

If (for a subsequence) $\gamma'_k = o(\gamma_k)$, deduce then with (142) that

$$J(w_k) \geq J^{\bar{\mu}}(w_k) \geq J(\bar{w}) + c_H \gamma''_k + o(\gamma_k) = J(\bar{w}) + c_H \gamma_k + o(\gamma_k), \tag{144}$$

which gives the desired contradiction to (134). So in the sequel (and this will be used at the end of the proof) we may assume that

$$\gamma_k = O(\gamma'_k). \tag{145}$$

By lemma 4.7, the restoration property (definition 4.2) is satisfied. So there exists $\hat{w}_k \in F(P)$ such that (η'_k being the initial-final state associated with w'_k):

$$\begin{cases} \text{(i)} & \|\hat{w}_k - w'_k\|_\infty = O(d(\eta'_k)); \\ \text{(ii)} & J(\hat{w}_k) = J^{\bar{\mu}}(w'_k) + O\left(\|w'_k - \bar{w}\|_P d(\eta'_k)\right). \\ \text{(iii)} & \hat{u}_k(t) = \bar{u}(t), \text{ for a.a. } t \in B_k. \end{cases} \tag{146}$$

Combining (142) and (146)(ii), obtain

$$J^{\bar{\mu}}(w_k) \geq J(\hat{w}_k) + c_H \gamma''_k + O\left(\|w'_k - \bar{w}\|_P d(\eta'_k)\right) + o(\gamma_k). \tag{147}$$

Since $\text{meas}(B_k) \to 0$, and hence, $\|u''_k - \bar{u}\|_1 = o(\|u''_k - \bar{u}\|_2)$, we have that

$$|\Phi(\eta'_k) - \Phi(\eta_k)| = O(\|u''_k - \bar{u}\|_1) = o((\gamma''_k)^{1/2}). \tag{148}$$

Consequently

$$d(\eta'_k) = d(\eta_k) + o((\gamma''_k)^{1/2}). \tag{149}$$

Since $\|w'_k - \bar{w}\|_P = O((\gamma'_k)^{1/2})$, we deduce from (147) that

$$J^{\bar{\mu}}(w_k) \geq J(\hat{w}_k) + c_H \gamma''_k + O\left((\gamma_k)^{1/2} d(\eta_k)\right) + o(\gamma_k). \tag{150}$$

Using (73), we deduce that, for large enough k,

$$J(w_k) \geq J(\hat{w}_k) + c_H \gamma''_k - \tfrac{1}{2}\bar{\mu}\Phi(\eta_k) + o(\gamma_k). \tag{151}$$

In view of (134), we obtain

$$J(\hat{w}_k) - \tfrac{1}{2}\bar{\mu}\Phi(\eta_k) \leq J(\bar{w}) + o(\gamma_k). \tag{152}$$

We will end the proof by obtaining a contradiction to (152). Set $\hat{\gamma}_k := \|\hat{u}_k - \bar{u}\|_2^2 + |\hat{y}_{k0} - \bar{y}_0|^2$. By (146)(i), we have that

$$\hat{\gamma}_k^{1/2} = (\gamma_k')^{1/2} + O(d(\eta_k')) = O((\gamma_k')^{1/2}) = O((\gamma_k)^{1/2}). \tag{153}$$

Since $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$ $(a, b \in \mathbb{R})$, by the first equality above, we have that for some $c > 0$ not depending on k:

$$\hat{\gamma}_k \geq \tfrac{1}{2}\gamma_k' - cd(\eta_k')^2. \tag{154}$$

This relation will be used later. Combining (37) and (153), obtain

$$J(\hat{w}_k) \geq J^{\bar{\mu}}(\hat{w}_k) = J(\bar{w}) + \int_0^T H_u(\bar{w}(t), \bar{p}(t))(\hat{u}_k(t) - \bar{u}(t))\mathrm{d}t + O(\gamma_k). \tag{155}$$

In view of (152), we deduce that

$$\int_0^T H_u(\bar{w}(t), \bar{p}(t))(\hat{u}_k(t) - \bar{u}(t))\mathrm{d}t \leq O(\gamma_k). \tag{156}$$

Consider the projection, for each time t, of the displacement $\hat{u}_k(t) - \bar{u}(t)$ over the local critical cone, and its difference with the projected direction:

$$v_k(t) := P_{C_t}(\hat{u}_k(t) - \bar{u}(t)); \quad \hat{v}_k := v_k - (\hat{u}_k - \bar{u}). \tag{157}$$

Note that v_k is measurable and that, in view of the restoration property, $\hat{u}_k = \bar{u}$ a.e. on B_k, and hence, $v_k = \hat{v}_k(t) = 0$ a.e. on B_k. If $i \in I_t$, since $g_i(\bar{u}(t)) = 0$ and $g_i(\hat{u}_k(t)) \leq 0$, we have a.e.

$$g_i'(\bar{u}(t))(\hat{u}_k(t) - \bar{u}(t)) \leq g_i(\hat{u}_k(t)) + O(|\hat{u}_k(t) - \bar{u}(t)|^2) \leq O(|\hat{u}_k(t) - \bar{u}(t)|^2). \tag{158}$$

By the definition of sets A_k and B_k, a.e. on $[0, T]$, we have that

$$|\hat{v}_k(t)| \leq \frac{1}{\varepsilon_k}\left([H_u(\bar{w}(t), \bar{p}(t))(\hat{u}_k(t) - \bar{u}(t))]_+ + \sum_{i \in I_t}[g_i'(\bar{u}(t))(\hat{u}_k(t) - \bar{u}(t))]_+\right), \tag{159}$$

so that by (153) and (156)-(159):

$$\|\hat{v}_k\|_1 = \|v_k - (\hat{u}_k - \bar{u})\|_1 = O\left(\frac{\gamma_k}{\varepsilon_k}\right). \tag{160}$$

Since a projection is nonexpansive, we also have

$$\text{(i)} \;\; |v_k(t)| \leq |\hat{u}_k(t) - \bar{u}(t)|, \text{ for a.a. } t; \quad \text{(ii)} \;\; \|v_k\|_\infty = O(\|\hat{u}_k - \bar{u}\|_\infty). \tag{161}$$

We now fix $\varepsilon_k := (\|u_k - \bar{u}\|_\infty + |y_{k0} - \bar{y}_0|)^{1/2}$. Using (146)(i), (149), and $\gamma_k^{1/2} = O(\varepsilon_k^2)$, and denoting by $\mathbf{1}_{A_k}(t)$ the characteristic function of the set A_k, obtain

$$\|(\hat{u}_k - u_k)\mathbf{1}_{A_k}\|_\infty = \|(\hat{u}_k - u_k')\mathbf{1}_{A_k}\|_\infty = O(d(\eta_k')) = O(\gamma_k^{1/2}) = O(\varepsilon_k^2), \tag{162}$$

so that in particular

$$\|\hat{u}_k - \bar{u}\|_\infty \leq \|\hat{u}_k - u_k\|_\infty + \|u_k - \bar{u}\|_\infty = O(\varepsilon_k^2). \tag{163}$$

Combining with (160), get

$$\int_0^T |\hat{u}_k(t) - \bar{u}(t)||\hat{v}_k(t)|\mathrm{d}t \leq \frac{\|\hat{u}_k - \bar{u}\|_\infty}{\varepsilon_k}\varepsilon_k\|\hat{v}_k\|_1 = O(\varepsilon_k\gamma_k) = o(\gamma_k). \quad (164)$$

By (157), (161) and (163), we have that $\|\hat{v}_k\|_\infty \leq 2\|\hat{u}_k - \bar{u}\|_\infty = O(\varepsilon_k^2)$. Therefore we obtain in a similar way

$$\|\hat{v}_k\|_2^2 \leq \|\hat{v}_k\|_\infty\|\hat{v}_k\|_1 = O\left(\frac{\|\hat{u}_k - \bar{u}\|_\infty}{\varepsilon_k}\right)\varepsilon_k\|\hat{v}_k\|_1 = o(\gamma_k), \quad (165)$$

$$\|v_k\|_2^2 = \|\hat{u}_k - \bar{u} + \hat{v}_k\|_2^2 = \|\hat{u}_k - \bar{u}\|_2^2 + o(\gamma_k). \quad (166)$$

Since $\hat{w}_k \in F(P)$, setting $\hat{e}_k := (\hat{u}_k, \hat{y}_{k0})$, and using (153), we have that (I_F was defined in (118))

$$G_i'(\bar{e})(\hat{u}_k - \bar{u}, \hat{y}_{k0} - \bar{y}_0) = G_i(\hat{e}_k) - G_i(\bar{e}) + O(\gamma_k)\begin{cases} = O(\gamma_k), & 1 \leq i \leq r_1, \\ \leq O(\gamma_k), & i \in I_F. \end{cases} \quad (167)$$

In view of (160), using $\gamma_k/\varepsilon_k = o(\gamma_k^{1/2})$, we have that

$$\begin{aligned} G_i'(\bar{e})(v_k, \hat{y}_{k0} - \bar{y}_0) = &\ G_i'(\bar{e})(\hat{u}_k - \bar{u}, \hat{y}_{k0} - \bar{y}_0) + o((\gamma_k)^{1/2}) = o(\gamma_k^{1/2}), \\ &\ 1 \leq i \leq r_1, \\ G_i'(\bar{e})(v_k, \hat{y}_{k0} - \bar{y}_0) = &\ G_i'(\bar{e})(\hat{u}_k - \bar{u}, \hat{y}_{k0} - \bar{y}_0) + o((\gamma_k)^{1/2}) \leq o(\gamma_k^{1/2}), \\ &\ i \in I_F. \end{aligned} \quad (168)$$

Also, when $(\lambda, \bar{\mu}) \in LM^L(\bar{w})$, using (152), (153) and (158), for some $c > 0$:

$$\begin{aligned} \bar{\mu}G'(\bar{e})(\hat{u}_k - \bar{u}, \hat{y}_{k0} - \bar{y}_0) &= -\hat{J}'(\bar{u}, \bar{y}_0)(\hat{u}_k - \bar{u}, \hat{y}_{k0} - \bar{y}_0) - \langle\lambda, g'(\bar{u})(\hat{u}_k - \bar{u})\rangle \\ &\geq -c\gamma_k. \end{aligned} \quad (169)$$

Therefore, using again (160) and $\gamma_k/\varepsilon_k = o(\gamma_k^{1/2})$ we get

$$-\bar{\mu}G'(\bar{e})(v_k, \hat{y}_{k0} - \bar{y}_0) \leq o(\gamma_k^{1/2}). \quad (170)$$

It follows with (168) that

$$G_i'(\bar{e})(v_k, \hat{y}_{k0} - \bar{y}_0) = o((\gamma_k')^{1/2}), \qquad \text{for all } i \in I_+. \quad (171)$$

Therefore, by the special qualification condition (78), there exists a critical direction (v_k', z_k') such that, setting $z_{k0} := \hat{y}_{k0} - \bar{y}_0$

$$\|v_k' - v_k\|_\infty + |z_{k0}' - z_{k0}| = o((\gamma_k)^{1/2}). \quad (172)$$

We recall that $\Omega(v, z_0)$ was defined in (125). Note that

$$\Omega(v_k, z_{k0}) := \max_{(\lambda,\bar{\mu}) \in LM^L(\bar{w})} \int_0^T H_{ww}^a(\bar{w}, \bar{p}, \lambda)(v_k, z_k)^2\mathrm{d}t + (\Phi^{\bar{\mu}})''(\bar{\eta})(z_{k0}, z_k(T))^2. \quad (173)$$

By (131), we obtain that

$$\Omega(v_k', z_{k0}') \geq \alpha\left(\|v_k'\|_2^2 + |z_{k0}'|^2\right), \quad (174)$$

and hence in view of (172)

$$\Omega(v_k, z_{k0}) \geq \alpha \left(\|v_k\|_2^2 + |z_{k0}|^2\right) + o(\gamma_k). \tag{175}$$

By (164)-(166), and since $LM^L(\bar{w})$ is bounded, we have

$$\Omega(v_k, z_{k0}) = \Omega(\hat{u}_k - \bar{u} + \hat{v}_k, z_{k0}) = \Omega(\hat{u}_k - \bar{u}, z_{k0}) + o(\gamma_k), \tag{176}$$

and hence, in view of (175) and (164)-(166) again:

$$\begin{aligned} \Omega(\hat{u}_k - \bar{u}, z_{k0}) &= \Omega(v_k, z_{k0}) + o\left(\gamma_k\right) \\ &\geq \alpha(\|v_k\|_2^2 + |z_{k0}|^2) + o\left(\gamma_k\right) = \alpha\hat{\gamma}_k + o\left(\gamma_k\right). \end{aligned} \tag{177}$$

For a given λ such that $(\lambda, \bar{\mu}) \in LM^L(\bar{w})$, adding to (37) (written for the sequence $\hat{w}_k$) the inequality $0 \geq \int_0^T \lambda(t)(g(\hat{u}_k(t)) - g(\bar{u}(t)))\,\mathrm{d}t$ and using $H_u^a(\bar{w}, \bar{p}, \lambda) = 0$, obtain

$$\begin{aligned} J^{\bar{\mu}}(\hat{w}_k) - J(\bar{w}) \geq \; & \tfrac{1}{2}\int_0^T H_{ww}^a(\bar{w}, \bar{p}, \lambda)((\hat{u}_k - \bar{u}), (\hat{y}_k - \bar{y}))^2 \mathrm{d}t \\ & + \tfrac{1}{2}(\Phi^{\bar{\mu}})''(\bar{\eta}))(\hat{\eta}_k - \bar{\eta})^2 + (\|\lambda\|_\infty + 1)o(\gamma_k), \end{aligned} \tag{178}$$

so that, maximizing w.r.t. the bounded set $LM^L(\bar{w})$, we obtain in view of (177)

$$J(\hat{w}_k) - J(\bar{w}) \geq J^{\bar{\mu}}(\hat{w}_k) - J(\bar{w}) \geq \Omega(\hat{u}_k - \bar{u}, z_{k0}) + o(\gamma_k) \geq \alpha\hat{\gamma}_k + o(\gamma_k). \tag{179}$$

Combining with (145) and (154), we get for some $\alpha' > 0$:

$$J(\hat{w}_k) - J(\bar{w}) \geq \tfrac{1}{2}\alpha\gamma_k' - \alpha' d(\eta_k')^2 + o(\gamma_k'). \tag{180}$$

By (149), $d(\eta_k')^2 \leq 2d(\eta_k)^2 + o(\gamma_k)$ and so using also (145)

$$J(\hat{w}_k) - J(\bar{w}) \geq \tfrac{1}{2}\alpha\gamma_k' - 2\alpha' d(\eta_k)^2 + o(\gamma_k'). \tag{181}$$

Using (73) and (145), we see that this gives a contradiction to (152), as was to be shown. □

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