A Linear Storage-Retrieval Policy for Robust Warehouse Management

Marcus Ang[†] • Yun Fong Lim[*] • Melvyn Sim[††]

[†] Singapore-MIT Alliance, National University of Singapore,

4 Engineering Drive 3, Singapore 117576, Singapore

[*] Lee Kong Chian School of Business, Singapore Management University,

50 Stamford Road, Singapore 178899, Singapore

[††] NUS Business School, Singapore-MIT Alliance, NUS Risk Management Institute, National University of Singapore,

1 Business Link, Singapore 117592, Singapore

smamatm@nus.edu.sg • yflim@smu.edu.sg • melvynsim@nus.edu.sg

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Abstract

Assigning products to and retrieving them from proper storage locations in a unit-load warehouse are crucial in minimizing its operating cost. The problem becomes intractable when the warehouse faces uncertain demand in a dynamic setting. We assume a factor-based demand model in which demand for each product in each period is affinely dependent on some uncertain factors. The distributions of these factors are only partially characterized. We introduce a robust optimization model that minimizes the worst-case expected total operating cost of a warehouse under distributional ambiguity. Under a linear decision rule, we can obtain a storage and retrieval policy by solving a moderate-size linear optimization problem. Surprisingly, despite imprecise specification of demand distributions, our computational studies suggest that the simple linear policy achieves close to the expected value given perfect demand information, and significantly outperforms existing heuristics in the literature.

Keywords: Dynamic programming, Transportation, Materials handling, Inventory-production: Uncertainty.

1 Introduction

In a global economy, companies can create a competitive advantage by paying substantial attention to supply chain design and operations. For some industries, effective supply chain management has become vital for their survival. A distribution center is a consolidation hub of various products in a supply chain. A large distribution center that supports a wide range of businesses may store hundreds of thousands of different products. These products pass through the distribution center in huge volume daily. Thus, the operations efficiency of distribution centers becomes crucial to the competence of a company's supply chain.

According to a report by Maltz and DeHoratius (2004), distribution centers are increasingly measured by their performance on value-added services and fulfillment quality. Value-added services include, for example, light assembly for manufacturing firms, pricing and labeling for retail stores, and customization to local context for electronics and telecommunication companies. Modern distribution centers tend to be larger and provide more value-added services than their traditional counterparts. As distribution centers get larger, their operating costs also increase.

Fulfillment quality includes order-picking responsiveness and accuracy. These performance measures also increase the labor requirement for a distribution center because products need to be picked faster and more accurately. Many distribution centers that support e-commerce pick and ship products immediately after they receive orders. In a service part distribution center, spare parts are stored for expensive capital equipment such as medical equipment or airplanes. Almost every order is urgent because it might imply that a critical equipment is unusable. The increase in requirements for modern distribution centers makes effective operations management of the centers increasingly important.

In an *unit-load* warehouse, all products are received, stored, and retrieved in unit-load (pallet) quantities. Each pallet carries items of the same product and is handled singly by a forklift. Unit-load warehouses can be found in the reserve areas of a large distribution center, where products are stored to replenish fast-pick areas in the same distribution center (see Bartholdi and Hackman (2007)). Another example of a unit-load warehouse is a 3rd party transshipment warehouse, which is a subcontractor that provides warehousing services for others.

To store a pallet to its assigned storage location, a forklift first moves the pallet from the receiving dock to the storage location. After inserting the pallet into its storage location, the forklift returns to the receiving dock. Similarly, to retrieve a pallet, a forklift first moves from the shipping dock to the corresponding storage location, extracts the pallet, and then moves it to the shipping dock. This travel pattern is known as *single-command* travel. We assume the insertion and extraction times of a pallet at its storage location are constant. This is reasonable because the difference in insertion (or extraction) time is negligible compared to the difference in total travel time to store and retrieve a pallet. The insertion and extraction times can therefore be considered as fixed costs and they are ignored in deciding where to store and retrieve pallets.

In practice, the number of pallets of each product arriving at the warehouse in each time period (say, every day) follows some predetermined schedule according to the supplier's production plan. In contrast, the number of pallets of each product leaving the warehouse, which is mainly driven by uncertain demand, is less predictable. Pallets arrive in batches and they are moved from the receiving dock to some predetermined storage locations. Each pallet is stored at its assigned storage location until it is requested. The pallet is then moved from the storage location to the shipping dock. The major operating costs of a unit-load warehouse are the travel times to move each pallet from the receiving dock to a storage location, and then to the shipping dock. It is therefore critical to find a good storage and retrieval policy that minimizes the total expected operating cost in the face of demand uncertainty.

With regard to storing products in a warehouse, two policies are commonly used. The *dedicated storage* policy reserves each storage location for an assigned product and no other products can be stored in that location. The advantage of using dedicated storage is that each product is stored at a fixed location and hence, workers can memorize its location. However, when the inventory of a product is depleted, its storage location cannot be reassigned to other products. This inefficient use of space is a major disadvantage of the dedicated storage policy. To overcome this problem, one can use the *shared storage* policy. With shared storage, an empty storage location can be assigned to any product. Hence, a product may be assigned to different storage locations over time. A warehouse that implements shared storage policy must rely on a computerized warehouse management system for tracking products.

Several heuristics have been proposed to minimize the operating cost of a unit-load warehouse. Hausman et al. (1976) define the turnover rate as the number of times a given product requires storage and retrieval per unit time. They find that the turnover-based policies, which essentially assign storage locations with the smallest travel time to products with the highest turnover rate, outperform a policy that randomly assigns storage locations to arriving pallets. Goetschalckx and Ratliff (1990) propose a policy that is based on the duration of stay (DOS) in the warehouse of each individual pallet. Their policy involves categorizing the storage locations in the warehouse into zones. Pallets that have the smallest DOS are stored in the zone with the smallest average travel time. A unit-load warehouse is perfectly balanced if, for every time period, the number of arriving pallets and the number of departing pallets with the same DOS are equal. They prove that the DOS policy is optimal for a perfectly balanced warehouse. For warehouses that are not perfectly balanced, they find that their policy outperforms other policies including the turnover-based policies proposed by Hausman et al. (1976). Van den Berg (1999) gives an excellent survey on planning and control of warehousing systems. The survey includes a literature review on the storage location assignment problem in unit-load warehouses.

A large unit-load warehouse may deal with hundreds of thousands of different products whose demand pattern is subject to seasonality effects, product correlation, influence of market factors among others. The demand for different products may peak at different times of a year. For example, the highest demand for barbecue grills occurs in spring and summer, whereas the demand for ski equipment peaks in winter. We consider a factor-based demand model in which the demand for each product is affinely dependent on some uncertain factors. One crucial issue in any practical optimization problem is how to properly account for data uncertainty. The traditional approach to describe uncertainty via probability distributions is impractical as such descriptions are almost never available in the real world.

In recent years, robust optimization has witnessed an explosive growth and has become a dominant approach to address optimization problem under uncertainty. Traditionally, the goal of robust optimization is to immunize uncertain mathematical optimization problems against infeasibility while preserving the tractability of the models. See, for instance, Soyster (1973), Ben-Tal and Nemirovski (1998, 1999, 2000), Bertsimas and Sim (2003, 2004), Bertsimas et al. (2003), Chen and Sim (2008), El-Ghaoui and Lebret (1997), El-Ghaoui et al. (1998), and Erdoğan and Iyengar (2006). Many robust optimization approaches have the following two important characteristics:

1. The model of data uncertainty in robust optimization permits distributional ambiguity in characterizing uncertain data. For instance, the distribution of data is unknown but is confined to a family of distributions that would generate the same descriptive statistics such as mean and covariance. Data uncertainty can also be completely distribution free and specified by an uncertainty set.

2. The solution (or approximate solution) to a robust optimization model can be obtained by solving a tractable deterministic mathematical optimization problem. Robust optimization methodology often decouples model formulation from the optimization engine, which enables the modeler to focus on modeling the actual problem and not to be hindered by algorithm design.

Robust optimization has also been implemented in a dynamic setting that involves decision making in stages. Notably, Bertsimas and Thiele (2006) propose a robust optimization solution to a multi-period inventory control problem. Using an open-loop inventory control approach where the replenishment policy is static, they show that the solutions can be obtained by solving a tractable linear optimization problem and that the optimum solution of their robust model has a base-stock structure. Similarly, Adida and Perakis (2006) handle demand uncertainty in a dynamic pricing and inventory control problem by formulating a deterministic robust optimization problem. To address the inadequacy of open-loop robust optimization models involving a multi-stage decision process, Ben-Tal et al. (2004) introduce the concept of *adjustable robust counterpart*, which permits decisions to be delayed until the information becomes available. Some interesting applications of the adjustable robust counterpart include Atamtürk and Zhang (2007) and Erera et al. (2007). Unfortunately, with the additional flexibility in modeling, adjustable robust counterpart models are generally NP-hard. The authors propose and advocate the use of a linear decision rule, which they call *affinely adjustable robust counterpart.* If the uncertainty set is a polyhedral, affinely adjustable robust counterpart becomes an attractive polynomial-size linear optimization problem. Ben-Tal et al. (2005) demonstrate that affinely adjustable robust counterpart can be remarkably effective in minimizing the worst-case objective of a multi-period inventory control problem.

Recently, Chen et al. (2008) and Chen and Sim (2008) propose several piecewise linear decision rules that improve upon linear ones for approximating stochastic linear optimization problems. See and Sim (2008) demonstrate the effectiveness of these rules in minimizing the expected objective of a multi-period inventory control problem under stochastic demand with correlation. Unlike linear decision rules, piecewise linear decision rules lead to second order cone optimization problems, which are polynomially time solvable but are less scalable than linear optimization problems.

Our approach to address the warehouse management problem is similar to the affinely adjustable robust counterpart where we restrict the storage and retrieval policy to a linear decision rule in order to obtain a tractable formulation. Specifically, our contributions over the related work of Ben-Tal et al. (2005) can be summarized as follows:

1. We propose a factor-based demand model in which the demand for each product is affinely dependent on some uncertain factors. The support set of the uncertain factors is a polytope and we call it the *factors support set.* The factors have uncertain means and the corresponding support set, which we call the *factor means support set*, is a polytope. In our proposed model, the warehouse manager seeks to minimize the worst-case *expected* total operating cost. In contrast, the model of demand uncertainty of Ben-Tal et al. (2005) is distribution free and the warehouse manager aims

to minimize the worst-case total operating cost. This is a special case of our uncertainty model in which the factor means support set is the same as the factors support set.

2. We characterize the factors support set that will ensure feasibility in the robust warehouse management problem under a linear decision rule. Such characterization is not required in the robust inventory control problem of Ben-Tal et al. (2005) because feasibility is guaranteed for any bounded support set, which is not the case in the warehouse management problem.

3. To understand the degree of conservativeness, we compare the expected operating cost achievable by a linear decision rule against the expected value given perfect demand information under an assumed demand distribution. We also benchmark numerically our approach with existing heuristics in the literature, namely the turnover-based and DOS policies. In contrast, the computational studies by Ben-Tal et al. (2005) are based on the worst-case attainable cost under a distribution-free demand model.

This paper is organized as follows. In Section 2, we introduce the formulation of the warehouse management problem and the factor-based demand model to account for the random demand. We present the robust optimization model for the warehouse management problem in Section 3. The linear storage and retrieval policy will be studied and its implementation will be illustrated using an example. The numerical experiments comparing the performance of the linear policy with the existing heuristics will be discussed in Section 4. Finally, we conclude the paper in Section 5.

Notations: Throughout this paper, we use $\boldsymbol{y'}$ to denote the transpose of vector $\boldsymbol{y}$. We denote an uncertain variable with a tilde sign such as $\tilde{z}$, and denote an uncertain vector with bold face lower case letters such as $\tilde{\boldsymbol{z}}$. The support set of an uncertain vector $\tilde{\boldsymbol{z}}$ is the smallest convex set containing all instances of $\tilde{\boldsymbol{z}}$. Given an uncertain vector $\tilde{\boldsymbol{z}}$ with support set W and function mappings $f, g : W \to \Re^m$, we use the notation $f(\tilde{\boldsymbol{z}}) \geq g(\tilde{\boldsymbol{z}})$ to represent state-wise dominance: $f(\boldsymbol{z}) \geq g(\boldsymbol{z})$ for all $\boldsymbol{z} \in W$. Similarly, $f(\tilde{\boldsymbol{z}}) = g(\tilde{\boldsymbol{z}})$ denotes state-wise equality: $f(\boldsymbol{z}) = g(\boldsymbol{z})$ for all $\boldsymbol{z} \in W$.

2 Problem formulation

We consider an unit-load warehouse using a single command travel. For each storage location, we define its *store cost* as the travel time of a standard forklift to move from the receiving dock to the storage location and return to the receiving dock. Similarly, the *retrieve cost* of a storage location is the travel time of a standard forklift to move from the shipping dock to the storage location and return to the shipping dock. All storage locations that have identical store and retrieve costs are said to belong to the same *class*. When a pallet is assigned to a class, it is stored at an arbitrary storage location in the class. Thus, we adopt a policy that is based on shared storage.

Let there be N classes of storage locations in the warehouse. The classes are indexed by $j = 1, \ldots, N$. To simply the analysis, we assume that the storage and retrieve costs are proportional to the number of pallets handled. Let s_j and r_j denote the unit store cost and retrieve cost of class j respectively. Assume class j has capacity c_j for $j = 1, \ldots, N$. We assume that the N-th class represents emergency storage, which has infinite capacity ($c_N = \infty$) but incurs high store and retrieve costs.

Let there be M products indexed by $i = 1, \ldots, M$. We divide the planning horizon into T periods, which are indexed by $t = 1, \ldots, T$. For each period t, we assume that all pallets arrive at the start of the period and they are ready to be assigned to storage locations. For all pallets that are ordered during period t, we assume that they are retrieved from the warehouse upon demand request at the end of the period. The objective is to minimize the total expected operating cost of the unit-load warehouse for the entire planning horizon. To simplify notations, we let $\mathcal{N} = \{1, \ldots, N\}$, $\mathcal{N}^- = \{1, \ldots, N-1\}$, $\mathcal{T} = \{1, \ldots, T\}$, $\mathcal{T}^+ = \{1, \ldots, T+1\}$ and $\mathcal{M} = \{1, \ldots, M\}$.

2.1 Deterministic demand

We first consider a deterministic version of the problem in which all demand information throughout the entire planning horizon is available at the start of the first period. Let a_i^t denotes the number of pallets of product i that arrive at the start of period t. Let v_{ij}^t be a nonnegative decision variable that determines the number of arriving pallets of product i assigned to class j in period t. To represent the assignment of pallets to different storage classes, we have

$$\sum_{j \in \mathcal{N}} v_{ij}^t = a_i^t, \qquad \text{for } i \in \mathcal{M}, t \in \mathcal{T}.$$

Similarly, let d_i^t denotes the number of pallets of product i that are requested in period t. Let w_{ij}^t be a nonnegative decision variable that determines the number of pallets of product i retrieved from class j in period t. We have

$$\sum_{j \in \mathcal{N}} w_{ij}^t = d_i^t, \qquad \text{for } i \in \mathcal{M}, t \in \mathcal{T}.$$

Let x_{ij}^t denotes the number of pallets of product i in storage location of class j at the start of period t. We assume that there is no initial inventory in the warehouse. Hence,

$$x_{ij}^1 = 0, \qquad \text{for } i \in \mathcal{M}, j \in \mathcal{N}.$$

We do not allow backlog of orders at all time, even after the planning horizon. Thus,

$$x_{ij}^t \geq 0, \qquad \text{for } i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}^+.$$

The inventory of product i in class j at the start of period $t+1$ is

$$x_{ij}^{t+1} \;=\; x_{ij}^t + v_{ij}^t - w_{ij}^t, \qquad \text{for } i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}.$$

For each class j, the initial inventory and the arrivals of products in period t must not exceed its capacity. We have the following capacity constraints

$$\sum_{i \in \mathcal{M}} (x_{ij}^t + v_{ij}^t) \;\leq\; c_j, \qquad \text{for } j \in \mathcal{N}^-, t \in \mathcal{T}. \tag{1}$$

Rightfully, the decision variables should be restricted to integers. However, in order to yield a tractable formulation, we relax the integrality constraints and formulate the linear optimization problem of minimizing the total operating cost of a unit-load warehouse as follows:

$$\begin{aligned}
Z_D = \min \quad & \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{M}} \sum_{j \in \mathcal{N}} (s_j v_{ij}^t + r_j w_{ij}^t) & \\
\text{s.t.} \quad & \sum_{j \in \mathcal{N}} v_{ij}^t = a_i^t, & i \in \mathcal{M}, t \in \mathcal{T}; \\
& \sum_{j \in \mathcal{N}} w_{ij}^t = d_i^t, & i \in \mathcal{M}, t \in \mathcal{T}; \\
& x_{ij}^{t+1} = x_{ij}^t + v_{ij}^t - w_{ij}^t, & i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}; \\
& x_{ij}^1 = 0, & i \in \mathcal{M}, j \in \mathcal{N}; \\
& \sum_{i \in \mathcal{M}} (x_{ij}^t + v_{ij}^t) \leq c_j, & j \in \mathcal{N}^-, t \in \mathcal{T}; \\
& v_{ij}^t, w_{ij}^t \geq 0, & i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}; \\
& x_{ij}^t \geq 0, & i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}^+.
\end{aligned} \tag{2}$$

We assume that any shortage will be handled by the suppliers and this will not incur any costs to the warehouse. Thus, there is always sufficient inventory to meet demand for every period. Equivalently,

$$\sum_{\tau=1}^{t} d_i^\tau \leq \sum_{\tau=1}^{t} a_i^\tau, \qquad \text{for } i \in \mathcal{M}, t \in \mathcal{T}. \tag{3}$$

Proposition 1 *Model (2) is feasible if and only if the inequalities (3) hold.*

Proof : See Appendix A.

Remark : Note that the initial conditions $x_{ij}^1 = 0$ for $i \in \mathcal{M}, j \in \mathcal{N}$ are not overly restrictive. The result can be extended to a more general initial setting, in which case, the inequalities (3) become

$$\sum_{\tau=1}^{t} d_i^{\tau} \leq \sum_{\tau=1}^{t} a_i^{\tau} + \sum_{j \in \mathcal{N}} x_{ij}^1, \qquad \text{for } i \in \mathcal{M}, t \in \mathcal{T},$$

where $x_{ij}^1 \geq 0$ for $i \in \mathcal{M}, j \in \mathcal{N}$.

2.2 Factor-based demand model

We adopt a *factor-based demand model* similar to the model of Sim and See (2008), in which the pallet demand at time period t is affinely dependent on some uncertain factors $\tilde{z}_k$, $k = 1, ..., K_t$, with $1 \leq K_1 \leq K_2 \leq ... \leq K_T$. Under the factor-based demand model, the uncertain factors, $\tilde{z}_k$, are realized sequentially. At the end of time period t, the factors, $\tilde{z}_k$, $k = 1, \ldots, K_t$ have already been unfolded. In progressing to the end of period $t+1$, the new factors $\tilde{z}_k$, $k = K_t + 1, \ldots, K_{t+1}$ are made available. For notational convenience, we define $\mathcal{K}_t = \{1, \ldots, K_t\}$, $\mathcal{K}_t^0 = \{0, \ldots, K_t\}$, $\tilde{\boldsymbol{z}}^t \triangleq (\tilde{z}_1, \ldots, \tilde{z}_{K_t})$, $\boldsymbol{z}^t \triangleq (z_1, \ldots, z_{K_t})$, $\tilde{\boldsymbol{z}} \triangleq \tilde{\boldsymbol{z}}^T$ and $\boldsymbol{z} \triangleq \boldsymbol{z}^T$. The demand at period t is an affine function of $\tilde{\boldsymbol{z}}^t$ as follows:

$$d_i^t(\tilde{\boldsymbol{z}}^t) \triangleq d_i^{t,0} + \sum_{k \in \mathcal{K}_t} d_i^{t,k} \tilde{z}_k, \qquad \text{for } t \in \mathcal{T}, i \in \mathcal{M}. \tag{4}$$

Demand that is affected by random noise or shocks can be represented by the factor-based demand model. Moreover, the correlation of the demand for different products across different periods can be captured by using appropriate values for $d_i^{t,k}$. In the trivial case of the independently distributed demand across products and periods, we have

$$d_i^t(\tilde{\boldsymbol{z}}^t) = d_i^{t,0} + \tilde{z}_{i+M(t-1)},$$

where $\tilde{z}_i$, $i = 1, ..., Mt$, are independently distributed factors. The factor-based demand model allows us to model autocorrelation. We may consider standard forecasting techniques such as an ARMA(p, q) demand process (see Box et al. (1994)) as follows:

$$d_i^t(\tilde{\boldsymbol{z}}^t) = \begin{cases} d_i^{t,0} & \text{if } t \leq 0 \\ \sum_{l=1}^{p} \phi_l d_i^{t-l}(\tilde{\boldsymbol{z}}^{t-1}) + \tilde{z}_t + \sum_{l=1}^{\min\{p,t-1\}} \theta_l \tilde{z}_{t-l} & \text{otherwise,} \end{cases}$$

where $\phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ are known constants. Indeed, it is easy to show by induction that $d_i^t(\tilde{\boldsymbol{z}}^t)$ can be expressed in the form of a factor-based demand model. Factor-based models have been used extensively in finance for modeling returns as affine functions of external factors, in which the coefficients of the factors can be determined statistically. In the same way, we can apply the factor-based demand model to characterize the influence of demands with external factors such as market outlook, oil prices and so forth. Effects of trend, seasonality, cyclic variation, and randomness can also be incorporated in this model.

In practice, it is impossible to obtain the true distribution of the uncertain factors. As such, we cater for modeling ambiguity and characterize the distribution of the uncertain factors as follows:

Assumption U:

The uncertain factors $\tilde{\boldsymbol{z}}$ are random variables with unknown distributions. They lie in a full dimensional polytope support set, W, which we call the *factors support set.* The factors have uncertain means and the corresponding support set, $\hat{W}$, which we call the *factor means support set*, is a polytope. We define $\mathcal{U}$ as the family of distributions of $\tilde{\boldsymbol{z}}$ such that $\mathrm{E}_{\mathcal{P}}(\tilde{\boldsymbol{z}}) \in \hat{W}$ for all $\mathcal{P} \in \mathcal{U}$.

Similar to Inequalities (3), we do not allow demand to exceed product inventory at any time period. Hence, the support sets W and $\hat{W}$ are subsets of the following set:

$$G \triangleq \left\{ \boldsymbol{z} \in \Re^{K_T} \ : \ \sum_{\tau=1}^{t} d_i^\tau(\boldsymbol{z}^\tau) \leq \sum_{\tau=1}^{t} a_i^\tau, d_i^t(\boldsymbol{z}^t) \geq 0 \text{ for } i \in \mathcal{M}, t \in \mathcal{T} \right\}. \tag{5}$$

Therefore, without loss of generality, we can define the factors support set as follows:

$$W \triangleq \left\{ \boldsymbol{z} \in \Re^{K_T} : \boldsymbol{z} \in G, \boldsymbol{z} \in S \right\},$$

where S is a polytope given by

$$S = \{ \boldsymbol{z} \in \Re^{K_T} \ : \ \exists \boldsymbol{u} \in \Re^{N_b} \ : \ \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} \leq \boldsymbol{q} \}$$

and $\boldsymbol{A} \in \Re^{N_a \times K_T}$,$\boldsymbol{B} \in \Re^{N_a \times N_b}$, $\boldsymbol{q} \in \Re^{N_a}$. Likewise, we can define the factor means support set as follows:

$$\hat{W} \triangleq \left\{ \boldsymbol{z} \in \Re^{K_T} : \boldsymbol{z} \in G, \boldsymbol{z} \in \hat{S} \right\},$$

where $\hat{S}$ is a polytope given by

$$\hat{S} = \{ \boldsymbol{z} \in \Re^{K_T} \ : \ \exists \boldsymbol{u} \in \Re^{N_b} \ : \ \hat{\boldsymbol{A}}\boldsymbol{z} + \hat{\boldsymbol{B}}\boldsymbol{u} \leq \hat{\boldsymbol{q}} \}$$

and $\hat{\boldsymbol{A}} \in \Re^{\hat{N}_a \times K_T}$,$\hat{\boldsymbol{B}} \in \Re^{\hat{N}_a \times \hat{N}_b}$, $\hat{\boldsymbol{q}} \in \Re^{\hat{N}_a}$.

Note that in classical robust optimization, the uncertainty sets used are typically simple geometric sets such as boxes, ellipsoids, or their intersections. Such uncertainty sets are not necessarily subset of G and can render the problem infeasible. For properness, we assume that W and $\hat{W}$ are nonempty. Note that $\hat{W} \subseteq W$ and hence, we may assume $\hat{S} \subseteq S$. When the factor means are completely unknown, we have $\hat{W} = W$. This becomes the adjustable robust counterpart model of Ben-Tal et al. (2005).

3 A robust optimization model

We consider a robust optimization model that takes into account the adjustability or recourse as information of the dynamic system unfolds. For each period t, the following sequence of events is repeated: At the start of the period, the pallets arrive according to a given schedule. A decision on where to store these pallets is made based on the information available captured in $\tilde{\boldsymbol{z}}^{t-1}$. These pallets are then moved to their assigned storage locations. After we know the demand in the period, the information available becomes $\tilde{\boldsymbol{z}}^t$, and a decision on where to retrieve pallets to fulfill the demand is made. Pallets are then retrieved from their storage locations.

We introduce the following adjustable or recourse variables.

1. $v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1})$: The number of arriving pallets of product i assigned to class j at the start of period t, after observing $\tilde{\boldsymbol{z}}^{t-1}$.

2. $w_{ij}^t(\tilde{\boldsymbol{z}}^t)$: The number of pallets of product i retrieved from class j after observing $\tilde{\boldsymbol{z}}^t$. This is the decision to be made after the demand in period t is realized.

3. $x_{ij}^{t+1}(\tilde{\boldsymbol{z}}^t)$: The number of pallets of product i in storage class j at the start of period $t+1$.

Under Assumption U, the exact demand distribution is not known. We assume that the warehouse manager is ambiguity averse and seeks to minimize the worst-case expected total operating cost over the family of distributions of $\tilde{\boldsymbol{z}}$ as follows:

$$
\begin{aligned}
Z_R = \min \quad & \max_{\mathcal{P}\in\mathcal{U}} \mathrm{E}_{\mathcal{P}}\left[\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{N}}\left(s_j v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) + r_j w_{ij}^t(\tilde{\boldsymbol{z}}^t)\right)\right] & \\
\text{s.t.} \quad & \sum_{j\in\mathcal{N}} v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) = a_i^t, & i\in\mathcal{M}, t\in\mathcal{T}; \\
& \sum_{j\in\mathcal{N}} w_{ij}^t(\tilde{\boldsymbol{z}}^t) = d_i^t(\tilde{\boldsymbol{z}}^t), & i\in\mathcal{M}, t\in\mathcal{T}; \\
& x_{ij}^{t+1}(\tilde{\boldsymbol{z}}^t) = x_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) + v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) - w_{ij}^t(\tilde{\boldsymbol{z}}^t), & i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T}; \\
& x_{ij}^1 = 0, & i\in\mathcal{M}, j\in\mathcal{N}; \\
& \sum_{i\in\mathcal{M}}\left(x_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) + v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1})\right) \le c_j, & j\in\mathcal{N}^-, t\in\mathcal{T}; \\
& v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}), w_{ij}^t(\tilde{\boldsymbol{z}}^t) \ge 0, & i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T}; \\
& x_{ij}^t(\tilde{\boldsymbol{z}}^t) \ge 0, & i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T}^+; \\
& v_{ij}^t, x_{ij}^t \in \mathcal{F}_{K_{t-1}}, & i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T}, \\
& w_{ij}^t \in \mathcal{F}_{K_t}, & i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};
\end{aligned} \tag{6}
$$

where $\mathcal{F}_p$ denotes a family of measurable functions that map from $\Re^p$ to $\Re$. Note that $x_{ij}^1(\tilde{\boldsymbol{z}}^0)$ is equivalent to x_{ij}^1. In Model (6), the optimal solutions v_{ij}^t and w_{ij}^t are functions representing the optimal policies for storage and retrieval respectively.

3.1 Linear storage and retrieval policy

It is generally intractable to obtain the optimal storage and retrieval policy that minimizes the worst-case expected total operating cost of the warehouse. Hence, we use decision rules, which are restricted classes of policies, with the aim of obtaining tractable formulations. The open-loop policy, or the zeroth decision rule, is a simple policy that does not respond to informational feedback. However, it is not difficult to see that the constraints in Problem (6) will be violated under the open-loop storage and retrieval policy in which v_{ij}^t and w_{ij}^t are restricted to scalar decision variables.

We next consider a linear decision rule. We let $\mathcal{L}_p$ be a family of affine functions that map from $\Re^p$ to $\Re$. For instance, $f\in\mathcal{L}_p$ means that $f:\Re^p\mapsto\Re$ and there exists $(f^0, f^1, \ldots, f^p)\in\Re^{p+1}$ such that

$$
f(\boldsymbol{z}) = f^0 + \sum_{j=1}^p f^j z_j.
$$

In contrast with arbitrary functions, a function $f\in\mathcal{L}_p$ can be completely characterized by a small collection of decision variables $f^0, f^1, \ldots, f^p$. Under Assumption U, by restricting the storage-retrieval policy to affine functions, the objective function becomes

$$
\begin{aligned}
& \max_{\mathcal{P}\in\mathcal{U}} \mathrm{E}_{\mathcal{P}}\left[\sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{N}}\Big(s_j v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) + r_j w_{ij}^t(\tilde{\boldsymbol{z}}^t)\Big)\right] \\
= \quad & \max_{\mathcal{P}\in\mathcal{U}} \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{N}}\Big(s_j v_{ij}^t(\mathrm{E}_{\mathcal{P}}(\tilde{\boldsymbol{z}}^{t-1})) + r_j w_{ij}^t(\mathrm{E}_{\mathcal{P}}(\tilde{\boldsymbol{z}}^t))\Big) \\
= \quad & \max_{\boldsymbol{z}\in\hat{W}} \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{N}}\Big(s_j v_{ij}^t(\boldsymbol{z}^{t-1}) + r_j w_{ij}^t(\boldsymbol{z}^t)\Big).
\end{aligned}
$$

Using the derived objective function and the linear decision rule, we propose the following optimization problem:

$$
\begin{aligned}
Z_{LR} = \min \quad & \max_{\boldsymbol{z}\in\hat{W}} \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{N}}\Big(s_j v_{ij}^t(\boldsymbol{z}^{t-1}) + r_j w_{ij}^t(\boldsymbol{z}^t)\Big) & \\
\text{s.t.} \quad & \sum_{j\in\mathcal{N}} v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) = a_i^t, & i\in\mathcal{M}, t\in\mathcal{T}; \\
& \sum_{j\in\mathcal{N}} w_{ij}^t(\tilde{\boldsymbol{z}}^t) = d_i^t(\tilde{\boldsymbol{z}}^t), & i\in\mathcal{M}, t\in\mathcal{T}; \\
& x_{ij}^{t+1}(\tilde{\boldsymbol{z}}^t) = x_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) + v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) - w_{ij}^t(\tilde{\boldsymbol{z}}^t), & i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T}; \\
& x_{ij}^1 = 0, & i\in\mathcal{M}, j\in\mathcal{N}; \qquad (7) \\
& \sum_{i\in\mathcal{M}}\Big(x_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) + v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1})\Big) \le c_j, & j\in\mathcal{N}^-, t\in\mathcal{T}; \\
& v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}), w_{ij}^t(\tilde{\boldsymbol{z}}^t) \ge 0, & i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T}; \\
& x_{ij}^t(\tilde{\boldsymbol{z}}^t) \ge 0, & i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T}^+; \\
& v_{ij}^t, x_{ij}^t \in \mathcal{L}_{K_{t-1}}, & i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T}; \\
& w_{ij}^t \in \mathcal{L}_{K_t}, & i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T}.
\end{aligned}
$$

Clearly, if Problems (6) and (7) are feasible, we have $Z_R \le Z_{LR}$. Chen et al. (2008) show that a feasible stochastic optimization problem can become infeasible under linear decision rule. Hence, even if Problem (6) is feasible, it is not clear whether there exists a linear policy that is feasible. Fortunately, this is not the case for our problem.

Theorem 1 *Under Assumption U, Problem (7) is feasible and its objective function Z_{LR} is finite. The parameters of the optimal linear storage-retrieval policy*

$$
\begin{aligned}
v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) &= v_{ij}^{t,0} + \sum_{k\in\mathcal{K}_{t-1}} v_{ij}^{t,k} z_k, & i\in\mathcal{M}, j\in\mathcal{N}; \\
w_{ij}^t(\tilde{\boldsymbol{z}}^t) &= w_{ij}^{t,0} + \sum_{k\in\mathcal{K}_t} w_{ij}^{t,k} z_k, & i\in\mathcal{M}, j\in\mathcal{N};
\end{aligned}
$$

can be computed by solving the following optimization problem:

$$
\begin{aligned}
Z_{LR} = \min \quad & g^0 + \max_{\boldsymbol{z}\in\hat{W}} \sum_{k\in\mathcal{K}_T} g^k z_k \\
\text{s.t.} \quad & g^k = \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{N}} \left(s_j v_{ij}^{t,k} + r_j w_{ij}^{t,k}\right), && k \in \mathcal{K}_T^0; \\
& \sum_{j\in\mathcal{N}} v_{ij}^{t,k} = \begin{cases} a_i^t, & \textit{if } k = 0, \\ 0, & \textit{otherwise}, \end{cases} && i \in \mathcal{M}, k \in \mathcal{K}_T^0, t \in \mathcal{T}; \\
& \sum_{j\in\mathcal{N}} w_{ij}^{t,k} = \begin{cases} d_i^{t,k}, & \textit{if } k \in \mathcal{K}_t^0, \\ 0, & \textit{otherwise}, \end{cases} && i \in \mathcal{M}, k \in \mathcal{K}_T^0, t \in \mathcal{T}; \\
& x_{ij}^{t+1,k} = x_{ij}^{t,k} + v_{ij}^{t,k} - w_{ij}^{t,k}, && i \in \mathcal{M}, j \in \mathcal{N}, k \in \mathcal{K}_T^0, t \in \mathcal{T}; \\
& x_{ij}^1 = 0, && i \in \mathcal{M}, j \in \mathcal{N}; \\
& h_j^{t,k} = \sum_{i\in\mathcal{M}} \left(x_{ij}^{t,k} + v_{ij}^{t,k}\right), && j \in \mathcal{N}, k \in \mathcal{K}_T^0, t \in \mathcal{T}; \\
& h_j^{t,0} + \sum_{k\in\mathcal{K}_T} h_j^{t,k} z_k \le c_j \quad \forall \boldsymbol{z} \in W, && j \in \mathcal{N}^-, t \in \mathcal{T}; \\
& v_{ij}^{t,0} + \sum_{k\in\mathcal{K}_T} v_{ij}^{t,k} z_k \ge 0 \quad \forall \boldsymbol{z} \in W, && i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}; \\
& w_{ij}^{t,0} + \sum_{k\in\mathcal{K}_T} w_{ij}^{t,k} z_k \ge 0 \quad \forall \boldsymbol{z} \in W, && i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}; \\
& x_j^{t,0} + \sum_{k\in\mathcal{K}_T} x_j^{t,k} z_k \ge 0 \quad \forall \boldsymbol{z} \in W, && i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}^+; \\
& v_{ij}^{t,k} = x_{ij}^{t,k} = h_j^{t,k} = 0, && i \in \mathcal{M}, j \in \mathcal{N}, k \in \mathcal{K}_T \backslash \mathcal{K}_t, t \in \mathcal{T}; \\
& w_{ij}^{t,k} = 0, && i \in \mathcal{M}, j \in \mathcal{N}, k \in \mathcal{K}_T \backslash \mathcal{K}_{t-1}, t \in \mathcal{T}.
\end{aligned} \tag{8}
$$

Proof : See Appendix B.

Remark : For the special case in which the factor means are exactly known, we have $\hat{W} = \{\mathrm{E}(\tilde{\boldsymbol{z}})\}$, we can translate the factors such that their means are zeros, that is, $\mathrm{E}(\tilde{\boldsymbol{z}}) = \mathbf{0}$.We can then simplify the objective function of Problem (8) to $\min g^0$. Furthermore, Z_{LR} becomes the optimal expected operating costs under linear decision rule.

Observe that the objective function and some of the constraints of Problem (8) contains embedded optimization problems involving the parameter $\boldsymbol{z}$ over the support sets $\hat{W}$ and W. These are known as robust counterparts in the context of robust optimization. See, for instance, Ben-Tal and Nemirovski (1998). Since the support sets W and $\hat{W}$ are polytopes, it is well known that the optimization problem (8) can be compactly represented as a linear optimization problem. For brevity, we present the derivation of the robust counterpart involving the factors support set, W. It is a straightforward extension to derive the robust counterpart involving the factor means support set, $\hat{W}$.

Proposition 2 *The variables $\boldsymbol{y}$ and r are feasible in the robust counterpart*

$$\boldsymbol{z}'\boldsymbol{y} \le r \quad \forall \boldsymbol{z} \in W,$$

or equivalently

$$\max_{\boldsymbol{z}\in W} \boldsymbol{z}'\boldsymbol{y} \le r,$$

if and only if there exists $\boldsymbol{\gamma} \in \Re^{N_a}$ $\boldsymbol{\alpha}^t, \boldsymbol{\beta}^t \in \Re^M$, $t \in \mathcal{T}$ feasible in

$$\begin{aligned}
&\sum_{t\in\mathcal{T}} \left(\bar{\boldsymbol{a}}^{t\prime}\boldsymbol{\alpha}^t + \boldsymbol{d}^{t,0\prime}\boldsymbol{\beta}^t \right) + \boldsymbol{\gamma}'\boldsymbol{q} \le r \\
&\sum_{t\in\mathcal{T}} \left(\bar{\boldsymbol{D}}^{t\prime}\boldsymbol{\alpha}^t - \boldsymbol{D}^{t\prime}\boldsymbol{\beta}^t \right) + \boldsymbol{A}'\boldsymbol{\gamma} = \boldsymbol{y} \\
&\boldsymbol{B}'\boldsymbol{\gamma} = \boldsymbol{0} \\
&\boldsymbol{\gamma} \ge \boldsymbol{0}, \boldsymbol{\beta}^t, \boldsymbol{\alpha}^t \ge \boldsymbol{0} \quad t \in \mathcal{T},
\end{aligned}$$

where

$$\boldsymbol{d}^{t,0} = \begin{pmatrix} d_1^{t,0} \\ \vdots \\ d_M^{t,0} \end{pmatrix}, \boldsymbol{a}^t = \begin{pmatrix} a_1^t \\ \vdots \\ a_M^t \end{pmatrix}, \bar{\boldsymbol{a}}^t = \sum_{\tau=1}^{t} (\boldsymbol{a}^\tau - \boldsymbol{d}^{\tau,0}),$$

$$\boldsymbol{D}^t = \begin{pmatrix} d_1^{t,1} & \dots & d_1^{t,K_t} & 0 & \dots & 0 \\ \vdots & \dots & \ddots & 0 & \dots & 0 \\ d_M^{t,1} & \dots & d_M^{t,K_t} & 0 & \dots & 0 \end{pmatrix} \in \Re^{M\times K_T}$$

and

$$\bar{\boldsymbol{D}}^t = \sum_{\tau=1}^{t} \boldsymbol{D}^\tau.$$

Proof : See Appendix C.

After transforming Problem (8) to a linear optimization problem using Proposition 2, the full formulation is heavy in notations and we present it in Appendix D.

Note that Chen et al. (2008) comprehensively discussed several linear-based decision rules that preserve the tractability of the problem. While it is possible to explore piecewise linear rules such as deflected and segregated linear decision rules, it would necessarily render the problem nonlinear (for deflected rule) or double the size (for segregated rule), which is less computationally attractive. Moreover, our computational studies suggest that the policy based on linear decision rule is close to optimal. As such, it may be superfluous to consider more sophisticated decision rules such as those proposed in Chen et al. (2008). In general, the replenishment policy can further be enhanced by

Class	Store cost	Retrieve cost	Capacity
1	1	1	15
2	1.5	1.5	30
3	2	2	45
4	3	3	60
5	100	100	∞

Table 1: Layout of a warehouse

folding horizon implementation in which the model is solved at every period using the updated demand information. Since more accurate information is available each time the model is resolved, the results will also improve.

3.2 An example

We illustrate the implementation of the linear policy with an example. We consider the problem with 3 products and 5 storage classes. Table 1 shows the layout of the warehouse. Note that storage class 5 represents an emergency storage area with high store and retrieve costs. We assume the emergency storage area has an infinite capacity. The number of arriving pallets and uncertain demand for each product in each period are given in Table 2. The uncertain factors, represented by $(\tilde{z}_1, \ldots, \tilde{z}_9)$ have support set

$$W = \{\boldsymbol{z} \ : \ -1 \leq z_k \leq 1, k = 1, \ldots, 9\}$$

and zero means so that

$$\hat{W} = \{\mathbf{0}\}.$$

We argue that $W \subseteq G$, since it is easy to verify that there is enough inventory to cope with maximum demand (corresponding to $\tilde{z}_k = 1$) for all products and at all time periods. Moreover, the robust counterpart

$$\sum_{k=1}^{9} z_k y_k \leq r \qquad \boldsymbol{z} \in W$$

is simply

$$\sum_{k=1}^{9} |y_k| \leq r$$

which can be easily transformed to standard linear constraints.

	$t=1$		$t=2$		$t=3$	
i	a_i^1	$\boldsymbol{d}_i^1(\tilde{\boldsymbol{z}}^1)$	a_i^2	$\boldsymbol{d}_i^2(\tilde{\boldsymbol{z}}^2)$	a_i^3	$\boldsymbol{d}_i^3(\tilde{\boldsymbol{z}}^3)$
1	35	$29+\tilde{z}_1$	35	$19+\tilde{z}_4$	35	$45+\tilde{z}_7$
2	35	$10+\tilde{z}_2$	35	$47+\tilde{z}_5$	35	$37+\tilde{z}_8$
3	10	$6+\tilde{z}_3$	10	$7+\tilde{z}_6$	25	$20+\tilde{z}_9$

Table 2: The number of arriving pallets and uncertain demand for each product in each period.

Using the parameters given, the optimization problem can be expressed concisely as Problem (9).

$$
\begin{array}{lll}
\min & \displaystyle\sum_{t=1}^{3}\sum_{i=1}^{3}\sum_{j=1}^{5}\left(s_j v_{ij}^{t,0} + r_j w_{ij}^{t,0}\right) & \\
\text{s.t.} & \displaystyle\sum_{j=1}^{5} v_{ij}^{t,k} = \begin{cases} a_i^t, & \text{if } k=0, \\ 0, & \text{otherwise} \end{cases}, & i=1,\ldots,3, k=1,\ldots,9, t=1,\ldots,3; \\
& \displaystyle\sum_{j=1}^{5} w_{ij}^{t,k} = \begin{cases} 1, & \text{if } k=(t-1)+i, \\ 0, & \text{otherwise,} \end{cases} & i=1,\ldots,3, k=1,\ldots,9, t=1,\ldots,3; \\
& x_{ij}^{t+1,k} = x_{ij}^{t,k} + v_{ij}^{t,k} - w_{ij}^{t,k}, & i=1,\ldots,3, k=1,\ldots,9, t=1,\ldots,3; \\
& x_{ij}^{1} = 0, & i=1,\ldots,3, t=1,\ldots,3; \\
& \displaystyle\sum_{i=1}^{3}\left(\left(x_{ij}^{t,0}+v_{ij}^{t,0}\right)+\sum_{k=1}^{9}\left|x_{ij}^{t,k}+v_{ij}^{t,k}\right|\right) \le c_j, & j=1,\ldots,4, t=1,\ldots,3; \\
& v_{ij}^{t,0} - \displaystyle\sum_{k=1}^{9}|v_{ij}^{t,k}| \ge 0, & i=1,\ldots,3, j=1,\ldots,5, t=1,\ldots,3; \\
& w_{ij}^{t,0} - \displaystyle\sum_{k=1}^{9}|w_{ij}^{t,k}| \ge 0, & i=1,\ldots,3, j=1,\ldots,5, t=1,\ldots,3; \\
& x_{ij}^{t,0} - \displaystyle\sum_{k=1}^{9}|x_{ij}^{t,k}| \ge 0, & i=1,\ldots,3, j=1,\ldots,5, t=1,\ldots,3; \\
& v_{ij}^{t,k} = 0, & i=1,\ldots,3, j=1,\ldots,5, \\
& & \quad k=3(t-1)+1,\ldots,9, t=1,\ldots,3; \\
& w_{ij}^{t,k} = 0, & i=1,\ldots,3, j=1,\ldots,5, \\
& & \quad k=3t+1,\ldots,9, t=1,2; \\
& x_{ij}^{t,k} = 0, & i=1,\ldots,3, j=1,\ldots,5, \\
& & \quad k=3(t-1)+1,\ldots,9, t=1,\ldots,3.
\end{array}
\tag{9}
$$

The implementation of the linear policy requires the solution for the decision variables $v_{ij}^{t,k}$ and $w_{ij}^{t,k}$, $i=1,\ldots,3$, $j=1,\ldots,5$, $k=1,\ldots,9$, $t=1,\ldots,3$. This can be computed by solving Problem (9). Note that we are only required to solve Problem (9) once. Let $\boldsymbol{v}^{t,k}=\{v_{ij}^{t,k}\}_{\substack{i=1,\ldots,3\\ j=1,\ldots,5}}$ and $\boldsymbol{w}^{t,k}=\{w_{ij}^{t,k}\}_{\substack{i=1,\ldots,3\\ j=1,\ldots,5}}$. For period 1, the store decision is represented by $\boldsymbol{v}^{1,k}$ for $k=1,\ldots,9$ computed from Problem (9) as follows:

$$\boldsymbol{v}^{1,0} = \begin{pmatrix} 10 & 0 & 5 \\ 20 & 10 & 0 \\ 0 & 14 & 5 \\ 5 & 11 & 0 \\ 0 & 0 & 0 \end{pmatrix}', \boldsymbol{v}^{1,k} = \mathbf{0}, \qquad \text{for } k = 1, \ldots, 9.$$

Thus, we store $10, 20, 0, 5, 0$ pallets of product 1 in storage locations $1, 2, 3, 4, 5$ respectively. Similarly, the store decisions are read from $\boldsymbol{v}^{1,k}, k = 1 \ldots, 9$ for product 2 and 3. Once demand for period 1, $\boldsymbol{d}_i^1(\tilde{\boldsymbol{z}}^1)$, is realized, we can compute the retrieve decisions based on $\boldsymbol{w}^{1,k}$ for $k = 1, \ldots, 9$ given by:

$$\boldsymbol{w}^{1,0} = \begin{pmatrix} 10 & 0 & 5 \\ 19 & 9.5 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}', \boldsymbol{w}^{1,1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}', \boldsymbol{w}^{1,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}', \boldsymbol{w}^{1,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}',$$

$$\boldsymbol{w}^{1,k} = \mathbf{0}, \qquad \text{for } k = 4, \ldots, 9.$$

The retrieve decision for product i in period 1 to location j is a function of uncertain factors, $\tilde{z}_1$,$\tilde{z}_2$ and $\tilde{z}_3$ given by

$$w_{ij}^1(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) = w_{ij}^{1,0} + w_{ij}^{1,1}\tilde{z}_1 + w_{ij}^{1,2}\tilde{z}_2 + w_{ij}^{1,3}\tilde{z}_3.$$

We demonstrate the implementation of the linear policy with a sample instance. Suppose in period 1 the realized demands for products 1, 2, and 3 are respectively 28, 11, and 6. The realized factors correspond to $z_1 = -1, z_2 = 1$, and $z_3 = 0$. Using product 1 as an illustration, we have

$$\begin{pmatrix} w_{11}^1(-1,1,0) \\ w_{12}^1(-1,1,0) \\ w_{13}^1(-1,1,0) \\ w_{14}^1(-1,1,0) \\ w_{15}^1(-1,1,0) \end{pmatrix} = \begin{pmatrix} 10 \\ 19 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}(-1) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}(1) + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}(0) = \begin{pmatrix} 10 \\ 18 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, according to the linear policy, we retrieve $10, 18, 0, 0, 0$ pallets of product 1 from storage locations $1, 2, 3, 4, 5$ respectively. In period 2, we have to make a decision to store based on the realization of the uncertain factors $\tilde{z}_1$,$\tilde{z}_2$ and $\tilde{z}_3$ given by the equation

$$v_{ij}^2(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) = v_{ij}^{2,0} + v_{ij}^{2,1}\tilde{z}_1 + v_{ij}^{2,2}\tilde{z}_2 + v_{ij}^{2,3}\tilde{z}_3,$$

which corresponds to the store decision for product i in period 2 to location j. From Problem (9), $\boldsymbol{v}^{2,k}$ for $k = 1, \ldots, 9$ are computed as follows:

$$\boldsymbol{v}^{2,0} = \begin{pmatrix} 1 & 9 & 5 \\ 15.5 & 12 & 1 \\ 9.5 & 14 & 0 \\ 9 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}', \boldsymbol{v}^{2,1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}', \boldsymbol{v}^{2,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0.5 & 0 & 0 \\ -0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}', \boldsymbol{v}^{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}',$$

$$\boldsymbol{v}^{2,k} = \mathbf{0}, \qquad \text{for } k = 4, \ldots, 9.$$

Thus, according to the linear policy, the store decision for product 1 in period 2 is

$$\begin{pmatrix} v_{11}^2(-1,1,0) \\ v_{12}^2(-1,1,0) \\ v_{13}^2(-1,1,0) \\ v_{14}^2(-1,1,0) \\ v_{15}^2(-1,1,0) \end{pmatrix} = \begin{pmatrix} 1 \\ 15.5 \\ 9.5 \\ 9 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} (-1) + \begin{pmatrix} 0 \\ 0.5 \\ -0.5 \\ 0 \\ 0 \end{pmatrix} (1) + \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} (0) = \begin{pmatrix} 0 \\ 16 \\ 9 \\ 10 \\ 0 \end{pmatrix},$$

which corresponds to storing $0, 16, 9, 10, 0$ pallets of product 1 in storage locations $1, 2, 3, 4, 5$ respectively. We see that by solving Problem (9) once, we can compute a set of store and retrieve decisions in matrix form that is dependent on the demand realization. The computation for the decision matrices is no more than solving a linear optimization problem.

4 Numerical studies

Although our model is set up to minimize the *worst-case* expected operating cost under distributional ambiguity, it is difficult to evaluate such an objective. Under an assumed distribution, we can approximate an expected operating cost using Monte Carlo simulations and compare the performance of the linear policy with that of the turnover-based and DOS heuristics. We use the *expected value given perfect information* ($EV|PI$) as our benchmark, which is a lower bound of the optimum expected operating cost. We define *percentage efficiency* as

$$\frac{EV|PI}{Z_X} \times 100\%,$$

where Z_X represents the expected cost by using heuristic X. A heuristic that works well would have a high percentage efficiency.

We investigate the effects of the number of products, the capacity of each storage class, the number of storage classes, and the variability of demand on the performance of various policies. Specifically, these policies include the static turnover-based heuristic (which sorts the products according to their mean turnover rates over the entire planning horizon), the dynamic turnover-based heuristic (which sorts the products according to their mean turnover rates in each period), the DOS heuristic, and the linear policy proposed in this paper.

For each numerical experiment, we assume that the receiving and shipping docks are located at the same point. Therefore, the store cost is equal to the retrieve cost for each storage class. The number of arriving pallets of each product in each period is predetermined according to the supplier's production schedule and it remains constant for all sampling scenarios. In contrast, the demand for each product in each period is random. We assume that the true distribution of the uncertain factors $\tilde{\boldsymbol{z}}$ follows a discrete uniform distribution $U(-p, p)$, where p is a scalar parameter for the demand. Hence, the factors support set is given by

$$W = \{\boldsymbol{z} \ : \ \|\boldsymbol{z}\|_\infty \leq p\}\,, \tag{10}$$

and the factor means support set is

$$\hat{W} = \{\mathbf{0}\}\,.$$

For any chosen parameter p, we ensure that there is enough inventory to cope with the demand for each product in each period. Hence, $W \subseteq G$. Note that if this condition is violated, $EV|PI$ would technically be infinite because there exist sampling scenarios in which the optimization problem is infeasible. In each experiment we create 1000 sampling scenarios. This adequately ensures that the standard errors obtained are less than 2% of the empirical averages. For each sampling scenario we generate the products' demands for all periods and compute the minimum operating cost by solving the deterministic version of the problem (Problem (2)). By taking the average of the minimum costs over all sampling scenarios we obtain an approximation of $EV|PI$. Likewise, we implement the static turnover-based heuristic, the dynamic turnover-based heuristic, and the DOS heuristic on each sampling scenario. The average cost over all sampling scenarios based on each heuristic is used to approximate its expected cost. Since the means of the uncertain factors are known, the optimal objective value under the linear decision rule, Z_{LR} is exactly the expected operating cost.

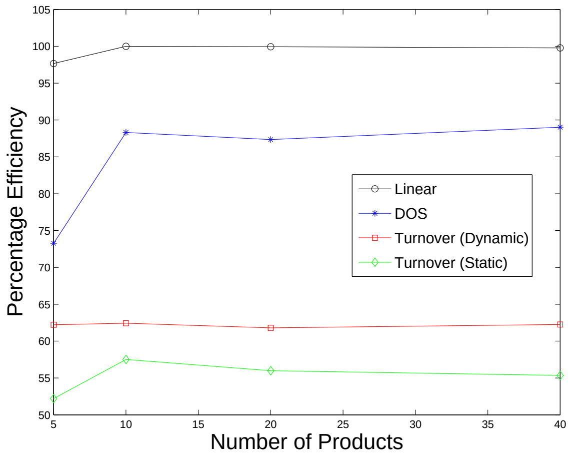

Figure 1: Percentage efficiency under different numbers of products

4.1 Effects of number of products

We examine the performance of the policies as the number of products varies. Consider Layout A in Table 9 in Appendix F. We assume a planning horizon of 5 periods and consider the problem with 5, 10, 20 and 40 products. The number of arriving pallets and demand for each product in each period are given in Tables 3, 4, 5, and 6 in Appendix E. Increasing the number of products will increase the number of pallets flowing through the warehouse and therefore all policies will eventually increase the use of the emergency storage class (Class 5). To avoid this trivial result, we scale the capacity of each storage class by scaling factors 0.8, 1, 2, and 4 for 5, 10, 20, and 40 products respectively. We set $p = 100$ for this numerical experiment.

Figure 1 shows the percentage efficiency of all policies. The linear policy significantly outperforms other policies for all numbers of products. It attains almost 100% efficiency as the number of products increases. The DOS policy, known to work well in a balanced system, outperforms the turnover-based policies. The static turnover-based policy has the lowest percentage efficiency in this experiment. The results suggest that the linear policy works very well compared with other policies for different numbers of products.

4.2 Effects of capacity of storage classes

We investigate the impact of the capacity of storage classes on the performance of different policies. Consider Layout A in Table 9 and assume there are 10 products in a planning horizon of 5 periods.

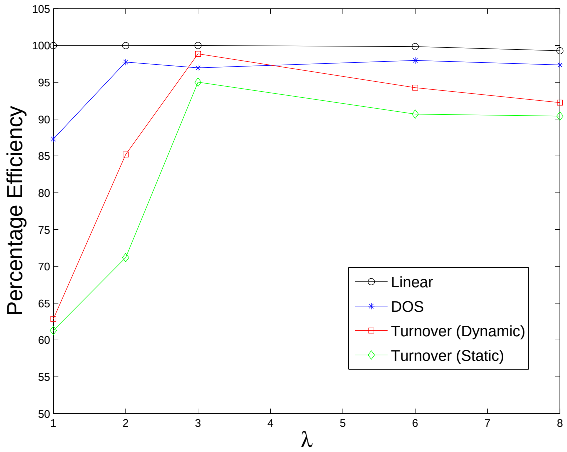

Figure 2: Percentage efficiency under different scaling factors

The number of arriving pallets and demand for each product in each period are given in Table 7 in Appendix E. We examine the performance of the policies as we increase the capacity of each storage class by a scaling factor λ. We set $p = 100$.

Figure 2 shows that the linear policy performs extremely well compared to $EV|PI$. For all scaling factors the linear policy attains almost 100% efficiency. As we increase the scaling factor λ, the performance of other policies improves and eventually attains above 90% efficiency. This is because the storage class with the lowest store and retrieve costs is large enough to store most of the arriving pallets in each period. This reduces the overall operating cost.

Figure 2 suggests that the performance of the linear policy is especially remarkable relative to other policies when the capacity of each storage class is tight ($\lambda \leq 2$). This is because the linear policy takes the capacity constraints of the storage classes into account (see Problem (8)). In contrast, other policies do not consider the capacity of storage classes.

4.3 Effects of number of storage classes

We investigate the impact of the number of storage classes on the performance of the policies. Consider the different layouts in Table 9 in Appendix F. The number of storage classes in these layouts ranges from 5 to 9. We create Layouts B, C, D, and E in Table 9 by increasing the store and retrieve costs of some storage locations in Layout A. Note that the total capacity of nonemergency storage classes remains constant for all layouts. We assume that there are 10 products in a planning horizon of 5

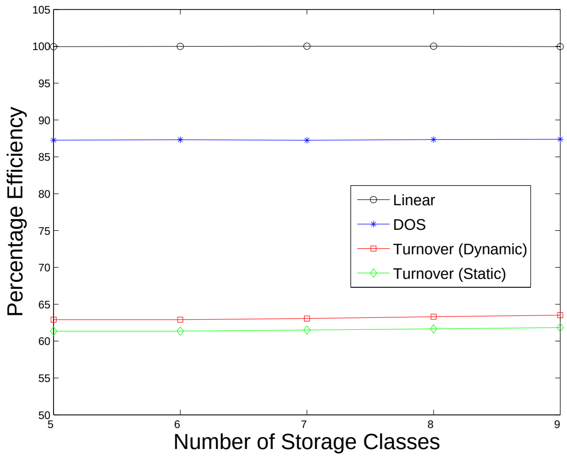

Figure 3: Percentage efficiency under different numbers of storage classes

periods. Table 7 shows the parameters for the arrivals and demand for each product in each period.

Figure 3 shows the percentage efficiency of the policies with different numbers of storage classes. The linear policy consistently and significantly outperforms other policies for different layouts. It attains almost 100% efficiency for all layouts in Table 9.

4.4 Effects of variance of demand

We expect the linear policy, which captures the variability of demand, to outperform other policies under various levels of demand variability. In this numerical experiment, we assume 10 products and a planning horizon of 5 periods. We consider Layout A in Table 9. Table 8 shows the number of arriving pallets and demand for each product in each period. We examine the performance of the policies under different levels of demand variability by varying the parameter p from 100 to 600.

Figure 4 shows the percentage efficiency of all policies. It is evident from the figure that the linear policy significantly outperforms other policies for all variability levels. The efficiency of the linear policy can be as high as 98% when $p = 100$. Although the efficiency of the linear policy decreases with p, it remains above 83% for a wide range of p. In contrast, the efficiency of the DOS and turnover-based policies, which ignore the demand variability, is consistently below 81% and 63% respectively. Figure 4 suggests that it is worthwhile to take demand variability into consideration as the resultant improvement in operating cost can be as high as 35%.

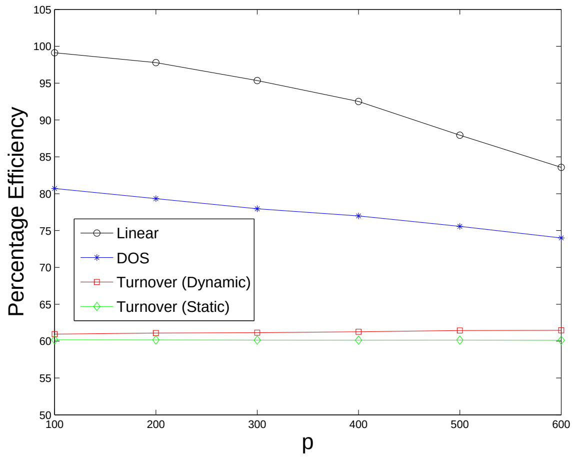

Figure 4: Percentage efficiency under different p

5 Conclusions

The major operating cost of a unit-load warehouse is the travel time to store and retrieve products. Therefore, efficient decisions on storage and retrieval of products are extremely important. Replenishments to the warehouse may not follow a simple rule because they typically depend on the production plans and status of the suppliers. The problem of minimizing the operating cost is further complicated by the fact that almost all products face stochastic demand in a dynamic setting. It is therefore challenging to find an efficient storage and retrieval policy for a unit-load warehouse.

Existing heuristics in the literature such as the turnover-based and duration-of-stay heuristics neglect the variability of both in-flow and out-flow of products. Furthermore, these heuristics do not consider the capacity of each storage class in the warehouse. Although these heuristics are easier to implement in practice, we suggest that by taking the variability of product flows and capacity constraints into account feasible solutions with much higher quality can be obtained.

We assume a factor-based demand model in which demand for each product in each period is affinely dependent on some uncertain factors. The distributions of these uncertain factors are only partially characterized. We introduce a robust optimization model that minimizes the worst-case expected total operating cost under distributional ambiguity of these uncertain factors. By restricting to linear policies, we obtain a storage and retrieval policy that minimizes the worst-case expected total operating cost by solving a moderate-size linear optimization problem.

We compare our linear policy with the turnover-based and duration-of-stay heuristics through a set of numerical experiments. Specifically, we investigate the performance of each policy under different numbers of products, different sizes of each storage class, different numbers of storage classes, and different levels of demand variability. Surprisingly, despite imprecise specification of demand distributions, our computational studies suggest that the linear policy achieves close (and extremely close for many cases) to the expected value given perfect demand information, which is a lower bound of the optimal expected operating cost. The linear policy significantly outperforms the turnover-based and duration-of-stay heuristics in all cases studied. Besides considering the randomness of demand, our results suggest that the inclusion of the capacity constraints of storage classes in the formulation contributes partly to the effectiveness of the linear policy.

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A Proof of Proposition 1

($\Rightarrow$) Suppose that (2) is feasible, then there exists a feasible solution $\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{w}$ such that

$$x_{ij}^{t+1} = x_{ij}^{t} + v_{ij}^{t} - w_{ij}^{t} \quad i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T} \tag{11}$$

From the third constraint when $t = 1$ and summing all the j-th terms for $j \in \mathcal{N}$, we have

$$\begin{aligned} \sum_{j \in \mathcal{N}} x_{ij}^{2} &= \sum_{j \in \mathcal{N}} (x_{ij}^{1} + v_{ij}^{1} - w_{ij}^{1}) \\ &= a_i^1 - d_i^1 \text{ (from the first and second constraint of (2))} \end{aligned} \tag{12}$$

From the constraint $x_{ij}^t \geq 0, i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}^+$, we conclude that inequality (3) is true for $t = 1$. Similarly, for $t = 2, ..., T$, we have

$$\begin{aligned} \sum_{\tau=1}^{t} \sum_{j \in \mathcal{N}} x_{ij}^{\tau+1} &= \sum_{\tau=1}^{t} \sum_{j \in \mathcal{N}} \left(x_{ij}^{\tau} + v_{ij}^{\tau} - w_{ij}^{\tau} \right) \\ &= \sum_{\tau=1}^{t} \sum_{j \in \mathcal{N}} x_{ij}^{\tau} + \sum_{\tau=1}^{t} \sum_{j \in \mathcal{N}} \left(v_{ij}^{\tau} - w_{ij}^{\tau} \right) \\ &= \sum_{\tau=1}^{t} \sum_{j \in \mathcal{N}} x_{ij}^{\tau} + \sum_{\tau=1}^{t} (a_i^{\tau} - d_i^{\tau}) \end{aligned} \tag{13}$$

Hence,

$$\begin{aligned} \sum_{\tau=1}^{t} (a_i^{\tau} - d_i^{\tau}) &= \sum_{\tau=1}^{t} \sum_{j \in \mathcal{N}} x_{ij}^{\tau+1} - \sum_{\tau=2}^{t} \sum_{j \in \mathcal{N}} x_{ij}^{\tau} \\ &= \sum_{j \in \mathcal{N}} x_{ij}^{t+1} \\ &\geq 0 \end{aligned} \tag{14}$$

Similarly, the last inequality is deduced from the non-negativity of x_{ij}^t for $i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}^+$. Thus we proved Inequality (3).

($\Leftarrow$) Suppose that $\sum_{\tau=1}^{t} a_i^{\tau} - \sum_{\tau=1}^{t} d_i^{\tau} \geq 0$ for $i \in \mathcal{M}, t \in \mathcal{T}$. Using the assumption that there is an emergency storage class, N, with infinite storage, the fifth constraint (capacity constraint) is satisfied, we have

$$v_{ij}^{t} = \begin{cases} a_i^t & \text{for } j = N, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$w_{ij}^{t} = \begin{cases} d_i^t & \text{for } j = N, \\ 0 & \text{otherwise,} \end{cases}$$

for $i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}$. Hence, $v_{ij}^t, w_{ij}^t \geq 0$ for $i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}$. We now prove the nonnegativity of x_{ij}^t. For $j \neq N$, it is clear that $x_{ij}^t = 0$ for all $i \in \mathcal{M}, t \in \mathcal{T}$. We now prove the case $j = N$. For $t = 1$,

$$\begin{aligned} x_{iN}^2 &= x_{iN}^1 + v_{iN}^1 - w_{iN}^1 \\ &= a_i^1 - d_i^1 \\ &\geq 0 \text{ (from Inequality (3) when } \tau = 1). \end{aligned}$$

For any $i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}$,

$$\begin{aligned} x_{iN}^{t+1} &= x_{ij}^t + v_{iN}^t - w_{iN}^t \\ &= x_{iN}^{t-1} + \sum_{\tau=t-1}^{t} (v_{iN}^\tau - w_{iN}^\tau) \\ &= x_{iN}^{t-1} + \sum_{\tau=t-1}^{t} (a_i^\tau - d_i^\tau) \\ &= x_{iN}^{t-2} + \sum_{\tau=t-2}^{t} (a_i^\tau - d_i^\tau) \\ &= \cdots \\ &= x_{iN}^1 + \sum_{\tau=1}^{t} (a_i^\tau - d_i^\tau) \\ &\geq 0 \text{ (from Inequality(3))} \end{aligned}$$

Inductively, $x_{iN}^t \geq 0, , i \in \mathcal{M}, t \in \mathcal{T}$. Thus, $x_{ij}^t \geq 0, i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}$. ■

B Proof of Theorem 1

Under Assumption U, we have $\tilde{\boldsymbol{z}}$ has support set W. For $i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}$, let

$$v_{ij}^t(\boldsymbol{z}^{t-1}) = \begin{cases} a_i^t & \text{for } j = N \\ 0 & \text{otherwise,} \end{cases}$$

and

$$w_{ij}^t(\boldsymbol{z}^t) = \begin{cases} d_i^t(\boldsymbol{z}^t) & \text{for } j = N \\ 0 & \text{otherwise,} \end{cases}$$

where $\boldsymbol{z} \in W$. Under the uncertainty set W, we have $d_i^t(\boldsymbol{z}^t) \geq 0$, hence $w_{ij}^t(\boldsymbol{z}^t) \geq 0$. The nonnegativity of $v_{ij}^t(\boldsymbol{z}^{t-1})$ follows from the nonnegativity of a_i^t. We now prove the constraint $x_{ij}^{t+1}(\boldsymbol{z}^t) \geq 0$. For $i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}$,

$$x_{ij}^{t+1}(\boldsymbol{z}^t) = x_{ij}^t(\boldsymbol{z}^{t-1}) + v_{ij}^t(\boldsymbol{z}^{t-1}) - w_{ij}^t(\boldsymbol{z}^t)$$

$$= \begin{cases} x_{ij}^t(\boldsymbol{z}^{t-1}) + a_i^t - d_i^t(\boldsymbol{z}^t) & \text{for } j = N \\ x_{ij}^t(\boldsymbol{z}^{t-1}) & \text{otherwise.} \end{cases}$$

Since $x_{ij}^1 = 0$, it is clear that $x_{ij}^{t+1}(\boldsymbol{z}^t) = 0$ for $j \neq N$. For $i \in \mathcal{M}$, $j = N$ and $t \in \mathcal{T}$,

$$\begin{aligned} x_{iN}^{t+1}(\boldsymbol{z}^t) &= x_{iN}^t(\boldsymbol{z}^{t-1}) + v_{iN}^t(\boldsymbol{z}^{t-1}) - w_{iN}^t(\boldsymbol{z}^t) \\ &= x_{iN}^t(\boldsymbol{z}^{t-1}) + a_N^t - d_N^t(\boldsymbol{z}^t) \\ &= x_{iN}^{t-1}(\boldsymbol{z}^{t-2}) + \sum_{\tau=t-1}^{t} (a_N^\tau - d_N^\tau(\boldsymbol{z}^\tau)) \\ &= \cdots \\ &= x_{iN}^1 + \sum_{\tau=1}^{t} (a_N^\tau - d_N^\tau(\boldsymbol{z}^\tau)) \\ &\geq 0 \text{ (since } \boldsymbol{z} \in W) \end{aligned}$$

Thus $x_{ij}^{t+1}(\boldsymbol{z}^t) \geq 0$ for $i \in \mathcal{M}, j \in \mathcal{N}, t \in \mathcal{T}$, and we have found a feasible solution for Z_{LR}. It is clear that the objective value is also finite.

Using the linear decision rule on the objective function of (7), we have

$$\begin{aligned} &\sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{M}} \sum_{j\in\mathcal{N}} \left(s_j v_{ij}^t(\boldsymbol{z}^{t-1}) + r_j w_{ij}^t(\boldsymbol{z}^t)\right) \\ =\ &\sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{M}} \sum_{j\in\mathcal{N}} \left(\left(s_j v_{ij}^{t,0} + \sum_{k\in\mathcal{K}_{t-1}} s_j v_{ij}^{t,k} z_k \right) + \left(r_j w_{ij}^{t,0} + \sum_{k\in\mathcal{K}_t} r_j w_{ij}^{t,k} z_k \right) \right) \\ =\ &\sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{M}} \sum_{j\in\mathcal{N}} \left(\left(s_j v_{ij}^{t,0} + r_j w_{ij}^{t,0} \right) + \sum_{k\in\mathcal{K}_T} \left(s_j v_{ij}^{t,k} + r_j w_{ij}^{t,k} \right) z_k \right) \end{aligned}$$

in which $v_{ij}^{t,k} = 0$ for $i \in \mathcal{M}, j \in \mathcal{N}, k \in \mathcal{K}_T \setminus \mathcal{K}_t, t \in \mathcal{T}$ and $w_{ij}^{t,k} = 0$ for $i \in \mathcal{M}, j \in \mathcal{N}, k \in \mathcal{K}_T \setminus \mathcal{K}_{t-1}, t \in \mathcal{T}$. Setting $g^k = \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{M}} \sum_{j\in\mathcal{N}} \left(s_j v_{ij}^{t,k} + r_j w_{ij}^{t,k} \right)$ for $k \in \mathcal{K}_T^0$, the objective function becomes

$$\begin{aligned} &\min \max_{z\in\hat{W}} \left(g^0 + \sum_{k\in\mathcal{K}_T} g^k \right) \\ &= \min \left(g^0 + \max_{z\in\hat{W}} \sum_{k\in\mathcal{K}_T} g^k \right). \end{aligned}$$

For $i \in \mathcal{M}$,

$$\begin{aligned} &\sum_{j\in\mathcal{N}} v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) = a_i^t, && t \in \mathcal{T}. \\ \Leftrightarrow\ &\sum_{j\in\mathcal{N}} \left(v_{ij}^{t,0} + \sum_{k\in\mathcal{K}_T} v_{ij}^{t,k} z_k \right) = a_i^t \qquad \forall \boldsymbol{z} \in W, && t \in \mathcal{T}. \\ \Leftrightarrow\ &\sum_{j\in\mathcal{N}} v_{ij}^{t,k} = \begin{cases} a_i^t, & \text{if } k = 0, \\ 0, & \text{otherwise,} \end{cases} && k \in \mathcal{K}_T^0, t \in \mathcal{T}. \end{aligned}$$

The last equivalence holds since the set W is full dimensional. Similarly, for $i \in \mathcal{M}$,

$$\begin{aligned}
& \sum_{j\in\mathcal{N}} w_{ij}^t(\tilde{\boldsymbol{z}}^t) = d_i^t(\tilde{\boldsymbol{z}}^t), && t \in \mathcal{T}. \\
\Leftrightarrow \quad & \sum_{j\in\mathcal{N}} \left(w_{ij}^{t,0} + \sum_{k\in\mathcal{K}_T} w_{ij}^{t,k} z_k \right) = d_i^{t,0} + \sum_{k\in\mathcal{K}_T} d_i^{t,k} z_k \qquad \forall \boldsymbol{z} \in W, && t \in \mathcal{T}. \\
\Leftrightarrow \quad & \sum_{j\in\mathcal{N}} w_{ij}^{t,k} = d_i^{t,k}, && k \in \mathcal{K}_T^0, t \in \mathcal{T}.
\end{aligned}$$

For $i \in \mathcal{M}, j \in \mathcal{N}$,

$$\begin{aligned}
& x_{ij}^{t+1}(\tilde{\boldsymbol{z}}^t) = x_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) + v_{ij}^t(\tilde{\boldsymbol{z}}^{t-1}) - w_{ij}^t(\tilde{\boldsymbol{z}}^t), && t \in \mathcal{T}. \\
\Leftrightarrow \quad & x_{ij}^{t+1,0} + \sum_{k\in\mathcal{K}_t} x_{ij}^{t+1,k} z_k \\
& = \left(x_{ij}^{t,0} + \sum_{k\in\mathcal{K}_t} x_{ij}^{t,k} z_k \right) + \left(v_{ij}^{t,0} + \sum_{k\in\mathcal{K}_{t-1}} v_{ij}^{t,k} z_k \right) - \left(w_{ij}^{t,0} + \sum_{k\in\mathcal{K}_t} w_{ij}^{t,k} z_k \right), && t \in \mathcal{T}. \\
\Leftrightarrow \quad & x_{ij}^{t+1,0} + \sum_{k\in\mathcal{K}_T} x_{ij}^{t+1,k} z_k \\
& = \left(x_{ij}^{t,0} + \sum_{k\in\mathcal{K}_T} x_{ij}^{t,k} z_k \right) + \left(v_{ij}^{t,0} + \sum_{k\in\mathcal{K}_T} v_{ij}^{t,k} z_k \right) - \left(w_{ij}^{t,0} + \sum_{k\in\mathcal{K}_T} w_{ij}^{t,k} z_k \right), && t \in \mathcal{T}. \\
\Leftrightarrow \quad & x_{ij}^{t+1,k} = x_{ij}^{t,k} + v_{ij}^{t,k} - w_{ij}^{t,k}, && k \in \mathcal{K}_T^0, t \in \mathcal{T}.
\end{aligned}$$

For $i \in \mathcal{M}, j \in \mathcal{N}$, since $v_{ij}^{t,k} = 0, w_{ij}^{t,k} = 0$ for $k \in \mathcal{K}_T \setminus \mathcal{K}_t, t \in \mathcal{T}$, we have $x_{ij}^{t+1,k} = x_{ij}^{t,k}$ for $k \in \mathcal{K}_T \setminus \mathcal{K}_t, t \in \mathcal{T}$. From the constraint $x_{ij}^1 = 0$ for $i \in \mathcal{M}, j \in \mathcal{N}$, we conclude that $x_{ij}^{t,k} = 0$ for $i \in \mathcal{M}, j \in \mathcal{N}, k \in \mathcal{K}_T \setminus \mathcal{K}_t, t \in \mathcal{T}$.

By the definition that $h_j^{t,k} = \sum_{i\in\mathcal{M}} x_{ij}^{t,k} + v_{ij}^{t,k}$, for $j \in \mathcal{N}, k \in \mathcal{K}_T^0, t \in \mathcal{T}$, we can draw the similar deduction that $h_j^{t,k} = 0$ for $j \in \mathcal{N}, k \in \mathcal{K}_T \setminus \mathcal{K}_t, t \in \mathcal{T}$.

Finally, for inequality constraints involving linear decision rule, we note that given $y \in \mathcal{L}_{K_t}$ the constraint

$$y(\tilde{\boldsymbol{z}}^t) \le r$$

is equivalent the following robust counterpart

$$y^0 + \sum_{k\in\mathcal{K}_T} y^k z_k \le r, \qquad \forall \boldsymbol{z} \in W.$$

Hence, we prove that under the linear storage-retrieval policy, we have Problem (8). ■

C Proof of Proposition 2

Note that with

$$\boldsymbol{d}^{t,0} = \begin{pmatrix} d_1^{t,0} \\ \vdots \\ d_M^{t,0} \end{pmatrix}, \boldsymbol{D}^t = \begin{pmatrix} d_1^{t,1} & \dots & d_1^{t,K_t} & 0 & \dots & 0 \\ \vdots & \dots & \ddots & 0 & \dots & 0 \\ d_M^{t,1} & \dots & d_M^{t,K_t} & 0 & \dots & 0 \end{pmatrix} \in \Re^{M \times K_T},$$

we can concisely represent the vector of uncertain demands as

$$\boldsymbol{d}^t(\tilde{\boldsymbol{z}}^t) = \begin{pmatrix} \boldsymbol{d}_1^t(\tilde{\boldsymbol{z}}^t) \\ \vdots \\ \boldsymbol{d}_M^t(\tilde{\boldsymbol{z}}^t) \end{pmatrix} = \boldsymbol{d}^{t,0} + \boldsymbol{D}^t \boldsymbol{z} \qquad \text{for } t \in \mathcal{T},$$

where $\boldsymbol{z} \triangleq \boldsymbol{z}^T$. Hence, we can express the factor support set, W as follows:

$$\begin{aligned} W &= \left\{ \boldsymbol{z} \mid \exists \boldsymbol{u} \;:\; \sum_{\tau=1}^{t} \boldsymbol{d}^{\tau}(\boldsymbol{z}^{\tau}) \le \sum_{\tau=1}^{t} \boldsymbol{a}^{\tau}, \boldsymbol{d}^t(\boldsymbol{z}^t) \ge \boldsymbol{0}, t \in \mathcal{T}, \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} \le \boldsymbol{q} \right\} \\ &= \left\{ \boldsymbol{z} \mid \exists \boldsymbol{u} \;:\; \sum_{\tau=1}^{t} \boldsymbol{D}^{\tau}\boldsymbol{z} \le \sum_{\tau=1}^{t} (\boldsymbol{a}^{\tau} - \boldsymbol{d}^{\tau,0}), \boldsymbol{d}^{t,0} + \boldsymbol{D}^t\boldsymbol{z} \ge \boldsymbol{0}, t \in \mathcal{T}, \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} \le \boldsymbol{q} \right\} \\ &= \left\{ \boldsymbol{z} \mid \exists \boldsymbol{u} \;:\; \bar{\boldsymbol{D}}^t \boldsymbol{z} \le \bar{\boldsymbol{a}}^t, -\boldsymbol{D}^t\boldsymbol{z} \le \boldsymbol{d}^{t,0}, t \in \mathcal{T}, \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} \le \boldsymbol{q} \right\}, \end{aligned}$$

where $\boldsymbol{a}^t = \begin{pmatrix} a_1^t \\ \vdots \\ a_M^t \end{pmatrix}$, $\bar{\boldsymbol{a}}^t = \sum_{\tau=1}^{t} (\boldsymbol{a}^{\tau} - \boldsymbol{d}^{\tau,0})$ and $\bar{\boldsymbol{D}}^t = \sum_{\tau=1}^{t} \boldsymbol{D}^{\tau}$. To obtain the robust counterpart, $\max_{\boldsymbol{z} \in W} \boldsymbol{z}'\boldsymbol{y} \le r$, we note that by strong linear programming duality, the primal problem

$$\begin{aligned} \max_{\boldsymbol{u},\boldsymbol{z}} \quad & \boldsymbol{y}'\boldsymbol{z} & \\ \text{s.t.} \quad & \bar{\boldsymbol{D}}^t \boldsymbol{z} \le \bar{\boldsymbol{a}}^t & t \in \mathcal{T} \\ & -\boldsymbol{D}^t \boldsymbol{z} \le \boldsymbol{d}^{t,0} & t \in \mathcal{T} \\ & \boldsymbol{A}\boldsymbol{z} + \boldsymbol{B}\boldsymbol{u} \le \boldsymbol{q}, & \end{aligned}$$

has the same objective value as the following dual problem,

$$\begin{aligned} \min \quad & \sum_{t \in \mathcal{T}} \left(\bar{\boldsymbol{a}}^{t\prime} \boldsymbol{\alpha}^t + \boldsymbol{d}^{t,0\prime} \boldsymbol{\beta}^t \right) + \boldsymbol{\gamma}'\boldsymbol{q} & \\ \text{s.t.} \quad & \sum_{t \in \mathcal{T}} \left(\bar{\boldsymbol{D}}^{t\prime} \boldsymbol{\alpha}^t - \boldsymbol{D}^{t\prime} \boldsymbol{\beta}^t \right) + \boldsymbol{A}'\boldsymbol{\gamma} = \boldsymbol{y} & \\ & \boldsymbol{B}'\boldsymbol{\gamma} = \boldsymbol{0} & \\ & \boldsymbol{\gamma} \ge \boldsymbol{0}, \boldsymbol{\beta}^t, \boldsymbol{\alpha}^t \ge \boldsymbol{0} & t \in \mathcal{T}. \end{aligned}$$

Hence, the robust counterpart, $\max_{\boldsymbol{z}\in W} \boldsymbol{z}'\boldsymbol{y} \leq r$ is feasible if and only if there exists $\boldsymbol{\gamma}$, $\boldsymbol{\alpha}^t, \boldsymbol{\beta}^t$, $t \in \mathcal{T}$ feasible in the dual problem and that

$$\sum_{t\in\mathcal{T}} \left(\bar{\boldsymbol{a}}^{t\prime}\boldsymbol{\alpha}^t + \boldsymbol{d}^{t,0\prime}\boldsymbol{\beta}^t \right) + \boldsymbol{\gamma}'\boldsymbol{q} \leq r.$$

■

D Formulation for computing the optimal linear policy

$$
\begin{aligned}
\min\quad & g^0 + \max_{\boldsymbol{z}\in\hat{W}} \sum_{k\in\mathcal{K}_t} g^k z_k \\
\text{s.t.}\quad & g^k = \sum_{t\in\mathcal{T}}\sum_{i\in\mathcal{M}}\sum_{j\in\mathcal{N}} \left(s_j v_{ij}^{t,k} + r_j w_{ij}^{t,k}\right), && k\in\mathcal{K}_T^0;\\
& \sum_{j\in\mathcal{N}} v_{ij}^{t,k} = \begin{cases} a_i^t, & \text{if } k=0,\\ 0, & \text{otherwise,}\end{cases} && i\in\mathcal{M}, k\in\mathcal{K}_T, t\in\mathcal{T};\\
& \sum_{j\in\mathcal{N}} w_{ij}^{t,k} = d_i^{t,k}, && i\in\mathcal{M}, k\in\mathcal{K}_T^0, t\in\mathcal{T};\\
& x_{ij}^{t+1,k} = x_{ij}^{t,k} + v_{ij}^{t,k} - w_{ij}^{t,k}, && i\in\mathcal{M}, j\in\mathcal{N}, k\in\mathcal{K}_T^0, t\in\mathcal{T};\\
& x_{ij}^1 = 0, && i\in\mathcal{M}, j\in\mathcal{N};\\
& h_j^{t,k} = \sum_{i\in\mathcal{M}} \left(x_{ij}^{t,k} + v_{ij}^{t,k}\right), && j\in\mathcal{N}, k\in\mathcal{K}_T^0, t\in\mathcal{T};\\
& \sum_{\tau\in\mathcal{T}} \left(\bar{\boldsymbol{a}}^{\tau\prime}\boldsymbol{\alpha}_{h_j^t}^{\tau} + \boldsymbol{d}^{\tau,0\prime}\boldsymbol{\beta}_{h_j^t}^{\tau}\right) + \boldsymbol{\gamma}_{h_j^t}'\boldsymbol{q} \le c_j - h_j^{t,0}, && j\in\mathcal{N}^-, t\in\mathcal{T};\\
& \sum_{\tau\in\mathcal{T}} \left(\bar{\boldsymbol{D}}^{\tau\prime}\boldsymbol{\alpha}_{h_j^t}^{\tau} - \boldsymbol{D}^{\tau\prime}\boldsymbol{\beta}_{h_j^t}^{\tau}\right) + \boldsymbol{A}'\boldsymbol{\gamma}_{h_j^t} = \boldsymbol{h}_j^{t,K_T}, && j\in\mathcal{N}^-, t\in\mathcal{T};\\
& \boldsymbol{B}'\boldsymbol{\gamma}_{h_j^t} = \boldsymbol{0}, && j\in\mathcal{N}^-, t\in\mathcal{T};\\
& \sum_{\tau\in\mathcal{T}} \left(\bar{\boldsymbol{a}}^{\tau\prime}\boldsymbol{\alpha}_{v_{ij}^t}^{\tau} + \boldsymbol{d}^{\tau,0\prime}\boldsymbol{\beta}_{v_{ij}^t}^{\tau}\right) + \boldsymbol{\gamma}_{v_{ij}^t}'\boldsymbol{q} \le -v_{ij}^{t,0}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& \sum_{\tau\in\mathcal{T}} \left(\bar{\boldsymbol{D}}^{\tau\prime}\boldsymbol{\alpha}_{v_{ij}^t}^{\tau} - \boldsymbol{D}^{\tau\prime}\boldsymbol{\beta}_{v_{ij}^t}^{\tau}\right) + \boldsymbol{A}'\boldsymbol{\gamma}_{v_{ij}^t} = \boldsymbol{v}_{ij}^{t,K_T}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& \boldsymbol{B}'\boldsymbol{\gamma}_{v_{ij}^t} = \boldsymbol{0}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& \sum_{\tau\in\mathcal{T}} \left(\bar{\boldsymbol{a}}^{\tau\prime}\boldsymbol{\alpha}_{w_{ij}^t}^{\tau} + \boldsymbol{d}^{\tau,0\prime}\boldsymbol{\beta}_{w_{ij}^t}^{\tau}\right) + \boldsymbol{\gamma}_{w_{ij}^t}'\boldsymbol{q} \le -w_{ij}^{t,0}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& \sum_{\tau\in\mathcal{T}} \left(\bar{\boldsymbol{D}}^{\tau\prime}\boldsymbol{\alpha}_{w_{ij}^t}^{\tau} - \boldsymbol{D}^{\tau\prime}\boldsymbol{\beta}_{w_{ij}^t}^{\tau}\right) + \boldsymbol{A}'\boldsymbol{\gamma}_{w_{ij}^t} = \boldsymbol{w}_{ij}^{t,K_T}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& \boldsymbol{B}'\boldsymbol{\gamma}_{w_{ij}^t} = \boldsymbol{0} && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& \sum_{\tau\in\mathcal{T}} \left(\bar{\boldsymbol{a}}^{\tau\prime}\boldsymbol{\alpha}_{x_{ij}^t}^{\tau} + \boldsymbol{d}^{\tau,0\prime}\boldsymbol{\beta}_{x_{ij}^t}^{\tau}\right) + \boldsymbol{\gamma}_{x_{ij}^t}'\boldsymbol{q} \le -x_{ij}^{t,0}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& \sum_{\tau\in\mathcal{T}} \left(\bar{\boldsymbol{D}}^{\tau\prime}\boldsymbol{\alpha}_{x_{ij}^t}^{\tau} - \boldsymbol{D}^{\tau\prime}\boldsymbol{\beta}_{x_{ij}^t}^{\tau}\right) + \boldsymbol{A}'\boldsymbol{\gamma}_{x_{ij}^t} = \boldsymbol{x}_{ij}^{t,K_T}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& \boldsymbol{B}'\boldsymbol{\gamma}_{x_{ij}^t} = \boldsymbol{0}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& v_{ij}^{t,k} = x_{ij}^{t,k} = h_j^{t,k} = 0, && i\in\mathcal{M}, j\in\mathcal{N}, k\in\mathcal{K}_T\backslash\mathcal{K}_t, t\in\mathcal{T};\\
& w_{ij}^{t,k} = 0, && i\in\mathcal{M}, j\in\mathcal{N}, k\in\mathcal{K}_T\backslash\mathcal{K}_{t-1}, t\in\mathcal{T}\\
& \boldsymbol{\gamma}_{h_j^t}, \boldsymbol{\gamma}_{v_{ij}^t}, \boldsymbol{\gamma}_{w_{ij}^t}, \boldsymbol{\gamma}_{x_{ij}^t} \ge \boldsymbol{0}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& \boldsymbol{\alpha}_{h_j^t}, \boldsymbol{\alpha}_{v_{ij}^t}, \boldsymbol{\alpha}_{w_{ij}^t}, \boldsymbol{\alpha}_{x_{ij}^t} \ge \boldsymbol{0}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};\\
& \boldsymbol{\beta}_{h_j^t}, \boldsymbol{\beta}_{v_{ij}^t}, \boldsymbol{\beta}_{w_{ij}^t}, \boldsymbol{\beta}_{x_{ij}^t} \ge \boldsymbol{0}, && i\in\mathcal{M}, j\in\mathcal{N}, t\in\mathcal{T};
\end{aligned}
\tag{15}
$$

where $\boldsymbol{h}_j^{t,K_T} = \begin{pmatrix} h_j^{t,1} \\ \vdots \\ h_j^{t,K_T} \end{pmatrix}$, $\boldsymbol{v}_{ij}^{t,K_T} = \begin{pmatrix} v_{ij}^{t,1} \\ \vdots \\ v_{ij}^{t,K_T} \end{pmatrix}$, $\boldsymbol{w}_{ij}^{t,K_T} = \begin{pmatrix} w_{ij}^{t,1} \\ \vdots \\ w_{ij}^{t,K_T} \end{pmatrix}$ and $\boldsymbol{x}_{ij}^{t,K_T} = \begin{pmatrix} x_{ij}^{t,1} \\ \vdots \\ x_{ij}^{t,K_T} \end{pmatrix}$.

E Arrival and demand parameters for numerical experiments

	$t=1$		$t=2$		$t=3$		$t=4$		$t=5$	
i	a_i^1	$\boldsymbol{d}_i^1(\tilde{\boldsymbol{z}}^1)$	a_i^2	$\boldsymbol{d}_i^2(\tilde{\boldsymbol{z}}^2)$	a_i^3	$\boldsymbol{d}_i^3(\tilde{\boldsymbol{z}}^3)$	a_i^4	$\boldsymbol{d}_i^4(\tilde{\boldsymbol{z}}^4)$	a_i^5	$\boldsymbol{d}_i^5(\tilde{\boldsymbol{z}}^5)$
1	3500	$2900+\tilde{z}_1$	3500	$1900+\tilde{z}_6$	3500	$4500+\tilde{z}_{11}$	3500	$2300+\tilde{z}_{16}$	3500	$5600+\tilde{z}_{21}$
2	3500	$1000+\tilde{z}_2$	3500	$5700+\tilde{z}_7$	3500	$3700+\tilde{z}_{12}$	3500	$2100+\tilde{z}_{17}$	3500	$3000+\tilde{z}_{22}$
3	1000	$600+\tilde{z}_3$	1000	$700+\tilde{z}_8$	1000	$800+\tilde{z}_{13}$	1000	$1800+\tilde{z}_{18}$	1000	$600+\tilde{z}_{23}$
4	1000	$800+\tilde{z}_4$	1000	$500+\tilde{z}_9$	1200	$1800+\tilde{z}_{14}$	1200	$1000+\tilde{z}_{19}$	1000	$500+\tilde{z}_{24}$
5	1000	$500+\tilde{z}_5$	1000	$1400+\tilde{z}_{10}$	1000	$1100+\tilde{z}_{15}$	1200	$500+\tilde{z}_{20}$	1000	$1700+\tilde{z}_{25}$

Table 3: The number of arriving pallets and uncertain demand for each product in each period for the 5-product case in Section 4.1.

	$t=1$		$t=2$		$t=3$		$t=4$		$t=5$	
i	a_i^1	$\boldsymbol{d}_i^1(\tilde{\boldsymbol{z}}^1)$	a_i^2	$\boldsymbol{d}_i^2(\tilde{\boldsymbol{z}}^2)$	a_i^3	$\boldsymbol{d}_i^3(\tilde{\boldsymbol{z}}^3)$	a_i^4	$\boldsymbol{d}_i^4(\tilde{\boldsymbol{z}}^4)$	a_i^5	$\boldsymbol{d}_i^5(\tilde{\boldsymbol{z}}^5)$
1	3500	$2900+\tilde{z}_1$	3500	$1900+\tilde{z}_{11}$	3500	$4500+\tilde{z}_{21}$	3500	$2300+\tilde{z}_{31}$	3500	$5600+\tilde{z}_{41}$
2	3500	$1000+\tilde{z}_2$	3500	$5700+\tilde{z}_{12}$	3500	$3700+\tilde{z}_{22}$	3500	$2100+\tilde{z}_{32}$	3500	$3000+\tilde{z}_{42}$
3	1000	$600+\tilde{z}_3$	1000	$700+\tilde{z}_{13}$	1000	$800+\tilde{z}_{23}$	1000	$1800+\tilde{z}_{33}$	1000	$600+\tilde{z}_{43}$
4	1000	$800+\tilde{z}_4$	1000	$500+\tilde{z}_{14}$	1200	$1800+\tilde{z}_{24}$	1200	$1000+\tilde{z}_{34}$	1000	$500+\tilde{z}_{44}$
5	1000	$500+\tilde{z}_5$	1000	$1400+\tilde{z}_{15}$	1000	$1100+\tilde{z}_{25}$	1200	$500+\tilde{z}_{35}$	1000	$1700+\tilde{z}_{45}$
6	1000	$500+\tilde{z}_6$	1000	$800+\tilde{z}_{16}$	1000	$800+\tilde{z}_{26}$	1000	$1100+\tilde{z}_{36}$	3500	$1300+\tilde{z}_{46}$
7	1000	$500+\tilde{z}_7$	1000	$600+\tilde{z}_{17}$	1000	$1200+\tilde{z}_{27}$	1000	$700+\tilde{z}_{37}$	3500	$1100+\tilde{z}_{47}$
8	1000	$700+\tilde{z}_8$	1000	$500+\tilde{z}_{18}$	1000	$1800+\tilde{z}_{28}$	1000	$500+\tilde{z}_{38}$	1000	$1400+\tilde{z}_{48}$
9	1000	$500+\tilde{z}_9$	1000	$1000+\tilde{z}_{19}$	1000	$1100+\tilde{z}_{29}$	1000	$1400+\tilde{z}_{39}$	1000	$700+\tilde{z}_{49}$
10	1000	$900+\tilde{z}_{10}$	1000	$500+\tilde{z}_{20}$	1000	$1200+\tilde{z}_{30}$	1000	$500+\tilde{z}_{40}$	1000	$1700+\tilde{z}_{50}$

Table 4: The number of arriving pallets and uncertain demand for each product in each period for the 10-product case in Section 4.1.

	$t=1$		$t=2$		$t=3$		$t=4$		$t=5$	
i	a_i^1	$\boldsymbol{d}_i^1(\tilde{\boldsymbol{z}}^1)$	a_i^2	$\boldsymbol{d}_i^2(\tilde{\boldsymbol{z}}^2)$	a_i^3	$\boldsymbol{d}_i^3(\tilde{\boldsymbol{z}}^3)$	a_i^4	$\boldsymbol{d}_i^4(\tilde{\boldsymbol{z}}^4)$	a_i^5	$\boldsymbol{d}_i^5(\tilde{\boldsymbol{z}}^5)$
1	3500	$2900+\tilde{z}_1$	3500	$1900+\tilde{z}_{21}$	3500	$4500+\tilde{z}_{41}$	3500	$2300+\tilde{z}_{61}$	3500	$4700+\tilde{z}_{81}$
2	3500	$1000+\tilde{z}_2$	4000	$5700+\tilde{z}_{22}$	3500	$3700+\tilde{z}_{42}$	3500	$2100+\tilde{z}_{62}$	3500	$2500+\tilde{z}_{82}$
3	1000	$600+\tilde{z}_3$	1000	$700+\tilde{z}_{23}$	1200	$800+\tilde{z}_{43}$	1000	$1700+\tilde{z}_{63}$	1000	$600+\tilde{z}_{83}$
4	1000	$800+\tilde{z}_4$	1000	$500+\tilde{z}_{24}$	1000	$1400+\tilde{z}_{44}$	1000	$900+\tilde{z}_{64}$	1000	$500+\tilde{z}_{84}$
5	1000	$500+\tilde{z}_5$	1000	$1400+\tilde{z}_{25}$	1000	$1100+\tilde{z}_{45}$	1000	$500+\tilde{z}_{65}$	1000	$1700+\tilde{z}_{85}$
6	1000	$500+\tilde{z}_6$	1000	$800+\tilde{z}_{26}$	1000	$800+\tilde{z}_{46}$	1000	$1100+\tilde{z}_{66}$	1000	$1300+\tilde{z}_{86}$
7	1000	$500+\tilde{z}_7$	1000	$600+\tilde{z}_{27}$	1000	$1200+\tilde{z}_{47}$	1000	$700+\tilde{z}_{67}$	1000	$1100+\tilde{z}_{87}$
8	1000	$700+\tilde{z}_8$	1000	$500+\tilde{z}_{28}$	1400	$1800+\tilde{z}_{48}$	1000	$500+\tilde{z}_{68}$	1000	$1400+\tilde{z}_{88}$
9	1000	$500+\tilde{z}_9$	1000	$1000+\tilde{z}_{29}$	1000	$1100+\tilde{z}_{49}$	1000	$1400+\tilde{z}_{69}$	1000	$700+\tilde{z}_{89}$
10	1000	$900+\tilde{z}_{10}$	1000	$500+\tilde{z}_{30}$	1000	$1200+\tilde{z}_{50}$	1000	$500+\tilde{z}_{70}$	1000	$1700+\tilde{z}_{90}$
11	3500	$2900+\tilde{z}_{11}$	3500	$1900+\tilde{z}_{31}$	3500	$4500+\tilde{z}_{51}$	3500	$2300+\tilde{z}_{71}$	3500	$5600+\tilde{z}_{91}$
12	3500	$1000+\tilde{z}_{12}$	3500	$5700+\tilde{z}_{32}$	3700	$3600+\tilde{z}_{52}$	3500	$2100+\tilde{z}_{72}$	3500	$3000+\tilde{z}_{92}$
13	1000	$600+\tilde{z}_{13}$	1000	$700+\tilde{z}_{33}$	1000	$800+\tilde{z}_{53}$	1000	$1800+\tilde{z}_{73}$	1000	$600+\tilde{z}_{93}$
14	1500	$800+\tilde{z}_{14}$	1000	$500+\tilde{z}_{34}$	1000	$1800+\tilde{z}_{54}$	1000	$1000+\tilde{z}_{74}$	1000	$500+\tilde{z}_{94}$
15	1500	$500+\tilde{z}_{15}$	1000	$1400+\tilde{z}_{35}$	1000	$1100+\tilde{z}_{55}$	1000	$500+\tilde{z}_{75}$	1000	$1700+\tilde{z}_{95}$
16	1000	$500+\tilde{z}_{16}$	1000	$800+\tilde{z}_{36}$	1000	$800+\tilde{z}_{56}$	1000	$1100+\tilde{z}_{76}$	1000	$1300+\tilde{z}_{96}$
17	1000	$500+\tilde{z}_{17}$	1000	$600+\tilde{z}_{37}$	1000	$1200+\tilde{z}_{57}$	1000	$700+\tilde{z}_{77}$	1000	$1100+\tilde{z}_{97}$
18	1000	$700+\tilde{z}_{18}$	1000	$500+\tilde{z}_{38}$	1300	$1800+\tilde{z}_{58}$	1000	$500+\tilde{z}_{78}$	1000	$1400+\tilde{z}_{98}$
19	1000	$500+\tilde{z}_{19}$	1000	$1000+\tilde{z}_{39}$	1000	$1100+\tilde{z}_{59}$	1000	$1400+\tilde{z}_{79}$	1000	$700+\tilde{z}_{99}$
20	1000	$900+\tilde{z}_{20}$	1000	$500+\tilde{z}_{40}$	1000	$1200+\tilde{z}_{60}$	1000	$500+\tilde{z}_{80}$	1000	$1700+\tilde{z}_{100}$

Table 5: The number of arriving pallets and uncertain demand for each product in each period for the 20-product case in Section 4.1.

	$t=1$		$t=2$		$t=3$		$t=4$		$t=5$	
i	a_i^1	$\boldsymbol{d}_i^1(\tilde{\boldsymbol{z}}^1)$	a_i^2	$\boldsymbol{d}_i^2(\tilde{\boldsymbol{z}}^2)$	a_i^3	$\boldsymbol{d}_i^3(\tilde{\boldsymbol{z}}^3)$	a_i^4	$\boldsymbol{d}_i^4(\tilde{\boldsymbol{z}}^4)$	a_i^5	$\boldsymbol{d}_i^5(\tilde{\boldsymbol{z}}^5)$
1	3500	$2900+\tilde{z}_1$	3500	$1900+\tilde{z}_{41}$	3500	$4500+\tilde{z}_{81}$	3800	$2300+\tilde{z}_{121}$	3800	$4700+\tilde{z}_{161}$
2	3500	$1000+\tilde{z}_2$	4000	$5700+\tilde{z}_{42}$	3500	$3700+\tilde{z}_{82}$	3800	$2100+\tilde{z}_{122}$	3800	$2500+\tilde{z}_{162}$
3	1000	$600+\tilde{z}_3$	1000	$700+\tilde{z}_{43}$	1200	$800+\tilde{z}_{83}$	1300	$1700+\tilde{z}_{123}$	1300	$600+\tilde{z}_{163}$
4	1000	$800+\tilde{z}_4$	1000	$500+\tilde{z}_{44}$	1000	$1400+\tilde{z}_{84}$	1300	$900+\tilde{z}_{124}$	1300	$500+\tilde{z}_{164}$
5	1500	$500+\tilde{z}_5$	1000	$1400+\tilde{z}_{45}$	1000	$1100+\tilde{z}_{85}$	1300	$500+\tilde{z}_{125}$	1300	$1500+\tilde{z}_{165}$
6	1000	$500+\tilde{z}_6$	1000	$800+\tilde{z}_{46}$	1000	$800+\tilde{z}_{86}$	1300	$1100+\tilde{z}_{126}$	1300	$1300+\tilde{z}_{166}$
7	1000	$500+\tilde{z}_7$	1000	$600+\tilde{z}_{47}$	1000	$1200+\tilde{z}_{87}$	1300	$700+\tilde{z}_{127}$	1300	$1100+\tilde{z}_{167}$
8	1000	$700+\tilde{z}_8$	1000	$500+\tilde{z}_{48}$	1400	$1800+\tilde{z}_{88}$	1300	$500+\tilde{z}_{128}$	1300	$1400+\tilde{z}_{168}$
9	1000	$500+\tilde{z}_9$	1000	$1000+\tilde{z}_{49}$	1000	$1100+\tilde{z}_{89}$	1300	$1300+\tilde{z}_{129}$	1300	$700+\tilde{z}_{169}$
10	1000	$900+\tilde{z}_{10}$	1000	$500+\tilde{z}_{50}$	1000	$1200+\tilde{z}_{90}$	1300	$500+\tilde{z}_{130}$	1300	$1700+\tilde{z}_{170}$
11	3500	$2900+\tilde{z}_{11}$	3500	$1900+\tilde{z}_{51}$	3500	$4500+\tilde{z}_{91}$	3800	$2300+\tilde{z}_{131}$	3800	$5600+\tilde{z}_{171}$
12	3500	$1000+\tilde{z}_{12}$	3500	$5700+\tilde{z}_{52}$	3700	$3600+\tilde{z}_{92}$	3800	$2100+\tilde{z}_{132}$	3800	$3000+\tilde{z}_{172}$
13	1000	$600+\tilde{z}_{13}$	1000	$700+\tilde{z}_{53}$	1000	$800+\tilde{z}_{93}$	1300	$1800+\tilde{z}_{133}$	1300	$600+\tilde{z}_{173}$
14	1500	$800+\tilde{z}_{14}$	1000	$500+\tilde{z}_{54}$	1000	$1800+\tilde{z}_{94}$	1300	$1000+\tilde{z}_{134}$	1300	$500+\tilde{z}_{174}$
15	1500	$500+\tilde{z}_{15}$	1000	$1400+\tilde{z}_{55}$	1000	$1100+\tilde{z}_{95}$	1300	$500+\tilde{z}_{135}$	1300	$1700+\tilde{z}_{175}$
16	1000	$500+\tilde{z}_{16}$	1000	$800+\tilde{z}_{56}$	1000	$800+\tilde{z}_{96}$	1300	$1100+\tilde{z}_{136}$	1300	$1300+\tilde{z}_{176}$
17	1000	$500+\tilde{z}_{17}$	1000	$600+\tilde{z}_{57}$	1000	$1200+\tilde{z}_{97}$	1300	$700+\tilde{z}_{137}$	1300	$1100+\tilde{z}_{177}$
18	1000	$700+\tilde{z}_{18}$	1000	$500+\tilde{z}_{58}$	1300	$1800+\tilde{z}_{98}$	1300	$500+\tilde{z}_{138}$	1300	$1400+\tilde{z}_{178}$
19	1000	$500+\tilde{z}_{19}$	1000	$1000+\tilde{z}_{59}$	1000	$1100+\tilde{z}_{99}$	1300	$1300+\tilde{z}_{139}$	1300	$700+\tilde{z}_{179}$
20	1000	$900+\tilde{z}_{20}$	1000	$500+\tilde{z}_{60}$	1000	$1200+\tilde{z}_{100}$	1300	$500+\tilde{z}_{140}$	1300	$1700+\tilde{z}_{180}$
21	2500	$1900+\tilde{z}_{21}$	3500	$1900+\tilde{z}_{61}$	3500	$4500+\tilde{z}_{101}$	3800	$2300+\tilde{z}_{141}$	3800	$4700+\tilde{z}_{181}$
22	2500	$1000+\tilde{z}_{22}$	4000	$4700+\tilde{z}_{62}$	3500	$3700+\tilde{z}_{102}$	3800	$2100+\tilde{z}_{142}$	3800	$2500+\tilde{z}_{182}$
23	1000	$600+\tilde{z}_{23}$	1000	$700+\tilde{z}_{63}$	1200	$800+\tilde{z}_{103}$	1300	$1700+\tilde{z}_{143}$	1300	$600+\tilde{z}_{183}$
24	1000	$800+\tilde{z}_{24}$	1000	$500+\tilde{z}_{64}$	1000	$1400+\tilde{z}_{104}$	1300	$900+\tilde{z}_{144}$	1300	$500+\tilde{z}_{184}$
25	1500	$500+\tilde{z}_{25}$	1000	$1400+\tilde{z}_{65}$	1000	$1100+\tilde{z}_{105}$	1300	$500+\tilde{z}_{145}$	1300	$1700+\tilde{z}_{185}$
26	1000	$500+\tilde{z}_{26}$	1000	$800+\tilde{z}_{66}$	1000	$800+\tilde{z}_{106}$	1300	$1100+\tilde{z}_{146}$	1300	$1300+\tilde{z}_{186}$
27	1000	$500+\tilde{z}_{27}$	1000	$600+\tilde{z}_{67}$	1000	$1200+\tilde{z}_{107}$	1300	$700+\tilde{z}_{147}$	1300	$1100+\tilde{z}_{187}$
28	1000	$700+\tilde{z}_{28}$	1000	$500+\tilde{z}_{68}$	1400	$1800+\tilde{z}_{108}$	1300	$500+\tilde{z}_{148}$	1300	$1400+\tilde{z}_{188}$
29	1000	$500+\tilde{z}_{29}$	1000	$1000+\tilde{z}_{69}$	1000	$1100+\tilde{z}_{109}$	1400	$1400+\tilde{z}_{149}$	1300	$700+\tilde{z}_{189}$
30	1000	$900+\tilde{z}_{30}$	1000	$500+\tilde{z}_{70}$	1000	$1200+\tilde{z}_{110}$	1300	$500+\tilde{z}_{150}$	1300	$1700+\tilde{z}_{190}$
31	3200	$2900+\tilde{z}_{31}$	3500	$1900+\tilde{z}_{71}$	3500	$4500+\tilde{z}_{111}$	3800	$2300+\tilde{z}_{151}$	3800	$4700+\tilde{z}_{191}$
32	3500	$1000+\tilde{z}_{32}$	3500	$5700+\tilde{z}_{72}$	3500	$3700+\tilde{z}_{112}$	3800	$2100+\tilde{z}_{152}$	3800	$2500+\tilde{z}_{192}$
33	1000	$600+\tilde{z}_{33}$	1000	$700+\tilde{z}_{73}$	1200	$800+\tilde{z}_{113}$	1300	$1700+\tilde{z}_{153}$	1300	$600+\tilde{z}_{193}$
34	1500	$800+\tilde{z}_{34}$	1000	$500+\tilde{z}_{74}$	1000	$1400+\tilde{z}_{114}$	1300	$900+\tilde{z}_{154}$	1300	$500+\tilde{z}_{194}$
35	1500	$500+\tilde{z}_{35}$	1000	$1400+\tilde{z}_{75}$	1000	$1100+\tilde{z}_{115}$	1300	$500+\tilde{z}_{155}$	1300	$1700+\tilde{z}_{195}$
36	1000	$500+\tilde{z}_{36}$	1000	$800+\tilde{z}_{76}$	1000	$800+\tilde{z}_{116}$	1300	$1100+\tilde{z}_{156}$	1300	$1300+\tilde{z}_{196}$
37	1000	$500+\tilde{z}_{37}$	1000	$600+\tilde{z}_{77}$	1000	$1200+\tilde{z}_{117}$	1300	$700+\tilde{z}_{157}$	1300	$1100+\tilde{z}_{197}$
38	1000	$700+\tilde{z}_{38}$	1000	$500+\tilde{z}_{78}$	1400	$1800+\tilde{z}_{118}$	1300	$500+\tilde{z}_{158}$	1300	$1400+\tilde{z}_{198}$
39	1000	$500+\tilde{z}_{39}$	1000	$1000+\tilde{z}_{79}$	1000	$1100+\tilde{z}_{119}$	1500	$1400+\tilde{z}_{159}$	1300	$700+\tilde{z}_{199}$
40	1000	$900+\tilde{z}_{40}$	1000	$500+\tilde{z}_{80}$	1000	$1200+\tilde{z}_{120}$	1300	$500+\tilde{z}_{160}$	1300	$1700+\tilde{z}_{200}$

Table 6: The number of arriving pallets and uncertain demand for each product in each period for the 40-product case in Section 4.1.

	$t=1$		$t=2$		$t=3$		$t=4$		$t=5$	
i	a_i^1	$\boldsymbol{d}_i^1(\tilde{\boldsymbol{z}}^1)$	a_i^2	$\boldsymbol{d}_i^2(\tilde{\boldsymbol{z}}^2)$	a_i^3	$\boldsymbol{d}_i^3(\tilde{\boldsymbol{z}}^3)$	a_i^4	$\boldsymbol{d}_i^4(\tilde{\boldsymbol{z}}^4)$	a_i^5	$\boldsymbol{d}_i^5(\tilde{\boldsymbol{z}}^5)$
1	4500	$2900+\tilde{z}_1$	4500	$1900+\tilde{z}_{11}$	4500	$4500+\tilde{z}_{21}$	4500	$2300+\tilde{z}_{31}$	4500	$5600+\tilde{z}_{41}$
2	4500	$1000+\tilde{z}_2$	4500	$5700+\tilde{z}_{12}$	4500	$3700+\tilde{z}_{22}$	4500	$2100+\tilde{z}_{32}$	4500	$3000+\tilde{z}_{42}$
3	2000	$600+\tilde{z}_3$	2000	$700+\tilde{z}_{13}$	2000	$800+\tilde{z}_{23}$	2000	$1800+\tilde{z}_{33}$	2000	$600+\tilde{z}_{43}$
4	2000	$800+\tilde{z}_4$	2000	$500+\tilde{z}_{14}$	2200	$1800+\tilde{z}_{24}$	2200	$1000+\tilde{z}_{34}$	2000	$500+\tilde{z}_{44}$
5	2000	$500+\tilde{z}_5$	2000	$1400+\tilde{z}_{15}$	2000	$1100+\tilde{z}_{25}$	2200	$500+\tilde{z}_{35}$	2000	$1700+\tilde{z}_{45}$
6	2000	$500+\tilde{z}_6$	2000	$800+\tilde{z}_{16}$	2000	$800+\tilde{z}_{26}$	2000	$1100+\tilde{z}_{36}$	2000	$1300+\tilde{z}_{46}$
7	2000	$500+\tilde{z}_7$	2000	$600+\tilde{z}_{17}$	2000	$1200+\tilde{z}_{27}$	2000	$700+\tilde{z}_{37}$	2000	$1100+\tilde{z}_{47}$
8	2000	$700+\tilde{z}_8$	2000	$500+\tilde{z}_{18}$	2000	$1800+\tilde{z}_{28}$	2000	$500+\tilde{z}_{38}$	2000	$1400+\tilde{z}_{48}$
9	2000	$500+\tilde{z}_9$	2000	$1000+\tilde{z}_{19}$	2000	$1100+\tilde{z}_{29}$	2000	$1400+\tilde{z}_{39}$	2000	$700+\tilde{z}_{49}$
10	2000	$900+\tilde{z}_{10},$	2000	$500+\tilde{z}_{20}$	2000	$1200+\tilde{z}_{30}$	2000	$500+\tilde{z}_{40}$	2000	$1700+\tilde{z}_{50}$

Table 7: The number of arriving pallets and uncertain demand for each product in each period for the 10-product case in Sections 4.2 and 4.3.

	$t=1$		$t=2$		$t=3$		$t=4$		$t=5$	
i	a_i^1	$\boldsymbol{d}_i^1(\tilde{\boldsymbol{z}}^1)$	a_i^2	$\boldsymbol{d}_i^2(\tilde{\boldsymbol{z}}^2)$	a_i^3	$\boldsymbol{d}_i^3(\tilde{\boldsymbol{z}}^3)$	a_i^4	$\boldsymbol{d}_i^4(\tilde{\boldsymbol{z}}^4)$	a_i^5	$\boldsymbol{d}_i^5(\tilde{\boldsymbol{z}}^5)$
1	4500	$3000+\tilde{z}_1$	4500	$2000+\tilde{z}_{11}$	4500	$4600+\tilde{z}_{21}$	4500	$2400+\tilde{z}_{31}$	4500	$5700+\tilde{z}_{41}$
2	4500	$1100+\tilde{z}_2$	4500	$5800+\tilde{z}_{12}$	4500	$3800+\tilde{z}_{22}$	4500	$2200+\tilde{z}_{32}$	4500	$3100+\tilde{z}_{42}$
3	2000	$700+\tilde{z}_3$	2000	$800+\tilde{z}_{13}$	2000	$900+\tilde{z}_{23}$	2000	$1900+\tilde{z}_{33}$	2000	$700+\tilde{z}_{43}$
4	2000	$900+\tilde{z}_4$	2000	$600+\tilde{z}_{14}$	2200	$1900+\tilde{z}_{24}$	2200	$1100+\tilde{z}_{34}$	2000	$600+\tilde{z}_{44}$
5	2000	$600+\tilde{z}_5$	2000	$1500+\tilde{z}_{15}$	2000	$1200+\tilde{z}_{25}$	2200	$600+\tilde{z}_{35}$	2000	$1800+\tilde{z}_{45}$
6	2000	$600+\tilde{z}_6$	2000	$900+\tilde{z}_{16}$	2000	$900+\tilde{z}_{26}$	2000	$1200+\tilde{z}_{36}$	2000	$1400+\tilde{z}_{46}$
7	2000	$600+\tilde{z}_7$	2000	$700+\tilde{z}_{17}$	2000	$1300+\tilde{z}_{27}$	2000	$800+\tilde{z}_{37}$	2000	$1200+\tilde{z}_{47}$
8	2000	$800+\tilde{z}_8$	2000	$600+\tilde{z}_{18}$	2000	$1900+\tilde{z}_{28}$	2000	$600+\tilde{z}_{38}$	2000	$1500+\tilde{z}_{48}$
9	2000	$600+\tilde{z}_9$	2000	$1100+\tilde{z}_{19}$	2000	$1200+\tilde{z}_{29}$	2000	$1500+\tilde{z}_{39}$	2000	$800+\tilde{z}_{49}$
10	2000	$1000+\tilde{z}_{10}$	2000	$600+\tilde{z}_{20}$	2000	$1300+\tilde{z}_{30}$	2000	$600+\tilde{z}_{40}$	2000	$1800+\tilde{z}_{50}$

Table 8: The number of arriving pallets and uncertain demand for each product in each period for the 10-product case in Section 4.4.

F Warehouse layouts for numerical experiments

Layout	Class	Store cost	Retrieve cost	Capacity
A	1	1	1	1500
	2	1.5	1.5	3000
	3	2	2	4500
	4	3	3	6000
	5	100	100	∞
B	1	1	1	1000
	2	1.5	1.5	3000
	3	2	2	4500
	4	3	3	6000
	5	4	4	500
	6	100	100	∞
C	1	1	1	1000
	2	1.5	1.5	2000
	3	2	2	4500
	4	3	3	6000
	5	4	4	500
	6	5	5	1000
	7	100	100	∞
D	1	1	1	1000
	2	1.5	1.5	2000
	3	2	2	3000
	4	3	3	6000
	5	4	4	500
	6	5	5	1000
	7	6	6	1500
	8	100	100	∞
E	1	1	1	1000
	2	1.5	1.5	2000
	3	2	2	3000
	4	3	3	4000
	5	4	4	500
	6	5	5	1000
	7	6	6	1500
	8	7	7	2000
	9	100	100	∞

Table 9: Layouts with different numbers of storage classes for Section 4.