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Sensitivity analysis of the optimal solutions to Huff-type competitive location and design problems

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Abstract A chain wants to set up a single new facility in a planar market where similar facilities of competitors, and possibly of its own chain, are already present. Fixed demand points split their demand probabilistically over all facilities in the market proportionally with their attraction to each facility, determined by the different perceived qualities of the facilities and the distances to them, through a gravitational or logit type model. Both the location and the quality (design) of the new facility are to be found so as to maximize the profit obtained for the chain. Several types of constraints and costs are considered.

Applying an interval analysis based global optimization method on several spatial patterns in a quasi-real-world environment, the behaviour of optimal solutions is investigated when changes are made in the basic model parameters. The study yields valuable insight for modelers into the impact of spatial pattern and various model parameters of the model on the resulting location and design decision. Spatial patterns differ in distribution of demand, of own and/or competing facilities, and of facility qualities. Studied model parameters include distance decay, income function, and push force effects, as well as investment restrictions and aggregation of demand.

Keywords Continuous location · Facility design · Competition · Global optimization · Sensitivity analysis

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1 Introduction

When a retail chain considers entering or extending its presence in a market the two major questions are ‘how’ and ‘where’. ‘How’ indicates the facility design part of the decision, required when several options exist in the type of outlet and/or the goods and services that will be available there. ‘Where’ clearly states that a location has also to be found. In order to be able to evaluate the market share resulting from the entry of the new facility, one needs to consider the way consumers behave in their choice between facilities offering similar goods/services, and many quite different proposals exist in the literature, as extensively explained by Drezner-Eiselt [10]. The goal pursued by the chain is to increase its profit, through an increase of its expected market share and without a too high investment in the facilities’ design; this leads to optimization questions of various difficulty levels, depending on the modelling assumptions, see e.g. the surveys of Eiselt et al. [14], Eiselt-Laporte [12] and Plastria [31]. Observe that in most of the literature only the location question is investigated, whereas we want to determine design and location simultaneously, since these two features cannot be separated. A remarkable exception is the work of Aboolian et al. [1], who consider a related problem, although in discrete space. In a non-competitive setting, Verter, Dasci and Laporte [5, 36, 37] have studied the problem of simultaneous optimization of the plant location and technology selection decisions in a multi-product environment. There is another major question to be considered when locating a new outlet, the ‘when’. However, this dynamic component is out of the scope of this research.

The general situation we investigate in this paper is static, without looking ahead in time towards possible later newcomers (like in Drezner-Drezner [7], Sáiz et al. [34]) or changes in the market (like in Drezner-Drezner [8], Averbakh-Berman [2]). We consider essential goods only, in other words, demand is assumed to be known and fixed (see Berman-Krass [3] for a discrete competitive location model with elastic (variable) demand, and McGarvey-Cavalier [28] for a planar location space). According to the terminology introduced by Hakimi in [19], we are studying a $(1|X_m)$-medianoid problem: given a set of m pre-existing facilities that are located at a set of points X_m, we want to determine where a competitor should locate a single new facility so as to maximize his profit. Notice, however, that in this paper some of the pre-existing facilities may belong to the locating chain, and that the objective is maximization of expected profit, instead of market share.

Within these restrictions and considering a single new facility only, we investigate one of the most complicated situations for designing and locating a facility: Huff-type customer behavior in a continuous region of the plane. This model assumes, following Reilly [33], that customers will patronize all available facilities offering the adequate service in a probabilistic way, with probability proportional to their attraction to this facility, as compared to the other existing facilities. We assume that attraction depends on the customer’s view of the quality offered by the facility and its distance to it, thereby generalizing the proposals of Huff [24], Nakanishi-Cooper [29] and Jain-Mahajan [25]. The chosen site for the new facility determines the distances from the potential customers, while the chosen design influences their quality assessments. Taking both effects together yields the probabilities with which the different customers will make use of the new facility, which then allows to evaluate the expected total market share of the new facility. No pricing policy is assumed in this paper. The influence of the prices in the patronizing behavior of the customers is assumed to be included in the perception of the quality of the facilities.

The resulting model is a highly nonlinear optimization problem, involving several modelling decisions and many corresponding parameters to be estimated, the influence of which remain virtually unexplored. It is also intuitively clear that the number, location and relative size or design, both of the existing chain outlets and of its competitors, should have a significant impact on the location decision for a new facility for the chain. The existing literature says almost nothing about this impact, and we may say that the model's behaviour is not really well understood. It is the aim of this study to make a contribution to this understanding, by investigating the changes in optimal design/location when the model environment and parameters change. To this end one must first of all be able to solve the problem optimally in a reliable way.

But even for a fixed design, the objective function has a complicated multimodal shape as can be seen for example in the graph shown by Drezner [6], and the situation is complicated further when considering, as we do here, the presence of locational constraints, unavoidable in any realistic description of a real-world problem. For such functions and/or constraints traditional analytic solution techniques or more model-specific iterative methods (see [6]) are unable to find guaranteed optimal solutions: what is obtained is a proposal of a site apparently close to a local optimal solution, without clear indication of the actual degree of (non)optimality, both in terms of objective value as in terms of spatial position. More precise information can then only be gathered through a more or less blind repetition of the method from varying starting solutions, and it is not uncommon that very different solutions are then produced, and an actual global optimum is not necessarily found, and, even if it is, one does not know so.

Additionally, the model typically contains many parameters which are quite difficult to estimate precisely. On the one hand, it is therefore important to know how stable the solution is with respect to these parameters, in order to be able to appreciate whether the uncertainty in certain parameters may be ignored or not. This calls for a sensitivity analysis of the optimal solution in terms of changes in the parameters. But how can one correctly appreciate the results of such an analysis without a clear distinction between jumps in the solution due to parameter modification rather than due to algorithm inaccuracy? Traditional local optimization approaches therefore are not acceptable for an adequate sensitivity analysis. One way of tackling this difficulty was described by Fernández-Pelegrín [15]: they incorporate the uncertainty on the parameter-values as interval-valued data, specifying both an upper and a lower bound on each uncertain parameter, and apply interval analysis based optimization tools, which construct a close approximation of the set of all possible optimal solutions for any setting of each parameter within its bounds.

On the other hand, recognizing that the model captures only approximately the real world situation, Plastria [30] argues that in practice a good idea of the full set of almost-optimal solutions is more useful than knowledge of an optimal solution, and proceeds to obtain such information by extending the global optimization method BSSS devised by Hansen et al. [22].

Both methods, as well as a more recent variant based on triangle subdivisions [9, 11], were described for a pure location problem only. In a recent paper [16] a solution strategy was developed for the full location-design problem using 'rigorous' global optimization based on interval analysis [see e.g. 21, 26, 32], and including several novel acceleration techniques. This approach yields a complete view of the set of close-to-optimal solutions to any desired degree of accuracy, and therefore lends itself better to the proposed sensitivity analysis. To the best of our knowledge, no other available local or global optimization method allows to obtain the same amount of reliable information. However, a number of new interpretation difficulties arise, as will be explained in the sequel.

This tool has allowed us to investigate the behaviour of the optimal solution of Huff-type competitive location and design problem under various circumstances. The experimental design was made in such a way as to maximize the information gathered while keeping the computational burden within reasonable limits. We have done all experiments on data-sets based on a single set of real-world demand data from the region of Murcia in South-eastern Spain, and recorded the variation of the optimal objective value and of the optimal solution set, both in design and in location.

Broadly speaking we have studied the effects of

- the market structure, comparing chain superiority with strong competition, in Section 4,
- the aggregation of demand, in Section 5,
- homogeneous versus widely different facility qualities, in Section 7,
- a restricted budget for initial investments, in Section 9.

We have also investigated the effect of the some more technical modelling parameters:

- the distance decay effect, in Section 8.1,
- the shape of the income valuation of the market share, in Section 8.2,
- the location-cost relative to the quality-cost, in Section 8.3,
- the costs relative to the income, in Section 8.4.

A precise description of these experiments and their results is given in separate sections, after the model has been fully formalized in the next section. The paper concludes with an overview of the experience gathered through these results and a number of recommendations for analysts and practitioners.

2 The Huff-type competition model

2.1 The formal model

A chain wants to set up a single new facility in a given area of the plane $S \subset \mathbb{R}^2$. There already exist m facilities in the vicinity offering the same good or product. The first k of these belong to the chain, the next $m-k$ facilities belong to competing chains. Since it does not matter in this model whether the competing facilities are of the same competing chain or of different competing chains, we will assume the former to be the case. Note that $k = 0$ means that the entering chain, which we will call 'the chain', is new to the market, whereas $k = m$ indicates simple expansion of the chain without competition. Existing facility $j = 1, \ldots, m$ is located at point f_j and has a known design, expressed by a quality $\alpha_j > 0$. The unknown location of the new facility will be denoted by x, and its quality by α.

Demand is concentrated at n demand points with known locations p_i and buying power w_i ($i = 1, \ldots, n$).

Distance between demand point i and new (resp. existing) facility (j) is denoted by d_{ix} (resp. d_{ij}), and are considered to be Euclidean throughout. Each demand point i has its own subjective perception of distance, obtained by application of some positive non-decreasing function g_i to actual distance. The inverse function $1/g_i$ is called the *distance decay function*. Several typical perceived distance functions g_i are found in literature, such as $g_i(d_{ix}) = e^{\lambda_i d_{ix}}$ (see Hodgson [23]) or $g_i(d_{ix}) = (d_{ix})^{\lambda_i}$ (see Drezner [6]), with $\lambda_i > 0$ a given parameter.

Attraction is modelled multiplicatively as quality divided by perceived distance, i.e. the attraction that demand point p_i feels for facility f_j is given by $\alpha_j/g_i(d_{ij})$, and for the new facility by $\alpha/g_i(d_{ix})$. This generalizes the 'law of retail gravitation' of Reilly [33], who considered perceived distance to be squared distance. In this paper we made the same assumption, except in Section 8.1 where other decay functions are tested. Quality was first estimated as store surface by Huff [24], and later incorporated several other store characteristics by Jain-Mahajan [25] and Nakanishi-Cooper [29] (for details see Drezner-Eiselt [10]).

In the spirit of Huff [24], we consider that the patronizing behaviour of customers is probabilistic, that is, demand points split their buying power among the facilities proportionally to the attraction they feel for them. By these assumptions the total market share captured by the chain is given by

$$M(x,\alpha) \equiv \sum_{i=1}^{n} w_i \frac{\dfrac{\alpha}{g_i(d_{ix})} + \displaystyle\sum_{j=1}^{k} \frac{\alpha_j}{g_i(d_{ij})}}{\dfrac{\alpha}{g_i(d_{ix})} + \displaystyle\sum_{j=1}^{m} \frac{\alpha_j}{g_i(d_{ij})}}.$$

The chain's market share gives rise to expected sales through a strictly increasing function $F(\cdot)$. Operating costs for the new facility are given by a function $G(x,\alpha)$ of both its location x and its quality α.

Profit of the chain is obtained as sales minus costs. Since operating costs of the existing facilities of the own chain are assumed to be constant, we take into account costs of the new facility only.

The problem to be solved may therefore be written as follows

$$\begin{cases} \max\ \Pi(x,\alpha) \equiv F(M(x,\alpha)) - G(x,\alpha) \\ \text{s.t.}\quad d_{ix} \geq d_i^{\min}\ \forall i \\ \qquad\ \alpha \in [\alpha_{\min}, \alpha_{\max}] \\ \qquad\ x \in S \subset \mathbb{R}^2 \end{cases} \tag{1}$$

where the set S denotes the feasible space wherein the new facility is to be located. The parameters $d_i^{\min} > 0$ and $\alpha_{\min} > 0$ are given thresholds, guaranteeing on the one hand that the new facility is not located on top of demand point i and on the other hand that it has a minimum level of quality. The parameter $\alpha_{\max}$ is the maximum quality that a facility may have in practice. Note that in case some legal protection rules exist to guarantee a minimal market to any installed facility, one could also incorporate a minimum allowed distance to competitors, or another kind of constraint on the location.

In this study F was chosen to be linear, $F(M(x,\alpha)) = c \cdot M(x,\alpha)$, meaning each unit of good sold yields the fixed income c, except in Section 8.2 where we test also when F is convex quadratic, $F(M(x,\alpha)) = c_1 + c_2 \cdot M(x,\alpha)^2$ ($c_1, c_2 > 0$), describing increasing returns in scale. The cost-function G is assumed to have the separable form $G(x,\alpha) = G_1(x) + G_2(\alpha)$, with $G_1(x) = \sum_{i=1}^{n} \frac{w_i}{(d_{ix})^{\varphi_{i0}} + \varphi_{i1}}$ and $G_2(\alpha) = e^{\frac{\alpha}{\beta_0} + \beta_1} - e^{\beta_1}$. Notice how $G_1(x)$ increases as x approaches any demand point, expressing that in centers of high population concentrations the operational cost of the facility will be higher due to the value of land and premises, thus raising the cost of buying or renting the site. Also the minimal distance to demand point constraints $d_{ix} \geq d_i^{\min} > 0$ ensure existence and differentiability of $G_1(x)$ at all *feasible* x. On the other hand, the cost of quality $G_2(\alpha)$ is an increasing convex function with $G_2(0) = 0$: the higher the quality of the facility, the higher the costs, at an increasing rate.

As discussed in Fernández et al. [16] many other income and cost functions may be considered and handled by the general algorithm proposed there.

2.2 Representation of the results

Results of an optimization problem may be understood in several ways. On the one hand there is the optimal value found, and on the other hand one may look at the optimal solution. In our location-design problem we are clearly more interested in the latter, since it is the choice of the site that matters most, although the actual profit figure is also important for judging the profitability of the project. Therefore we want to represent both types of results, so as to be able to judge the sensitivity of these results when the parameter settings change.

Theoretically this seems relatively simple, particularly for the optimal values: these are single values which are easily compared. But this does not take into account that the outcome of a traditional local optimization algorithm usually only gives an approximation for the optimal value, very often without any information about the precision of the approximation or, even worse, without guarantee of having found (an approximation of) a globally optimal value. However, with the global optimization method we have used, we do have quite precise information about the global optimal value: during the optimization stage both an upper and lower bound on the global optimal value are available, and gradually improved until their difference is below a prespecified precision ε. The resulting upper bound is therefore guaranteed to be an overestimation within ε-accuracy of the actual global optimal value. In our computations we used $\varepsilon = 0.05$. It follows that when we compare the 'optimal' objective values obtained for two different parameter settings, the conclusions can be trusted, as long as they differ by significantly more than 0.05 units.

The situation is quite more complicated in variable space, when we look at the optimal sites and corresponding qualities found. Traditional methods simply give an approximate solution, which might be close to a local optimal solution (but not necessarily a globally optimal one), without information about how far it might be from such a solution (except possibly some very fuzzy indication one might derive from a Lipschitz constant, which is often awfully overestimated), nor about the possible existence of other (near)-optimal solutions. The second stage of our global optimization algorithm is aimed at producing an outer approximation (up to a precision η) of the set R_δ of δ-optimal solutions, i.e. with objective value at least a fraction $1-\delta$ of the optimal value. To be more precise, it yields a list of solution-boxes, and for each box an upper and lower bound on the objective values at any solution within the box. The final result is the list of all boxes lying fully within this δ-optimal region and those on its 'boundary', i.e. probably containing solutions which are not δ-optimal, but not too far, as indicated by the second precision parameter η. These solutions are all guaranteed to lie within $R_{\delta+\eta(1-\delta)}$. This is illustrated graphically in Figure 1 for a

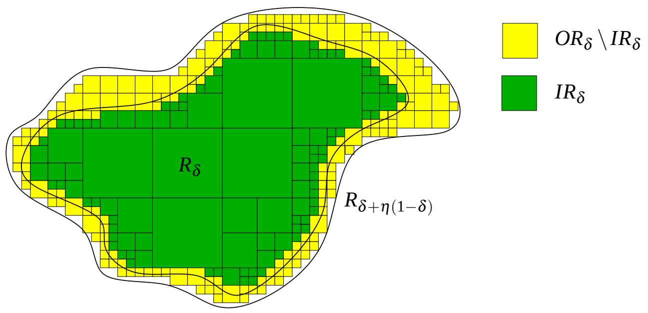


Fig. 1 Inner and outer approximation of the near optimal region.

two-dimensional problem. The first (inner) set IR_δ contains all boxes for which the lower bound is at least a fraction $1-\delta$ of the (approximate) global optimal value, the second set (surrounding) $OR_\delta \setminus IR_\delta$ consists of boxes with upper bound at least reaching this fraction $1-\delta$, but lower bound below it, but not by more than η% (for details, see [30]). This implies that in case the objective is quite flat around a global optimum these boxes may cover a non negligible area. The advantage of this approach is that in case alternative near-global optimal solutions exist anywhere else in the area, this will immediately be apparent from the results, since the lists of boxes then define several disconnected regions. In our computational tests we used $\delta = 0.01$ and $\eta = 0.002$.

A more precise reporting of the near-optimal region calls for a graphical representation of the resulting lists of boxes in variable space, which leads to new difficulties, due to their three-dimensional shape. We resolve this by reporting the information in three two-dimensional maps, each showing the interactions between some pair of variables. This is obtained by adding two window-panes on the right and bottom of the basic spatial map, allowing to view all three 2-dimensional projections of the 3-dimensional solution set; see e.g. Figure 4. The map itself shows the 2-dimensional spatial part, without the quality (x_1 increases from left (0) to right (10) and x_2 increases from top (0) to bottom (10)). The right-hand pane shows the quality and vertical space part (quality increases from left (0.5) to right (5)). The bottom pane shows the quality and horizontal space part (quality increase from top (0.5) to bottom (5)).

Note that often the spatial solution set is quite small or falls almost exactly on the boundary of some forbidden region, and is therefore not well visible. One may however know where it is, by combining the regions shown in the two side-panes.

3 Base data

The real-world data consists of an area around the city of Murcia in South-Eastern Spain. The city of Murcia is the capital of the Autonomous Region of Murcia (A.R.M.), a province with 11314 Km^2 and over one million inhabitants. More than half of the inhabitants of A.R.M. live in the city of Murcia or in the villages close to it.

A working radius of 25 km around the main city was considered suitable for being used in a Huff-type model. This choice was motivated by two concerns. On the one hand, customers are assumed to patronize *all* existing facilities, so none of the facilities should be too far away from a population center (e.g. within a 15 minutes drive, say), because otherwise none of its inhabitants would go there. On the other hand, in the region at hand, increasing the radius of the circle to 30 km increases the number of covered inhabitants only slightly, whereas reducing the radius to 20 km decreases it considerably.

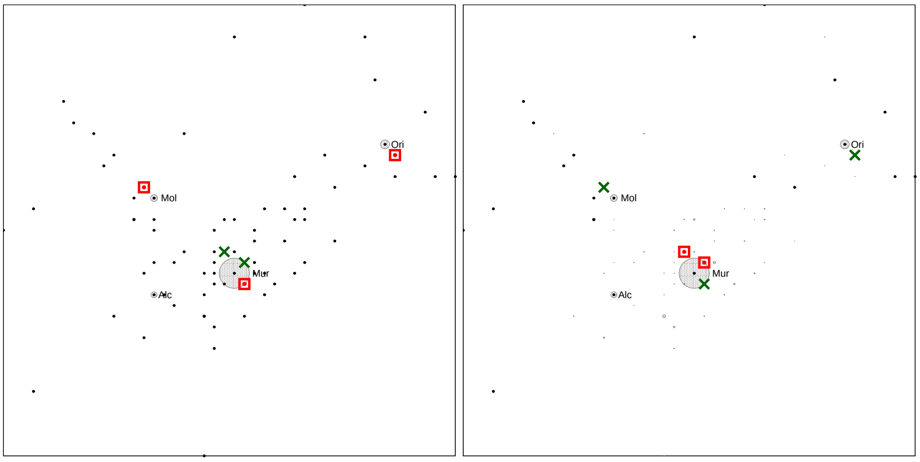
Within the circle of radius 25 km, centered at Murcia, live 557746 inhabitants of A.R.M. and another 74812 inhabitants from the neighboring province of Alicante, on the East of A.R.M. (see Figure 2). These 632558 inhabitants form our primary set of customers. They are distributed over 71 population centers, with population varying between 1138 and 178013 inhabitants. In this study we have considered each population center as a demand point, with buying power proportional to its total population (one unit of buying power per 17.800 inhabitants). Their position and population can be seen in Figure 3(a): each demand point is shown as a black dot, and also a gray circle which radius proportional to the buying power. In Figure 3(b) the aggregated case is depicted, as discussed in Section 5. Note that here the gray circles also show the forbidden regions, but contrary to (a), do not represent the demand proportionally.

Fig. 2 Population centers in the neighborhood of the city of Murcia.

Our basic data deals with the location of supermarkets. Currently there are five supermarkets present in the area: two from a first chain, the 'Small Chain', and three from another chain, the 'Large Chain'. Figure 3 shows the location of each supermarket as viewed from some chain's point of view: firms belonging to the own chain are marked by a $\times$, and firms from the other chain are shown by a $\boxdot$ marker on the map.

The feasible set S was taken exactly as depicted in Figure 3, i.e. the smallest rectangle containing all demand points. This is approximately a square centered at Murcia and of sides close to 45 km. It is reduced further by some forbidden regions around each population center as defined by $d_i^{\min}$ (see below).

The coordinates of the population centers and the supermarkets were obtained with the Geographical Information System called VisualMap (see [38]), and were rescaled from coordinates $[200,245] \times [243,285]$ to an approximate standard square $[0,10] \times [0,9.3]$, thus the units correspond approximately to 4.5 Km. Note that, as usual in computer graphics, the origin is at the top-left, and vertical axis runs downwards. The population data were obtained from the *Regional Center of Statistics of Murcia* (see [4]) and correspond to the year 2002, rescaled to $]0,10]$. The minimum distance $d_i^{\min}$ at which the new facility must be from the



(a) Full problem for the Small Chain. (b) Aggregated problem for the Large Chain.

Fig. 3 Existing facilities, demand points (black) and forbidden regions (gray).

population center i was chosen to be $\frac{w_i}{\xi}$, where ξ is set to 30. This means that for a demand point with 1500 inhabitants ($w_i = 0.067$) $d_i^{\min} = 0.00223$ ($\sim$10m), and for the largest demand point (Murcia, $w_i = 10$) $d_i^{\min} = 0.\dot{3}$ ($\sim$1.5km). All final data are given in Appendix.

Approximate values for the quality-parameters α_j for the five existing supermarkets were obtained through a small survey among 10 persons who had visited all five supermarkets. Each respondent was asked to rank the supermarkets in increasing order of their perceived quality, and to indicate the difference in quality between any two consecutive supermarkets in their ranking by a score between 1 (small) to 4 (big). That information yielded individual marks for all supermarkets, by starting from a lowest mark of 1, and adding each difference-score to obtain the mark of the next supermarket in their ranking. Finally these marks were rescaled to the interval $[3,4]$, because, according to all the respondents, in a proposed scale from 0.5 to 5 all considered supermarkets have a quality above the mean (2.75), but they all still have a large margin for improvement. The qualities α_j we finally considered, as given in appendix, are the average rescaled marks over the ten respondents. The optimization of quality was done in the interval $[\alpha_{\min}, \alpha_{\max}] = [0.5, 5]$. In that scale, 1 is considered to be the quality of small-size low-quality supermarkets, and 5 the quality of very big-size big-quality supermarkets. The range 3 to 4 for the existing facilities shows that all these supermarkets are of similar type (size and quality above average) with small differences.

Due to the lack of real data from the chains (they consider those data sensitive for them, and are not willing to make them public), the other parameters have been validated in an ad hoc way to obtain ‘reasonable’ results. In particular, we used the following initial settings: the income per unit sold $c = 12$, in the location cost function G_1 we chose the parameter $\varphi_{i0} = 2$ for all i, while value of φ_{i1} decreases as the population increases (the actual values can be found in the appendix). Finally, the parameters of the cost of quality function G_2 were initially set to $\beta_0 = 7$ and $\beta_1 = 3.75$.

4 Impact of market structure

The basic data described above have been used to define three quite different competitive market structures:

***Newcomer* :** a new chain enters the market with one new facility, which has to compete with all five existing facilities,

***Small Chain* :** the smaller chain wants to strengthen its two existing facilities by adding a new one to compete on a more equal basis against the three facilities of the Large Chain,

***Large Chain* :** the larger chain wants to extend its dominance in the market by moving from the current 3 against 2 to a 4 against 2 facility situation.

The results obtained in these three problem instances are described in Table 1 and depicted in Figures 4-6 at the left.

All the computational results in this and later sections have been obtained under Linux on a Pentium IV with 2.66GHz CPU and 2GB memory. The algorithm has been implemented in C++ using the interval arithmetic in the PROFIL/BIAS library [27], and the automatic differentiation of the C++ Toolbox library [20].

Table 1 Results for the basic problems.

		Newcomer	**Small Chain**	**Large Chain**
Optimum (best found)		44.93	210.39	242.96
δ-opt interval		[44.39, 45.02]	[207.87, 210.81]	[240.05, 243.44]
Market share		$0 \to 8.47$	$15.77 \to 18.70$	$19.74 \to 21.14$
Best point found		(4.82, 6.11, 5.00)	(8.40, 3.18, 1.43)	(3.29, 6.48, 0.52)
Hull of results:	x_1	[4.78, 4.88]	[8.32, 8.57]	[3.07, 3.57]
	x_2	[5.99, 6.19]	[2.98, 3.22]	[6.19, 6.70]
	α	[4.56, 5.00]	[0.69, 2.47]	[0.50, 1.70]
Second region:	x_1		[3.25, 3.31]	[4.77, 5.45]
	x_2		[4.26, 4.36]	[5.61, 6.26]
	α		[1.34, 2.08]	[2.01, 4.23]
Volume of the δ-opt		1.7e-04	2.1e-02	1.8e-01
Time in seconds		137.07	165.55	340.32

In the table we first see the maximal value found for the profit of the locating chain, which is guaranteed to be within $\varepsilon = 0.05$ units of the global optimal value. Next we indicate the interval of objective values reached within the $\delta = 1\%$-optimal region, up to a precision of $\eta = 0.2\%$.

Bear in mind that the objective values concern the total income from the chain minus the costs of the new facility. Due to the quite different sizes of the locating chain, this value is not directly comparable between the three cases.

A much better point of comparison is obtained when looking at the change in market shares after entry of the new facility, which is indicated on the third line of the table. The original market shares were estimated by applying the model parameter settings to the location of the existing facilities only, using our estimation of their quality as given in the appendix. The Newcomer's single facility would be able to obtain a market share of 8.5 out of a total of approx. 35.5, i.e. 24%, a quite amazing success. The Small Chain would see its market share expand from its (estimated) current value of 44% to 53%, thus doing slightly better than its competitor with the same number of facilities. This shows, incidentally, that

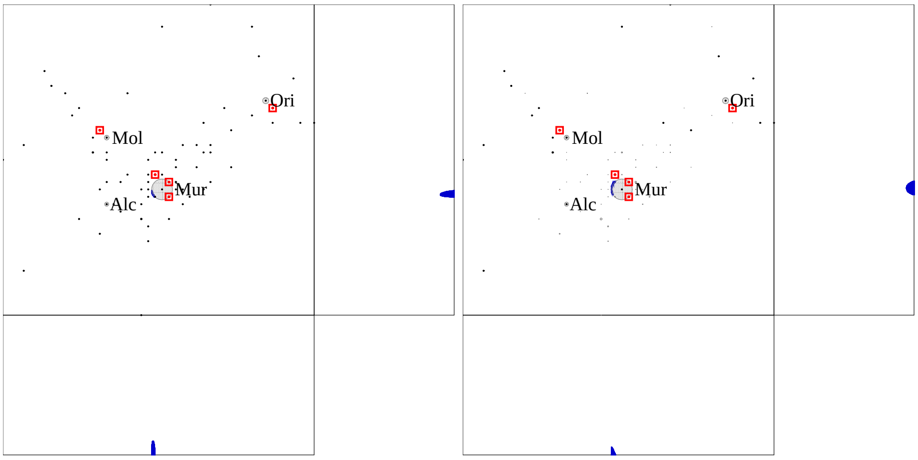


Fig. 4 Near optimal regions for the Newcomer with unaggregated and aggregated demand.

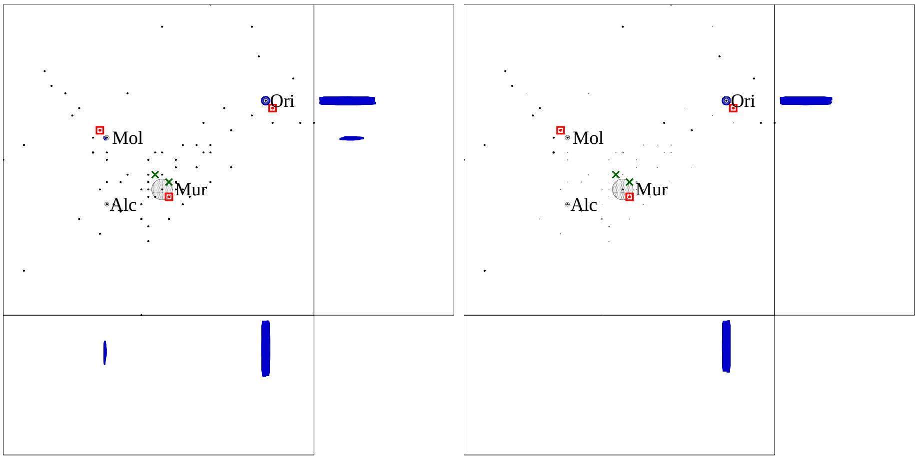


Fig. 5 Near optimal regions for the Small Chain with unaggregated and aggregated demand.

the Large Chain was not optimally located. Expansion of the Large Chain increases its current market share of 56% to only 59%, which is rather low considering that this results from twice as many facilities as the competition.

This may be explained when looking at the corresponding best solutions found on the fourth row of the table, indicated as the triplet (horizontal position x_1, vertical position x_2, and quality α), or in the graphical representation in Figures 4-6 at the left.

The Newcomer's best strategy turns out to be to go for the largest demand concentration, although the competition there is quite fierce. That is why its quality should be taken as high as possible: the best quality is at the maximum value of 5. The facility should be sited next to the central and main city of Murcia, at the southwestern boundary of its forbidden region, which is as far as possible from both competitors already present around the city. Since the competitors do not have quality better than 4, it obtains almost one third of Murcia's market, and a quite large part of the demand points on the South-West. This comes with an

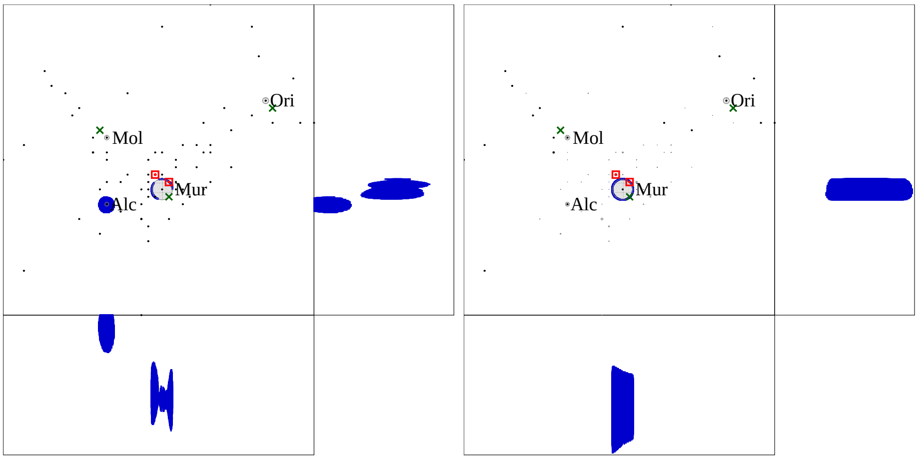


Fig. 6 Near optimal regions for the Large Chain with unaggregated and aggregated demand.

important cost of around 57, though, as can be evaluated by the difference between sales ($12 * 8.5 = 102$) and actual profit (45).

The Small Chain, however, is already well and exclusively present around Murcia, and can therefore better aim at the next largest and cheaper city, Orihuela at the North East of Murcia, in direct competition with the existing facility of the Large Chain next to it. At a low quality around 1.4 it quite easily grabs a good part of its competitor's market. This explains the relatively better gain/cost ratio as compared to the Newcomer's strategy.

For the Large Chain the most interesting place is at Alcantarilla, only slightly to the West of Murcia. This region is currently badly served, and here minimum quality is sufficient to take over the south-eastern market.

The next rows in the table give much more precise information about the near-optimal location and qualities: for each of the three basic variables of the problem it indicates the 'hull', i.e. the range within which it varies within the near-optimal region. One must, however, be quite careful with this hull: the near-optimal region may be disconnected, which happens if there exist more than one local optimum yielding almost the same global optimal value. In such a case the 'naive' hull would enclose all of these connected components, and would be uninformative, since it would yield wide overestimations of the range, but the disconnectedness would not be apparent. Therefore the hull is then given for each connected component separately.

One might argue that this should not happen very easily; though many local optima may exist, there usually is a clear winner, and only rarely more than one near-global optimum. This is perhaps true for pure location problems, and is then restricted to some exceptional circumstances such as almost symmetrical situations. Consideration of the quality issue as part of the decision, as is done here, changes the setting, however: this independent choice variable offers the additional freedom to allow to more easily obtain similar close to optimal profits at different sites. And this is what is observed here. This phenomenon happens in two of the three instances. Both the Small Chain and the Large Chain have two almost equivalent possibilities, which appear as the "second region" in the table, and are also clearly visible in the left part of Figures 5 and 6. No more than two connected components were observed in our experiments, but with other data such may certainly appear.

The penultimate row in each table gives the total volume of the three-dimensional boxes defining the near-optimal region. The ratio between the third root of this value and the size of the near optimal objective interval gives a more precise impression of the flatness of the objective function around the optimum value, although boundary effects on quality somewhat cloud the issue. Still it is quite clear that the Large Chain has the flattest objective, the Small Chain case comes next, and in the Newcomer's case the objective seems quite steep.

Finally, in the last row we give the required computational time to find the near-optimal regions, which varied from a few seconds to several minutes.

As already observed, and clearly visible in the figures, the two last cases are quite interesting since they show there are two near-optimal regions which are quite far apart. The Small Chain has two almost equivalent options, first to locate at the North-East of Murcia, more precisely at Orihuela, or to locate at the North West, at Molina. In both cases it will be in direct competition with the facilities of the Large Chain in these same areas.

Similarly, the Large Chain also has two main options which are almost equivalent, but of quite different type: either locating an additional facility around the main city Murcia, increasing its presence against the two-facility strong competition there, or rather grabbing the badly served market around the region's fourth city Alcantarilla, West of Murcia. The second and third largest markets (Orihuela and Molina) are not of interest to this chain since it already well covers them with its existing facilities. Observe also that several positions around Murcia's border are available, needing slightly different but rather upper average quality, whereas Alcantarilla is already conquered at almost minimum quality.

These last two examples demonstrate the power of global optimization: *all* (nearly) global optima are found. This is extremely useful when alternative location criteria are to be considered. A usual local optimization method might have found any of these global optima or perhaps a worse local optimum, but will probably not have found out that there are also almost equivalent solutions totally elsewhere.

5 The impact of aggregation

We have also constructed a reduced data set by aggregation of the demand. The aim is to study to what extent the aggregation may influence the solution of the problem. This is a very relevant question in practice, where the data may be available at different levels of aggregation at very different costs. One should then have some idea if it is worth the effort/money to obtain the more detailed unaggregated data, and if this can affect the results considerably. Such errors incurred by aggregation have been studied intensively for several types of location models, see e.g. Francis et al. [17]. But for competitive Huff-like location models no theoretical or empirical information seems to be available as yet, even in the most recent survey Francis et al. [18]. And the question has never been considered before with quality as part of the decision.

5.1 Aggregation design

An aggregated data set was obtained as follows. Several population centers do not have a city hall, so for public administrative tasks they depend on another town's city hall. The 21 towns with city hall form our reduced set of demand points, with population obtained by aggregating all population centers at the town on which they administratively depend. The resulting reduced demand distribution is shown in Figure 3(b). This way of aggregating is quite typical. Our two sources of data, [4] and [38] give the data grouped by city hall, as in our aggregated case, while for the larger cities they are subdivided by population centers, as in our non-aggregated case.

In the aggregated problems, exactly the same forbidden regions were kept as in the corresponding unaggregated instances, including the forbidden regions around the population centers which are not demand points anymore after aggregation. Therefore the results are directly comparable with the unaggregated instances, allowing to see the effect of the concentration of the demand by the aggregation.

5.2 Results

Three aggregated versions were considered, considering the same market structures as in previous section, but now with the aggregated demand. The results obtained for these instances are described in Table 2 and are depicted in Figures 4-6 at the right.

Table 2 Results for the aggregated problems.

	Aggr. Newcomer	**Aggr. Small Chain**	**Aggr. Large Chain**
Optimum (best found)	53.11	210.19	246.90
δ-opt interval	[53.00, 53.75]	[207.67, 210.61]	[243.95, 247.40]
Market share	$0 \rightarrow 9.00$	$15.52 \rightarrow 18.62$	$19.70 \rightarrow 23.73$
Best point found	(4.78, 5.92, 5.00)	(8.53, 3.05, 1.36)	(4.79, 6.02, 3.54)
Hull of results: x_1	[4.78, 4.90]	[8.33, 8.56]	[4.76, 5.45]
x_2	[5.70, 6.12]	[3.00, 3.21]	[5.61, 6.30]
α	[4.75, 5.00]	[0.69, 2.33]	[2.15, 4.93]
Volume of the δ-opt	8.4e-05	1.6e-02	5.3e-02
Time in seconds	48	57	190

For the Newcomer the figure seems to show almost exactly the same results after aggregation: the best site is at Murcia at the western boundary of the forbidden region and using top quality almost exclusively. A closer look, in particular at the right pane, shows that the vertical position is less precisely determined than in the unaggregated case. The optimal profit, as found in the table, has increased by 18%, though. Both phenomena are explained by the demand at Murcia which has now more than doubled, whereas it was spread around this city without the aggregation. On the one hand this yields a less precise positioning of the optimum, and yields a higher income evaluation. This latter is also clearly shown by the increase in market share. One may also observe that the objective has become steeper around the optimum.

For both the Small Chain and the Large Chain one observes that only one of the two alternative near-optimal solution regions have remained when the demand is aggregated. In the first instance it is the former global optimum at Orihuela that remains as only good choice. The former second option around Molina was only very small, and now differs too much in value from the global optimum to remain visible at the $\delta = 1\%$ level. This indicates that the differences between local optima have been sharpened in value by the aggregation. The slightly smaller volume of the near-optimal region, together with a virtually unchanged objective value also indicates that the objective has become somewhat sharper at the optimum.

For the Large Chain, however, it is the second option in the central Murcia that has taken the clear lead. This may be explained by the fact that Murcia has grown the most, and by far, through the aggregation, and thus has gained more in competitive interest than the smaller market around Alcantarilla. The estimation of the gain in market share has indeed been strongly influenced by the aggregation in this case.

This last result is quite interesting, because it shows that the decision may be strongly influenced by the fact that aggregated data were used.

As a final observation, note that the computational times have been more or less halved. This may be seen as basically due to the reduction of 71 to 21 demand points, which has proportionally reduced each objective function evaluation.

6 Comparing lists of boxes

The remainder of the paper is devoted to several types of sensitivity analysis, every time carried out for each of the three unaggregated instances and their three aggregated counterparts, yielding six base instances, varying in installed base of the locating firm and in the dispersion of demand.

In each analysis a particular model parameter is modified sequentially, and the intention is to measure the corresponding change in optimal value and in optimal solution. However, as explained in Section 2.2, these results are not simple values or points, but lists of boxes. In order to compare the sets of optimal solutions for two or more instances of our model, we must be able to judge the 'likeness' of the outcomes we obtain in each case. Therefore we need a dissimilarity measure between lists of (pairwise non-overlapping) boxes.

According to the standard notation for interval analysis, lower case is reserved for scalar quantities or vectors, while intervals are indicated by boldface, where vector intervals are boxes (set theoretic products of intervals). A list of boxes $\mathscr{L}$ in $\mathbb{R}^d$ is then a finite set of non-overlapping d-dimensional boxes. Clearly any box $x = (x_i)_{i=1,\dots,d}$ has a volume $vol(x)$, and the volume of a list is the sum of volumes of its boxes: $vol(\mathscr{L}) = \sum_{x\in\mathscr{L}} vol(x)$.

Let us introduce first the following asymmetric connector distance from a box x to the box y:

$$\Delta(x,y) = \max_{x\in x} \min_{y\in y} \|x-y\|$$

where $\|.\|$ is the standard Euclidean norm. We then define the following dissimilarity measure for two lists $\mathscr{L}$ and $\mathscr{L}'$:

$$diss(\mathscr{L},\mathscr{L}') = \sum_{x\in\mathscr{L}} \frac{vol(x)}{vol(\mathscr{L})} \min_{y\in\mathscr{L}'} \Delta(x,y) + \sum_{y\in\mathscr{L}'} \frac{vol(y)}{vol(\mathscr{L}')} \min_{x\in\mathscr{L}} \Delta(y,x).$$

Applied to two single boxes (singleton lists of boxes), this formula yields the symmetrization of Δ as follows

$$diss(x,y) = \Delta(x,y) + \Delta(y,x).$$

For instance, in dimension 1, i.e. for two (disjoint) intervals on a line, this corresponds to the sum of the lengths of the two arrows in Figure 7. It is similar to the traditional Hausdorff distance between the intervals, but uses a sum instead of a maximum. This means that the one-dimensional dissimilarity roughly doubles the distances between the corresponding bounds.

Figure 8 illustrates this distance measure for two-dimensional lists of boxes. At the left we have the simplest cases of two singleton lists. In the top the situation of two non-overlapping boxes is shown twice to differentiate between the two terms of the measure:

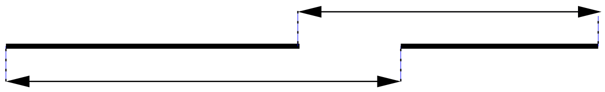


Fig. 7 The dissimilarity measure for two intervals

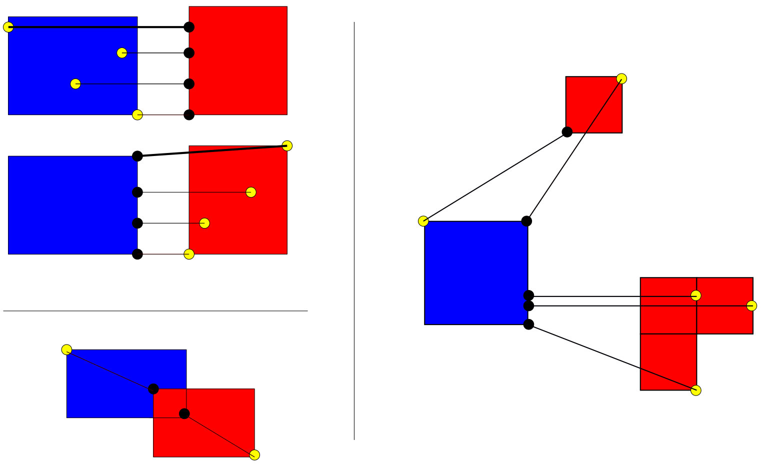


Fig. 8 The dissimilarity measure for two-dimensional boxes and lists of boxes

for each box separately we take an active connector from this box to the other box, i.e. the largest (indicated by the thick connecting line) among all the distances of some point of it to the other box (for some points these are shown as a thin connecting line), and then sum the resulting lengths. At the bottom left only the active connectors are shown for two overlapping boxes.

An example with two lists of boxes (one being singleton, one consisting of four boxes) is shown at the right hand side. For each box only the shortest of all active connectors to all boxes of the other list is used. Finally for each list a convex sum over all its boxes of the lengths of these active connectors is made, with coefficients proportional to the volume of the boxes, and these two results are summed. This means that large boxes contribute proportionally more to the dissimilarity than small boxes, and that we make some kind of integral over the whole list union.

This is not a metric on the set of all lists of boxes, but still has some nice properties. Noting that $\Delta(x,y) = 0$ iff $x \subset y$, we see that $diss(x,y) = 0$ iff $x = y$. It then also readily follows that $diss(\mathscr{L},\mathscr{L}') = 0$ iff $\cup\mathscr{L} = \cup\mathscr{L}'$, a quite interesting property. Unfortunately, however, nonzero distance does not necessarily remain unchanged when one list is replaced by another with the same union, as shown by the following counterexample in dimension 2. As a first list $\mathscr{L}_1$ take the single point (i.e. a singleton list with a degenerate box) $(0,0)$ and as second list $\mathscr{L}_2$ the single square box $[1,3]\times[-1,1]$. This yields $diss(\mathscr{L}_1,\mathscr{L}_2) = 1+\sqrt{10}$. But a vertical central split of the box of $\mathscr{L}_2$ yields the alternate list $\mathscr{L}_2' = \{[1,2]\times[-1,1],[2,3]\times[-1,1]\}$ for which $diss(\mathscr{L}_1,\mathscr{L}_2') = 1+\frac{1}{2}(\sqrt{5}+\sqrt{10})$. Note that any horizontal split of $\mathscr{L}_2$ does not change this dissimilarity.

Other properties are that a homothety of all data will multiply the dissimilarity by its factor, and any translation of one of the lists will add (or subtract) at most the translation vector's length to the dissimilarity. A full study falls outside the scope of this paper, but it should be clear that lists describing close shapes will be much less dissimilar than lists of very different shape or lying far apart, while the dissimilarity also says something about the complexity of the list. Therefore this dissimilarity is quite appropriate to our purpose to be able to compare in a synthetic way the outcome from optimizing two (apparently slightly different) objectives, as obtained by changing some parameter setting.

This dissimilarity measure allows to compare the results in variable space. However, for objective values one should not only be able to see the amount of change, but also the direction of the change, either improvement or worsening. Since the results come as intervals of values known to bracket the corresponding optimal value, we have to compare two such intervals. However, since the bracket has an (almost) fixed relative size, both extremes will either have increased, and this will be indicated by a $+$, or worsened, indicated by $-$. In the few cases when the extremes change with different signs, which only happen when the change is very small, no sign notations are used.

7 Impact of facility quality range

The first sensitivity analysis concerns the influence of the quality of each existing facility, as given by the parameters α_j. These were varied in several ways as described by the following scenarios:

Original In the base data, as given in the Appendix, all existing qualities lie in the range $[3,4]$. Since the new facility's quality is allowed to vary between 0.5 and 5, this means that it may be chosen as having a quality quite different from the qualities already present in the market.

Rescaled In this scenario, we rescaled all the α_j to the interval $[1,4.5]$, to make the existing facilities' range of qualities closer to the new facility's possibilities.

S>L Here the Small Chain has much higher quality facilities than the Large Chain. This is obtained by setting $\alpha_j = 4$ for all the existing facilities of the Small Chain and $\alpha_j = 2$ for all the existing facilities of the Large Chain.

S<L This is the opposite situation, where the largest chain offers the highest quality: Large Chain's existing facilities are now of quality 4 and those of the Small Chain of quality 2.

Closer better Now all existing facilities close to Murcia (two of the Large Chain and one of the Small Chain) have high quality 4, and those far from Murcia quality only 2.

Closer worse This final scenario inverses the previous scheme and gives lowest quality 2 to all existing facilities in the surroundings of the city of Murcia, but high quality 4 to all those far from the main city.

Table 3 gives an overview of the comparative results for each of the six considered cases: Newcomer, Small Chain, Large Chain, each both in unaggregated and in aggregated form. This table is organised as follows.

In all cases, unaggregated and aggregated, the top row recalls the obtained range of objective values of the original problem reached within the near-optimal region given by the final list of solution-boxes. All other rows show dissimilarities between the newly found list of boxes enclosing the optimal solution set with another list of boxes, and this in four ways: '3D' considers both quality and location variables together, while 'Loc' (resp. 'Qual') considers only the locational aspect (resp. only the quality aspect) by using the projections of all boxes on the 2-dimensional location space (resp. 1-dimensional quality space), and finally 'Obj' shows the dissimilarity in objective value ranges, using the sign convention

Table 3 Sensitivity to the quality of the existing facilities.

	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.
	Newcomer [44.39, 45.02]				Small Chain [207.87, 210.81]				Large Chain [240.05, 243.44]			
rescaled	0.001	0.006	0.007	+8.2	0.042	0.002	0.118	+42.2	0.079	0.066	0.202	−60.3
S>L	0.002	0.008	0.010	+27.9	0.005	0.001	0.019	+82.0	1.274	0.814	2.283	−47.7
S<L	0.000	0.000	0.001	+12.8	0.005	0.017	0.079	−111.3	0.081	0.177	0.104	+77.4
closer better	0.000	0.002	0.002	+2.5	0.003	0.000	0.011	+9.9	0.251	0.844	0.264	−2.3
closer worse	0.078	0.002	0.064	+44.1	0.010	0.023	0.132	−37.7	1.282	0.564	1.983	+34.9
	Aggr. Newcom. [53.0, 53.75]				Aggr. Small Ch. [207.7, 210.6]				Aggr. Large Ch. [243.9, 247.4]			
rescaled	0.001	0.000	0.005	+4.6	3.089	0.001	3.125	+45.9	0.002	0.034	0.081	−55.5
S>L	0.001	0.000	0.002	+30.6	0.916	0.001	0.923	+88.2	0.003	0.146	0.251	−42.9
S<L	0.000	0.000	0.001	+19.2	2.058	1.160	2.263	−121.4	0.056	0.080	0.157	+78.8
closer better	0.000	0.000	0.002	−0.2	0.586	0.000	0.590	+2.5	0.000	0.001	0.004	−0.7
closer worse	0.002	0.000	0.004	+59.6	0.003	0.002	0.008	−30.5	0.001	0.039	0.071	+38.9

explained in Section 6. In the rows ‘rescaled’, ‘S>L’, ‘S<L’, ‘closer better’ and ‘closer worse’ the comparison is made with the solution of the original problem.

We first observe that in most cases nothing much happens with respect to location. None of the quite important variations in the quality of the existing facilities have significant influence on the optimal spatial solution. In particular the Small Chain with unaggregated demand systematically keeps its two almost equivalent options, a quite remarkable fact. The best solution for the Newcomer is also remarkably stable throughout.

In the Small Chain case with aggregated demand, however, we see appreciable dissimilarities appear with the corresponding base case. Two of such results are shown in Figure 9. The first remark is that the original optimal solution around Orihuela shown at the right of Figure 5 always still exists.

‘rescaled’ When rescaling the qualities one observes the reappearance of the second near optimal region around Molina, that had been lost by the aggregation of demand (compare with Figure 5). This site is in direct competition with the existing Large Chain’s facility in the same area, that, by rescaling of its original quality of 3.1, has now a quality around 1. In other words, the range of interesting qualities for the entering facility is still spread around its direct competitor’s quality.

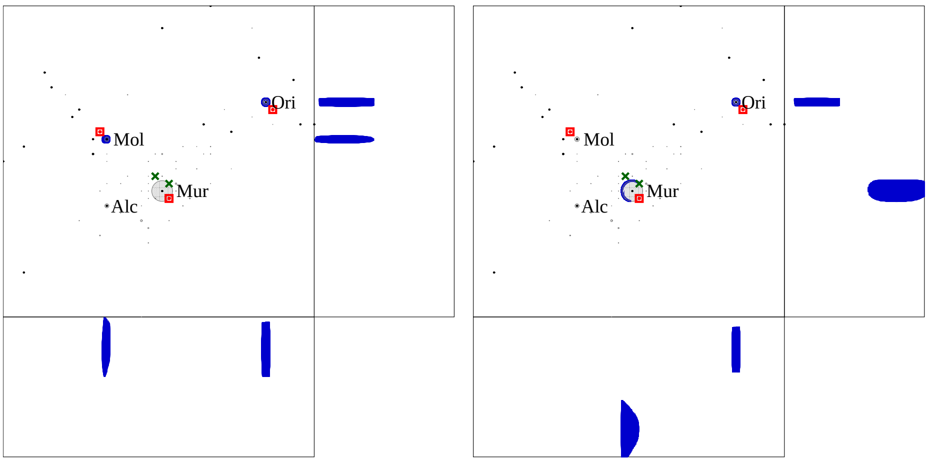


Fig. 9 Near optimal regions for the Small Chain with aggregated demand: left ‘rescaled’, right ‘S<L’

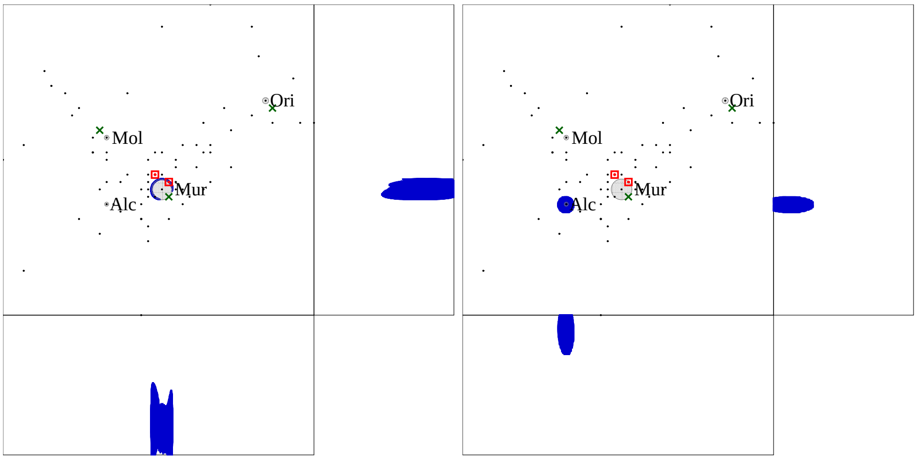


Fig. 10 Near optimal regions for the Large Chain with unaggregated demand: left ‘S>L’, right ‘closer better’

‘S<L’ When the competing Large Chain has the better quality facilities we see a totally different second possibility appearing for the Small Chain: locate a third facility around Murcia, now of competitively high quality, to grab a larger share of this most important demand point from the competitor. Apparently the existing two facilities are at too low comparative quality to do this job correctly.

For the Large Chain under unaggregated demand some smaller changes arise in the optimal solution, in particular in the scenarios ‘S>L’, ‘closer better’ and ‘closer worse’ (see Figure 10).

‘S>L’ When the competing (small) chain has the higher quality facilities the Large Chain’s best option is to go into direct competition for the largest demand in Murcia, helping out its comparatively weak facility there, and doing so at high quality. The alternative option appearing in the base case is not present anymore.

‘closer better’ When there are already three high quality facilities near Murcia, part of its demand can be gained only with the highest quality, but that is too expensive, thus in this situation it is much better to go for Alcantarilla.

‘closer worse’ Finally, when the own facility and both competing ones close to Murcia have a lower quality, it becomes interesting for the Large Chain to add a second facility around Murcia with a rather high quality, because this sufficiently grabs its opponent’s market share.

We conclude that for a very small player quality of existing facilities does not strongly influence its decision: it systematically goes towards the large demand areas where it is not yet present, taking over part of the competitor’s market. A medium size player should be somewhat more careful in its appreciation of the facility qualities present in the market. Particularly when demand is quite concentrated it may otherwise miss some chances. But for chains having many and dispersed facilities (as the Large Chain has) almost any new facility may lead to a lot of cannibalization (loss of market share of the existing facilities of the chain), and therefore the qualities of own and competing facilities should be as accurate as possible if a correct decision is to be made. This seems to be particularly the case in a dispersed market.

With respect to optimal objective values, however, one observes quite strong variations in general, even when the optimal solution itself didn't change much. Keep in mind that what is given in the table is the *change* in objective value as compared to the original near-optimal interval, and expressed by the dissimilarity. In most cases we see clear increases or decreases by relatively large amounts (although our dissimilarity roughly doubles the distances). Profit may increase by almost 50% compared to the original case (see the Newcomer in scenario 'closer worst') or reduced by over 25% (see the Small Chain in case 'S<L'). This implies that the evaluation of the expected profit is extremely sensitive to the existing facilities' quality assessment. Since any small chain has to attain a given minimum level of increase in profit in order to survive, qualities of the existing facilities will have to be estimated very precisely, if one wants to have confidence in the estimation of future profit.

From the cases 'S>L' and 'S<L', we can see that the chain that increases the quality of its facilities also has an increase in profit, and vice versa, when the competitor increases quality, profit drops. These increases are almost perfectly proportional to the number of existing facilities with higher quality, but the drops are not: the Small Chain looses relatively much more profit when the Large Chain increases its 3 facilities' quality, than the Large Chain in the opposite situation.

One may also observe, that when existing facilities close to the global optimum increase (decrease) their quality, the optimal profit of the new facility decreases (increases) accordingly.

To understand the case of rescaled qualities, we should recall the original qualities of the existing facilities. In the case of the Small Chain, the qualities are 3.25 and 4, whereas for the Large Chain they are 3.1, 3.5 and 3.15. When we rescaled [3,4] to [1,4.5] the lower bound 3 goes down to 1, but the upper bound 4 goes up to only 4.5. Thus, the rescaled facilities of the Large Chain have lost more quality, when compared to the facilities of the Small Chain. This explains why the Newcomer and the Small Chain increase their profit, whereas the Large Chain sees a decrease.

8 Impact of some modeling features

We now move to a short study of three more technical choices to be made when applying the model. This includes a number of features for which several alternative proposals may be found in the literature, or which depend on a particular setting of some technical parameter. The purpose of this short study is to measure experimentally the impact of changes in these features so as to obtain some indications which may guide the analyst in its choice.

8.1 Distance decay

In this section we examine the impact of the shape of the distance effect decay functions $g_i(d_{ix}) = d_{ix}^{\lambda}$ in the solution of the problems, more particularly with respect to the choice of the exponent λ. The three examined cases were $\lambda = 1, 2,$ and 4. The case $\lambda = 2$ is considered as the original problem. Thus, like in the previous table, in the second row we give the interval giving the range of the objective function at the solution boxes offered by the algorithm (when $\lambda = 2$). Next, we have solved the problems setting $\lambda = 1$ and $\lambda = 4$. As before the dissimilarity between the final box-lists are given in location, quality and full 3D terms, as well as the change in objective. For all the cases, comparison is with the original case $\lambda = 2$.

Table 4 Results for $g(x) = x^{\lambda}, \lambda = 1, 2, 4$.

	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.
	Newcomer [44.39, 45.02]				Small Chain [207.87, 210.81]				Large Chain [240.05, 243.44]			
$\lambda = 1$	0.032	0.033	0.088	$+5.9$	0.006	0.101	0.128	-3.3	0.107	0.019	0.179	-27.7
$\lambda = 4$	0.001	0.142	0.145	$+0.4$	0.130	0.098	0.293	-7.5	0.368	1.093	0.504	$+37.0$
	Aggr. Newcomer [53.0, 53.75]				Aggr. Small Chain [207.67, 210.61]				Aggr. Large Chain [243.95, 247.40]			
$\lambda = 1$	0.020	0.002	0.037	+2.9	0.006	0.090	0.106	+4.3	0.007	0.000	0.011	-41.2
$\lambda = 4$	0.019	0.004	0.033	+2.8	0.005	0.157	0.225	-13.7	0.044	0.001	0.077	+39.4

From Table 4 we can see that the solutions are only slightly affected by the changes in the parameter λ. The only change that deserves to be mentioned appears in the unaggregated Large Chain when $\lambda = 4$: Murcia is no longer in the region of δ-optimality. That region becomes very large around Alcantarilla with minimal quality. Indeed, for distances larger than 1, attraction becomes much smaller as compared to the original ($\lambda = 2$) case shown in Figure 6 left. This implies that the facilities near Murcia have almost no attractiveness for the demand at Alcantarilla, so a new facility there wins all its demand.

8.2 Income function

We now analyse the impact of the shape of the expected sales function F which converts the market share into income. We have considered two different types of functions in this sensitivity analysis.

First we have considered the linear function $F(M) = c \cdot M$, as in the base instances where $c = 12$. We have pushed c upwards by 30% to $c \uparrow = 15.6$, and downwards by 30% to $c \downarrow = 8.4$. Then we also considered convex functions of type $F(M) = c_1 + c_2 \cdot M^2$, where income increases more at higher market shares. This may be seen as a brand effect: larger market shares (so better known chains) enhance the consumption per unit. Initially we have set the parameters so as to obtain a similar scale around the observed optimal market shares as for the linear case. However, due to the quite different market share sizes this had to be adapted to each of the three market structures. This led to the instances 'quad', using the coefficients $c_1 = 10.599, c_2 = 1.535$ for the Newcomer, $c_1 = 106.985, c_2 = 0.334$ for

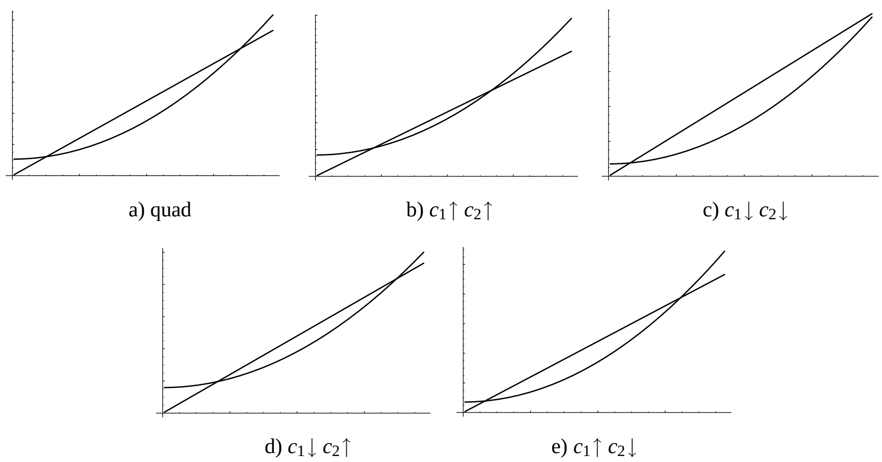


Fig. 11 Comparing the different income functions tested.

Table 5 Results for $F(M) = c \cdot M$, $F(M) = c_1 + c_2 \cdot M^2$.

	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.
	Newcomer [44.39,45.02]				Small Chain [207.87,210.81]				Large Chain [240.05,243.44]			
c ↑	0.000	0.049	0.043	+60.6	0.010	0.048	0.170	+134.7	1.274	0.733	2.168	+166.2
c ↓	0.002	0.664	0.700	−44.5	0.126	0.032	0.183	−102.5	0.267	1.189	0.387	−116.5
quad	0.001	0.152	0.136	+37.9	0.000	0.000	0.003	−1.4	0.001	0.003	0.030	−1.1
$c_1 \uparrow c_2 \uparrow$	0.001	0.149	0.133	+70.3	0.006	0.014	0.066	+128.4	0.024	0.084	0.131	+157.2
$c_1 \downarrow c_2 \uparrow$	0.001	0.148	0.132	+28.5	0.003	0.002	0.008	+83.9	0.063	0.130	0.094	+107.3
$c_1 \uparrow c_2 \downarrow$	0.001	0.154	0.138	+52.7	0.002	0.007	0.015	−49.0	0.071	0.092	0.235	−60.2
$c_1 \downarrow c_2 \downarrow$	0.002	0.155	0.139	+10.9	0.127	0.003	0.135	−93.5	0.268	1.094	0.346	−110.2
	Aggr. Newcom. [53.0,53.75]				Aggr. Small Ch. [207.7,210.6]				Aggr. Large Ch. [243.9,247.4]			
c ↑	0.000	0.005	0.010	+64.5	0.532	0.032	0.584	+134.0	0.000	0.067	0.135	+172.7
c ↓	0.001	0.180	0.202	-49.1	0.001	0.030	0.049	-102.2	0.730	0.292	1.357	-127.1
quad	0.002	0.053	0.065	+47.	0.000	0.001	0.003	-1.0	0.001	0.002	0.012	+0.3
$c_1 \uparrow c_2 \uparrow$	0.001	0.051	0.063	+81.8	0.347	0.013	0.371	+127.1	0.000	0.014	0.046	+162.6
$c_1 \downarrow c_2 \uparrow$	0.001	0.051	0.062	+35.9	0.055	0.002	0.059	+82.1	0.001	0.005	0.014	+102.7
$c_1 \uparrow c_2 \downarrow$	0.002	0.055	0.066	+64.0	0.001	0.007	0.009	-46.9	0.003	0.052	0.110	-55.0
$c_1 \downarrow c_2 \downarrow$	0.003	0.055	0.067	+18.0	0.002	0.003	0.008	-91.9	0.002	0.001	0.006	-114.6

the Small Chain, and $c_1 = 131.121, c_2 = 0.273$ for the Large Chain. Then we have varied each of these parameters upwards and downwards, c_1 by 50%, and c_2 by 10%, leading to four possible combinations '$c_1\uparrow$ $c_2\uparrow$', '$c_1\downarrow$ $c_2\uparrow$', '$c_1\uparrow$ $c_2\downarrow$','$c_1\downarrow$ $c_2\downarrow$'. Figure 11 shows and compares the general shapes of the income functions pictorially.

In the first parts of the table (rows $c\uparrow$, $c\downarrow$ and quad), the comparison is made to the original case, and in the second parts taking the quadratic F function as original.

The spatial influence of the function F is small, in many cases negligible, in fact, particularly for the Newcomer. However, in the few cases where some influence is observed (see for instance Large Chain in the cases '$c\uparrow$' and '$c\downarrow$') we observe a clear effect: when the parameter c is increased, the optimal quality of the new facility also increases, while when c is decreased, the optimal quality decreases as well. This is not surprising: since we are going to earn more (resp. less) money from the market share that we capture, we can spend more (resp. less) money to improve the quality of the new facility, and our investment implies an increase (resp. decrease) of our profit since we are going to get more (resp. less) market share.

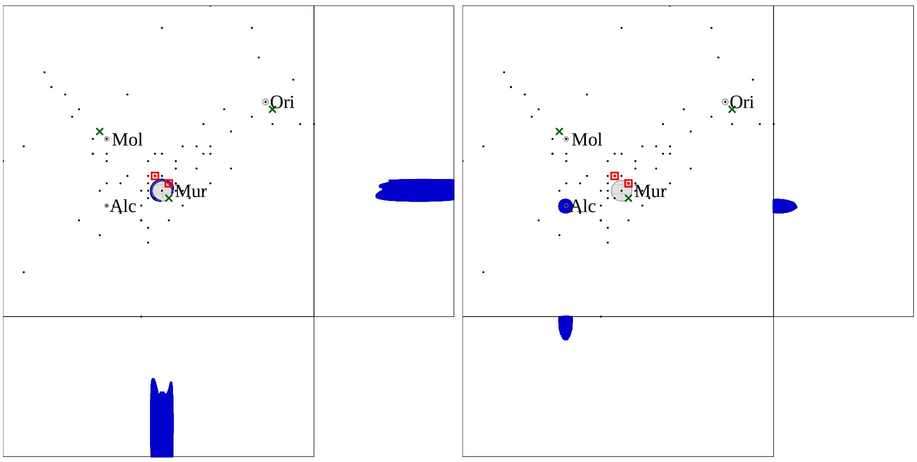


Fig. 12 Results for the Large Chain when c is increased and decreased.

Sometimes, this may even lead to a change in the optimal location, either when the quality of the facility cannot be modified further, or when the variation in the quality leads to a high variation in the market share captured. Consider for example the Large Chain. In the original setting (Figure 6) the δ-optimal location-design is in Murcia with medium quality or in Alcantarilla (west of Murcia) with low quality. However, on the left in Figure 12, the δ-optimal location is only in Murcia with a high quality, while for the case '$c\downarrow$' (see right side of Figure 12) the δ-optimal region is only around Alcantarilla. Thus, we can say that the higher the income obtained from the market share, the higher the quality of the new facility will be (and this, in turn, may provoke a change in the optimal location), and vice versa.

The conclusions are similar for the quadratic income function. The constant term c_1 seems to play a somewhat lesser role, though, compared to the effect of the quadratic term's coefficient c_2.

8.3 Location cost vs quality cost

Next we studied the influence of the relative importance between the two cost function elements, the location cost given by G_1 and the cost of quality, as described by G_2. This was done using the modified cost function $G(x,\alpha) = \mu_x G_1(x) + \mu_\alpha G_2(\alpha)$, with $\mu_\alpha = 2 - \mu_x$. The original case corresponds to $\mu_x = \mu_\alpha = 1$, and the test values analyzed were μ_x equal to 0.9, 0.7 ... 0.1 and 1.1, 1.3, ..., 1.9. The results are given in Table 6, following a structure similar to the previous tables.

One may expect the following phenomenon. When μ_x is very small, quality is quite expensive, so rather low quality settings will be chosen. When μ_x increases, and thus μ_α decreases, it becomes cheaper to increase quality, if possible, which will increase sales as long as the site does not change, thus increasing profits. This is exactly what happens in both Newcomer instances and in the unaggregated Small Chain instance, where the best location remains stable throughout.

Table 6 Cost sensitivity results changing parameter μ_x ($\mu_\alpha = 2 - \mu_x$).

μ_x	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.
1.0	Newcomer [44.39,45.02]				Small Chain [207.87,210.81]				Large Chain [240.05,243.44]			
0.1	0.008	3.363	3.400	−34.2	0.022	0.180	0.507	−4.7	0.265	1.461	0.557	+8.1
0.3	0.005	2.583	2.619	−29.9	0.013	0.126	0.345	−4.3	0.264	1.399	0.510	+6.3
0.5	0.002	1.682	1.717	−24.1	0.005	0.076	0.185	−3.5	0.264	1.300	0.446	+4.4
0.7	0.001	0.612	0.649	−16.4	0.002	0.032	0.070	−2.5	0.261	1.165	0.373	+2.6
0.9	0.000	0.043	0.037	−6.2	0.000	0.004	0.010	−1.0	0.051	0.182	0.079	+0.9
1.1	0.000	0.020	0.018	+6.4	0.000	0.004	0.011	+1.1	0.007	0.033	0.043	0.0
1.3	0.000	0.075	0.065	+19.1	0.002	0.076	0.150	+4.1	1.277	0.944	2.466	+8.1
1.5	0.001	0.104	0.090	+31.9	0.010	0.436	0.773	+8.6	1.279	1.929	3.612	+20.2
1.7	0.001	0.119	0.104	+44.6	0.031	1.816	2.170	+16.8	1.278	2.605	4.243	+33.0
1.9	0.002	0.126	0.114	+57.4	0.033	4.419	4.519	+31.7	1.278	2.921	4.498	+45.7
1.0	Aggr. Newcom. [53.0,53.75]				Aggr. Small Ch. [207.7,210.6]				Aggr. Large Ch. [243.9,247.4]			
0.1	0.002	2.779	2.798	−46.0	2.017	0.176	2.293	−5.0	3.092	4.365	5.452	−7.1
0.3	0.001	1.970	1.992	−39.1	1.473	0.122	1.672	−4.4	1.435	1.791	3.195	−8.3
0.5	0.001	1.045	1.073	−30.6	0.730	0.073	0.851	−3.6	0.618	0.708	1.542	−9.4
0.7	0.000	0.190	0.208	−19.9	0.195	0.030	0.244	−2.4	0.200	0.204	0.478	−8.0
0.9	0.000	0.007	0.010	−6.8	0.000	0.004	0.007	−0.9	0.000	0.011	0.021	−3.2
1.1	0.000	0.002	0.004	+6.8	0.000	0.004	0.009	+1.1	0.000	0.006	0.021	+3.9
1.3	0.000	0.011	0.017	+20.5	0.001	0.064	0.106	+3.8	0.000	0.146	0.252	+14.4
1.5	0.001	0.021	0.027	+34.2	0.004	0.299	0.462	+7.8	0.000	0.571	0.780	+28.0
1.7	0.002	0.027	0.035	+47.9	0.408	1.252	1.795	+14.0	0.000	0.891	1.137	+41.7
1.9	0.003	0.031	0.044	+61.5	0.677	3.518	3.965	+26.6	0.000	1.079	1.329	+55.4

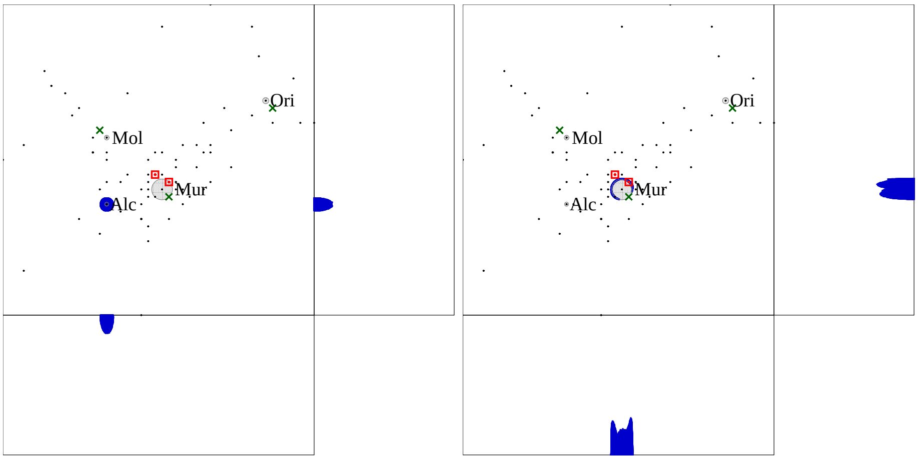


Fig. 13 The results for Large Chain for $\mu_x = 0.5$ (left) and $\mu_x = 1.5$ (right).

For the Newcomer the maximum quality is almost reached in the base case (see Figure 4). When μ_x increases above 1 both site and quality remain fixed, resulting in a corresponding linear increase in profit. When μ_x decreases, however, we see a steady decrease of optimal quality and a less than linear decrease in profit.

For the Small Chain the location remains stable too, but now quality starts at around minimum value, and very slowly increases towards the still low 1.4 in the basic case. This slow increase goes on but accelerates towards the end, not quite reaching the maximum of 5, even when quality is cheapest. As the quality was quite similar at the two δ-optimal sites, Molina and Orihuela, both places remain optimal with the only change in quality. The same holds in the aggregated case, although at the extremes values of μ_x Molina also becomes part of the δ-optimal region, as in the unaggregated case.

For the Large Chain, in the base case, the optimal solution is at Alcantarilla with low quality, and also at Murcia with relatively high quality. When μ_x decreases Murcia is no longer a suitable place to locate: the high quality needed there becomes too expensive, see Figure 13 left for $\mu_x = 0.5$. When μ_x increases, however, Alcantarilla is no longer a suitable place, because a location in Murcia at the higher quality that now becomes affordable obtains more and more profit, see Figure 13 right for $\mu_x = 1.5$. Note also that this shift in optimal site may lead to increasing profits in both directions.

With the concentrated demand this last effect is more pronounced around the same shift in best site, but it occurs already at the much lower value $\mu_x = 0.5$.

8.4 Cost vs Income

We also studied the influence of the cost function in an overall sense, as compared to the income from sales. To this end we used as cost function $G(x,\alpha) = \mu\big(G_1(x) + G_2(\alpha)\big)$ and varied the value of μ. The base case corresponds to $\mu = 1$, and we tested a few lower and higher values, as shown in Table 7.

When costs are negligible compared to income from sales, i.e. for low values of μ, one can easily choose high quality and expensive sites to increase sales, resulting in higher

Table 7 Cost sensitivity results changing parameter μ in $G(x,\alpha) = \mu\big(G_1(x) + G_2(\alpha)\big)$.

	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.
	Newcomer [44.39,45.02]				Small Chain [207.87,210.81]				Large Chain [240.05,243.44]			
0.5	0.002	0.090	0.077	+56.4	0.044	0.847	1.379	+19.9	1.279	1.875	3.568	+44.8
0.7	0.001	0.065	0.055	+33.8	0.019	0.115	0.345	+9.6	1.277	0.941	2.454	+22.9
0.9	0.000	0.017	0.015	+11.3	0.002	0.004	0.021	+2.9	0.031	0.039	0.102	+4.9
1.1	0.000	0.039	0.034	−11.1	0.125	0.004	0.135	−2.7	0.257	1.021	0.300	−2.1
1.3	0.002	0.664	0.701	−31.1	0.126	0.032	0.183	−7.6	0.267	1.189	0.387	−6.4
1.5	0.008	1.774	1.807	−48.6	0.127	0.083	0.264	−12.0	0.272	1.321	0.467	−10.6
	Aggr. Newcom. [53.0,53.75]				Aggr. Small Ch. [207.7,210.6]				Aggr. Large Ch. [243.9,247.4]			
0.5	0.001	0.015	0.022	+54.1	2.357	0.360	2.679	+15.9	0.000	0.545	0.760	+47.8
0.7	0.000	0.010	0.013	+32.4	0.920	0.064	1.013	+8.7	0.000	0.146	0.250	+26.3
0.9	0.000	0.001	0.003	+10.8	0.023	0.004	0.032	+2.7	0.000	0.006	0.021	+7.9
1.1	0.000	0.005	0.008	−10.8	0.000	0.004	0.007	−2.6	0.000	0.011	0.022	−7.2
1.3	0.001	0.180	0.202	−31.8	0.001	0.030	0.049	−7.3	0.731	0.292	1.357	−17.9
1.5	0.003	1.105	1.130	−50.4	0.002	0.076	0.117	−11.7	3.162	4.282	5.449	−21.6

profits. When μ increases this will have the opposite effect, usually lowering quality, thus sales, and possibly a move to another cheaper site, all resulting in a clear loss in profit.

In the aggregated Small Chain case, for lower μ values the solution moves towards more expensive sites until Molina also becomes part of the δ-optimal region. For the unaggregated Large Chain with low μ the solution moves fully to Murcia with increasing quality, whereas for high μ the solution moves to Alcantarilla at low quality. In the aggregated case for high μ the solution also moves to Alcantarilla with low quality. Finally, one observes that at the extreme values of μ the solution of the unaggregated and aggregated cases are almost the same.

9 The impact of budget restrictions

In our final study we investigated the influence of a limited budget on the optimal solution. This was done as follows. The optimal solutions (x^*, α^*) of the base cases yielded a corresponding maximally needed budget $B^* = G(x^*, \alpha^*)$. Then, we added a budget constraint of the form $G(x,\alpha) \leq B$, and solved these problems by reducing the available budget B to 95%, 90%, 85%, …, 55% of B^* for the corresponding base case. The results are summarized in Table 8, all values shown as dissimilarities as compared to the corresponding base case near-optimal region.

Note that when B increases, we obtain a relaxation of the problem at hand, so maximal profit cannot decrease. Therefore each ‘Obj’ column in the table should show decreasing negative values.

When the budget decreases slightly, the only effect observed is that quality is lowered in order to stay within the budget, resulting in a corresponding loss in sales, thus in profit. But at some point the budget becomes so small that lowering quality does not suffice anymore and one also has to move to cheaper places. This typically means moving away from the population centers. In the Newcomer and Small Chain instances, for example, the optimal location remains fixed, and we observe only a reduction in the quality of the new facility as B decreases. Additionally, the Small Chain looses one of its two near-optimal locations in the move from 100% to 95%.

This spatial outward effect is fully due to the locational cost part in our model, and may become so large that the facility is pushed completely towards the boundaries of the feasible region. This happens for example for the Large Chain case at very low budgets, as illustrated in Figure 14. The inner region is now unfeasible due to the added constraint, and one obtains

Table 8 Results on budget restriction.

	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.	Loc.	Qual.	3D	Obj.
	Newcomer [44.39,45.02]				Small Chain [207.87,210.81]				Large Chain [240.05,243.44]			
95%	0.000	0.081	0.098	−0.4	0.127	0.192	0.443	−0.1	0.490	1.711	0.905	−4.5
90%	0.000	0.400	0.436	−1.2	0.127	0.261	0.517	−0.2	0.986	1.551	1.318	−10.6
85%	0.001	0.876	0.904	−2.4	0.128	0.311	0.584	−0.5	4.914	1.445	4.732	−14.3
80%	0.001	1.365	1.391	−4.1	0.128	0.391	0.667	−0.9	5.037	1.326	5.199	−15.5
75%	0.002	1.906	1.932	−6.4	0.129	0.482	0.761	−1.6	5.155	1.327	5.381	−16.0
70%	0.003	2.458	2.484	−9.3	0.129	0.586	0.868	−2.5	5.271	1.420	5.592	−16.9
65%	0.005	3.023	3.050	−12.9	0.130	0.713	1.000	−3.8	5.525	1.439	5.826	−16.9
60%	0.008	3.619	3.646	−17.3	0.131	0.881	1.173	−5.5	5.851	1.497	6.158	−16.9
55%	0.012	4.247	4.273	−22.8	0.132	1.049	1.339	−7.9	6.281	1.526	6.610	−16.9
	Aggr. Newcom. [53.0,53.75]				Aggr. Small Ch. [207.7,210.6]				Aggr. Large Ch. [243.9,247.4]			
95%	0.000	0.186	0.224	−1.1	0.001	0.203	0.325	−0.1	0.000	0.331	0.299	−0.2
90%	0.000	0.605	0.630	−2.7	0.002	0.269	0.391	−0.3	0.000	0.464	0.419	−0.4
85%	0.000	1.082	1.102	−4.7	0.002	0.344	0.478	−0.6	0.000	0.617	0.566	−1.0
80%	0.000	1.568	1.583	−7.2	0.003	0.400	0.549	−1.2	0.000	0.763	0.742	−1.8
75%	0.000	2.062	2.075	−10.3	0.003	0.494	0.645	−2.0	0.001	0.970	0.982	−3.0
70%	0.000	2.603	2.615	−14.0	0.004	0.607	0.765	−3.1	0.001	1.275	1.308	−4.5
65%	0.001	3.137	3.158	−18.5	0.005	0.752	0.923	−4.6	0.001	1.643	1.709	−6.3
60%	0.001	3.701	3.724	−23.8	0.005	0.891	1.070	−6.7	0.362	2.113	2.438	−8.6
55%	0.002	4.295	4.318	−30.9	0.046	1.052	1.214	−9.5	0.914	2.625	3.531	−11.4

a wide boundary region at low quality. Observe that the sheer size of this near-optimal region indicates a very flat objective function. This flatness is also the reason why the optimal profit is almost insensitive for a whole range of low budgets.

Overall the profit losses due to budget restrictions may be considered as reasonable in absolute terms in all cases. However, the relative losses due to budget constraints for the Newcomer move up to 25%. The Small Chain looses the least. This is because it starts out in the unconstrained base case by choosing a not too expensive location (Orihuela) at rather

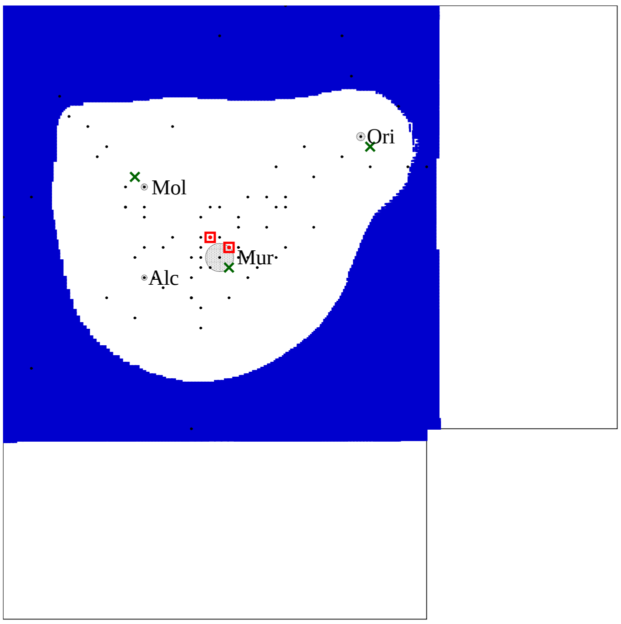


Fig. 14 Results for Large Chain when the budget is restricted by 55%.

low quality. Thus, when the budget decreases it can cope by a small decrease of its quality, and still capture most of its original market share. In contrast, the Newcomer has to decrease its quality by much more, so it looses more market share, while the Large Chain has to move out to much cheaper locations because reducing quality cannot compensate for the reduction in budget.

10 Conclusions

Not all results collected have led to significant conclusions, but we felt it important to give the reader access to the full study results. It should be considered as a first broad exploratory step towards a better understanding of Huff-type competitive models. In view of the relatively modest size of each particular sensitivity analysis, the conclusions remain, of course, tentative. But they have allowed to prove that several parameters of the model play a significant role, and should therefore be estimated with care in real applications in order to have confidence in the decisions suggested by the model.

Let us terminate by summarizing the main conclusions we consider to have obtained.

– First of all, it is very important to decide the location and the quality simultaneously, because each one may influence the other considerably (see also [35]).
– Our experiments have shown the importance of using global optimization methods, in particular methods that guarantee that the found optimum is global and that construct a region of near-optimality. This region is a great help to the decision maker to choose the best location and quality, in particular by indicating possible alternatives.
– All studied parameters have a direct impact on profit evaluations, and most of them turned out to be able to sometimes strongly modify the best choice. Therefore estimation of data and parameters should always be done carefully, particularly when the profit forecasts are requested to be accurate.
– The most important technical parameters seem to be those influencing the tradeoff between costs and income. The precise shape of the distance decay in attraction or the income functions has less influence. In particular the shape of the income function transforming market share into income has a strong impact on the evaluated profit.
– When demand is aggregated there is an almost systematic tendency to underestimate optimal profits. A more important effect is that the objective function usually becomes sharper with aggregation, and therefore may lead to the disappearance of good alternate optimal solutions.
– A newcomer in the market is working in a considerably more hilly environment, and usually only a single best option appears, remaining quite insensitive to data changes.
– For cases when the owner's chain is already present, wrongly approximated data can cause wrong estimation not only in profit, but in location and design as well. Thus, apart from accurate approximation of the data and parameters, the decision maker has to look not only at the globally optimal location and design, but also at the near-optimal regions. We observed that the Large Chain has the flattest function, and thus larger near-optimal regions, often showing more than one almost equivalent options, the best choice among which may vary with relatively small parameter changes. For these larger chains one may also recommend to compute a larger near-optimal region, and select from the possible new insights according to the chain's requirements and facilities.

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Appendix: the basic data

Existing facilities

Name	Location		Qualities
j	x_1 coord.	x_2 coord.	α_j
E1	3.11	4.05	3.1
E2	5.33	6.19	3.5
E3	8.67	3.33	3.15
C1	5.33	5.71	4.0
C2	4.89	5.48	3.25

Demand points

Name	Location		Buying power	Aggregated		Aggreg.
i	x_1 coord.	x_2 coord.	w_i	w_i	φ_{i1}	φ_{i1}
Abanilla	6.67	0.00	0.339	0.339	0.806	0.806
Albudeite	0.00	5.00	0.076	0.076	0.527	0.527
Alcantarilla	3.33	6.43	1.966	1.966	1.301	1.301
Alguazas	2.89	4.29	0.409	0.409	0.849	0.849
Archena	1.56	2.62	0.737	0.864	0.998	1.042
Algaida	2.00	2.86	0.127		0.611	
Beniel	7.33	4.05	0.503	0.503	0.899	0.899
Campos del Río	0.67	4.52	0.115	0.115	0.594	0.594
Ceutí	2.22	3.57	0.443	0.443	0.868	0.868
Fortuna	5.11	0.71	0.404	0.404	0.846	0.846
Librilla	0.67	8.57	0.225	0.225	0.718	0.718
Lorquí	2.44	3.33	0.330	0.330	0.800	0.800
Molina	3.33	4.29	2.282	2.720	1.354	1.419
Ribera	3.33	5.00	0.090		0.552	
Romeral	4.00	2.86	0.281		0.765	
Torrealta	3.33	4.76	0.067		0.507	
Murcia	5.11	5.95	10.000	21.226	2.000	2.433
Albatalia	4.67	5.48	0.123		0.605	
Alberca	4.67	7.14	0.557		0.925	
Algezares	5.33	6.90	0.252		0.742	
Aljucer	4.89	6.19	0.395		0.841	
Alquerias	6.67	4.76	0.272		0.758	
Arboleja	4.67	5.71	0.123		0.606	
Beniajan	6.00	6.19	0.505		0.900	
Cabezo de Torres	5.11	4.76	0.586		0.938	
Casillas	5.56	5.24	0.179		0.673	
Cobatillas	6.22	4.52	0.106		0.580	
Corvera	4.44	10.00	0.111		0.588	
Churra	4.89	4.76	0.223		0.717	
Dolores	5.56	5.95	0.269		0.755	
Era Alta	4.67	6.19	0.154		0.645	
Esparragal	5.78	4.52	0.191		0.686	
Garres	5.78	6.43	0.297		0.777	
Guadalupe	4.00	5.48	0.226		0.720	
Javali Nuevo	3.11	5.95	0.188		0.683	
Javali Viejo	3.33	5.71	0.113		0.590	
Llano de Brujas	6.22	5.24	0.260		0.748	
Monteagudo	5.56	5.00	0.210		0.704	
Nonduermas	3.78	6.67	0.127		0.611	
Ñora	3.78	5.71	0.199		0.694	
Palmar	4.44	6.90	0.961		1.073	
Puebla de Soto	3.56	6.43	0.086		0.546	
Puente Tocinos	5.56	5.71	0.752		1.004	
Puntal	4.67	5.00	0.233		0.725	
Raal	6.67	4.52	0.299		0.779	
Ramos	6.67	5.71	0.146		0.636	
Raya	4.44	5.95	0.124		0.607	
Rincon de Seca	4.67	5.95	0.121		0.602	
San Benito	5.33	6.19	0.499		0.897	
San Gines	4.44	6.43	0.110		0.586	
San Jose de la Vega	5.78	5.95	0.180		0.674	
Sangonera la Seca	2.44	6.90	0.246		0.737	
Sangonera la Verde	3.11	7.38	0.457		0.875	
Santa Cruz	6.44	4.76	0.121		0.602	
Santiago y Zaraiche	4.89	5.48	0.201		0.696	
Santo Angel	4.67	7.62	0.271		0.757	
Torreagüera	6.44	5.95	0.345		0.810	

Demand points						
Name	Location		Buying power	Aggregated		Aggreg.
i	x_1 coord.	x_2 coord.	w_i	w_i	φ_{i1}	φ_{i1}
Zarandona	5.11	5.48	0.320		0.793	
Zeneta	7.33	5.24	0.090		0.552	
Santomera	6.44	3.81	0.681	0.681	0.977	0.977
Torres de Cotillas	2.89	4.76	0.938	0.938	1.066	1.066
Villanueva	1.33	2.14	0.089	0.089	0.551	0.551
Benferri	8.22	1.67	0.064	0.064	0.500	0.500
Bigastro	9.56	3.81	0.295	0.295	0.775	0.775
Jacarilla	10.00	3.81	0.090	0.090	0.553	0.553
Orihuela	8.44	3.10	2.978	3.428	1.454	1.509
Arneva	8.67	3.81	0.091		0.555	
Desamparados	8.00	3.57	0.095		0.562	
Aparecida	7.11	3.33	0.106		0.579	
Murada	8.00	0.71	0.157		0.649	
Redován	9.33	2.38	0.326	0.326	0.797	0.797