COUNTER EXAMPLE TO A CONJECTURE ON INFEASIBLE INTERIOR-POINT METHODS

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Abstract. Based on extensive computational evidence (hundreds of thousands of randomly generated problems) the second author conjectured that $\bar{\kappa}(\zeta) = 1$ (Conjecture 5.1 in [1]), which is a factor of $\sqrt{2n}$ better than has been proved in [1], and which would yield an $O(\sqrt{n})$ iteration full-Newton step infeasible interior-point algorithm. In this paper we present an example showing that $\bar{\kappa}(\zeta)$ is in the order of $\sqrt{n}$, the same order as has been proved in [1]. In other words, the current best iteration bound for infeasible interior-point algorithms cannot be improved.

Key words. linear optimization, infeasible interior-point algorithm, full-Newton step method, conjecture

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1. Introduction. We consider the LO (Linear Optimization) problem in the standard form

$$\min\{c^T x : \ Ax = b, \ x \geq 0\}, \tag{P}$$

with its dual problem

$$\max\{b^T y : \ A^T y + s = c, \ s \geq 0\}. \tag{D}$$

Here $A \in R^{m\times n}$, b, $y \in R^m$, and c, x, $s \in R^n$. Without loss of generality we assume that $\operatorname{rank}(A) = m$. The vectors x, y, and s are the vectors of variables.

We assume throughout that there exists an optimal solution (x^*, y^*, s^*) and a positive number ζ such that

$$0 \leq x^* \leq \zeta e, \ 0 \leq s^* \leq \zeta e, \ x^* s^* = 0.^1$$

For any ν with $0 < \nu \leq 1$ we consider the perturbed problem (P_ν), defined by

$$\min\{(c - \nu(c - \zeta e))^T x : \ Ax = b - \nu(b - A\zeta e), \ x \geq 0\}, \tag{P_ν}$$

and its dual problem (D_ν), which is given by

$$\max\{(b - \nu(b - A\zeta e))^T y : \ A^T y + s = c - \nu(c - \zeta e), \ s \geq 0\}. \tag{D_ν}$$

Note that if $\nu = 1$, then $x = \zeta e$ yields a strictly feasible solution of (P_ν) and $(y, s) = (0, \zeta e)$ a strictly feasible solution of (D_ν). We conclude that if $\nu = 1$, then (P_ν) and (D_ν) satisfy the IPC (interior-point condition).

Let (P) and (D) be feasible and $0 < \nu \leq 1$. Then problems (P_ν) and (D_ν) satisfy the IPC (Theorem 5.13 in [2], see also Lemma 3.1 in [1]), and hence their central paths exists. This means that the system

$$\begin{aligned} b - Ax &= \nu(b - A\zeta e), \qquad & x > 0, \\ c - A^T y - s &= \nu(c - \zeta e), \qquad & s > 0, \\ xs &= \nu\zeta^2 e & \end{aligned}$$

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[1] Here and follows, e denotes the all-one vector of length n, and if $x, \ s \in R^n$, then xs denotes the componentwise (or Hadamard) product of the vectors x and s.

has a unique solution for every $\nu > 0$. In what follows this unique solution is denoted by $(x(\mu,\nu), y(\mu,\nu), s(\mu,\nu))$. These are the μ-centers ($\mu = \nu\zeta^2$) of the perturbed problems (P_ν) and (D_ν). We define

$$\kappa(\zeta,\nu) := \frac{\sqrt{\|x(\mu,\nu)\|^2 + \|s(\mu,\nu)\|^2}}{\zeta\sqrt{2n}},\ 0 < \nu \le 1,\ \mu = \nu\zeta^2,$$

and

$$\bar{\kappa}(\zeta) = \max_{0<\nu\le 1} \kappa(\zeta,\nu).$$

Then the total number of inner iterations (Section 4.7 in [1]) is bounded above by

$$16\bar{\kappa}(\zeta)\sqrt{n}\log\frac{\max\{n\zeta^2, \|b - A\zeta e\|, \|c - \zeta e\|\}}{\epsilon}.$$

In [1], the second author proved that $\bar{\kappa}(\zeta) \le \sqrt{2n}$, and based on extensive computational evidence (hundreds of thousands of randomly generated problems), he made the following conjecture.

Conjecture 1.1 (Conjecture 5.1 in [1]). *If (P) and (D) are feasible and $\zeta \ge \|x^* + s^*\|_\infty$ for some pair of optimal solutions x^* and (y^*, s^*), then $\bar{\kappa}(\zeta) = 1$.*

2. Counter example. Due to the choice of the optimal solution (x^*, y^*, s^*), we have

$$\begin{aligned} Ax^* = b, &\quad 0 \le x^* \le \zeta e, \\ A^T y^* + s^* = c, &\quad 0 \le s^* \le \zeta e, \\ x^* s^* = 0. & \end{aligned} \tag{2.1}$$

To simplify notation in the rest of this section, we denote $x := x(\nu)$, $y := y(\nu)$ and $s := s(\nu)$. Then x, y and s are uniquely determined by the system

$$\begin{aligned} b - Ax = \nu(b - A\zeta e), &\quad x > 0, \\ c - A^T y - s = \nu(c - \zeta e), &\quad s > 0, \\ xs = \nu\zeta^2 e. & \end{aligned}$$

Using (2.1) we get the equivalent system

$$\begin{aligned} Ax^* - Ax = \nu(Ax^* - A\zeta e), &\quad x > 0, \\ A^T y^* + s^* - A^T y - s = \nu(A^T y^* + s^* - \zeta e), &\quad s > 0, \\ xs = \nu\zeta^2 e. & \end{aligned}$$

We rewrite this system as

$$\begin{aligned} A(x^* - x - \nu x^* + \nu\zeta e) = 0, &\quad x > 0, \\ A^T(y^* - y - \nu y^*) = s - s^* + \nu s^* - \nu\zeta e, &\quad s > 0, \\ xs = \nu\zeta^2 e. & \end{aligned} \tag{2.2}$$

Hence the maximal value that $\bar{\kappa}(\zeta)$ can attain is obtained by solving the problem

$$\max\left\{\frac{\sqrt{\|x\|^2 + \|s\|^2}}{\zeta\sqrt{2n}} : (2.1) \text{ and } (2.2)\right\}. \tag{2.3}$$

In this problem we maximize over all possible values of A, b, c, ζ, ν, x^*, y^*, s^*, x, y, and s satisfying (2.1) and (2.2). Note that if (2.1) and (2.2) are satisfied, then after replacing x^*, y^*, s^*, x, y, s, b, and c by x^*/ζ, y^*/ζ, s^*/ζ, x/ζ, y/ζ, s/ζ, b/ζ, and c/ζ, respectively, we get a solution of (2.1) and (2.2) with $\zeta = 1$, and in that case the value of the objective function in (2.3) does not change. Hence, without loss of generality we may assume below that $\zeta = 1$.

Our aim is to construct a feasible solution for (2.1) and (2.2) whose objective value $\frac{\sqrt{\|x\|^2+\|s\|^2}}{\sqrt{2n}}$ is of the same order as $\sqrt{n}$, thus showing that the order of the theoretical bound for $\bar{\kappa}(\zeta)$ in [1] is sharp. This will be done by first constructing suitable vectors x^*, y^*, s^*, x, y, s such that, for some fixed value of $\nu \in (0,1)$,

$$0 \leq x^* \leq e, \quad 0 \leq s^* \leq e, \quad x^* s^* = 0, \quad x > 0, \quad s > 0, \quad xs = \nu e, \tag{2.4}$$

and such that the objective value in (2.3) is of the same order as $\sqrt{n}$. After this we will construct A, b and c such that (2.1) and (2.2) are satisfied (for $\zeta = 1$). It follows that the constructed (x, y, s) is just the μ-center of the perturbed problem pair (P_ν) and (D_ν) with $\mu = \nu\mu^0 = \nu\zeta^2 = \nu$. Hence will suffice to falsify the conjecture.

Using that the row space of a matrix and its null space are orthogonal, we relax for the moment the first two equations in the system (2.2) to

$$(x^* - x - \nu x^* + \nu e)^T (s - s^* + \nu s^* - \nu e) = 0, \quad x > 0, \quad s > 0. \tag{2.5}$$

Since x^* and s^* are orthogonal, we may rewrite the above equation as follows.

$$x^T \left[\frac{1-\nu}{\nu} s^* + e \right] + s^T \left[\frac{1-\nu}{\nu} x^* + e \right] = (1-\nu) e^T (x^* + s^*) + n(1+\nu). \tag{2.6}$$

At this stage we choose a fixed value of $\nu \in (0,1)$ and x^* and s^* such that their positive entries are small enough to have

$$\frac{1-\nu}{\nu} s^* + e \approx e, \quad \frac{1-\nu}{\nu} x^* + e \approx e, \quad (1-\nu) e^T (x^* + s^*) + n(1+\nu) \approx n(1+\nu).$$

Then it follows from (2.6) that

$$x^T e + s^T e \approx n(1+\nu).$$

Yet we choose

$$x_i = s_i = \sqrt{\nu}, \quad \text{for } i > 1, \tag{2.7}$$

leaving x_1 and s_1 free for the moment. This gives

$$x_1 + s_1 + 2(n-1)\sqrt{\nu} \approx n(1+\nu),$$

or, equivalently,

$$x_1 + s_1 \approx (n-1)\left(1 - \sqrt{\nu}\right)^2 + (1+\nu). \tag{2.8}$$

Our aim is to make x and s be the μ-centers of the perturbed problems corresponding to $\mu = \nu\mu^0 = \nu\zeta^2$, and then to compute $\kappa(\zeta, \nu)$. This holds if $xs = \mu e$. Since $\zeta = 1$, and because of (2.7), this holds if $x_1 s_1 = \nu$. We may easily check that there exists x_1 and s_1 which satisfy (2.8) and $x_1 s_1 = \nu$. Hence

$$x_1^2 + s_1^2 = (x_1 + s_1)^2 - 2x_1 s_1 \approx \left[(n-1)\left(1-\sqrt{\nu}\right)^2 + (1+\nu)\right]^2 - 2\nu,$$

Thus we obtain

$$\|x\|^2 + \|s\|^2 \approx \left[(n-1)\left(1-\sqrt{\nu}\right)^2 + (1+\nu)\right]^2 - 2\nu + 2(n-1)\nu,$$

whence

$$\kappa(1,\nu) = \frac{\sqrt{\|x\|^2 + \|s\|^2}}{\sqrt{2n}} \approx \frac{\sqrt{\left[(n-1)\left(1-\sqrt{\nu}\right)^2 + (1+\nu)\right]^2 + 2(n-2)\nu}}{\sqrt{2n}}. \qquad (2.9)$$

Note that for fixed ν $(0 < \nu < 1)$ the last expression is of the same order as $\sqrt{n}$. E.g., for $\nu = \frac{1}{4}$ it equals $\sqrt{\frac{n+16}{32}}$, and if ν approaches zero then it becomes $\sqrt{\frac{n}{2}}$.

Until now the vectors x^*, y^*, s^*, x, y, s only satisfy (2.4) and (2.5). It remains to show that there exist A, b and c such that (2.1) and (2.2) are satisfied. This is easy. We take for A any matrix whose row space is equal to the orthogonal complement of the linear space generated by the vector $x^* - x - \nu x^* + \nu e$. Then the vector $s^* - s - \nu s^* + \nu e$ belongs to the row space of A, and hence there exists a vector y such that $A^T y = s^* - s - \nu s^* + \nu e$. Taking $y^* = 0$ it follows that (2.2) holds. Finally, taking $b = Ax^*$ and $c = A^T y^* + s^*$, also (2.1) holds. Thus we have shown the existence of a feasible solution of (2.3) for which the $\kappa(\zeta,\nu)$ has the order of $\sqrt{n}$, and hence $\bar{\kappa}(\zeta)$ will be at least of this order.

Just to add some numerical evidence to the above analysis we applied the above described construction for several values of n and ν. We took for x^* and s^* randomly generated nonnegative and orthogonal vectors, whose positive entries are uniformly distributed in $(0, 1/1000)$. For the computation of x_1 and s_1 we used (2.6), instead of its approximation (2.8). As a consequence x and s are the μ-centers of the perturbed problems (P_ν) and (D_ν) with $\mu = \nu\zeta^2 = \nu$, and $\kappa(1,\nu)$ is well-approximated by (2.9). The resulting values of $\kappa(\zeta,\nu)$ and $\bar{\kappa}(\zeta)$ with $\zeta = 1$ are listed in Table 2.1.

	$n=40$		$n=160$		$n=640$	
	$\kappa(1,\nu)$	$\bar{\kappa}(1)$	$\kappa(1,\nu)$	$\bar{\kappa}(1)$	$\kappa(1,\nu)$	$\bar{\kappa}(1)$
$\nu = 1/4$	1.3214	2.2956	2.3419	4.4009	4.521	8.6913
$\nu = 1/400$	4.0048	4.1137	7.9881	8.1898	15.975	16.363
$\nu = 1/40000$	3.9883	4.1027	7.9373	8.2063	15.898	16.389

Table 2.1
Typical values of $\kappa(1,\nu)$ and $\bar{\kappa}(1)$ for some values of n and ν.

It is clear from Table 2.1 that Conjecture 1.1 ([1, Conjecture 5.1]) is false.

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