On a time consistency concept in risk averse multi-stage stochastic programming

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Abstract. In this paper we discuss time consistency of multi-stage risk averse stochastic programming problems. We approach the concept of time consistency from an optimization point of view. That is, at each state of the system optimality of a decision policy should not involve states which cannot happen in the future. We also discuss a relation of this concept of time consistency to deriving dynamic programming equations. Finally, we argue that some risk averse approaches to multi-stage programming are time consistent while some others are not.

Key Words. Stochastic programming, time consistency, risk averse optimization, dynamic programming, coherent risk measures

1 Introduction

In recent years risk averse stochastic optimization attracted considerable attention from theoretical and applied points of view (see, e.g., [2, 7, 11, 14, 15, 16]). Starting with pioneering paper by Artzner et al [1], coherent risk measures were investigated in numerous studies (see [8] and references therein). It seems that there is a general agreement now of how to model static risk averse stochastic programming problems. A dynamical setting is more involved and several approaches to modelling dynamic risk measures were suggested by various authors (e.g., [2, 3, 4, 6, 7, 9, 12, 15, 16]).

A basic concept of multi-stage stochastic programming is the requirement of *nonanticipativity*. That is, our decisions should be a function of the history of the data process available at the time when decisions are made. This requirement is necessary for the designed policy to be implementable in the sense that at every state of the system it produces a decision based on available information. The nonanticipativity is a feasibility type constraint indicating which policies are implementable and which are not. On top of that one has to define a criterion for choosing an optimal policy. In the risk neutral setting the optimization is performed on average. Under the nonanticipativity constraint this allows to write the corresponding dynamic programming equations for an optimal policy. In risk averse settings the situation is more subtle. This motivated an introduction of various concepts of time invariance and time consistency by several authors at various degrees of generality and abstraction (cf., [2, 3, 4, 5, 10, 13, 16]).

In this paper we discuss an approach to time consistency tailed to conceptual optimality of a decision policy. If we are currently at a certain state of the system, then we know the past, and hence it is reasonable to require that our decisions should be based on that information. This is the nonanticipativity constraint. If we believe in the considered model, we also have an idea which scenarios could and which cannot happen in the future. Therefore it is also reasonable to consider the requirement that at every state of the system our "optimal" decisions should *not* depend on scenarios which we already know cannot happen in the future. We call this principle the *time consistency* requirement. This time consistency requirement is closely related to, although is not the same, as the so-called Bellman's principle used to derive dynamic programming equations. The standard risk neutral formulation of multi-stage stochastic programming problems satisfies this principle. On the other hand, some approaches to risk averse stochastic programming satisfy while others do not this requirement. It should be mentioned that if the time consistency property does not hold, it does not mean that the corresponding policies are not implementable. We only would like to point out that there is an additional consideration, associated with a chosen optimality criterion, which is worthwhile to keep in mind.

This paper is organized as follows. In the next section we formalize the concept of time consistency. We also discuss its relation to deriving dynamic programming equations. In section 3 we give examples and show that some approaches to risk averse multi-stage stochastic programming are time consistent while some others are not.

2 Basic analysis

Consider a T-stage scenario tree representing evolution of the corresponding data process. Such scenario tree represents a finite number of possibilities of what can happen in the future. Considering the case with finite number of scenarios will allow us to avoid some technical complications and concentrate on conceptual issues. At stage (time) $t = 1$ we have one root node denoted ξ_1. At stage $t = 2$ we have as many nodes as many different realizations of data may occur. Each of them is connected with the root node by an arc. A generic node at time $t = 2$ is denoted ξ_2, etc at the later stages. By Ω_t we denote the set of all nodes at stage $t = 1, ..., T$. One can view a node $\xi_t \in \Omega_t$ as a state of the system at time t. Children nodes of a node $\xi_t \in \Omega_t$ are nodes which can

happen at the next stages $t+1, ..., T$ if we are currently at state (node) ξ_t. A scenario, representing a particular realization (sample path) of the data process, is a sequence $\xi_1, ..., \xi_T$ of nodes such that $\xi_t \in \Omega_t$ and $\xi_{t+1} \in \Omega_{t+1}$ is a child of the node ξ_t. Note that, at the moment, we do not assume any probabilistic structure of the process $\xi_1, ..., \xi_T$. For $1 \le s \le t \le T$ denote $\xi_{[s,t]} := \{\xi_s, ..., \xi_t\}$. In particular, $\xi_{[1,t]}$ represents history of the process from the root node to node ξ_t.

Our decisions should be adapted to the time structure of the process described by the considered scenario tree. The decision process has the form:

$$\text{decision } (x_1) \rightsquigarrow \text{observation } (\xi_2) \rightsquigarrow \text{decision } (x_2) \rightsquigarrow$$
$$..... \rightsquigarrow \text{observation } (\xi_T) \rightsquigarrow \text{decision } (x_T).$$

The values of the decision vector $x_t \in \mathbb{R}^{n_t}$, chosen at stage t, may depend on the information (data) available up to time t, but not on the results of future observations. This is the basic requirement of *nonanticipativity*. The nonanticipativity requirement means that our decisions $x_t = x_t(\xi_{[1,t]})$ are functions of the history $\xi_{[1,t]}$ of the process up to time t. At the first stage $t=1$ there is only one (root) node and x_1 is independent of the data. We also have feasibility constraints which in an abstract form can be written as

$$x_1 \in \mathcal{X}_1, \ x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \ t = 2, ..., T, \tag{2.1}$$

where $\mathcal{X}_1 \subset \mathbb{R}^{n_1}$ is a (deterministic) set and $\mathcal{X}_t : \mathbb{R}^{n_{t-1}} \times \Omega_t \rightrightarrows \mathbb{R}^{n_t}$, $t = 2, ..., T$, are point-to-set mappings (multifunctions). A decision policy $x_t = x_t(\xi_{[1,t]})$ should be *feasible*, i.e., should satisfy constraints (2.1) for all realizations (scenarios) of the data process $\xi_1, ..., \xi_T$. (In case the data process is modelled as a random (stochastic) process, not necessarily finitely supported, the feasibility constraints should be satisfied w.p.1.)

So far we didn't say anything about optimality of our decisions. Let us look carefully at what this means. At every state (node) $\xi_t \in \Omega_t$ of the system, at time t, we have information about the past, i.e., we know history $\xi_{[1,t]}$ of the process from the root node to the current node ξ_t. We also have information (if we believe in the model) what states in the future cannot happen. It is natural to consider the conceptual requirement that an optimal decision at state ξ_t should not depend on states which do not follow ξ_t, i.e., cannot happen in the future. That is, optimality of our decision at state ξ_t should only involve future children nodes of state ξ_t. We call this principle *time consistency*.

In order to formalize this concept of time consistency we need to say what do we optimize (say minimize) at every state of the process. We assume that to every node $\xi_t \in \Omega_t$ and decision vector x_t corresponds a real number (cost function) $f_t(x_t, \xi_t)$. Denote by $\mathcal{Z}_t$ the space of all functions $Z : \Omega_t \to \mathbb{R}$. The space $\mathcal{Z}_t$ is a linear space of (finite) dimension $|\Omega_t|$. We can also associate with the process ξ_t the corresponding sequence (a filtration) $\mathcal{F}_1 \subset \cdots \subset \mathcal{F}_T$ of sigma algebras on the set Ω_T. That is, $\mathcal{F}_T$ is the sigma algebra formed by all subsets of Ω_T, $\mathcal{F}_{T-1}$ is the sigma subalgebra of $\mathcal{F}_T$ generated by sets of Ω_T which are children nodes of nodes at stage $t = T - 1$, etc. Finally $\mathcal{F}_1 = \{\emptyset, \Omega_T\}$. We can view $\mathcal{Z}_t$ as the space of $\mathcal{F}_t$-measurable functions $Z : \Omega_T \to \mathbb{R}$, and hence view $\mathcal{Z}_t$ as a subspace of $\mathcal{Z}_{t+1}$.

With every time period $t = 1, ..., T - 1$, we need to associate a (real valued) objective function $\mathbb{F}_t (Z_t, ..., Z_T \,|\, \xi_{[1,t]})$ of $(Z_t, ..., Z_T) \in \mathcal{Z}_t \times \cdots \times \mathcal{Z}_T$ and $\xi_{[1,t]}$. That is, if we are currently at node ξ_t, and hence know history $\xi_{[1,t]}$, then we have an objective for future optimization starting from ξ_t. At stage $t = 2, ..., T - 1$, optimality is understood in terms of the following (minimization) problem

$$\begin{aligned} \min_{x_t, ..., x_T} \quad & \mathbb{F}_t \big(f_t(x_t, \xi_t), ..., f_T(x_T, \xi_T) \,|\, \xi_{[1,t]}\big) \\ \text{s.t.} \quad & x_\tau \in \mathcal{X}_\tau(x_{\tau-1}, \xi_\tau), \ \tau = t, ..., T. \end{aligned} \tag{2.2}$$

Optimization in (2.2) is performed over feasible policies $x_\tau = x_\tau(\xi_{[t,\tau]})$, $\tau = t, ..., T$, considered as functions of the process $\xi_{[t,\tau]}$ conditional on $\xi_{[1,t]}$ being known and decision x_{t-1} being made. At the first stage $t = 1$ we solve the problem

$$\begin{aligned}
\operatorname*{Min}_{x_1,...,x_T} \quad & \mathbb{F}_1\big(f_1(x_t, \xi_1), ..., f_T(x_T, \xi_T)\big) \\
\text{s.t.} \quad & x_\tau \in \mathcal{X}_\tau(x_{\tau-1}, \xi_\tau), \ \tau = 1, ..., T,
\end{aligned} \tag{2.3}$$

where for $t = 1$ by $\mathcal{X}_t(x_{t-1}, \xi_t)$ we mean the set $\mathcal{X}_1$.

Of course, this procedure should be consistent over time. That is, the following property should hold:

(A1) If $1 \le t_1 < t_2 \le T$ and $\bar{x}_\tau(\xi_{[t_1,\tau]})$, $\tau = t_1, ..., T$, is an optimal solution of problem (2.2) for $t = t_1$, conditional on a realization $\xi_1, ..., \xi_{t_1}$ of the process, then $\bar{x}_\tau(\xi_{[t_1,\tau]})$, $\tau = t_2, ..., T$, is an optimal solution of problem (2.2) for $t = t_2$, conditional on a realization $\xi_1, ..., \xi_{t_1}, \xi_{t_1+1}, ..., \xi_{t_2}$ of the process.

Note that in the above condition (A1), the optimal policy $\bar{x}_\tau(\xi_{[t_1,\tau]})$ becomes a function of $\xi_{[t_2,\tau]}$, for $\tau \ge t_2$, after we observe the realization $\xi_1, ..., \xi_{t_2}$ of the process and hence know $\xi_{t_1+1}, ..., \xi_{t_2}$.

The optimal value of problem (2.2) is a function of $\xi_{[1,t]}$ and x_{t-1}, which we denote $V_t(x_{t-1}, \xi_{[1,t]})$. We can write problem (2.2) in the form

$$\operatorname*{Min}_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \left\{ \begin{aligned} \operatorname*{inf}_{x_{t+1},...,x_T} \quad & \mathbb{F}_t\big(f_t(x_t, \xi_t), f_{t+1}(x_{t+1}, \xi_{t+1}), ..., f_T(x_T, \xi_T) \,|\, \xi_{[1,t]}\big) \\ \text{s.t.} \quad & x_\tau \in \mathcal{X}_\tau(x_{\tau-1}, \xi_\tau), \ \tau = t+1, ..., T. \end{aligned} \right\} \tag{2.4}$$

In order to write dynamic equations we need the following condition.

(A2) For $t = 1, ..., T-1$, the optimal value inside the parentheses in (2.4) can be written in the form

$$\phi_t\big(f_t(x_t, \xi_t), V_{t+1}(x_t, \xi_{[1,t+1]}) \,|\, \xi_{[1,t]}\big), \tag{2.5}$$

where $\phi_t(\cdot, \cdot \,|\, \cdot)$ is a real valued function.

Under the above condition (A2), problem (2.4) takes the form

$$\operatorname*{Min}_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \phi_t\big(f_t(x_t, \xi_t), V_{t+1}(x_t, \xi_{[1,t+1]}) \,|\, \xi_{[1,t]}\big). \tag{2.6}$$

Consequently, we can write the following dynamic programming equations for the problem (2.3):

$$V_T(x_{T-1}, \xi_{[1,T]}) = \operatorname*{inf}_{x_{T-1} \in \mathcal{X}_T(x_{T-1}, \xi_T)} \mathbb{F}_T\big(f_T(x_T, \xi_T) | \xi_{[1,T]}\big), \tag{2.7}$$

and for $t = T-1, ..., 1$,

$$V_t(x_{t-1}, \xi_{[1,t]}) = \operatorname*{inf}_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \phi_t\big(f_t(x_t, \xi_t), V_{t+1}(x_t, \xi_{[1,t+1]}) \,|\, \xi_{[1,t]}\big). \tag{2.8}$$

That is, a policy $\bar{x}_t = \bar{x}_t(\xi_{[1,t]})$ is optimal iff it satisfies the above equations (2.8). It follows that condition (A2) implies condition (A1).

3 Examples and a discussion

Consider the classical (risk neutral) multistage stochastic programming problem written in the nested formulation form

$$\underset{x_1 \in \mathcal{X}_1}{\text{Min}} \; f_1(x_1) + \mathbb{E}\left[\underset{x_2 \in \mathcal{X}_2(x_1,\xi_2)}{\inf} f_2(x_2,\xi_2) + \mathbb{E}\left[\cdots + \mathbb{E}\left[\underset{x_T \in \mathcal{X}_T(x_{T-1},\xi_T)}{\inf} f_T(x_T,\xi_T)\right]\right]\right]. \tag{3.1}$$

Here the process $\xi_1, \xi_2, ..., \xi_T$ is equipped with a probability structure such that it becomes a stochastic process. Define

$$\mathbb{F}_t\left(Z_t, ..., Z_T | \xi_{[1,t]}\right) := \mathbb{E}\left[Z_t + \cdots + Z_T | \xi_{[1,t]}\right]. \tag{3.2}$$

Property (A2) follows here with

$$\phi_t\left(f_t(x_t,\xi_t), V_{t+1}(x_t,\xi_{[1,t+1]}) \,|\, \xi_{[1,t]}\right) := \mathbb{E}\left[f_t(x_t,\xi_t) + V_{t+1}(x_t,\xi_{[1,t+1]}) \,|\, \xi_{[1,t]}\right]. \tag{3.3}$$

That is, in the considered sense the risk neutral problem (3.1) is time consistent. Note that problem (3.1) can be written in the equivalent form:

$$\begin{aligned} \underset{x_1,x_2,...,x_T}{\text{Min}} \quad & \mathbb{E}\left[f_1(x_1) + f_2(x_2,\xi_2) + \cdots + f_T(x_T,\xi_T)\right] \\ \text{s.t.} \quad & x_1 \in \mathcal{X}_1, \; x_t \in \mathcal{X}_t(x_{t-1},\xi_t), \; t = 2, ..., T, \end{aligned} \tag{3.4}$$

where the optimization is performed over policies $x_t = x_t(\xi_{[1,t]})$ satisfying w.p.1 the feasibility constraints (2.1).

A way of extending risk neutral formulation (3.4) to a risk averse setting is the following. Let $\mathcal{Z} := \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_T$ and $\varrho : \mathcal{Z} \to \mathbb{R}$ be, say a coherent, risk measure. Consider the problem

$$\begin{aligned} \underset{x_1,x_2,...,x_T}{\text{Min}} \quad & \varrho\big(f_1(x_1), f_2(x_2,\xi_2), \ldots, f_T(x_T,\xi_T)\big) \\ \text{s.t.} \quad & x_1 \in \mathcal{X}_1, \; x_t \in \mathcal{X}_t(x_{t-1},\xi_t), \; t = 2, ..., T, \end{aligned} \tag{3.5}$$

where, similar to (3.4), the optimization is performed over policies $x_t = x_t(\xi_{[1,t]})$ satisfying the feasibility constraints. This framework, in various forms of abstraction, was considered by several authors with ϱ called multiperiod risk measure (e.g., [2, 11, 16]).

Of course, for $\varrho(Z_1, ..., Z_T) := \mathbb{E}(Z_1 + ... + Z_T)$, formulation (3.5) coincides with (3.4), and satisfies the time consistency principle. Let now the multiperiod risk measure be defined as the absolute deviation risk measure:

$$\varrho(Z_1, ..., Z_T) := \mathbb{E}(Z_1 + ... + Z_T) + \lambda \mathbb{E}\big|Z_1 + ... + Z_T - \mathbb{E}(Z_1 + ... + Z_T)\big|, \tag{3.6}$$

where λ is a positive constant (for $\lambda \in [0, 1/2]$ this function $\varrho : \mathcal{Z} \to \mathbb{R}$ satisfies the axioms of coherent risk measures). For this multiperiod risk measure and $T > 2$ the corresponding problem (3.5) does not satisfy the time consistency principle and it is not clear how to write the associated dynamic programming equations.

As another example consider the following multiperiod risk measure

$$\varrho(Z_1, ..., Z_T) := Z_1 + \rho_2(Z_2) + ... + \rho_T(Z_T), \tag{3.7}$$

where $\rho_t : \mathcal{Z}_t \to \mathbb{R}$, $t = 2, ..., T$, are (real valued) risk measures. In that case problem (3.5) takes the form

$$\begin{aligned} \underset{x_1,x_2,...,x_T}{\text{Min}} \quad & f_1(x_1) + \rho_2[f_2(x_2,\xi_2)] + ... + \rho_T\left[f_T(x_T,\xi_T)\right] \\ \text{s.t.} \quad & x_1 \in \mathcal{X}_1, \; x_t \in \mathcal{X}_t(x_{t-1},\xi_t), \; t = 2, ..., T. \end{aligned} \tag{3.8}$$

If each ρ_t is given by Conditional Value-at-Risk, i.e., $\rho_t := \mathsf{CVaR}_{\alpha_t}$, $t = 2, ..., T$, with

$$\mathsf{CVaR}_{\alpha_t}(Z_t) := \inf_{r \in \mathbb{R}} \left\{ r + \alpha_t^{-1} \mathbb{E}[Z_t - r]_+ \right\}, \ \alpha_t \in (0,1], \tag{3.9}$$

then the corresponding multiperiod risk measure ϱ becomes a particular example of the polyhedral risk measures discussed in [7]. In that case we can write problem (3.8) as the multistage program

$$\begin{aligned}
\underset{(r,x_1),x_2,...,x_T}{\text{Min}} \quad & r_2 + ... + r_T + f_1(x_1) + \alpha_2^{-1} \mathbb{E}\left\{ [f_2(x_2, \xi_2) - r_2]_+ \right\} + \\
& ... + \alpha_T^{-1} \mathbb{E}\left\{ [f_T(x_T, \xi_T) - r_T]_+ \right\} \\
\text{s.t.} \quad & x_1 \in \mathcal{X}_1, \ x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \ t = 2, ..., T,
\end{aligned} \tag{3.10}$$

where $r = (r_2, ..., r_T)$. Problem (3.10) can be viewed as a standard multistage stochastic program with $x_1, r_2, ..., r_T$ being first stage decision variables.

Dynamic programming equations for problem (3.10) can be written as follows. At the last stage we solve problem

$$\begin{aligned}
\underset{x_T}{\text{Min}} \quad & \alpha_T^{-1} [f_T(x_T, \xi_T) - r_T]_+ \\
\text{s.t.} \quad & x_T \in \mathcal{X}_T(x_{T-1}, \xi_T).
\end{aligned} \tag{3.11}$$

Its optimal value is denoted $V_T(x_{T-1}, r_T, \xi_T)$. At stage $t = 2, ..., T-1$, the value function $V_t(x_{t-1}, r_t, ..., r_T, \xi_{[1,t]})$ is given by the optimal value of problem

$$\begin{aligned}
\underset{x_t}{\text{Min}} \quad & \alpha_t^{-1} [f_t(x_t, \xi_t) - r_t]_+ + \mathbb{E}\left\{ V_{t+1}(x_t, r_t, ..., r_T, \xi_{[1,t+1]}) \big| \xi_{[1,t]} \right\} \\
\text{s.t.} \quad & x_t \in \mathcal{X}_t(x_{t-1}, \xi_t).
\end{aligned} \tag{3.12}$$

At the first stage we need to solve the problem

$$\underset{x_1 \in \mathcal{X}_1, r_2, ..., r_T}{\text{Min}} \quad r_2 + ... + r_T + f_1(x_1) + \mathbb{E}[V_2(x_1, r_2, ..., r_T, \xi_2)]. \tag{3.13}$$

Although it was possible to write dynamic programming equations for problem (3.10), please note that decision variables $r_2, ..., r_T$ are decided at the first stage and their optimal values depend on all scenarios starting at the root node at stage $t = 1$. Consequently optimal decisions at later stages depend on scenarios other than following a considered node, and hence formulation (3.10) (as well as (3.8)) is *not time consistent*.

Let us discuss now an approach to risk averse formulation based on *conditional risk mappings* (cf., [6, 12, 15]). That is, consider the following nested formulation

$$\underset{x_1 \in \mathcal{X}_1}{\text{Min}} \ f_1(x_1) + \rho_{2|\xi_{[1,1]}}\left[\underset{x_2 \in \mathcal{X}_2(x_1, \xi_2)}{\inf} f_2(x_2, \xi_2) + ... + \rho_{T|\xi_{[1,T-1]}}\left[\underset{x_T \in \mathcal{X}_T(x_{T-1}, \xi_T)}{\inf} f_T(x_T, \xi_T) \right] \right], \tag{3.14}$$

where $\rho_{t+1|\xi_{[1,t]}} : \mathcal{Z}_{t+1} \to \mathcal{Z}_t$ are conditional risk mappings (cf., [15]). Of course, for

$$\rho_{t+1|\xi_{[1,t]}}(Z_{t+1}) := \mathbb{E}\left[Z_{t+1} | \xi_{[1,t]} \right], \ t = 1, ..., T-1, \tag{3.15}$$

the above nested problem (3.14) coincides with the risk neutral problem (3.1).

From the point of view of the nested formulation (3.14), two main properties of conditional risk mappings are the *monotonicity*: for any $Z_{t+1}, Z'_{t+1} \in \mathcal{Z}_{t+1}$ such that $Z_{t+1} \leq Z'_{t+1}$ it holds that

$$\rho_{t+1|\xi_{[1,t]}}(Z_{t+1}) \leq \rho_{t+1|\xi_{[1,t]}}(Z'_{t+1}), \tag{3.16}$$

and the *translation equivariance*: if $Z_{t+1} \in \mathcal{Z}_{t+1}$ and $Z_t \in \mathcal{Z}_t$, then

$$\rho_{t+1|\xi_{[1,t]}}(Z_{t+1} + Z_t) = \rho_{t+1|\xi_{[1,t]}}(Z_{t+1}) + Z_t. \tag{3.17}$$

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Apart from these two properties, conditional risk mappings are assumed to satisfy conditions of convexity and positive homogeneity. However, the above monotonicity and translation equivariance properties alone allow to write the corresponding dynamic programming equations (see below).

We have here

$$\mathbb{F}_t \left(Z_t, ..., Z_T | \xi_{[1,t]} \right) := \tilde{\rho}_t \left[Z_t + \cdots + Z_T \,|\, \xi_{[1,t]} \right], \tag{3.18}$$

where

$$\tilde{\rho}_t[\,\cdot\,|\xi_{[1,t]}] := \rho_{t+1|\xi_{[1,t]}} \circ \cdots \circ \rho_{T|\xi_{[1,T-1]}}[\,\cdot\,] \tag{3.19}$$

are composite risk measures. Moreover, condition (A2) holds with

$$\phi_t \big(f_t(x_t, \xi_t), V_{t+1}(x_t, \xi_{[1,t+1]}) \,|\, \xi_{[1,t]} \big) := \rho_{t+1|\xi_{[1,t]}} \left[f_t(x_t, \xi_t) + V_{t+1}(x_t, \xi_{[1,t+1]}) \right]. \tag{3.20}$$

This formulation is time consistent and the corresponding dynamic programming equations are (see [15])

$$V_t(x_{t-1}, \xi_{[1,t]}) = \inf_{x_t \in \mathcal{X}_t(x_{t-1}, \xi_t)} \rho_{t+1|\xi_{[1,t]}} \left[f_t(x_t, \xi_t) + V_{t+1}(x_t, \xi_{[1,t+1]}) \right]. \tag{3.21}$$

Nested formulation (3.14) can be written in the form

$$\begin{aligned} \underset{x_1, x_2, ..., x_T}{\text{Min}} \quad & \tilde{\rho}_1 \left[f_1(x_1) + f_2(x_2, \xi_2) + ... + f_T(x_T, \xi_T) \right] \\ \text{s.t.} \quad & x_1 \in \mathcal{X}_1, \ x_t \in \mathcal{X}_t(x_{t-1}, \xi_t), \ t = 2, ..., T, \end{aligned} \tag{3.22}$$

where $\tilde{\rho}_1 = \rho_{2|\xi_{[1,1]}} \circ \cdots \circ \rho_{T|\xi_{[1,T-1]}}$ and optimization is performed over feasible policies $x_t = x_t(\xi_{[1,t]})$. If the conditional risk mappings are given as conditional expectations, of the form (3.15), then $\tilde{\rho}_1(\cdot) = \mathbb{E}(\cdot)$ is the expectation operator. Unfortunately, in general it is quite difficult to write the composite function $\tilde{\rho}_1(\cdot)$ explicitly.

We discuss below an example of portfolio selection. Nested formulation of multistage portfolio selection can be written as

$$\begin{aligned} \text{Min} \quad & \rho_1 \big[\cdots \rho_{T-1} \big[\rho_T[W_T] \big] \big] \\ \text{s.t.} \quad & W_{t+1} = \sum_{i=1}^n \xi_{i,t+1} x_{i,t}, \ \sum_{i=1}^n x_{i,t} = W_t, \ x_t \geq 0, \ t = 0, ..., T-1, \end{aligned} \tag{3.23}$$

where ρ_t are conditional risk mappings. Note that in order to formulate this as a minimization problem we changed the sign of ξ_{it}. Suppose that the random process ξ_t is *stagewise independent*, i.e., ξ_{t+1} is independent of $\xi_{[1,t]}$, $t = 1, ..., T-1$. Let us write dynamic programming equations. At the last stage we have to solve problem

$$\begin{aligned} \underset{x_{T-1} \geq 0, W_T}{\text{Min}} \quad & \rho_T \left[W_T | W_{T-1} \right] \\ \text{s.t.} \quad & W_T = \sum_{i=1}^n \xi_{iT} x_{i,T-1}, \ \sum_{i=1}^n x_{i,T-1} = W_{T-1}. \end{aligned} \tag{3.24}$$

Since W_{T-1} is a function of $\xi_{[T-1]}$, by the stagewise independence we have that ξ_T is independent of W_{T-1}. It follows by positive homogeneity of ρ_T that the optimal value of (3.24) is $V_{T-1}(W_{T-1}) = W_{T-1}\nu_{T-1}$, where ν_{T-1} is the optimal value of

$$\begin{aligned} \underset{x_{T-1} \geq 0, W_T}{\text{Min}} \quad & \rho_T[W_T] \\ \text{s.t.} \quad & W_T = \sum_{i=1}^n \xi_{iT} x_{i,T-1}, \ \sum_{i=1}^n x_{i,T-1} = 1, \end{aligned} \tag{3.25}$$

and an optimal solution of (3.24) is $\bar{x}_{T-1}(W_{T-1}) = W_{T-1} x_{T-1}^*$, where x_{T-1}^* is an optimal solution of (3.25). And so on we obtain that the optimal policy here is myopic. Note that the composite risk measure $\rho_1 \big[\cdots \rho_{T-1} \big[\rho_T[\,\cdot\,] \big] \big]$ can be quite complicated.

The alternative approach is to write problem

$$
\begin{aligned}
&\text{Min} \quad \rho[W_T] \\
&\text{s.t.} \quad W_{t+1} = \sum_{i=1}^{n} \xi_{i,t+1} x_{i,t}, \ \sum_{i=1}^{n} x_{i,t} = W_t, \ x_t \geq 0, \ t = 0, ..., T-1,
\end{aligned}
\tag{3.26}
$$

for an explicitly defined (real valued) risk measure ρ. In particular, let $\rho := \mathsf{CVaR}_\alpha$. Then problem (3.26) becomes

$$
\begin{aligned}
&\text{Min} \quad r + \alpha^{-1}\mathbb{E}[W_T - r]_+ \\
&\text{s.t.} \quad W_{t+1} = \sum_{i=1}^{n} \xi_{i,t+1} x_{i,t}, \ \sum_{i=1}^{n} x_{i,t} = W_t, \ x_t \geq 0, \ t = 0, ..., T-1,
\end{aligned}
\tag{3.27}
$$

where $r \in \mathbb{R}$ is the (additional) first stage decision variable.

The respective dynamic programming equations become as follows. The last stage value function $V_{T-1}(W_{T-1}, r)$ is given by the optimal value of problem

$$
\begin{aligned}
&\underset{x_{T-1} \geq 0, W_T}{\text{Min}} \quad \alpha^{-1}\mathbb{E}[W_T - r]_+ \\
&\qquad \text{s.t.} \qquad W_T = \sum_{i=1}^{n} \xi_{iT} x_{i,T-1}, \ \sum_{i=1}^{n} x_{i,T-1} = W_{T-1}.
\end{aligned}
\tag{3.28}
$$

And so on, at stage $t = T-2, ..., 1$, we consider problem

$$
\begin{aligned}
&\underset{x_t \geq 0, W_{t+1}}{\text{Min}} \quad \mathbb{E}\{V_{t+1}(W_{t+1}, r)\} \\
&\qquad \text{s.t.} \qquad W_{t+1} = \sum_{i=1}^{n} \xi_{i,t+1} x_{i,t}, \ \sum_{i=1}^{n} x_{i,t} = W_t,
\end{aligned}
\tag{3.29}
$$

whose optimal value is denoted $V_t(W_t, r)$. Finally, at stage $t = 0$ we solve the problem

$$
\begin{aligned}
&\underset{x_0 \geq 0, W_1, r}{\text{Min}} \quad r + \mathbb{E}[V_1(W_1, r)] \\
&\qquad \text{s.t.} \qquad W_1 = \sum_{i=1}^{n} \xi_{i1} x_{i0}, \ \sum_{i=1}^{n} x_{i0} = W_0.
\end{aligned}
\tag{3.30}
$$

In formulation (3.28) the objective function $\alpha^{-1}\mathbb{E}[W_T - r]_+$ can be viewed as a utility function (or rather disutility function since we formulate this as a minimization rather than maximization problem). However, note again that r there is a first stage decision variable and formulation (3.28) is not time consistent.

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