Minimal Spanning Trees with Conflict Graphs

Andreas Darmann ** Ulrich Pferschy * Joachim Schauer *

Abstract

For the classical minimum spanning tree problem we introduce disjunctive constraints for pairs of edges which can not be both included in the spanning tree at the same time. These constraints are represented by a conflict graph whose vertices correspond to the edges of the original graph. Edges in the conflict graph connect conflicting edges of the original graph. It is shown that the problem becomes strongly NP-hard even if the connected components of the conflict graph consist only of paths of length two. On the other hand, for conflict graphs consisting of disjoint edges (i.e. paths of length one) the problem remains polynomially solvable.

Keywords: minimal spanning tree, conflict graph.

1 Introduction

In this paper we consider an extension of the minimum spanning tree problem (MST). In addition to the well studied problem of finding a minimum spanning tree in a weighted, undirected connected graph, there exist incompatibilities for certain pairs of edges. This means that from each such conflicting pair at most one edge can occur in the spanning tree. It is natural to represent these symmetric conflict relations by means of an undirected *conflict graph*, where every vertex of the conflict graph corresponds uniquely to an edge in the original graph and an edge in the conflict graph implies that the two adjacent vertices, i.e. edges in the original graph, cannot occur together in an MST solution.

For a formal definition of this *minimum spanning tree problem with conflict graph* (MSTCG), let $G = (V, E)$ be an undirected connected graph with n vertices and m edges, where each edge e has associated a weight $w(e)$ (w is a weight function $w : E \to \mathbb{R}$). Furthermore, an undirected graph $\bar{G} = (E, \bar{E})$ represents a conflict graph where each of the m vertices corresponds uniquely to an edge $e \in E$ of G. An edge $\bar{e} = (i, j) \in \bar{E}$ implies that the two vertices incident to $\bar{e}$ – that

**University of Graz, Institute of Public Economics, Universitaetsstr. 15, A-8010 Graz, Austria, andreas.darmann@uni-graz.at

*University of Graz, Department of Statistics and Operations Research, Universitaetsstr. 15, A-8010 Graz, Austria, {pferschy, joachim.schauer}@uni-graz.at

is, the two edges $i, j \in E$ – cannot occur together in a spanning tree of G. In contrast to G, $\bar{G}$ is not necessarily connected and may contain isolated vertices (i.e. edges of G which can be combined with every other edge in the minimum spanning tree solution). MSTCG asks for a minimum spanning tree T in G, given that adjacent vertices in $\bar{G}$ are not both together included in T.

For a set of vertices $F \subseteq V$ in G let $E(F)$ be the set of edges in G that have both of its endpoints in F. Then MSTCG can be stated by the following ILP formulation:

$$(MSTCG) \qquad \min \sum_{e \in E} w(e) * x_e \tag{1}$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = n - 1 \tag{2}$$

$$\sum_{e \in E(F)} x_e \leq |F| - 1 \quad \forall \emptyset \neq F \subseteq V \tag{3}$$

$$x_e + x_f \leq 1 \quad \forall (e, f) \in \bar{E} \tag{4}$$

$$x_e \in \{0, 1\} \quad \forall e \in E \tag{5}$$

Obviously, (1)–(3) is a classical ILP-model for MST and (4) adds the conflict constraints.

In this paper we will characterize the complexity of MSTCG and identify the graph classes for the conflict graph $\bar{G}$ where the problem changes from polynomially solvable to strongly $\mathcal{NP}$-hard. These are graphs whose connected components are edges resp. paths of length two (we define the length of a path as the number of edges in the path). For obvious illustrative reasons we introduce the following terminology.

Definition 1 *A* 2*-ladder is an undirected graph whose components are paths of length one, i.e. edges connecting pairs of vertices.*

Definition 2 *A* 3*-ladder is an undirected graph whose components are paths of length two.*

It will be shown in Section 2 that MSTCG is already strongly $\mathcal{NP}$-hard if the underlying conflict graph is a 3-ladder. In fact, $\mathcal{NP}$-hardness holds even if all edge weights are restricted to $\{0, 1\}$. On the other hand, it can be shown by a matroid intersection argument in Section 3 that the problem remains polynomially solvable for a 2-ladder as a conflict graph.

In contrast to the latter result, it should be noted that the shortest path problem with pairwise disjoint forbidden pairs of edges (i.e. with a 2-ladder conflict graph) is known to be strongly $\mathcal{NP}$-hard [GJ79]. Results of the same flavour were recently derived for the classical 0-1 knapsack problem with conflict graphs. While this problem is strongly $\mathcal{NP}$-hard for arbitrary conflict graphs, it was shown in [PS08] that pseudopolynomial algorithms (and hence also fully polynomial approximation schemes) exist if the given conflict graph is a tree, a graph of bounded treewidth or a chordal graph. Bin packing problems with special classes of conflict graphs were considered from an approximation point of view by [JO97] and [Jan98]. Complexity results for different classes of conflict graphs for a scheduling problem under makespan minimization are given in [BJ93]. Further references on combinatorial optimization problems with conflict graphs can be found in [PS08].

2 A strongly $\mathcal{NP}$-hardness result for MSTCG

In this section we show that MSTCG is already strongly $\mathcal{NP}$-hard if the conflict graph $\bar{G}$ is a 3-ladder. E.g. for $e_1, e_2, e_3 \in E$ let a component of $\bar{G}$ be made up of the path $(e_1 e_2 e_3)$. Then, in terms of the underlying graph G, a feasible spanning tree for MSTCG that contains e_2 must include neither edge e_1 nor edge e_3. However, a feasible tree that contains e_1 must not contain e_2, but it may contain e_3.

2.1 The graphs G_{3-SAT} and $\bar{G}_{3-SAT}$

We reduce an $\mathcal{NP}$-complete subproblem of 3-SAT on the decision problem corresponding to MSTCG with a 3-ladder as conflict graph.
Let I be an arbitrary instance of $3-SAT$ with k clauses $C_1, \ldots, C_k$ and n variables $x_1, \ldots, x_n$, such that each literal x_i occurs in ℓ_i clauses and its negation $\bar{x_i}$ occurs in $\bar{\ell}_i$ clauses. We restrict ourselves to instances with $\ell_i + \bar{\ell}_i \leq 5$. The decision problem whether or not there exists a satisfying truth assignment for I is $\mathcal{NP}$-complete [GJ79]. Considering this decision problem, we construct an instance of MSTCG by defining a graph G_{3-SAT} and a conflict graph $\bar{G}_{3-SAT}$ as described in the two following subsections.

2.1.1 Construction of G_{3-SAT}

Unless otherwise stated, each edge of G_{3-SAT} has zero weight. The graph G_{3-SAT} is being built as follows (see Figure 1): For each variable x_i, $1 \leq i \leq n$, we introduce

- the edges $x_i = (a_i, b_i)$ and $\bar{x}_i = (\bar{a}_i, \bar{b}_i)$ corresponding to the literals with the same label,
- a vertex i that is connected to b_i and $\bar{b}_i$ via the edges y_i and $\bar{y}_i$ respectively and

- one path of length ℓ_i starting in vertex i and ending in a_i. Let the edges of this path starting at i be called $w_{i0}, w_{i1}, w_{i2}, \ldots$. Each vertex of this path is connected to vertex b_i by the edges $z_{i1}, z_{i2}, \ldots$, where z_{ij} is adjacent to $w_{i(j-1)}$, $j \geq 1$. An analogous path is defined for i and $\bar{a}_i$ with the corresponding connections to $\bar{b}_i$.

For each clause C_j, $1 \leq j \leq k$,

- a vertex labelled C_j is introduced,
- for each x_i contained in clause C_j we insert a path of length 4 consisting of edges $(e_{ij}\, f_{ij}\, g_{ij}\, h_{ij})$ starting in vertex a_i and ending in vertex C_j. Furthermore, a shortcut is constructed by joining vertex a_i to the vertex incident to f_{ij} and g_{ij} via an edge Δ_{ij}.

  Analogously, the path $(\bar{e}_{ij}\, \bar{f}_{ij}\, \bar{g}_{ij}\, \bar{h}_{ij})$ connects $\bar{a}_i$ with C_j if literal $\bar{x}_i$ is contained in C_j with an analogous shortcut $\bar{\Delta}_{ij}$.

Finally, edges connecting the parts of G_{3-SAT} described above are introduced. For $1 \leq i \leq n-1$ we introduce an edge connecting vertex i with vertex $i+1$.

A weight of 1 is associated with all edges g_{ij} and $\bar{g}_{ij}$. Note that these are the only edges of non-zero weight in G_{3-SAT}.

2.1.2 Construction of $\bar{G}_{3-SAT}$

The conflict graph $\bar{G}_{3-SAT}$ on the edges of G_{3-SAT} is defined in the following way. Denote the clauses containing literal x_i resp. $\bar{x}_i$ by $C_{x_{i0}} \ldots C_{x_{i(\ell_i-1)}}$ resp. $C_{\bar{x}_{i0}} \ldots C_{\bar{x}_{i(\bar{l}_i-1)}}$, where the order in which theses clauses are chosen is arbitrary but fixed. Then we introduce in $\bar{G}_{3-SAT}$ the edge $(w_{i0}, f_{ix_{i0}})$ and the paths $(z_{i1}, w_{i1}, f_{ix_{i1}}), \ldots, (z_{i\ell_i-1}, w_{i\ell_i-1}, f_{ix_{i\ell_i-1}})$. Again we construct equivalent components of $\bar{G}_{3-SAT}$ for the clauses $C_{\bar{x}_{i0}} \ldots C_{\bar{x}_{i(\bar{\ell}_i-1)}}$ containing literal $\bar{x}_i$.

Furthermore we add the edges $(x_i, \bar{x}_i)$, the edges $(\Delta_{ix_{i0}}, g_{ix_{i0}}), \ldots, (\Delta_{ix_{i(\ell i-1)}}, g_{ix_{i(\ell_i-1)}})$ and the edges $(\bar{\Delta}_{i\bar{x}_{i0}}, \bar{g}_{i\bar{x}_{i0}}), \ldots, (\bar{\Delta}_{i\bar{x}_{i(\bar{\ell}_i-1)}}, \bar{g}_{i\bar{x}_{i(\bar{\ell}_i-1)}})$. This procedure is performed for all variables.

Remark 1 Note that $\bar{G}_{3-SAT}$ is not a 3-ladder. To be more precise, $\bar{G}_{3-SAT}$ consists of a subgraph being a 3-ladder, a subgraph which is made up of components consisting of a single edge, and of isolated vertices. However, by introducing "dummy edges" $\bar{G}_{3-SAT}$ can easily be transformed into a 3-ladder.

2.2 MSTCG with a 3-ladder conflict graph is strongly $\mathcal{NP}$-hard

Theorem 1 *Let $G = (V, E)$ be an undirected graph and let the conflict graph $\bar{G} = (E, \bar{E})$ be a 3-ladder. Then MSTCG is $\mathcal{NP}$-hard even if $w(e) \in \{0, 1\}\, \forall e \in E$.*

Proof.
Let I be an instance of 3-SAT where each variable occurs in at most 5 clauses. Let the instance I' of $MSTCG$ be defined by the graph G_{3-SAT} and the conflict graph $\bar{G}_{3-SAT}$ constructed as described in Section 2.1. Let τ be the set of feasible solutions of I'.
Let $T \in \tau$. It is easy to see that T must have a weight of at least k: The set $G_j := \{g_{qj} | 1 \leq q \leq n\} \cup \{\bar{g}_{qj} | 1 \leq q \leq n\}$ is a cut set, i.e. it separates the vertex C_j from the rest of the graph. Therefore, every spanning tree must contain at least one edge from G_j. Since each of the mentioned edges has a weight of 1 we have $w(T) \geq k$.
We proof the theorem by showing that the following equivalence holds:

$$\exists \text{ a satisfying truth assignment for } I \Longleftrightarrow \exists\, T \in \tau : w(T) \leq k$$

"$\Longrightarrow$": Given a satisfying truth assignment t_I for instance I we construct a feasible solution T of I' with $w(T) = k$. Let $T := \emptyset$ and let $X := \{x_{i_1}, \ldots, x_{i_r}\}$ and $\bar{X} = \{\bar{x}_{k_1}, \ldots, \bar{x}_{k_s}\}$ be the sets of literals set "TRUE" under t_I (recall that setting literal $\bar{x}_{k_j}$ "TRUE" means to set variable x_{k_j} "FALSE", $1 \leq j \leq s$).

$$T := T \cup X \cup \bar{X}$$

$$T := T \cup y_i \cup \bar{y}_i \quad \forall i \ \in \{1, \ldots, n\}$$
$$T := T \cup (i, i+1) \quad \forall i \ \in \{1, \ldots, n-1\}$$

Label all clauses C_j "unmarked". Let $C(x_i)$ (resp. $C(\bar{x}_i)$) be the set of all clauses containing x_i (resp. $\bar{x}_i$). We complete T by performing the following algorithmic statements:

for $i \in \{i_1, \ldots, i_r\}$:
    $T := T \cup \bar{w}_{i0}$
    for $j \in \{1, \ldots, \ell_i\}$:
        $T := T \cup z_{ij}$
    for $j' \in \{1, \ldots, \bar{\ell}_i\}$:
        $T := T \cup \bar{w}_{ij'}$
    for $C_j \ \in C(x_i)$:
        $T := T \cup \{e_{ij}, f_{ij}, h_{ij}\}$
        if C_j is "unmarked": (a)
            $T := T \cup \{g_{ij}\}$
            Label C_j "marked".

for $C_j \in C(\bar{x}_i)$:
$T := T \cup \{\bar{\Delta}_{ij}, \bar{e}_{ij}, \bar{h}_{ij}\}$

for $k \in \{k_1, \ldots, k_s\}$:
$T := T \cup w_{k0}$
for $j \in \{1, \ldots, \bar{\ell}_k\}$:
$T := T \cup \bar{z}_{kj}$
for $j' \in \{1, \ldots, \ell_k\}$:
$T := T \cup w_{kj'}$
for $C_j \in C(\bar{x}_k)$:
$T := T \cup \{\bar{e}_{kj}, \bar{f}_{kj}, \bar{h}_{kj}\}$
if C_j is "unmarked": (a)
$T := T \cup \{\bar{g}_{kj}\}$
Label C_j "marked".
for $C_j \in C(x_k)$:
$T := T \cup \{\Delta_{kj}, e_{kj}, h_{kj}\}$

It is easy to check that T is a feasible solution of I'. Since t_I is a satisfying truth assignment for I, each clause C_j contains a literal set "TRUE". As mentioned before, in order to reach a vertex C_j at least one edge with weight 1 has to be added to T. By (a) in the construction of T for each vertex C_j exactly one such edge is contained in T. Thus we get $w(T) = k$.

"$\Longleftarrow$": Let $T \in \tau$ with $w(T) = k$. Since each vertex C_j has to be reached, at least one edge $G_j = \{g_{qj} | 1 \leq q \leq n\} \cup \{\bar{g}_{qj} | 1 \leq q \leq n\}$ has to be contained in T, $1 \leq j \leq k$. However, since each edge of G_j has weight 1, exactly one edge of G_j is contained in T, $1 \leq j \leq k$. Recall that each vertex C_j corresponds to a clause with the same label. Let for vertex (clause) C_j the edge g_{ij} be the unique edge in T that is an element of G_j. We will now show that this implies $x_i \in T$ and $\bar{x}_i \notin T$:
Let γ be the vertex incident to both g_{ij} and f_{ij}. As T is a tree there has to be a unique simple path p in T between vertex γ and vertex 1. Due to the fact that g_{ij} is the only edge in T that is contained in the set G_j the path p cannot pass through vertex C_j. Analogously, p cannot pass through any of the vertices $C_r \neq C_j$, $1 \leq r \leq k$. Furthermore, the edge Δ_{ij} cannot be contained in p because the edge (Δ_{ij}, g_{ij}) is a component of the conflict graph $\bar{G}_{3-SAT}$. Thus $f_{ij}, e_{ij} \in p$ (and hence $f_{ij}, e_{ij} \in T$) has to hold.

Now assume $x_i \notin T$. Then in order to reach vertex 1 the path p must contain $w_{i(\ell_i-1)}$. Note that $w_{i(\ell_i-1)} \in T$ implies $z_{i(\ell_i-1)} \notin T$ because of $\bar{G}_{3-SAT}$. Analogously, we have $w_{ik} \in T$ and $z_{ik} \notin T$ for $1 \leq k \leq \ell_i - 2$ and $w_{i0} \in T$ as well. Thus we have $w_{ik} \in T$ for all $0 \leq k \leq \ell_i - 1$ which contradicts $f_{ij} \in T$ due to the construction of $\bar{G}_{3-SAT}$. Hence $x_i \in T$, and again by $\bar{G}_{3-SAT}$ we get $\bar{x_i} \notin T$.

Summarizing the facts we have that for each vertex C_j exactly one edge x_i, resp. $\bar{x}_i$, is contained in T that is connected to C_j by the path $(e_{ij}, f_{ij}, g_{ij}, h_{ij})$, resp.

$(\bar{e}_{ij}, \bar{f}_{ij}, \bar{g}_{ij}, \bar{h}_{ij})$. That is, for each clause at least one of the literals the clause is made up of is contained in the tree. Thus, the truth assignment that sets a variable x_i “TRUE” if $x_i \in T$ and x_i “FALSE” if $\bar{x}_i \in T$ constitutes a satisfying truth assignment for I. □

Since MSTCG is strongly $\mathcal{NP}$-hard given the conflict graph is a 3-ladder, MSTCG is also strongly $\mathcal{NP}$-hard in case the conflict graph is a path. Finally both results obviously imply that MSTCG is strongly $\mathcal{NP}$-hard for general conflict graphs.

Corollary 1 *Given the conflict graph is a path, MSTCG is strongly $\mathcal{NP}$-hard.*

Corollary 2 *MSTCG is strongly $\mathcal{NP}$-hard.*

3 MSTCG with disjoint conflicting pairs of edges is in $\mathcal{P}$

In this section we focus on the MSTCG where the conflict graph is a 2-ladder, i.e. the conflict graph represents pairwise disjoint forbidden pairs of edges of E. We will first give a representation of the set τ of feasible solutions of this problem by using matroid intersection. With the help of that representation, Edmonds' famous matroid-intersection theorem (Edmonds [Edm79], cf. [Sch03]) yields that an optimal solution of MSTCG with disjoint conflicting pairs of edges can be computed in polynomial time.

Let us consider the ILP-formulation of our problem. Condition (3) of this formulation induces the well-known *graphic-matroid*. To be more precise, the graphic matroid $\mathcal{M}_1 = (E, \mathcal{I}_1)$ is being formed by all subsets of E which do not form a cycle [Sch03]. More formally, $\mathcal{I}_1$ is the set of all subsets of E such that (3) is satisfied. Obviously a base of $\mathcal{M}_1$ consists of $n-1$ edges (a base corresponds to a spanning tree) and thus (2) is satisfied by all bases of $\mathcal{M}_1$. However, the fact that the conflict graph $\bar{G}$ is being made up of disjoint conflicting pairs of edges of G induces a matroid as well. As we will show below, defining $\mathcal{I}_2$ as the set of all subsets of E that do not contain a conflicting pair of edges (represented by $\bar{G}$) yields a matroid $\mathcal{M}_2 = (E, \mathcal{I}_2)$. This matroid will be called *conflict-free matroid*. More formally, the conflict-free matroid $\mathcal{M}_2 = (E, \mathcal{I}_2)$ is defined by

$$\mathcal{I}_2 := \{E' \subseteq E | \nexists (e, f) \in \bar{E} : \{e, f\} \subseteq E').$$

Lemma 1 *$\mathcal{M}_2 = (E, \mathcal{I}_2)$ is a matroid.*

Proof.
Obviously, $\emptyset \in \mathcal{I}_2$ is satisfied.
Let J be an element of $\mathcal{I}_2$ and I any subset of J. Then I cannot include a conflicting pair as J does not and hence $I \in \mathcal{I}_2$ holds. Thus $J \in \mathcal{I}_2$ implies

$I \in \mathcal{I}_2$ for all $I \subseteq J$.
Let $I, J \in \mathcal{I}_2$ and $|I| < |J|$, then we have to show that $(I \cup \{z\}) \in \mathcal{I}_2$ for some $z \in J \setminus I$:

- *Case 1:* $I \subset J$. This case is trivial.
- *Case 2:* $I \cap J = \emptyset$. Assume that no such edge z exists: Then for every edge $e \in J$ the set $(I \cup \{e\})$ is not in $\mathcal{I}_2$. This implies that there exists an edge $e^c \in I$ such that (e, e^c) is a conflicting pair in $(I \cup \{e\})$. But since $\bar{G}$ is a 2-ladder for each edge $e' \in E$ there is at most one edge that is in conflict with e'. Hence, every edge $e^c \in I$ can belong to only one conflicting pair (e, e^c) with $e \in J$. Thereby, we get a contradiction to $|I| < |J|$.
- *Case 3:* $I \cap J \neq \emptyset$. Let $I' := I \setminus (I \cap J)$ and $J' := J \setminus (I \cap J)$. Then we have $I', J' \in \mathcal{I}_2$, $|I'| < |J'|$ and $I' \cap J' = \emptyset$. From *Case 2* we get that there is a $z \in J'$ such that $(I' \cup \{z\}) \in \mathcal{I}_2$. It follows from $z \in J$ that z can not be in conflict with any edge in $I \cap J$ and hence $(I \cup \{z\}) \in \mathcal{I}_2$.

□

Clearly, any feasible solution of MSTCG with disjoint conflicting pairs of edges corresponds to a common base of the graphic matroid and the conflict-free matroid. Thus, an optimal solution of MSTCG is a common base of $\mathcal{M}_1$ and $\mathcal{M}_2$ with minimum weight. As a consequence, Edmonds' weighted matroid intersection algorithm gives the answer on the question of the computational complexity of MSTCG with disjoint conflicting pairs of edges.

Theorem 2 *(Edmonds [Edm79], cf. [Sch03])*
Let S be a set and let $c : S \to \mathbb{R}$. Given two matroids $M_1 = (S, I_1)$ and $M_2 = (S, I_2)$, a common base of M_1 and M_2 with minimum weight can be found in strongly polynomial time.

Since any optimal solution of MSTCG corresponds to a minimum-weight common base of the graphic matroid and the conflict-free matroid the above theorem yields the following result.

Theorem 3 *MSTCG with disjoint conflicting pairs of edges can be solved in strongly polynomial time.*

References

[BJ93] H. L. Bodlaender and K. Jansen. On the complexity of scheduling incompatible jobs with unit-times. In *MFCS '93: Proceedings of the 18th International Symposium on Mathematical Foundations of Computer Science*, pages 291–300, London, UK, 1993. Springer.

[Edm79] J. Edmonds. Matroid intersection. *Annals of Discrete Mathematics*, 4:39–49, 1979.

[GJ79] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman & Co., 1979.

[Jan98] K. Jansen. An approximation scheme for bin packing with conflicts. In *SWAT '98: Proceedings of the 6th Scandinavian Workshop on Algorithm Theory*, pages 35–46, London, UK, 1998. Springer.

[JO97] K. Jansen and S. Öhring. Approximation algorithms for time constrained scheduling. *Information and Copmutation*, 132(2):85–108, 1997.

[PS08] U. Pferschy and J. Schauer. The knapsack problem with conflict graphs. Optimization Online 2008-10-2128, 2008.

[Sch03] A. Schrijver. *Combinatorial Optimization, Polyhedra and efficiency*, volume B. Springer, 2003.

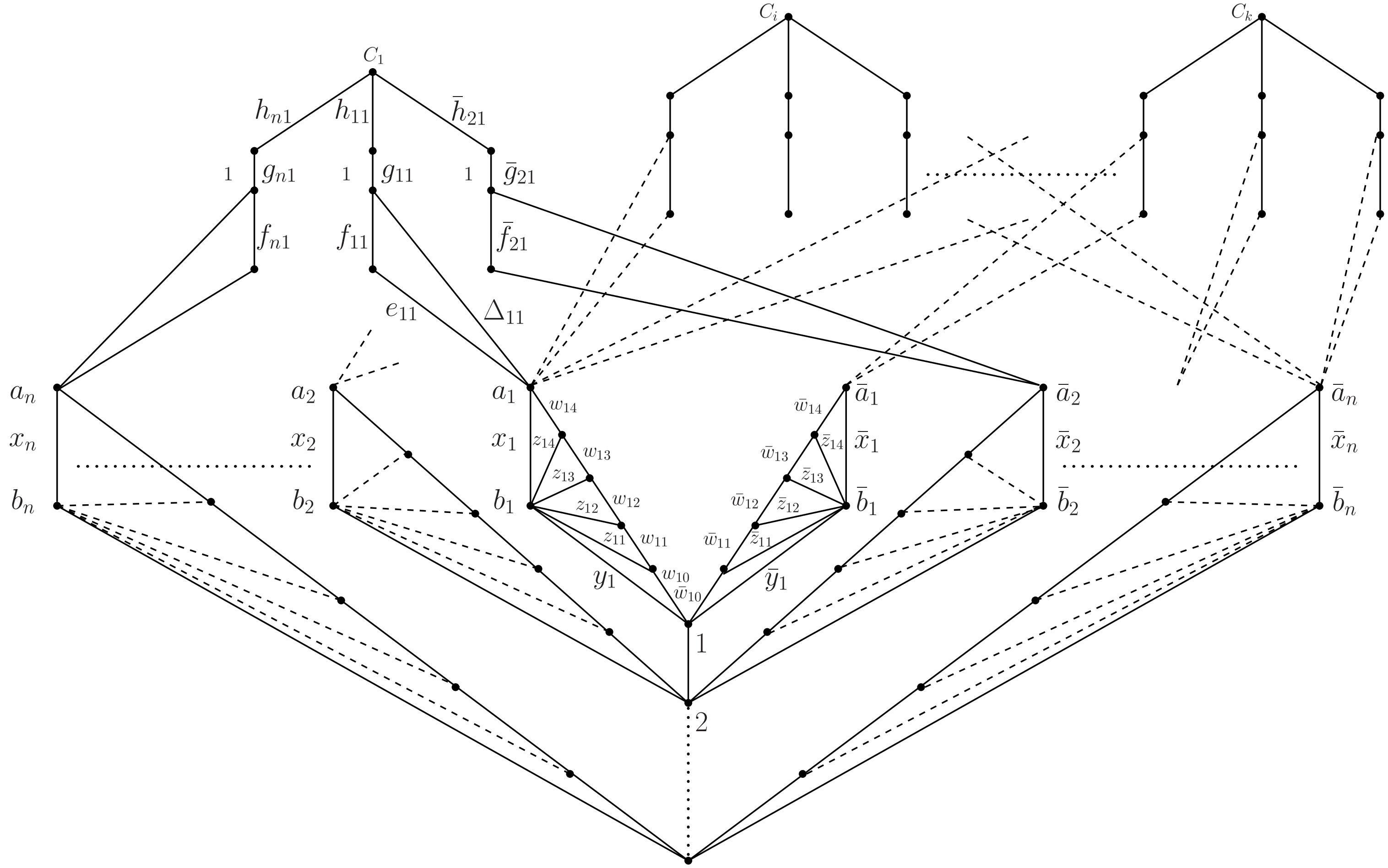

Figure 1: The graph G_{3-SAT}