

# A SECOND DERIVATIVE SQP METHOD : LOCAL CONVERGENCE \*

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**Abstract.** Gould and Robinson (NAR 08/18, Oxford University Computing Laboratory, 2008) gave global convergence results for a second-derivative SQP method for minimizing the exact  $\ell_1$ -merit function for a *fixed* value of the penalty parameter. To establish this result, we used the properties of the so-called Cauchy step, which was itself computed from the so-called predictor step. In addition, we allowed for the computation of a variety of (optional) SQP steps that were intended to improve the efficiency of the algorithm.

Although we established global convergence of the algorithm, we did not discuss certain aspects that are critical when developing software capable of solving general optimization problems. In particular, we must have strategies for updating the penalty parameter and better techniques for defining the positive-definite matrix  $B_k$  used in computing the predictor step. In this paper we address both of these issues. We consider two techniques for defining the positive-definite matrix  $B_k$ —a simple diagonal approximation and a more sophisticated *limited*-memory BFGS update. We also analyze a strategy for updating the penalty parameter based on *approximately* minimizing the  $\ell_1$ -penalty function over a sequence of increasing values of the penalty parameter.

Algorithms based on exact penalty functions have certain desirable properties. To be practical, however, these algorithms must be guaranteed to avoid the so-called Maratos effect. We show that a *nonmonotone* variant of our algorithm avoids this phenomenon and, therefore, results in asymptotically superlinear local convergence; this is verified by preliminary numerical results on the Hock and Shittkowski test set.

**Key words.** Nonlinear programming, nonlinear inequality constraints, sequential quadratic programming,  $\ell_1$ -penalty function, nonsmooth optimization

**AMS subject classifications.** 49J52, 49M37, 65F22, 65K05, 90C26, 90C30, 90C55

**1. Introduction.** In [19], we presented a sequential inequality/equality constrained quadratic programming algorithm (an SIQP/SEQP “hybrid”) for solving the problem

$$(\ell_1\text{-}\sigma) \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \phi(x) = f(x) + \sigma \| [c(x)]^- \|_1,$$

where the constraint vector  $c(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and the objective function  $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$  are assumed to be twice continuously differentiable,  $\sigma$  is a positive scalar known as the penalty parameter, and we have used the notation  $[v]^- = \min(0, v)$  for a generic vector  $v$  (the minimum is understood to be component-wise). The motivation for solving this problem is that solutions of problem  $(\ell_1\text{-}\sigma)$  correspond (under certain assumptions) to solutions of the nonlinear programming problem

$$(\text{NP}) \quad \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \quad \text{subject to} \quad c(x) \geq 0.$$

An outline of this paper is as follows. In Section 2 we provide methods for defining the positive-definite matrix associated with the so-called predictor step subproblem, while in Section 3 we discuss a strategy for updating the penalty parameter. In Section 4 we discuss the local convergence properties of a *nonmonotone* variant of the algorithm described in [19] that culminates with two rate-of-convergence results. The first applies when the so-called SQP step is computed from an *equality* constrained subproblem based on the predictor step, while the second applies when the SQP step is computed from an *inequality* constrained subproblem based on the so-called Cauchy step [19, Sections 2.3.1 and 2.3.2]. Finally, in Section 5, we provide preliminary numerical results for the proposed algorithm.

Before proceeding, we catalogue essential notation and provide an outline of the algorithm presented in [19]. The outline is relatively brief and, therefore, we recommend a careful reading of [19] since this paper is essentially a continuation of that work.

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**1.1. Notation and definitions.** Most of our notation is standard. We let  $e$  denote the vector of all ones whose dimension is determined by the context. A local solution of (NP) is denoted by  $x^*$ ;  $g(x)$  is the gradient of  $f(x)$ , and  $H(x)$  its (symmetric) Hessian; the matrix  $H_j(x)$  is the Hessian of  $c_j(x)$ ;  $J(x)$  is the  $m \times n$  Jacobian matrix of the constraints with  $i$ th row  $\nabla c_i(x)^T$ . The Lagrangian function associated with (NP) is  $\mathcal{L}(x, y) = f(x) - y^T c(x)$ . The Hessian of the Lagrangian with respect to  $x$  is  $\nabla_{xx}^2 \mathcal{L}(x, y) = H(x) - \sum_{j=1}^m y_j H_j(x)$ .

For a general vector  $v$ , the notation  $[v]^- = \min(0, v)$  is used, where the minimum is understood to be component-wise, and  $\text{diag}(v)$  represents a diagonal matrix whose  $i$ th diagonal entry is  $v_i$ ; given two general vectors  $v$  and  $w$ , the notation  $v \cdot w$  represents the vector whose  $i$ th component is  $v_i w_i$ ; given a general symmetric matrix  $A$  the notation  $A \succeq \lambda$  means that the smallest eigenvalue of  $A$  is bigger than or equal to  $\lambda$ ; and given a set of matrices  $A_1, A_2, \dots, A_p$  for some  $p \geq 1$ , we define  $\text{diag}(A_1, A_2, \dots, A_p)$  to be the block-diagonal matrix whose  $i$ th block is  $A_i$ .

Given a vector  $v \in \mathbb{R}^n$  and scalar  $\varepsilon$  we define  $\mathcal{B}_\varepsilon(v) = \{x \in \mathbb{R}^n : \|x - v\|_2 < \varepsilon\}$  and  $\bar{\mathcal{B}}_\varepsilon(v) = \{x \in \mathbb{R}^n : \|x - v\|_2 \leq \varepsilon\}$  to be the open and the closed ball centered at  $v$  of radius  $\varepsilon$ , respectively.

We often consider problem functions evaluated at a specific point  $x_k$ . To simplify notation we define the following:  $f_k = f(x_k)$ ,  $c_k = c(x_k)$ ,  $g_k = g(x_k)$  and  $J_k = J(x_k)$ . Given a pair of values  $(x_k, y_k)$ , we let  $H_k$  and  $B_k$  denote symmetric approximations to  $H(x_k, y_k)$  in which  $B_k$  is required additionally to be positive definite. Similar notation is used for a solution  $x^*$ ; we define  $f^* = f(x^*)$ ,  $c^* = c(x^*)$ ,  $g^* = g(x^*)$ , and  $J^* = J(x^*)$ . When a primal-dual solution  $(x^*, y^*)$  is given, we define  $H^* = H(x^*, y^*)$ .

Given the symmetric matrix  $H_k$ , we define

$$M_k^H(s) \stackrel{\text{def}}{=} M_k^H(s; x_k) = f_k + g_k^T s + \frac{1}{2} s^T H_k s + \sigma \| [c_k + J_k s]^- \|_1 \quad (1.1)$$

to be the *faithful* model of  $\phi$  and

$$\Delta M_k^H(s) \stackrel{\text{def}}{=} \Delta M_k^H(s; x_k) = M_k^H(0; x_k) - M_k^H(s; x_k) \quad (1.2)$$

to be the *change* in the faithful model. We mention that this notation does not allude to their dependence on the penalty parameter  $\sigma$ .

Given a solution  $x^*$  to problem (NP), we use the indexing sets  $\mathcal{A} \stackrel{\text{def}}{=} \{i : c_i(x^*) = 0\}$  and  $\mathcal{I} \stackrel{\text{def}}{=} \{i : c_i(x^*) > 0\}$ , which are the set of active and inactive constraints, respectively, at  $x^*$ . Given a generic vector  $v$ , a generic matrix  $V$ , and a generic indexing set  $S$ , the notation  $v_S$  and  $V_S$  will denote the rows of  $v$  and  $V$  that correspond to the indices in  $S$ ; if  $v$  and  $V$  are functions of  $x$ , then we sometimes write  $v_S(x)$  and  $V_S(x)$  instead of  $[v(x)]_S$  and  $[V(x)]_S$ .

We use the following definitions related to a solution of problem (NP).

**DEFINITION 1.1** (First-order KKT point). *We say that the point  $(x^*, y^*)$  is a first-order KKT point for problem (NP) if*

$$g^* - J^{*T} y^* = 0, \quad c^* \geq 0, \quad y^* \geq 0, \quad \text{and} \quad c^* \cdot y^* = 0. \quad (1.3)$$

**DEFINITION 1.2** (Second-order sufficient conditions). *A point  $(x^*, y^*)$  satisfies the second-order sufficient conditions for problem (NP) if  $(x^*, y^*)$  is a first-order KKT point and if there exists  $\lambda_{\min}^H > 0$  such that  $s^T H^* s \geq \lambda_{\min}^H s^T s$  for all  $s$  satisfying  $J_A^* s = 0$ .*

**DEFINITION 1.3** (Strict complementarity). *We say that strict complementarity holds at a KKT point  $(x^*, y^*)$  for problem (NP) if  $y_A^* > 0$ .*

**DEFINITION 1.4** (Linear independent constraint qualification). *We say that the linear independent constraint qualification (LICQ) holds at a KKT point  $(x^*, y^*)$  for problem (NP) if  $J_A^*$  has full row rank.*

**DEFINITION 1.5.** *We say that the strong second-order sufficient conditions hold at a point  $(x^*, y^*)$  if it satisfies Definitions 1.1 – 1.4.*

**1.2. Algorithm Overview.** We now give a brief description of the algorithm we proposed in [19] for minimizing problem  $(\ell_1\text{-}\sigma)$ ; the algorithm has been restated as Algorithm 1.1. First, the user supplies an initial guess  $(x_0, y_0)$  of a solution to problem  $(\ell_1\text{-}\sigma)$ . Next, “success” parameters  $0 < \eta_s \leq \eta_{vs} < 1$ , a maximum allowed predictor trust-region radius  $\Delta_U$ , expansion and contraction factors  $0 < \eta_c < 1 < \eta_e$ , sufficient model decrease and approximate Cauchy point tolerances  $0 < \eta \leq \eta_{ACP} < 1$ , and SQP trust-region radius factor  $\tau_f$  are defined. With parameters set, the main “do-while” loop begins. First, the problem functions are evaluated at the current point  $(x_k, y_k)$ . Next, a symmetric positive-definite matrix  $B_k$  is defined and the predictor step  $s_k^P$  is computed as a solution to

$$\underset{s \in \mathbb{R}^n}{\text{minimize}} \quad f_k + g_k^T s + \frac{1}{2} s^T B_k s + \sigma \| [c_k + J_k s]^- \|_1 \quad \text{subject to} \quad \|s\|_\infty \leq \Delta_k^P. \quad (1.4)$$

By introducing elastic variables [15], this problem is equivalent to

$$\underset{s \in \mathbb{R}^n}{\text{minimize}} \quad f_k + g_k^T s + \frac{1}{2} s^T B_k s + \sigma e^T v \quad \text{subject to} \quad c_k + J_k s + v \geq 0, \quad v \geq 0, \quad \|s\|_\infty \leq \Delta_k^P. \quad (1.5)$$

Strategies for defining the positive-definite matrix  $B_k$  are discussed in Section 2. Next, we define the Cauchy step as  $s_k^{CP} = \alpha_k s_k^P$ , where  $\alpha_k$  is the solution to

$$\underset{0 \leq \alpha \leq 1}{\text{minimize}} \quad M_k^H(\alpha s_k^P), \quad (1.6)$$

and then compute  $\Delta M_k^H(s_k^{CP})$ . We then have the option of computing an SQP step  $s_k^S$  as the solution of any of the subproblems discussed in [19, Section 3.2]. The trial step computation is completed by defining the full step  $s_k$  so that the condition

$$\Delta M_k^H(s_k) \geq \eta \Delta M_k^H(s_k^{CP}) \quad (1.7)$$

is satisfied for some constant  $0 < \eta < 1$  independent of  $k$  (see [19, Section 2.3] for more details). Next, we evaluate  $\phi(x_k + s_k)$  and  $\Delta M_k^H(s_k)$  and compute the ratio  $r_k$  of actual versus predicted decrease in the merit function.

The strategy for updating the trust-region radii and for accepting or rejecting candidate steps is identical to that used by Fletcher [12] and is determined by the ratio  $r_k$  (except we have the added responsibility of updating the SQP trust-region radius). More precisely, if the ratio  $r_k$  is larger than  $\eta_{vs}$ , then we believe that the model is a very accurate representation of the merit function within the current trust-region; therefore, we increase the predictor trust-region radius with the belief that the current trust-region radius may be overly restrictive. If the ratio is greater than  $\eta_s$ , then we believe the model is sufficiently accurate and we keep the predictor trust-region radius fixed. Otherwise, the ratio indicates that there is poor agreement between the model  $M_k^H$  and the merit function and, therefore, we decrease the predictor trust-region radius with the hope that the model will accurately capture the behavior of the merit function over the smaller trust-region. As for step acceptance or rejection, we accept any iterate for which  $r_k$  is positive, since this indicates that the merit function has decreased. We note that the update used for the dual vector  $y_{k+1}$  is not important for proving global convergence, so no specific update is provided. However, the update to  $y_k$  is crucial when considering local convergence and the multiplier vector from the SQP subproblem is the most obvious candidate. We consider this further in Section 4. Finally, the SQP trust-region radius is defined to be a constant multiple of the predictor trust-region radius, although the condition  $\Delta_{k+1}^S \leq \tau_f \cdot \Delta_{k+1}^P$  for some constant  $\tau_f$  is also sufficient.

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**Algorithm 1.1.** Minimizing the  $\ell_1$ -penalty function

Input:  $(x_0, y_0)$

Set parameters  $0 < \eta_s \leq \eta_{vs} < 1$ ,  $\Delta_U > 0$ ,  $0 < \eta \leq \eta_{ACP} < 1$ , and  $\tau_f > 0$ .

Set expansion and contraction factors  $0 < \eta_c < 1 < \eta_e$ .

```

k ← 0
do
  Evaluate  $f_k, g_k, c_k, J_k$ , and then compute  $\phi_k$ .
  Define  $B_k$  and  $H_k$  to be symmetric approximations to  $H(x_k, y_k)$  with  $B_k$  positive definite.
  Solve problem (1.4) for  $s_k^p$ .
  Solve problem (1.6) for  $s_k^{CP}$  and compute  $\Delta M_k^H(s_k^{CP})$ .
  Possibly compute an SQP step  $s_k^S$ .
  Define a full step  $s_k$  that satisfies condition (1.7).
  Evaluate  $\phi(x_k + s_k)$  and  $\Delta M_k^H(s_k)$ .
  Compute  $r_k = (\phi_k - \phi(x_k + s_k)) / \Delta M_k^H(s_k)$ .
  if  $r_k \geq \eta_{vs}$  [very successful]
     $\Delta_{k+1}^p \leftarrow \min(\eta_e \cdot \Delta_k^p, \Delta_U)$  [increase predictor radius]
  else if  $r_k \geq \eta_s$  [successful]
     $\Delta_{k+1}^p \leftarrow \Delta_k^p$  [keep predictor radius]
  else [unsuccessful]
     $\Delta_{k+1}^p \leftarrow \eta_c \cdot \Delta_k^p$  [decrease predictor radius]
  end
  if  $r_k > 0$  [accept step]
     $x_{k+1} \leftarrow x_k + s_k$ 
     $y_{k+1} \leftarrow$  arbitrary
  else [reject step]
     $x_{k+1} \leftarrow x_k$ 
     $y_{k+1} \leftarrow y_k$ 
  end
   $\Delta_{k+1}^s \leftarrow \tau_f \cdot \Delta_{k+1}^p$  [update SQP radius]
   $k \leftarrow k + 1$ 
end do

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The following global convergence result applies to Algorithm 1.1.

**THEOREM 1.6.** [19, Theorem 4.1] *Let  $f$  and  $c$  be twice continuously differentiable functions, and let  $\{x_k\}$ ,  $\{H_k\}$ ,  $\{B_k\}$ ,  $\{\Delta_k^p\}$ , and  $\{\Delta_k^s\}$ , be sequences generated by Algorithm 1.1. Assume that the following conditions hold:*

1.  $\{x_k\}_{k \geq 0} \subset \mathcal{B} \subset \mathbb{R}^n$ , where  $\mathcal{B}$  is a closed and bounded set; and
2. there exists positive constants  $b_B$  and  $b_H$  such that  $\|B_k\|_2 \leq b_B$  and  $\|H_k\|_2 \leq b_H$  for all  $k \geq 0$ .

*Then, either  $x_K$  is a first-order point for problem  $(\ell_1\text{-}\sigma)$  for some  $K \geq 0$ , or there exists a subsequence of  $\{x_k\}$  that converges to a first-order solution of problem  $(\ell_1\text{-}\sigma)$ .*

**2. Defining  $B_k$ .** The definition of the positive-definite matrix  $B_k$  is critical in developing an efficient algorithm. In Section 2.1 we consider defining  $B_k$  as a diagonal matrix. Although this approach is simple to implement and cheap to compute, it can not be expected to perform well in general. A more promising idea is to *update*  $B_k$  using the well-known BFGS formula. This is an appealing alternative, but we must be cautious since the matrix  $H(x_k, y_k)$  is generally indefinite and, therefore, the traditional BFGS update may result in an indefinite matrix [2,33]. In Section 2.2 we consider a *limited-memory* BFGS update since we are interested in solving large-scale problems.

**2.1. A diagonal approximation.** Given scalars  $\nu_i > 0$  for  $i = 1 : n$ , we define the diagonal matrix  $B_{k+1} = \text{diag}(\nu_1, \nu_2, \dots, \nu_n)$ . Possibly the simplest choice is

$$\nu_i = \max \left( \left| \left( s_k^T H(x_k, y_k) s_k \right) / \left( s_k^T s_k \right) \right|, \varepsilon \right) \quad (2.1)$$

for all  $i$ , where  $\varepsilon$  is a small pre-defined positive constant ( $\varepsilon$  has this meaning for the remainder of this section). This strategy approximates the magnitude of the curvature of  $H(x_k, y_k)$  in the previous direction  $s_k$ .

A second possibility is to utilize more of the matrix  $H(x_k, y_k)$ . Given any value  $0 \leq r \leq n$  we may define

$$\nu_i = \max \left( \frac{1}{r_u - r_l + 1} \sum_{j=r_l}^{r_u} |[H(x_k, y_k)]_{ij}|, \varepsilon \right), \quad (2.2)$$

where  $r_l = \max(i - r, 1)$  and  $r_u = \min(n, i + r)$ . In other words,  $\nu_i$  is the average of the absolute values of the elements of  $H(x_k, y_k)$  in row  $i$  within band-width  $r$ . We note that if  $r = 0$ , then the curvature of  $B_{k+1}$  and  $H(x_k, y_k)$  will agree in those standard co-ordinate directions for which  $H(x_k, y_k)$  is sufficiently positive definite as determined by the parameter  $\varepsilon$ .

**2.2. A limited-memory BFGS update.** Symmetric positive-definite approximations based on equations (2.1) and (2.2) are cheap to compute, but can not be expected to approximate  $H(x_k, y_k)$  very well. An attractive alternative is to define the matrix  $B_{k+1}$  from the positive-definite matrix  $B_k$  by using a limited-memory quasi-Newton BFGS update. This approach uses a fixed number of vectors, say  $l$ , to define a positive-definite approximation to  $H(x_k, y_k)$  based on the most recent  $l$  iterations (for more details see [2, 28]). If we define  $d_k = \nabla_x \mathcal{L}(x_k + s_k, y_{k+1}) - \nabla_x \mathcal{L}(x_k, y_{k+1})$ , then we may write the update as

$$B_k = B_k^0 + \sum_{i=k-l}^{k-1} (q_i q_i^T - p_i p_i^T), \quad (2.3)$$

where  $B_k^0$  denotes *any* initial positive-definite approximation to  $H(x_k, y_k)$  and

$$p_i = \frac{B_i s_i}{(s_i^T B_i s_i)^{1/2}}, \quad q_i = \frac{d_i}{(d_i^T s_i)^{1/2}}, \quad \text{and} \quad B_i = B_k^0 + \sum_{j=k-l}^{i-1} (q_j q_j^T - p_j p_j^T). \quad (2.4)$$

Note that in these definitions we have assumed that  $k \geq l - 1$  so that there are  $l$  vectors to use. This formula is relatively simple, but one must be careful. It is tempting to store the vector-pairs  $(p_i, q_i)$ . However, as equation (2.4) illustrates, the vector  $p_i$  is defined from  $B_i$  and the matrix  $B_i$  changes from iteration to iteration since the ‘‘oldest’’ vector-pair  $(s_i, d_i)$  is removed from the set of  $l$  vector-pairs. Hence, the vector  $p_i$  must be recomputed at each iteration. The relationships given by equation (2.4) suggest how this may be done since

$$B_i s_i = B_k^0 s_i + \sum_{j=k-l}^{i-1} [(q_j^T s_i) q_j - (p_j^T s_i) p_j]. \quad (2.5)$$

Algorithm 2.1, which is [28, Procedure 7.6], computes the vector-pair  $(p_i, q_i)$  recursively.

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**Algorithm 2.1.** Computing the vector-pairs  $(p_i, q_i)$

**for**  $i = k - l, k - l + 1, \dots, k - 1$

$$q_i \leftarrow d_i / (d_i^T s_i)^{1/2}$$

$$p_i \leftarrow B_k^0 s_i + \sum_{j=k-l}^{i-1} [(q_j^T s_i) q_j - (p_j^T s_i) p_j]$$

$$p_i \leftarrow p_i / (s_i^T p_i)^{1/2}$$

**end (for)**

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During the  $k$ th iteration, Algorithm 2.1 computes the values  $q_i$  for  $k - l \leq i \leq k - 1$  and  $q_j^T s_i$  for all  $k - l \leq j \leq i - 1$ . However, since  $q_i$  only depends on the data  $(d_i, s_i)$ , only the value  $q_{k-1}$  and values  $q_j^T s_{k-1}$  ( $k - l \leq j \leq k - 2$ ) need to be computed (the other quantities should be stored from previous iterations).

Once the vector-pairs  $(p_i, q_i)$  have been computed, we set  $B_k = B_k^0 - PP^T + QQ^T$  where we have defined  $P = [p_{k-l} \ p_{k-l+1} \ \dots \ p_{k-1}]$  and  $Q = [q_{k-l} \ q_{k-l+1} \ \dots \ q_{k-1}]$ . The predictor subproblem (1.5) then becomes

$$\begin{aligned} & \underset{s \in \mathbb{R}^n, v \in \mathbb{R}^m}{\text{minimize}} && f_k + g_k^T s + \frac{1}{2} s^T (B_k^0 - PP^T + QQ^T) s + \sigma e^T v \\ & \text{subject to} && c_k + J_k s + v \geq 0, \quad v \geq 0, \quad \|s\|_\infty \leq \Delta_k^P. \end{aligned} \quad (2.6)$$

If we define the  $2l$  extra variables

$$w_a = P^T s \quad \text{and} \quad w_b = Q^T s, \quad (2.7)$$

then problem (2.6) is equivalent to

$$\begin{aligned} & \underset{s, v, w_a, w_b}{\text{minimize}} && f_k + g_k^T s + \frac{1}{2} (s^T B_k^0 s - w_a^T w_a + w_b^T w_b) + \sigma e^T v \\ & \text{subject to} && c_k + J_k s + v \geq 0, \quad P^T s = w_a, \quad Q^T s = w_b, \quad v \geq 0, \quad \|s\|_\infty \leq \Delta_k^P. \end{aligned} \quad (2.8)$$

As a function of  $(s, v, w_a, w_b)$ , the Hessian associated with subproblem (2.8) is given by

$$B_k^A = \begin{pmatrix} B_k^0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}, \quad (2.9)$$

which is *not* positive definite. This may seem strange since problem (2.8) is equivalent to the *strictly convex* QP (2.6) (assuming that the updated matrix was positive definite). However, if the current iterate is feasible for subproblem (2.8), then any step that maintains linear feasibility is guaranteed to be a direction of positive curvature even though  $B_k^A$  is indefinite. To see this, suppose that  $(s, v, w_a, w_b)$  is a feasible point so that  $w_a = P^T s$  and  $w_b = Q^T s$ . Furthermore, suppose that  $P^T(s + \Delta s) = w_a + \Delta w_a$  and  $Q^T(s + \Delta s) = w_b + \Delta w_b$ . Simplification yields  $P^T \Delta s = \Delta w_a$  and  $Q^T \Delta s = \Delta w_b$ . It then follows that

$$\begin{aligned} (\Delta s, \Delta v, \Delta w_a, \Delta w_b)^T B_k^A (\Delta s, \Delta v, \Delta w_a, \Delta w_b) &= \Delta s^T B_k^0 \Delta s - \Delta w_a^T \Delta w_a + \Delta w_b^T \Delta w_b \\ &= \Delta s^T B_k^0 \Delta s - \Delta s^T P P^T \Delta s + \Delta s^T Q Q^T \Delta s \\ &= \Delta s^T (B_k^0 - P P^T + Q Q^T) \Delta s \\ &= \Delta s^T B_k \Delta s > 0, \end{aligned}$$

since  $B_k$  is positive definite by construction. A great advantage in using subproblem (2.8) is that the Hessian matrix has essentially the same sparsity as  $B_k^0$ . In contrast, the Hessian matrix associated with subproblem (2.6) is generally dense since it uses a sum of rank-1 updates. Note, however, that the  $2l$  extra constraints (2.7) are generally dense; fortunately a limited number of dense constraints can be accommodated easily by modern sparse QP solvers such as QPA and QPB from the GALAHAD library [18].

Until this point we have assumed that the limited-memory BFGS update results in a positive-definite matrix. However, it is well-known that this is true if and only if the quantity  $d_k^T s_k > 0$  and this is not guaranteed to hold. We say that the vector-pair  $(s_k, d_k)$  will result in a sufficiently positive-definite update if

$$s_k^T d_k \geq \eta_{\text{SPD}} s_k^T B_k s_k \quad (2.10)$$

for some positive scalar  $0 < \eta_{\text{SPD}} < 1$ . Since we want to reserve the notation  $s_k$  for the solution of  $k$ th iterate of Algorithm 1.1, we use the notation  $(\bar{s}_k, \bar{d}_k)$  to denote the (possibly) modified values of  $(s_k, d_k)$  that satisfy condition (2.10).

When the estimate  $d_k^T s_k \geq \eta_{\text{SPD}} s_k^T B_k s_k$  fails, the simplest course of action is to skip the update. Since we do not advocate this option, we use a *damping* technique introduced by Powell [33], which is guaranteed to produce acceptable values for  $\bar{d}_k$  and  $\bar{s}_k$  by perturbing  $d_k$  if necessary. This is accomplished by defining the “damped” vectors

$$\bar{d}_k = \theta_k d_k + (1 - \theta_k) B_k s_k, \quad (2.11a)$$

$$\bar{s}_k = s_k, \quad (2.11b)$$

where the *damping factor*  $\theta_k$  is defined as

$$\theta_k = \begin{cases} 1 & \text{if } s_k^T d_k \geq \eta_{\text{SPD}} s_k^T B_k s_k, \\ (1 - \eta_{\text{SPD}}) s_k^T B_k s_k / (s_k^T B_k s_k - s_k^T d_k) & \text{if } s_k^T d_k < \eta_{\text{SPD}} s_k^T B_k s_k. \end{cases} \quad (2.12)$$

If  $\theta_k \neq 1$ , it can be verified by computation that  $\bar{s}_k^T \bar{d}_k = \eta_{\text{SPD}} s_k^T B_k s_k$ . Note that if  $\theta_k = 0$ , then  $\bar{d}_k = B_k s_k$ ,  $p_k = q_k$ , and  $B_{k+1} = B_k$ .

We finish this section by briefly mentioning other strategies that could be implemented. First, we could approximate the *reduced* Hessian of the Lagrangian since it is this matrix that is known to be positive definite at a minimizer satisfying the second-order sufficient conditions (see Gill, Murray and Saunders [15]). It will be shown in Section 4 that if  $x^*$  is a solution to problem (NP) that satisfies the strong-second order sufficient conditions, then the active constraints at  $x^*$  will ultimately be identified by the predictor step  $s_k^p$ . If one examines the various SQP subproblems discussed in [19, Section 2], then it is reasonable to expect that the solution of the  $k$ th SQP subproblem will be “close enough” to the null space of the active constraints to be a direction of positive curvature. This observation suggests that we define

$$\bar{s}_k = s_k^s \quad \text{and} \quad \bar{d}_k = \nabla_x \mathcal{L}(x_{k+1}, y_{k+1}) - \nabla_x \mathcal{L}(x_k^p, y_{k+1}), \quad (2.13)$$

where  $x_k^p = x_k + s_k^p$  and  $s_k^s$  is the solution to any of the SQP subproblems considered in [19, Section 2.3]. The quantity  $\bar{s}_k^T \bar{d}_k$  approximates the curvature of the reduced Hessian and is likely to be positive definite in the neighborhood of a solution. Note that this strategy requires an extra evaluation of the first derivatives at  $x_k^p$ . Second, we could approximate the curvature of the Augmented Lagrangian. This has been studied by Han [21], Tapia [40], Byrd, Tapia, and Zhang [7], but we mention here the approach used in the software package SNOPT [15]. The idea is to use the augmented Lagrangian function to define a perturbation  $\Delta d$  of  $d_k$  such that  $(d_k + \Delta d)^T s_k = \eta_{\text{SPD}} s_k^T B_k s_k$ . To obtain  $\Delta d$ , we consider the augmented Lagrangian function [22,32]:

$$\mathcal{L}_{\mathcal{A}}(s, y, \Omega(\omega)) = f(x) - c(x)^T y + \frac{1}{2} c(x) \Omega(\omega) c(x), \quad (2.14)$$

where  $\omega \equiv (\omega_1, \dots, \omega_n)^T \in \mathbb{R}^m$  and  $\Omega(\omega) = \text{diag}(\omega_1, \dots, \omega_n)$ . The gradient of the augmented Lagrangian with respect to  $x$  is given by

$$\nabla_x \mathcal{L}_{\mathcal{A}}(x, y, \Omega(\omega)) = g(x) - J(x)^T (y - \Omega(\omega) c(x)), \quad (2.15)$$

so that

$$\nabla_x \mathcal{L}_{\mathcal{A}}(x_{k+1}, y_{k+1}, \Omega(\omega)) - \nabla_x \mathcal{L}_{\mathcal{A}}(x_k, y_{k+1}, \Omega(\omega)) = d_k + \Delta d(\omega), \quad (2.16)$$

where  $\Delta d(\omega) = J(x_{k+1})^T \Omega(\omega) c(x_{k+1}) - J_k^T \Omega(\omega) c_k$ . The authors of SNOPT [15] suggest computing an  $\omega$  of minimal norm so that  $\bar{s}_k \stackrel{\text{def}}{=} s_k$  and  $\bar{d}_k \stackrel{\text{def}}{=} d_k + \Delta d(\omega)$  satisfy condition (2.10) by solving the problem

$$\underset{\omega \in \mathbb{R}^n}{\text{minimize}} \quad \|\omega\|_2^2 \quad \text{subject to} \quad a^T \omega = b, \quad \omega \geq 0, \quad (2.17)$$

where  $b = \eta_{\text{SPD}} s_k^T B_k \bar{s}_k - d_k^T \bar{s}_k$ ,  $a_i = c_i(x_{k+1}) t_i - c_i(x_k) r_i$ ,  $t = J(x_{k+1}) \bar{s}_k$ , and  $r = J_k \bar{s}_k$ . If no solution exists or if the norm of the solution is considered too large, then a different strategy should be used.

**3. Updating the penalty parameter.** The following theorem clarifies why it is essential to incorporate a strategy for updating the penalty parameter (see [9, 10, 31] for more details).

**THEOREM 3.1.** [9, Theorem 14.5.1] *Suppose that  $f$  and  $c_i$  are twice continuously differentiable for  $1 \leq i \leq m$  and that  $x^*$  and  $y^*$  are vectors such that  $c(x^*) \geq 0$  and  $\sigma \geq \|y^*\|_\infty$ . Then if  $(x^*, y^*)$  satisfies the second-order sufficient conditions for problem (NP),  $x^*$  also satisfies the second-order sufficient conditions for problem  $(\ell_1-\sigma)$ . In addition, if  $\sigma > \|y^*\|_\infty$  then the second-order sufficient conditions for the two problems are equivalent.*

The updating scheme that we now discuss is based on the simple idea of calculating a sequence of approximate solutions for problem  $(\ell_1-\sigma)$ . After each approximate solution is computed, we check the constraint violation and if sufficient improvement is not obtained, then the penalty parameter is increased with the intent of driving the constraint violation to zero. Since the penalty parameter is now allowed to change over a sequence of iterations, we let  $\sigma_k$  denote the penalty parameter during the  $k$ th iterate. We accept the vector-pair  $(x_k, \pi_k)$  as an approximate solution for problem  $(\ell_1-\sigma)$  if it satisfies

$$\varepsilon_k^D \geq \frac{\|g_k + \sigma_k J_k^T \pi_k\|_\infty}{1 + \|g_k\|_\infty} \quad (3.1a)$$

$$[\pi_k]_i = \begin{cases} [-\frac{\varepsilon_k^C}{\sigma_k}, \frac{\varepsilon_k^C}{\sigma_k}] & \text{if } [c_k]_i > \varepsilon_k^P, \\ [-1 - \varepsilon_k^C, \frac{\varepsilon_k^C}{\sigma_k}] & \text{if } -\varepsilon_k^P \leq [c_k]_i \leq \varepsilon_k^P, \\ [-1 - \varepsilon_k^C, -1 + \varepsilon_k^C] & \text{if } [c_k]_i < -\varepsilon_k^P, \end{cases} \quad (3.1b)$$

where  $\varepsilon_k^P$ ,  $\varepsilon_k^D$ , and  $\varepsilon_k^C$  denote the  $k$ th primal, dual, and complementary-slackness tolerances, respectively, for problem  $(\ell_1-\sigma)$ . These conditions are based on the optimality conditions for an exact minimizer  $(x, \pi)$ , which are given by  $g(x) + \sigma_k J(x)^T \pi = 0$  for  $\pi \in \partial \|c(x)^-\|_1$  (see [12, Section 14.3] for more details). Perhaps the most natural way of generating a vector  $\pi_k$  is to use the multipliers from either subproblem (1.5) or any of the *elastic* SQP subproblems considered in [19, Section 3.3]. If we define  $y_k$  to be any of those choices, then the optimality conditions for their respective subproblems (assuming that the TR constraint is inactive) suggest the definition  $\pi_k = -y_k/\sigma_k$ . Provided the sequence  $\{y_k\}$  converges to a Lagrange multiplier vector for the elastic version of problem  $(\ell_1-\sigma)$ , this strategy will eventually produce a vector-pair  $(x_k, \pi_k)$  satisfying equation (3.1). A second way of generating a vector  $\pi_k$  is by defining it to be a solution to the optimization problem

$$\underset{\pi \in \mathbb{R}^m}{\text{minimize}} \quad \frac{1}{2} \|g_k + \sigma_k J_k^T \pi\|_2^2 \quad \text{subject to } \pi \text{ satisfying equation (3.1b)}. \quad (3.2)$$

This will also eventually result in a vector-pair  $(x_k, \pi_k)$  satisfying equation (3.1), regardless of the predictor and SQP multipliers. Therefore, for a fixed value of the penalty parameter, we have a guaranteed method for computing a vector pair  $(x_k, \pi_k)$  that is an approximate critical point to problem  $(\ell_1-\sigma)$ .

Algorithm 3.1 provides the pseudo-code for updating the penalty parameter as well as the additional parameter initiations that must be made.

**Algorithm 3.1.** Updating  $\sigma$  based on an approximate critical point to problem  $(\ell_1-\sigma)$ .

**begin** (additions to preamble of Algorithm 1.1)

Choose  $\sigma_0 > 0$ ,  $\eta_0 > 0$ ,  $0 < \varepsilon_c < 1$ ,  $0 < \varepsilon_0^P < \varepsilon_c \eta_0$ ,  $0 < \eta_c < 1$ , and  $1 < \sigma_e$ .

Set  $\varepsilon_0^D = \varepsilon_0^P$  and  $\varepsilon_0^C = \varepsilon_0^P$ .

**end** (additions to preamble of Algorithm 1.1)

**if**  $(x_k, \pi_k)$  satisfies condition (3.1) **then**

**if**  $c(x_k) \geq -\eta_k e$

$\eta_{k+1} \leftarrow \eta_c \eta_k$

$\varepsilon_{k+1}^P \leftarrow \varepsilon_c \eta_{k+1}$

[an approximate critical point]

[successful]

[decrease  $\eta_k$ ]

[ensure that  $\varepsilon_k^P$  is less than  $\eta_k$ ]

```

         $\varepsilon_{k+1}^D \leftarrow \varepsilon_{k+1}^P, \quad \varepsilon_{k+1}^C \leftarrow \varepsilon_{k+1}^D$ 
         $\sigma_{k+1} \leftarrow \sigma_k$ 
    else                                                                 [unsuccessful]
         $\eta_{k+1} \leftarrow \eta_k$ 
         $\varepsilon_{k+1}^P \leftarrow \varepsilon_c \varepsilon_k^P$ 
         $\varepsilon_{k+1}^D \leftarrow \varepsilon_{k+1}^P, \quad \varepsilon_{k+1}^C \leftarrow \varepsilon_{k+1}^D$ 
         $\sigma_{k+1} \leftarrow \sigma_e \sigma_k$                                                                  [increase  $\sigma_k$ ]
    end if
end if
else                                                                 [not an approximate critical point]
     $\eta_{k+1} \leftarrow \eta_k, \quad \varepsilon_{k+1}^P \leftarrow \varepsilon_k^P, \quad \varepsilon_{k+1}^D \leftarrow \varepsilon_k^D, \quad \varepsilon_{k+1}^C \leftarrow \varepsilon_k^C, \quad \sigma_{k+1} \leftarrow \sigma_k$ 
end if

```

---

For simplicity, we have defined  $\varepsilon_k^D = \varepsilon_k^C = \varepsilon_k^P$ . However, all that is required is that  $\lim_{k \rightarrow \infty} \varepsilon_k^P = \lim_{k \rightarrow \infty} \varepsilon_k^D = \lim_{k \rightarrow \infty} \varepsilon_k^C = 0$ .

For numerical considerations, it is generally not desirable to let the penalty parameter grow “too large”. However, there are two situations in which the penalty parameter should converge to infinity. The first is when the user supplies an optimization problem that is not well-defined. It is possible that the user may formulate a set of nonlinear constraints  $c(x) \geq 0$  for which no feasible point exists. Detecting this situation is difficult and is equivalent to showing that the *global* solution of

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad \|[c(x)]^-\|_1 \quad (3.3)$$

is strictly positive. The second situation occurs when the iterates converge to a critical point of problem (3.3) for which  $\|[c(x)]^-\|_1 > 0$ . This undesirable situation may occur for *all* penalty methods, but it is rarely encountered in practice. Barring these two situations and under reasonable assumptions, Theorem 3.3 below shows that the penalty parameter remains uniformly bounded and that we can expect to generate an approximate solution to problem (NP) in a finite number of iterations. We use the following definition.

DEFINITION 3.2. *A point  $x$  is a first-order critical point for problem (3.3) if it satisfies*

$$J(x)^T y = 0 \quad (3.4)$$

for some  $y \in \partial \|[c(x)]^-\|_1$ .

For given primal, dual, and complementary-slackness tolerances  $\tau_p$ ,  $\tau_d$ , and  $\tau_c$ , respectively, we say that a vector-pair  $(x_k, y_k)$  is an approximate solution to problem (NP) if it satisfies

$$\frac{\|g_k - J_k^T y_k\|_\infty}{1 + \|g_k\|_\infty} \leq \tau_d, \quad (3.5a)$$

$$c_k \geq -\tau_p e, \quad (3.5b)$$

$$y_k \geq -\tau_c e, \quad (3.5c)$$

$$\max(|c_k|, |y_k|) \leq \tau_c e, \quad (3.5d)$$

where condition (3.5d) should be interpreted component-wise.

THEOREM 3.3. *Let all the assumptions of Theorem 1.6 hold and let  $\{x_k\}$  be the sequence of iterates generated by Algorithm 1.1 with penalty parameter update given by Algorithm 3.1. Assume that at all limit points  $x_*$  of  $\{x_k\}$ , the Jacobian of active constraints has full row rank and if  $x_*$  is a first-order critical point for problem (3.3) then  $\|[c(x_*)]^- \|_1 = 0$ . Then*

- (i) *the penalty parameter remains uniformly bounded; and*

- (ii) if  $\tau_p$ ,  $\tau_d$ , and  $\tau_c$  denote positive primal, dual, and complementary-slackness tolerances, respectively, for problem (NP), then the algorithm described in this theorem terminates in a finite number of iterations with an approximate solution to problem (NP) as given by (3.5), where  $y_k \stackrel{\text{def}}{=} -\sigma_k \pi_k$  and  $(x_k, \pi_k)$  is an approximate solution to  $(\ell_1\text{-}\sigma)$  as given by (3.1) for the value  $\sigma_k$ .

*Proof.* We first note that

$$\lim_{k \rightarrow \infty} \varepsilon_k^p = \lim_{k \rightarrow \infty} \varepsilon_k^d = \lim_{k \rightarrow \infty} \varepsilon_k^c = 0, \quad (3.6)$$

since Algorithm 1.1 is guaranteed to generate an infinite sequence  $\{(x_k, \pi_k)\}$  of approximate critical points to problem  $(\ell_1\text{-}\sigma)$ .

We now prove part (i) by contradiction. Suppose that  $\{\sigma_k\} \rightarrow \infty$ . Examination of Algorithm 3.1 implies the existence of a subsequence  $K_0 \subseteq \mathbb{N}$  such that for each  $k \in K_0$  the vector-pair  $(x_k, \pi_k)$  is an approximate critical point to problem  $(\ell_1\text{-}\sigma)$  as given by equation (3.1) for which  $c_k \not\geq -\eta_k e$ . Since  $\{\pi_k\}$  is bounded and  $\{x_k\}$  belongs to the compact set  $\mathcal{B}$ , we may pass to a further subsequence  $K_1$  so that  $\lim_{k \in K_1} (x_k, \pi_k) = (x_*, \pi_*)$ . Equation (3.6) and condition (3.1b) then imply that  $\pi_* \in \partial \|[c(x_*)]^- \|_1$ , while condition (3.1a) and equation (3.6) imply

$$\lim_{k \in K_1} \frac{\|g_k + J_k^T \pi_k\|_\infty}{1 + \|g_k\|_\infty} \leq \lim_{k \in K_1} \frac{\varepsilon_k^d}{\sigma_k} = 0. \quad (3.7)$$

Since  $g$  is continuous by assumption and  $\lim_{k \in K_1} x_k = x_*$ , we know that  $\lim_{k \in K_1} g_k/\sigma_k = 0$  and we may conclude from equation (3.7) that

$$J(x_*)^T \pi_* = 0. \quad (3.8)$$

Thus,  $x_*$  is a first-order critical point for problem (3.3), and it follows from the assumptions of this theorem that  $\|[c(x_*)]^- \|_1 = 0$  so that  $c(x_*) \geq 0$ . Define the index set of active constraints at  $x_*$  to be  $\mathcal{A}_* \stackrel{\text{def}}{=} \mathcal{A}(x_*)$ . Since  $\pi_* \in \partial \|[c(x_*)]^- \|_1$  and  $c(x_*) \geq 0$ , it follows that  $[\pi_*]_i = 0$  for all  $i \notin \mathcal{A}_*$ , and therefore

$$J_{\mathcal{A}_*}(x_*)^T [\pi_*]_{\mathcal{A}_*} = 0. \quad (3.9)$$

Since  $J_{\mathcal{A}_*}(x_*)$  has full row rank by assumption, it follows that  $[\pi_*]_{\mathcal{A}_*} = 0$  and therefore  $\pi_* = 0$ . Since  $\lim_{k \in K_1} \pi_k = \pi_* = 0$ ,  $\lim_{k \rightarrow \infty} \sigma_k = \infty$ , and  $\lim_{k \rightarrow \infty} \varepsilon_k^c = 0$ , we conclude from (3.1b) that  $c(x_k) \geq -\varepsilon_k^p e$  for all  $k \in K_1$  sufficiently large. However, since  $k \in K_1$  we also know that  $c(x_k) \not\geq -\eta_k e$ . Combining these two inequalities, we have  $\eta_k < \varepsilon_k^p$  which contradicts how Algorithm 3.1 constructs the sequence  $\{\varepsilon_k^p\}$ . Therefore,  $\{\sigma_k\}$  is uniformly bounded.

We now prove part (ii) by contradiction. Suppose that Algorithm 1.1 does not terminate in a finite number of iterations. We may then define the subsequence  $K_2 \subseteq \mathbb{N}$  such that for all  $k \in K_2$  the vector-pair  $(x_k, \pi_k)$  is an approximate solution to problem  $(\ell_1\text{-}\sigma)$  as given by equation (3.1). Since  $\{\sigma_k\}$  is uniformly bounded from part (i) and since  $\sigma_k$  is increased by a constant factor when it is increased, there exists a number  $\sigma_b$  such that  $\sigma_k = \sigma_b$  for all  $k$  sufficiently large. Consideration of Algorithm 3.1 then implies that  $\lim_{k \rightarrow \infty} \eta_k = 0$ . These observations and equation (3.6) imply the existence of an integer  $k_c$  such that the following estimates hold for all  $k \geq k_c$ :

$$\sigma_k = \sigma_b, \quad \varepsilon_k^d \leq \tau_d, \quad \eta_k \leq \min(\tau_p, \tau_c), \quad \varepsilon_k^c \leq \min(\tau_c, 1) \quad \text{and} \quad \varepsilon_k^p \leq \tau_c. \quad (3.10)$$

We then define the subsequence  $K_3$  to be the subsequence of  $K_2$  for which  $k \geq k_c$ .

We claim that the vector-pair  $(x_k, y_k)$  is an approximate solution to problem (NP) for all  $k \in K_3$ , where the auxiliary vector sequence  $\{y_k\}$  is defined by  $y_k = -\sigma_k \pi_k$ . Condition (3.5a) follows from equation (3.1a) and equation (3.10) since

$$\frac{\|g_k - J_k^T y_k\|_\infty}{1 + \|g_k\|_\infty} = \frac{\|g_k + \sigma_b J_k^T \pi_k\|_\infty}{1 + \|g_k\|_\infty} \leq \varepsilon_k^d \leq \tau_d \quad (3.11)$$

for all  $k \in K_3$ . Next, since all  $k \in K_3$  are successful iterates by definition, we have

$$c_k \geq -\eta_k e \geq -\tau_p e, \quad (3.12)$$

where the second inequality follows from equation (3.10). Thus, condition (3.5b) is satisfied. Condition (3.5c) may be verified from equations (3.1b) and (3.10) since

$$y_k = -\sigma_b \pi_k \geq -\sigma_b \frac{\varepsilon_k^C}{\sigma_b} = -\varepsilon_k^C \geq -\tau_c \quad (3.13)$$

for all  $k \in K_3$ . Finally, we verify condition (3.5d). Let  $k \in K_3$  and consider the  $i$ th component of  $c_k$  and  $\pi_k$ . If  $|[c_k]_i| \leq \tau_c$ , then condition (3.5d) is satisfied. Therefore, we assume that  $|[c_k]_i| > \tau_c$ . Since  $k \in K_3$ , it follows from equation (3.10) that

$$[c_k]_i \geq -\eta_k \geq -\tau_c \quad (3.14)$$

and, therefore,

$$[c_k]_i > \tau_c \geq \varepsilon_k^P, \quad (3.15)$$

where the second inequality follows from equation (3.10). Condition (3.1b) and equation (3.10) then imply

$$|[y_k]_i| = \sigma_b |[ \pi_k ]_i | \leq \varepsilon_k^C \leq \tau_c. \quad (3.16)$$

This verifies condition (3.5d). We have shown that the vector-pair  $(x_k, y_k)$  is an approximate solution to problem (NP) with tolerances  $\tau_p$ ,  $\tau_d$ , and  $\tau_c$  for all  $k \in K_3$ . This is a contradiction and, therefore, the algorithm must terminate with an approximate solution to problem (NP) in a finite number of iterations.  $\square$

We close this section by mentioning two potential drawbacks associated with using Algorithm 3.1. First, if the initial penalty parameter is substantially smaller than the threshold value required to guarantee convergence (see Theorem 3.1), then Algorithm 3.1 may be laborious since it is based on computing a *sequence* of approximate minimizers of the merit function. We also note that when the penalty parameter is too small, the merit function may not even have a well-defined minimizer [5, Example 1]. Second, even if the merit function does have a well-defined minimizer, there may not exist a strictly decreasing path that connects a poor initial point  $x_0$  to this minimizer [5, Example 2]. A possible way of avoiding these situations is to *dynamically* update the penalty parameter based on linear infeasibility. The so-called “steering” method is based on this idea and has been studied by Byrd et al. [4,5]. Their algorithm is composed of essentially two stages that we now briefly describe using our notation. If we denote the current penalty parameter by  $\sigma_c$ , then the first stage is to compute a step  $s_\infty$  that locally minimizes the linearized constraint violation; this can be viewed as essentially solving the predictor subproblem with penalty parameter  $\sigma = \infty$ . The second stage is to compute a predictor step  $s_k^P$  and a new penalty parameter  $\sigma_N$  that satisfy the following conditions: (i) the decrease in the linearized constraint violation obtained from  $s_k^P$  must be at least a fixed multiple of the decrease obtained from  $s_\infty$ ; and (ii) the decrease in the faithful model must respect the progress made by  $s_k^P$  on the linearized infeasibility by satisfying

$$\Delta M_k^B(s_k^P) \geq \varepsilon_a \sigma_N (\|[c_k]^- \|_1 - \|[c_k + J_k s_k^P]^- \|_1), \quad (3.17)$$

where the constant  $\varepsilon_a$  satisfies  $0 < \varepsilon_a < 1$  (note that  $\Delta M_k^B(s_k)$  depends on  $\sigma_N$  although the notation does not make this explicit). The authors present three compelling examples that elucidate the strengths of this approach. For this approach to be beneficial, however, the additional cost must be offset by the “superior” values for the penalty parameter. This *dynamic* strategy is used in an SLQP method that is part of the KNITRO software package [41] and the authors report results that are superior to static penalty updating strategies.

We take the stance that both approaches should be available to the user. If the user has no information about the size of the multipliers, then our experience has been that steering is generally superior to finding a sequence of approximate minimizers. However, if a reasonable estimate for the size of the multipliers is known in advance, then steering is likely to be less efficient because of the potential overhead associated with the method.

**4. Local convergence.** This section considers the local convergence properties of Algorithm 4.1, which is a nonmonotone implementation of Algorithm 1.1. The update to the Lagrange multiplier vector  $y_k$  is now critical and we must consider the sequence of vector-pairs  $(x_k, y_k)$ . To simplify notation, we let  $w$  denote the combined  $x$  and  $y$  vectors, i.e.,  $w = (x, y)$ , and we write  $w_k = (x_k, y_k)$  for the current estimate of a solution  $w^* = (x^*, y^*)$ ,  $w_k^P = (x_k^P, y_k^P)$  for the solution to the predictor subproblem (1.5), and  $w_k^S = (x_k^S, y_k^S)$  for the solution of the SQP subproblem (the precise definition of  $y_k^S$  depends on which SQP subproblem is used).

The primary result of this section is that if a successful iterate of Algorithm 4.1 gets close enough to a local minimizer  $w^*$  of problem (NP) that satisfies the strong second-order sufficient conditions, then the sequence of iterates converges to  $w^*$  with convergence properties derived from Newton's Method for zero-finding applied to the function

$$F_N(x, y_A) = \begin{pmatrix} g(x) - J_A(x)^T y_A \\ c_A(x) \end{pmatrix}. \quad (4.1)$$

We accomplish this by first showing that if  $w_k$  is close enough to  $w^*$ , then the predictor step accurately predicts the optimal active set and that the trust-region constraint is inactive. We then show that specific SQP steps also identify the optimal active set and that their associated trust-region constraints are inactive. Since these steps are then equivalent to one step of Newton's Method for zero-finding applied to  $F_N$ , we deduce that  $w_{k+1}$  is closer to  $w^*$  than was  $w_k$ . This process is then repeated and results in the value  $w_{k+2}$ . Since Algorithm 4.1 is a nonmonotone approach, the analysis given by Conn, Gould and Toint [9, Section 15.3.2] shows that the  $\ell_1$ -merit function will accept the value  $x_{k+2}$  and it follows that convergence may be described using classical results for Newton's Method applied to the function  $F_N$ .

Algorithm 4.1 generates trial steps in exactly the same way as does Algorithm 1.1. In fact, if every iteration is successful, then the two algorithms are identical. However, if a failure occurs then Algorithm 4.1 still accepts the step with the hope that the next iterate will be successful; we say that a "nonmonotone phase" has been entered. If we enter a nonmonotone phase, the ratio  $r_k$  of actual to predicted decrease in the merit function is computed based on the trial point  $x_k + s_k$  and the *best-known* point, i.e., the solution estimate directly *before* the nonmonotone phase was entered. If the number of consecutive failures reaches the maximum number allowed (as denoted by the parameter *max\_fails*), then the algorithm reverts to the best-known point, reduces the predictor trust-region radius, and then tries again. In less precise terms, the algorithm has "gone back in time" and proceeds as if we were using Algorithm 1.1 until the next failure occurs.

We have also changed the update to the predictor trust-region radius. The new update ensures that the radius following every successful/very successful iteration is at least as large as some pre-defined positive number  $\Delta_R$ . We will see that this strategy allows us to prove that the trust-region constraints are eventually inactive; more complicated alternatives are briefly outlined in [9, Chapter 15].

One final modification to Algorithm 1.1 is the introduction of the vector  $y_k^F$ . Lemma 4.8 shows that we may choose  $y_k^F$  to be the multipliers from the predictor step, but any estimate satisfying the conditions  $y_k^F - y^* = O(\|x_k - x^*\|_2)$  and  $[y_k^F]_{\mathcal{I}} = 0$  may be used. We then define  $H_k$  to be any symmetric approximation to  $H(x_k, y_k^F)$ , but for the local convergence results given by Theorems 4.7 and 4.9 we choose  $H_k \equiv H(x_k, y_k^F)$ .

It may easily be verified that [19, Theorems 4.3, 4.4, and 4.7] are still true with these changes. Thus, Algorithm 4.1 is globally convergent.

**Algorithm 4.1.** Nonmonotone algorithm.

Input:  $(x_0, y_0)$

Set parameters  $0 < \eta_s \leq \eta_{vs} < 1$ ,  $0 < \Delta_R \leq \Delta_U$ ,  $0 < \eta \leq \eta_{ACP} < 1$ ,  $\tau_f \geq 1$ , and  $0 \leq \text{max\_fails} \in \mathbb{N}$ .

Set expansion and contraction factors  $0 < \eta_c < 1 < \eta_e$  and fail counter  $\text{fails} = 0$ .

$k \leftarrow 0$

**do**

Evaluate  $f_k, g_k, c_k, J_k$  and then compute  $\phi_k$ .

Define  $B_k$  to be a symmetric positive definite approximation to  $H(x_k, y_k)$ .

Solve problem (1.4) for predictor step and multipliers  $(s_k^P, y_k^P)$ .

Define  $y_k^F$  to be any multiplier estimate for which  $y_k^F - y^* = O(\|x_k - x^*\|_2)$  and  $[y_k^F]_{\mathcal{I}} = 0$ .

Define  $H_k$  to be a symmetric approximation to  $H(x_k, y_k^F)$ .

Solve problem (1.6) for  $s_k^{CP}$  and compute  $\Delta M_k^H(s_k^{CP})$ .

Compute an SQP step and multipliers  $(s_k^S, y_k^S)$  (optional).

Define a full step  $s_k$  that satisfies condition (1.7).

Evaluate  $\phi(x_k + s_k)$  and  $\Delta M_k^H(s_k)$ .

**if**  $fails = 0$  **then**

$$r_k \leftarrow (\phi(x_k) - \phi(x_k + s_k)) / \Delta M_k^H(s_k) \quad [\text{standard definition}]$$

**else**

$$r_k \leftarrow (\phi_R - \phi(x_k + s_k)) / \Delta_R^H \quad [\text{change in } \phi \text{ based on point } x_R]$$

**end if**

**if**  $r_k \geq \eta_{vs}$  **then**

[very successful]

$$x_{k+1} \leftarrow x_k + s_k$$

$$y_{k+1} \leftarrow y_k^S \quad (y_{k+1} \leftarrow y_k^F \text{ if SQP step not computed})$$

$$\Delta_{k+1}^P \leftarrow \min(\max(\eta_e \cdot \Delta_k^P, \Delta_R), \Delta_U)$$

[increase  $\Delta_k^P$  and ensure  $\Delta_k^P \geq \Delta_R$ ]

$$fails \leftarrow 0$$

**else if**  $r_k \geq \eta_s$  **then**

[successful]

$$x_{k+1} \leftarrow x_k + s_k$$

$$y_{k+1} \leftarrow y_k^S \quad (y_{k+1} \leftarrow y_k^F \text{ if SQP step not computed})$$

$$\Delta_{k+1}^P \leftarrow \max(\Delta_k^P, \Delta_R)$$

[ensure  $\Delta_k^P$  is bigger than  $\Delta_R$ ]

$$fails \leftarrow 0$$

**else**

[failure]

**if**  $fails \leq max\_fails$  **then**

**if**  $fails = 0$  **then**

[save current point]

$$x_R \leftarrow x_k, \quad y_R \leftarrow y_k, \quad \phi_R \leftarrow \phi_k, \quad \Delta_R^H \leftarrow \Delta M_k^H(s_k)$$

$$\Delta_R^P \leftarrow \Delta_k^P$$

$$\Delta_{k+1}^P \leftarrow \eta_c \Delta_k^P \quad (\text{optional})$$

**end if**

$$x_{k+1} \leftarrow x_k + s_k$$

$$\Delta_{k+1}^P \leftarrow \Delta_k^P$$

$$fails \leftarrow fails + 1$$

**else**

[revert to saved point]

$$x_{k+1} \leftarrow x_R, \quad y_{k+1} \leftarrow y_R$$

$$\Delta_{k+1}^P \leftarrow \eta_c \Delta_R^P$$

[decrease  $\Delta_k^P$ ]

$$fails \leftarrow 0$$

**end if**

**end if**

$$\Delta_{k+1}^S \leftarrow \tau_f \cdot \Delta_{k+1}^P$$

[update SQP radius]

$$k \leftarrow k + 1$$

**end do**

---

**4.1. Optimal active set identification.** The analysis that ensues requires a notion of “uniformity” for the underlying KKT systems within a neighborhood of a solution  $w^*$ . This is generally not an issue for systems involving  $H_k$  since it is reasonable to expect that if  $w_k$  converges to  $w^*$  then  $H_k$  will converge to  $H^*$ ; this certainly occurs if  $H_k \equiv H(x_k, y_k)$  or  $H_k \equiv H(x_k, y_k^F)$ . A similar statement does not hold for systems involving  $B_k$  since  $B_k$  is generally not a continuous function of  $w$ . Moreover, we certainly can not expect the positive-definite matrix  $B_k$  to converge to  $H^*$  since  $H^*$  is normally indefinite. The optimality conditions for problem (NP) suggest that we need the matrices  $H_k$  and  $B_k$  to be positive definite when restricted to the null space of the active constraints (note that  $B_k$  is positive definite by construction); this is essentially the uniformity that we need. To develop a general framework, we define the following sets that depend on the minimizer  $w^*$ :

$$S(x; x^*) = \{M \in \mathbb{R}^{n \times n} : M = M^T, \|M\|_2 \leq \beta_{\max} \text{ and } s^T M s \geq \lambda_{\min} s^T s \ \forall s \text{ satisfying } J_{\mathcal{A}}(x)s = 0\} \quad (4.2)$$

and

$$S_{\varepsilon} = \cup_{w \in \bar{\mathcal{B}}_{\varepsilon}(w^*)} S(x; x^*) \quad (4.3)$$

for given real numbers  $\beta_{\max}$  and  $\lambda_{\min} > 0$ . Using this definition, we now state a result that supplies the required uniformity.

LEMMA 4.1. *If  $w^*$  is a KKT point for problem (NP) that satisfies the LICQ, then*

- (i) *for any  $0 \leq \varepsilon < \infty$  the set*

$$S_{\varepsilon} = \cup_{w \in \bar{\mathcal{B}}_{\varepsilon}(w^*)} S(x; x^*)$$

*is compact;*

- (ii) *if  $\nu_1 \leq \nu_2$ , then  $S_{\nu_1} \subseteq S_{\nu_2}$ ;*

- (iii) *there exists a positive number  $\varepsilon_1$  such that if  $w \in \mathcal{B}_{\varepsilon_1}(w^*)$  and  $M \in S_{\varepsilon_1}$ , then  $s^T M s \geq (\lambda_{\min}/2)s^T s$  for all  $s$  satisfying  $J_{\mathcal{A}}(x)s = 0$ .*

*If in addition, strict complementarity holds at  $w^*$ , then*

- (iv) *there exists a positive number  $\varepsilon_2$  such that  $\varepsilon_2 \leq \varepsilon_1$  and numbers  $\beta_0 > 0$  and  $\beta > 0$  such that if  $w \in \mathcal{B}_{\varepsilon_2}(w^*)$  and  $M \in S_{\varepsilon_2}$ , then  $J_{\mathcal{A}}(x)$  has full row rank,  $c_{\mathcal{I}}(x) > 0$ ,  $y_{\mathcal{A}} > 0$ , and the matrices*

$$\bar{K}_M(x) = \begin{pmatrix} M & J_{\mathcal{A}}(x)^T \\ J_{\mathcal{A}}(x) & 0 \end{pmatrix} \text{ and } K_M(w) = \begin{pmatrix} M & -J_{\mathcal{A}}(x)^T & -J_{\mathcal{I}}(x)^T \\ \text{diag}(y_{\mathcal{A}})J_{\mathcal{A}}(x) & 0 & 0 \\ 0 & 0 & \text{diag}(c_{\mathcal{I}}) \end{pmatrix}$$

*are nonsingular and satisfy*

$$\|\bar{K}_M(x)^{-1}\|_2 \leq \beta_0 \text{ and} \quad (4.4a)$$

$$\|K_M(w)^{-1}\|_2 \leq \beta; \quad (4.4b)$$

- (v) *if  $w \in \mathcal{B}_{\varepsilon_2}(w^*)$  and  $M \in S_{\varepsilon_2}$ , then it follows that*

$$s = O(\|x - x^*\|_2), \quad \pi_{\mathcal{A}} - y_{\mathcal{A}}^* = O(\|x - x^*\|_2), \quad \text{and} \quad \pi - y^* = O(\|x - x^*\|_2), \quad (4.5)$$

*where  $s$  and  $\pi_{\mathcal{A}}$  satisfy*

$$\bar{K}_M(x) \begin{pmatrix} s \\ -\pi_{\mathcal{A}} \end{pmatrix} \equiv \begin{pmatrix} M & J_{\mathcal{A}}(x)^T \\ J_{\mathcal{A}}(x) & 0 \end{pmatrix} \begin{pmatrix} s \\ -\pi_{\mathcal{A}} \end{pmatrix} = - \begin{pmatrix} g(x) \\ c_{\mathcal{A}}(x) \end{pmatrix}, \quad (4.6)$$

*and  $\pi$  is obtained from  $\pi_{\mathcal{A}}$  by “scattering” the components of  $\pi_{\mathcal{A}}$  into a zero-vector of length  $m$  as indicated by  $\mathcal{A}$ .*

*Proof.* We first prove part (i). Since it is clear that  $S_\varepsilon$  is bounded, we only show that  $S_\varepsilon$  is closed. Let  $\{M_k\}$  be a sequence in  $S_\varepsilon$  such that  $\lim_{k \rightarrow \infty} M_k = \bar{M}$ . This implies the existence of a sequence  $\{w_k\} \in \bar{\mathcal{B}}_\varepsilon(w^*)$  such that  $M_k \in S(x_k; x^*)$  and  $v^T M_k v \geq \lambda_{\min} v^T v$  for all  $v$  such that  $J_{\mathcal{A}}(x_k)v = 0$ . The set  $\bar{\mathcal{B}}_\varepsilon(w^*)$  is compact and, therefore, we can pass to a subsequence  $K_1$  such that  $\lim_{k \in K_1} w_k = \bar{w} \in \bar{\mathcal{B}}_\varepsilon(w^*)$ . Since  $J$  is continuous and  $J_{\mathcal{A}}(x^*)$  has full row rank, [6, Theorem 2.3] implies the existence of a locally *continuous* null space basis function  $Z(\cdot)$  such that  $J_{\mathcal{A}}(x_k)Z(x_k) = 0$ ,  $\lim_{k \in K_1} Z(x_k) = \bar{Z}$ , and  $J_{\mathcal{A}}(\bar{x})\bar{Z} = 0$ . This implies that  $Z(x_k)^T M_k Z(x_k) \succeq \lambda_{\min}$  and upon taking limits that  $\bar{Z}^T \bar{M} \bar{Z} \succeq \lambda_{\min}$ . Since it is clear that  $\bar{M}$  is symmetric and satisfies  $\|\bar{M}\|_2 \leq \beta_{\max}$ , we have  $\bar{M} \in S(\bar{x}; x^*) \subseteq S_\varepsilon$ . Thus,  $S_\varepsilon$  is closed.

Part (ii) follows immediately from the definitions of  $S_{\nu_1}$  and  $S_{\nu_2}$ .

We now prove part (iii). If part (iii) was not true, then there would exist a monotonically decreasing and strictly positive sequence  $\{\delta_k\} \rightarrow 0$  and associated sequences  $\{w_k\}$ ,  $\{s_k\}$ , and  $\{M_k\}$  such that  $w_k \in \mathcal{B}_{\delta_k}(w^*)$ ,  $M_k \in S_{\delta_k} \subseteq S_{\delta_1}$ ,  $J_{\mathcal{A}}(x_k)s_k = 0$ ,  $\|s_k\|_2 = 1$  and  $s_k^T M_k s_k < \lambda_{\min}/2$ . It follows from these properties, part (i), and the fact that the sequence  $\{s_k\}$  belongs to a compact set, that there exists a subsequence  $K_2$ , a matrix  $M^* \in S_{\delta_1}$  and a unit vector  $s^*$  such that

$$\lim_{k \in K_2} w_k = w^*, \quad \lim_{k \in K_2} M_k = M^*, \quad \lim_{k \in K_2} s_k = s^*, \quad J_{\mathcal{A}}^* s^* = 0, \quad \text{and} \quad s^{*T} M^* s^* \leq \lambda_{\min}/2. \quad (4.7)$$

Since  $M_k \in S_{\delta_k}$  and  $\{\delta_k\} \rightarrow 0$ , there also exists a sequence  $\{\hat{x}_k\} \rightarrow x^*$  such that  $s^T M_k s \geq \lambda_{\min} s^T s$  for all  $s$  satisfying  $J_{\mathcal{A}}(\hat{x}_k)s = 0$ . Using the same argument as in the first paragraph of this proof, we find that  $Z^{*T} M^* Z^* \succeq \lambda_{\min}$ , where the columns of  $Z^*$  form a basis for the null space of  $J_{\mathcal{A}}^*$ . This contradicts (4.7) and thus (iii) must be true.

To show that part (iv) holds, we first note that strict complementarity and the LICQ imply that there exists a number  $\varepsilon_s$  such that  $0 < \varepsilon_s \leq \varepsilon_1$  and

$$c_i(x) \geq \frac{1}{2}c_i^* > 0 \text{ for } i \in \mathcal{I}, \quad y_i \geq \frac{1}{2}y_i^* > 0 \text{ for } i \in \mathcal{A}, \quad \text{and } J_{\mathcal{A}}(x) \text{ has full row rank} \quad (4.8)$$

for all  $w \in \mathcal{B}_{\varepsilon_s}(w^*)$ . Under the current assumptions, it follows from parts (ii), (iii) and [1, Lemma 1.27] that

$$\text{the matrix } \bar{K}_M(x) \text{ is nonsingular for all } w \in \mathcal{B}_{\varepsilon_s}(w^*) \text{ and } M \in S_{\varepsilon_s}. \quad (4.9)$$

Assume that (4.4a) does not hold for any  $\varepsilon_2 \leq \varepsilon_s$  so that there exists a monotonically decreasing sequence  $\{\delta_k\} \rightarrow 0$  such that  $0 < \delta_k \leq \varepsilon_s$ , and associated sequences  $\{w_k\} \in \mathcal{B}_{\delta_k}(w^*)$  and  $\{M_k\} \in S_{\delta_k} \subseteq S_{\varepsilon_s}$  such that

$$\|\bar{K}_{M_k}(x_k)^{-1}\|_2 \geq k \quad \text{for all } k \geq 0. \quad (4.10)$$

Since  $\{\delta_k\} \rightarrow 0$  and  $S_{\varepsilon_s}$  is compact as a result of part (i), there exists a subsequence  $K_3$  such that  $\lim_{k \in K_3} w_k = w^*$  and  $\lim_{k \in K_3} M_k = M^* \in S_{\varepsilon_s}$ . It then follows from (4.9) that  $\bar{K}_{M^*}(x^*)$  is nonsingular. Since  $\lim_{k \in K_3} \bar{K}_{M_k}(x_k) = \bar{K}_{M^*}(x^*)$ , [16, Theorem 8.64] implies that the singular values of  $\bar{K}_{M_k}(x_k)$  are uniformly bounded away from zero for  $k \in K_3$  sufficiently large. Therefore,  $\|\bar{K}_{M_k}(x_k)^{-1}\|_2$  must be bounded above for all  $k \in K_3$ , which contradicts (4.10). Thus, (4.4a) holds for some  $\varepsilon_1 \geq \varepsilon_s \geq \varepsilon_2 > 0$  and  $\beta_0 > 0$ . It also follows from (4.8) that  $J_{\mathcal{A}}(x)$  has full row rank,  $c_{\mathcal{I}}(x) > 0$ , and  $y_{\mathcal{A}} > 0$  for all  $w \in \mathcal{B}_{\varepsilon_2}(w^*)$ .

We now show that equation (4.4b) holds for  $\varepsilon_2$ . Let  $w \in \mathcal{B}_{\varepsilon_2}(w^*)$  and  $M \in S_{\varepsilon_2}$ . Equation (4.8) implies that the matrices

$$N_F = \begin{pmatrix} I & 0 & J_{\mathcal{I}}(x)^T \text{diag}(c_{\mathcal{I}}(x))^{-1} \\ 0 & \text{diag}(y_{\mathcal{A}})^{-1} & 0 \\ 0 & 0 & I \end{pmatrix}, \quad N_M = \begin{pmatrix} M & J_{\mathcal{A}}(x)^T & 0 \\ J_{\mathcal{A}}(x) & 0 & 0 \\ 0 & 0 & \text{diag}(c_{\mathcal{I}}(x)) \end{pmatrix},$$

and  $N_S = \text{diag}(I, -I, I)$  are *nonsingular*; they satisfy  $N_F K_M(w) N_S = N_M$  so that  $\|K_M(w)^{-1}\|_2 \leq \|N_M^{-1}\|_2 \|N_F\|_2$ . It is also clear from equation (4.8) that the quantity  $\|N_F\|_2$  is bounded for all  $w \in \mathcal{B}_{\varepsilon_2}(w^*)$ ,

so to bound  $\|K_M(w)^{-1}\|_2$  we must bound  $\|N_M^{-1}\|_2$ , but it is sufficient to bound  $\|\bar{K}_M(x)^{-1}\|_2$  due to equation (4.8). The result follows from equation (4.4a) and, therefore, there exists a number  $\beta > 0$  such that  $\|K_M(w)^{-1}\|_2 \leq \beta$  for all  $w \in \mathcal{B}_{\varepsilon_2}(w^*)$  and  $M \in S_{\varepsilon_2}$ .

Finally, we prove part (v). Let  $w_k \in \mathcal{B}_{\varepsilon_2}(w^*)$ , and  $M \in S_{\varepsilon_2}$ . Since  $c_{\mathcal{A}}^* = 0$ , it follows that system (4.6) is equivalent to

$$\begin{pmatrix} M & J_{\mathcal{A}}(x)^T \\ J_{\mathcal{A}}(x) & 0 \end{pmatrix} \begin{pmatrix} s \\ y_{\mathcal{A}}^* - \pi_{\mathcal{A}} \end{pmatrix} = - \begin{pmatrix} g(x) - J_{\mathcal{A}}(x)^T y_{\mathcal{A}}^* \\ c_{\mathcal{A}}(x) - c_{\mathcal{A}}^* \end{pmatrix}. \quad (4.11)$$

Equation (4.4a), norm inequalities, and Taylor expansions for  $g(x)$ ,  $c_{\mathcal{A}}(x)$ , and  $J_{\mathcal{A}}(x)$  at the point  $x^*$  yield  $s = O(\|x - x^*\|_2)$  and  $\pi_{\mathcal{A}} - y_{\mathcal{A}}^* = O(\|x - x^*\|_2)$ . The fact that  $\pi - y^* = O(\|x - x^*\|_2)$  follows since  $\pi_{\mathcal{I}} = 0$  by construction and  $y_{\mathcal{I}}^* = 0$  from the optimality conditions for problem (NP).  $\square$

Our next aim is to prove a result concerning active set identification. Given a vector  $w$ , we define the function

$$F_{\text{KKT}}(w) = \begin{pmatrix} g(x) - J(x)^T y \\ c(x) \cdot y \end{pmatrix}. \quad (4.12)$$

LEMMA 4.2. *Let  $w^*$  be a solution to problem (NP) that satisfies strict complementarity and the LICQ. Then there exist numbers  $\mu > 0$  and  $\beta > 0$  such that if  $w_k \in \mathcal{B}_{\mu/2}(w^*)$ ,  $M \in S_{\mu/2}$ , and  $4\beta\|F_{\text{KKT}}(w_k)\|_2 \leq \mu$  then there exists a unique closest minimizer  $(x_k(M), y_k(M)) = w_k(M)$  to the point  $w_k$  for the problem*

$$\begin{aligned} & \underset{x \in \mathbb{R}^n}{\text{minimize}} && \frac{1}{2}(x - x_k)^T M(x - x_k) + g_k^T(x - x_k) \\ & \text{subject to} && c_k + J_k(x - x_k) \geq 0 \end{aligned} \quad (4.13)$$

with the following properties:

- (i)  $\|x_k(M) - x_k\|_{\infty} \leq \|w_k(M) - w_k\|_2 \leq 2\beta\|F_{\text{KKT}}(w_k)\|_2$ ;
- (ii) the set of constraints active at  $x_k(M)$  for problem (4.13) are the same as the indices in  $\mathcal{A}$ ;
- (iii) the solution  $w_k(M)$  satisfies strict complementarity; and
- (iv)  $J_{\mathcal{A}}(x_k)$  has full row rank.

*Proof.* We begin by letting  $\varepsilon_1 \geq \varepsilon_2 > 0$  and  $\beta > 0$  be the constants guaranteed by Lemma 4.1. Given any vector-pair  $(w, \bar{w})$  and symmetric matrix  $M$ , we define

$$F_M(w; \bar{w}) = \begin{pmatrix} M(x - \bar{x}) + g(\bar{x}) - J(\bar{x})^T y \\ (c(\bar{x}) + J(\bar{x})(x - \bar{x})) \cdot y \end{pmatrix}. \quad (4.14)$$

Differentiating (4.14) we have

$$F'_M(w; \bar{w}) = \begin{pmatrix} M & -J(\bar{x})^T \\ \text{diag}(y)J(\bar{x}) & \text{diag}(c(\bar{x}) + J(\bar{x})(x - \bar{x})) \end{pmatrix}.$$

Choosing  $(w, \bar{w}) = (w^*, w^*)$  we have

$$F'_M(w^*; w^*) = \begin{pmatrix} M & -J_{\mathcal{A}}^{*T} & -J_{\mathcal{I}}^{*T} \\ \text{diag}(y_{\mathcal{A}}^*)J_{\mathcal{A}} & 0 & 0 \\ 0 & 0 & \text{diag}(c_{\mathcal{I}}^*) \end{pmatrix}$$

since optimality conditions at  $w^*$  imply  $c_{\mathcal{A}}^* = 0$  and  $y_{\mathcal{I}}^* = 0$ . It follows from (4.4a) with the choice  $w = w^*$  that the matrix  $F'_M(w^*; w^*)$  is nonsingular and satisfies

$$\|F'_M(w^*; w^*)^{-1}\| \leq \beta \text{ for all } M \in S_{\varepsilon_2}. \quad (4.15)$$

Next, choose a number  $\mu$  such that  $0 < \mu \leq \varepsilon_2$  and if  $w$  and  $\bar{w}$  are contained in  $B_{\mu}(w^*)$ , then the following conditions are satisfied:

- C1. if  $c_i^* > 0$  then  $[c(\bar{x}) + J_k(x - \bar{x})]_i > 0$ ;
- C2. if  $y_i^* > 0$  then  $y_i > 0$ ;
- C3.  $\|F'_M(w; \bar{w}) - F'_M(w^*; w^*)\|_2 \leq 1/(2\beta)$  (this estimate holds for all  $M$ );

Let  $w_k \in \bar{\mathcal{B}}_{\mu/2}(w^*)$  and  $M \in S_{\mu/2}$ . Since  $\mu < \mu/2 \leq \varepsilon_2 \leq \varepsilon_1$ , it follows from parts (ii) and (iii) of Lemma 4.1 that  $J_A(x_k)$  has full row rank and that estimate (4.15) holds for  $M$ ; thus part (iv) is true. Using the argument by Robinson [35, Lemma 1], we now show that  $F_M(w; w_k)$  has a unique zero in  $\bar{\mathcal{B}}_{\mu/2}(w_k)$ . Note that

$$\bar{\mathcal{B}}_{\mu/2}(w_k) \subset \mathcal{B}_\mu(w^*) \quad (4.16)$$

since if  $w \in \bar{\mathcal{B}}_{\mu/2}(w_k)$  then

$$\|w - w^*\|_2 \leq \|w - w_k\|_2 + \|w_k - w^*\|_2 < \mu/2 + \mu/2 \leq \mu.$$

Define the function

$$T_M(w) = w - F'_M(w^*; w^*)^{-1} F_M(w; w_k) \quad (4.17)$$

so that

$$T'_M(w) = I - F'_M(w^*; w^*)^{-1} F'_M(w; w_k) = F'_M(w^*; w^*)^{-1} (F'_M(w^*; w^*) - F'_M(w; w_k)).$$

It follows that

$$\|T'_M(w)\|_2 \leq \beta \|F'_M(w^*; w^*) - F'_M(w; w_k)\|_2 \leq \frac{1}{2} \quad (\text{use (4.15), (4.16), and C3})$$

for all  $w \in \bar{\mathcal{B}}_{\mu/2}(w_k)$ , which implies that  $T_M$  is a contraction. It also follows that

$$\|T_M(w_k) - w_k\|_2 \leq \beta \|F_M(w_k; w_k)\|_2 \quad (\text{use (4.17) and (4.15)}). \quad (4.18)$$

Using the triangle inequality, the fact that  $T_M(w)$  is a contraction with contraction factor  $1/2$ , equation (4.18), and the assumption that  $4\beta \|F_M(w_k; w_k)\|_2 \leq \mu$ , we have that for all  $w \in \bar{\mathcal{B}}_{\mu/2}(w_k)$  the estimate

$$\|T_M(w) - w_k\|_2 \leq \|T_M(w) - T_M(w_k)\|_2 + \|T_M(w_k) - w_k\|_2 \leq \frac{1}{2} \|w - w_k\|_2 + \beta \|F_M(w_k; w_k)\|_2 \leq \mu/2,$$

which implies  $T_M : \bar{\mathcal{B}}_{\mu/2}(w_k) \rightarrow \bar{\mathcal{B}}_{\mu/2}(w_k)$ . We may now apply the well-known fixed point result [37, Theorem 9.23]), which states that  $T_M$  has a unique fixed point  $w_k(M)$  in  $\bar{\mathcal{B}}_{\mu/2}(w_k)$  and that

$$\begin{aligned} \|x_k(M) - x_k\|_\infty &\leq \|x_k(M) - x_k\|_2 \leq \|w_k(M) - w_k\|_2 \quad (\text{use norm inequalities}) \\ &\leq 2 \|T_M(w_k) - w_k\|_2 \quad (\text{estimate from fixed-point theorem}) \\ &\leq 2\beta \|F_M(w_k; w_k)\|_2 \quad (\text{use (4.18)}), \end{aligned}$$

which proves part (i). Since  $w_k(M)$  is a fixed point for  $T_M(w)$ , equation (4.17) implies that

$$F_M(w_k(M); w_k) = 0. \quad (4.19)$$

Thus  $w_k(M)$  satisfies the *equality* conditions for being a first-order KKT point for problem (4.13). We now show that the point  $w_k(M)$  is actually a first-order KKT point for problem (4.13). Since  $w_k(M) \in \bar{\mathcal{B}}_{\mu/2}(w_k) \subset \mathcal{B}_\mu(w^*)$ , we may deduce the following: if  $y_i^* > 0$  then C2 implies  $[y_k(M)]_i > 0$  and then (4.19) implies  $[c_k + J_k(x_k(M) - x_k)]_i = 0$ ; and if  $c_i^* > 0$  then C1 implies  $[c_k + J_k(x_k(M) - x_k)]_i > 0$  and then (4.19) implies  $[y_k(M)]_i = 0$ . Since strict complementarity holds at  $w^*$  by assumption, one of these two cases must hold and, therefore,  $w_k(M)$  is a first-order KKT point for the problem (4.13) that satisfies strict complementarity and correctly identifies the optimal active set; this establishes parts (ii) and (iii). The fact that  $x_k(M)$  is a minimizer follows from parts (ii) and (iii) of Lemma 4.1. Finally,  $w_k(M)$  is the unique closest solution since any other solution would be a KKT point and, therefore, a zero of the function  $F_M(w; w_k)$ . However,  $w_k(M)$  is the unique zero inside  $\bar{\mathcal{B}}_{\mu/2}(w_k)$ .  $\square$

**4.2. Local descent properties.** In this section we show that, in a neighborhood of a solution  $w^*$ , directions related to the traditional SQP step are descent directions for the underlying model functions; this result is critical for proving that Algorithm 4.1 has a fast rate of convergence. We use the following definition.

DEFINITION 4.3. *Given a vector  $v \in \mathbb{R}^n$  and a subspace  $\mathcal{V} \subseteq \mathbb{R}^n$ , we define*

$$\theta(v, \mathcal{V}) = \begin{cases} \tan^{-1}(\|v_R\|_2/\|v_N\|_2) & \text{if } \|v_N\|_2 \neq 0; \\ \pi/2 & \text{otherwise.} \end{cases} \quad (0 \leq \theta \leq \pi/2) \quad (4.20)$$

to be the angle between  $v$  and  $\mathcal{V}$ , where  $v = v_N + v_R$  is the unique orthogonal decomposition of  $v$  such that  $v_N \in \mathcal{V}$  and  $v_R \perp \mathcal{V}$ .

The next result essentially says how close a vector  $s$  must be to the null space of the active constraints to guarantee positive curvature in a neighborhood of a solution.

LEMMA 4.4. *Let  $w^*$  be a solution to problem (NP) that satisfies the LICQ. Then, there exists a number  $\varepsilon_2 > 0$  such that if  $w$ ,  $s$ , and  $M$  satisfy  $w \in \mathcal{B}_{\varepsilon_2}(w^*)$ ,  $M \in \mathcal{S}_{\varepsilon_2}$ , and*

$$\theta(s, \text{null}(J_{\mathcal{A}}(x))) \leq \bar{\theta} \stackrel{\text{def}}{=} \min\left(\pi/4, \tan^{-1}\left(\frac{\lambda_{\min}}{24\beta_{\max}}\right)\right), \quad (4.21)$$

then  $s^T M s \geq (\lambda_{\min}/8)s^T s$ .

*Proof.* Let  $\varepsilon_2$  be defined as in part (iv) of Lemma 4.1 so that  $J_{\mathcal{A}}(x)$  has full row-rank for all  $w \in \mathcal{B}_{\varepsilon_2}(w^*)$ . Suppose that  $w \in \mathcal{B}_{\varepsilon_2}(w^*)$ ,  $M \in \mathcal{S}_{\varepsilon_2}$ , and  $s$  satisfy (4.21). If we write  $s = s_N + s_R$  for  $s_N \in \text{null}(J_{\mathcal{A}}(x))$  and  $s_R \in \text{range}(J_{\mathcal{A}}(x)^T)$ , it follows from (4.20) and (4.21) that  $\theta = \theta(s, \text{null}(J_{\mathcal{A}}(x)))$  satisfies

$$\frac{\|s_R\|_2}{\|s_N\|_2} = \tan(\theta) \leq 1. \quad (4.22)$$

Using the orthogonal decomposition of  $s$ , parts (ii) and (iii) of Lemma 4.1, the Cauchy-Schwarz inequality, definition of  $\beta_{\max}$ , and equations (4.22) and (4.21), we have

$$\begin{aligned} \frac{s^T M s}{s^T s} &= \frac{s_N^T M s_N + s_R^T M s_R + 2s_N^T M s_R}{\|s_N\|_2^2 + \|s_R\|_2^2} \\ &\geq \frac{(\lambda_{\min}/2)s_N^T s_N - \beta_{\max}\|s_R\|_2^2 - 2\beta_{\max}\|s_R\|_2\|s_N\|_2}{\|s_N\|_2^2 + \|s_R\|_2^2} \\ &\geq \lambda_{\min}/4 - \beta_{\max} \tan^2(\theta) - 2\beta_{\max} \tan(\theta) \\ &\geq \lambda_{\min}/4 - 3\beta_{\max} \tan(\theta) \geq \lambda_{\min}/8, \end{aligned}$$

which completes the proof.  $\square$

We now show that in the neighborhood of a solution  $w^*$ , the (unique) solution to

$$\underset{s \in \mathbb{R}^n}{\text{minimize}} \quad g(x)^T s + \frac{1}{2} s^T M s \quad \text{subject to} \quad c_{\mathcal{A}}(x) + J_{\mathcal{A}}(x)s = 0 \quad (4.23)$$

satisfies a certain ‘‘descent’’ property for the underlying models (under certain assumptions).

LEMMA 4.5. *Let  $w^*$  be a minimizer for problem (NP) that satisfies the LICQ and strict complementarity and suppose that  $\sigma > \|y^*\|_{\infty}$ . It follows that there exist positive numbers  $c_2$  and  $\varepsilon_3$  such that if  $w \in \mathcal{B}_{\varepsilon_3}(w^*)$  and  $M \in \mathcal{S}_{\varepsilon_3}$  then problem (4.23) is well-defined and the solution  $s_T$  satisfies*

$$(g(x) + \sigma J(x)^T z)^T s_T < -c_2 \|s_T\|_2^2 \quad \text{for} \quad z = \begin{cases} 0 & \text{if } c(x) \geq 0; \\ -1 & \text{otherwise.} \end{cases} \quad (4.24)$$

*Proof.* Strict complementarity implies the existence of a scalar  $\kappa_s > 0$  such that

$$y_A^* \geq \kappa_s e > 0. \quad (4.25)$$

We define  $\bar{\theta}$  as in Lemma 4.4 and choose positive scalars  $\kappa_j$  and  $\varepsilon_3$  so that the following hold for all  $w \in \mathcal{B}_{\varepsilon_3}(w^*)$  and  $M \in S_{\varepsilon_3}$ :

1.  $\varepsilon_3 \leq \varepsilon_2$ , where  $\varepsilon_2$  is defined in Lemma 4.4;
2.  $\|J(x)^T\|_2 \|(J(x)J(x)^T)^{-1}\|_2 \leq \kappa_j$ ;
3. the system

$$\begin{pmatrix} M & J_{\mathcal{A}}(x)^T \\ J_{\mathcal{A}}(x) & 0 \end{pmatrix} \begin{pmatrix} s \\ -q \end{pmatrix} = - \begin{pmatrix} g(x) - J_{\mathcal{A}}(x)^T y_{\mathcal{A}} \\ c_{\mathcal{A}}(x) \end{pmatrix} \quad (4.26)$$

has a unique solution  $(s, q)$  that satisfies

- a.  $(\kappa_s/2)e \leq y_{\mathcal{A}} + q \leq \sigma(1 - \kappa_\sigma)e$  for some  $\kappa_\sigma > 0$ ;
- b.  $\|s\|_2 \leq \min(1, c_1)$ , where

$$c_1 = \frac{\kappa \sin(\bar{\theta})}{2\kappa_j \beta_{\max}} > 0 \quad \text{and} \quad \kappa = \min\left(\frac{\kappa_s}{2}, \sigma\kappa_\sigma\right) > 0; \quad \text{and} \quad (4.27)$$

- c. if  $c_i^* > 0$  then  $c_i(x) + \nabla c_i(x)^T s > 0$ .

Condition 2 can be satisfied since  $J_{\mathcal{A}}^*$  has full row rank. Condition 1 is well-defined since the assumptions of this theorem imply that the assumptions of Lemma 4.4 hold. Since  $\varepsilon_3 \leq \varepsilon_2$ , parts (ii), (iii) and (iv) of Lemma 4.1 combined with [1, Lemma 1.27] guarantee that problem (4.23) has a unique solution, say  $s_T$ , and the optimality conditions show that  $(s_T, q_T)$  satisfies system (4.26), where  $q_T$  is the *step* from  $y$  to the Lagrange multiplier vector for problem (4.23). Note that we can make the solution  $(s_T, q_T)$  arbitrarily small in norm since the target vector in system (4.26) converges to zero as  $w$  converges to  $w^*$ . This observation, equation (4.25), and the assumption  $\sigma > \|y^*\|_\infty$ , guarantee that we can satisfy conditions 3a and 3b for some  $\kappa_\sigma > 0$ .

Now let  $w \in \mathcal{B}_{\varepsilon_3}(w^*)$ ,  $M \in S_{\varepsilon_3}$ , and  $(s_T, q_T)$  denote the solution to problem (4.23) so that it satisfies system (4.26). For convenience we “scatter” the vector  $q_T$ , which has length equal to the size of the indexing set  $\mathcal{A}$ , into a vector  $q_{\text{TF}} \in \mathbb{R}^m$  so that  $[q_{\text{TF}}]_i = 0$  if  $i \notin \mathcal{A}$ . We also partition the constraints up into four types: I, II, III, and IV (see Figure 1); 3a and the properties of  $s_T$  guarantee that these are the only possibilities. Note that  $\nabla c_i(x)^T s_T < 0$  for  $i \in \text{I}$ ,  $\nabla c_i(x)^T s_T = 0$  for  $i \in \text{II}$  and  $\nabla c_i(x)^T s_T > 0$  for  $i \in \text{III}$ . It then follows from system (4.26), the definitions of  $q_{\text{TF}}$  and  $z$ , condition 3a, and the definition of  $\kappa$  that

$$\begin{aligned} (g(x) + \sigma J(x)^T z)^T s_T &= -s_T^T M s_T + (J_{\mathcal{A}}(x) s_T)^T [y + q_{\text{TF}}]_{\mathcal{A}} + \sigma z^T (J(x) s_T) \\ &= -s_T^T M s_T + \sum_{i \in \text{I}} (\nabla c_i(x)^T s_T) [y + q_{\text{TF}}]_i + \sum_{i \in \text{III}} (\nabla c_i(x)^T s_T) [y + q_{\text{TF}} - \sigma e]_i \\ &\leq -s_T^T M s_T + \frac{\kappa_s}{2} \sum_{i \in \text{I}} (\nabla c_i(x)^T s_T) - \sigma \kappa_\sigma \sum_{i \in \text{III}} (\nabla c_i(x)^T s_T) \\ &\leq -s_T^T M s_T - \kappa \sum_{i \in \text{I} \cup \text{III}} |\nabla c_i(x)^T s_T| = -s_T^T M s_T - \kappa \|J_{\mathcal{A}}(x) s_T\|_1. \end{aligned} \quad (4.28)$$

We now develop a lower bound on  $\|J_{\mathcal{A}}(x) s_T\|_1$ .

Let  $s_T = s_T^R + s_T^N$  be the orthogonal decomposition of  $s_T$  such that  $s_T^R \in \text{range}(J_{\mathcal{A}}(x)^T)$  and  $s_T^N \in \text{null}(J_{\mathcal{A}}(x))$ . It follows that there exists a vector  $r$  such that  $J_{\mathcal{A}}(x)^T r = s_T^R$  and, therefore,

$$\|s_T^R\|_2 \leq \|J_{\mathcal{A}}(x)^T\|_2 \|r\|_2 \quad \text{and} \quad J_{\mathcal{A}}(x) s_T = J_{\mathcal{A}}(x) s_T^R = J_{\mathcal{A}}(x) J_{\mathcal{A}}(x)^T r. \quad (4.29)$$

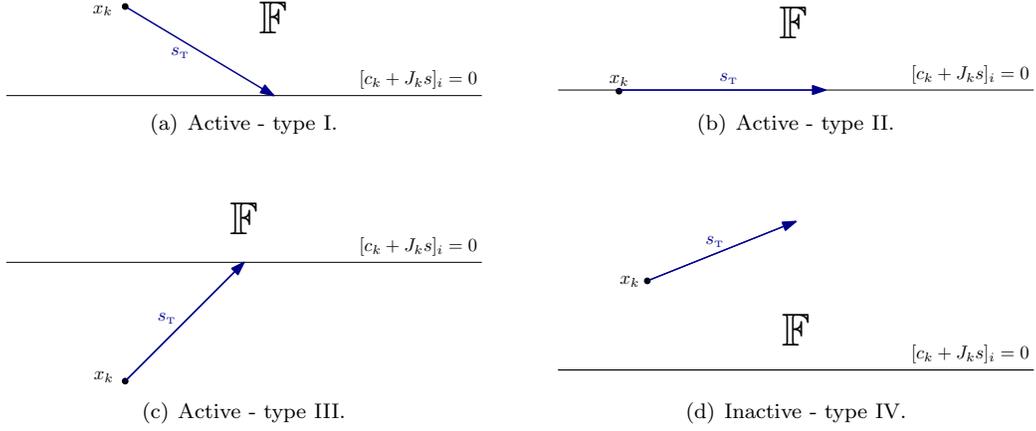


FIG. 1. The only four possibilities in a small enough neighborhood of the solution  $w^*$ . (a) For type I, we have  $c_i(x_k) > 0$ ,  $c_i^* = 0$ , and  $\nabla c_i(x_k)^T s_T < 0$ . (b) For type II, we have  $c_i(x_k) = 0$ ,  $c_i^* = 0$ , and  $\nabla c_i(x_k)^T s_T = 0$ . (c) For type III, we have  $c_i(x_k) < 0$ ,  $c_i^* = 0$ , and  $\nabla c_i(x_k)^T s_T > 0$ . (d) For type IV, we have  $c_i^* > 0$ .

Using the nonsingularity of  $J_{\mathcal{A}}(x)J_{\mathcal{A}}(x)^T$  and norm inequalities, we have

$$\|r\|_2 \leq \left\| (J_{\mathcal{A}}(x)J_{\mathcal{A}}(x)^T)^{-1} \right\|_2 \|J_{\mathcal{A}}(x)s_T\|_2. \quad (4.30)$$

This inequality, equation (4.29) and condition 2, imply

$$\|J_{\mathcal{A}}(x)s_T\|_2 \geq \frac{\|r\|_2}{\left\| (J_{\mathcal{A}}(x)J_{\mathcal{A}}(x)^T)^{-1} \right\|_2} \geq \frac{\|s_T^R\|_2}{\left\| (J_{\mathcal{A}}(x)J_{\mathcal{A}}(x)^T)^{-1} \right\|_2 \|J_{\mathcal{A}}(x)^T\|_2} \geq \frac{\|s_T^R\|_2}{\kappa_J}. \quad (4.31)$$

Using this inequality, norm inequalities, and the fact that  $\|s_T^R\|_2 = \sin(\theta)\|s_T\|_2$ , we have

$$\|J_{\mathcal{A}}(x)s_T\|_1 \geq \|J_{\mathcal{A}}(x)s_T\|_2 \geq (\sin(\theta)\|s_T\|_2)/\kappa_J. \quad (4.32)$$

Combining this with equation (4.28) we have

$$(g(x) + \sigma J(x)^T z)^T s_T \leq -s_T^T M s_T - (\kappa \sin(\theta)\|s_T\|_2)/\kappa_J. \quad (4.33)$$

We consider two cases. First suppose that  $s_T^T M s_T \geq (\lambda_{\min}/8)s_T^T s_T$ . Then it immediately follows from equation (4.33) that

$$(g(x) + \sigma J(x)^T z)^T s_T \leq -(\lambda_{\min}/8)\|s_T\|_2^2. \quad (4.34)$$

Next, suppose that  $s_T^T M s_T < (\lambda_{\min}/8)s_T^T s_T$ . Lemma 4.4 then implies that  $0 < \bar{\theta} < \theta$  and, therefore,  $0 < \sin(\bar{\theta}) < \sin(\theta)$ . We can then use this fact, equation (4.33), the Cauchy-Schwarz inequality, the definition of  $\beta_{\max}$ , and condition 3b to conclude

$$\begin{aligned} (g(x) + \sigma J(x)^T z)^T s_T &\leq \|s_T\|_2^2 \|M\|_2 - (\kappa \sin(\bar{\theta})\|s_T\|_2)/\kappa_J \\ &\leq \|s_T\|_2 (\beta_{\max}\|s_T\|_2 - (\kappa \sin(\bar{\theta}))/\kappa_J) \\ &\leq -\frac{\kappa \sin(\bar{\theta})}{2\kappa_J} \|s_T\|_2 \leq -\frac{\kappa \sin(\bar{\theta})}{2\kappa_J} \|s_T\|_2^2. \end{aligned} \quad (4.35)$$

If we define

$$c_2 = \min \left( \frac{\lambda_{\min}}{8}, \frac{\kappa \sin(\bar{\theta})}{2\kappa_j} \right) > 0, \quad (4.36)$$

then it follows from equations (4.34) and (4.35) that

$$(g(x) + \sigma J(x)^T z)^T s_{\mathcal{T}} \leq -c_2 \|s_{\mathcal{T}}\|_2^2, \quad (4.37)$$

which completes the proof.  $\square$

With a little more effort, we can show that the step from the Cauchy step  $s_k^{\text{CP}}$  to the solution of problem (4.23) is a descent direction for the underlying models. Since the Cauchy step is computed from the predictor step, it is imperative that we choose  $B_k$  so that  $s_k^{\text{P}}$  has desirable properties. The results in Section 4.1 suggest that we make the following assumption.

ASSUMPTION 4.1. *There exists a number  $\lambda_{\min}^B > 0$  such that the sequence of positive-definite matrices  $\{B_k\}$  defined in Algorithm 4.1 satisfies*

$$s^T B_k s \geq \lambda_{\min}^B s^T s \text{ for all } s \in \mathbb{R}^n \text{ and all } k \geq 0.$$

We now show that in the neighborhood of a solution  $w^*$ , the (unique) solution to

$$\underset{s \in \mathbb{R}^n}{\text{minimize}} \quad (g_k + M s_k^{\text{CP}})^T s + \frac{1}{2} s^T M s \quad \text{subject to} \quad c_{\mathcal{A}}(x_k) + J_{\mathcal{A}}(x_k)(s_k^{\text{CP}} + s) = 0 \quad (4.38)$$

is a descent direction for the underlying model determined by the matrix  $M$  (under certain assumptions).

LEMMA 4.6. *Let  $w^*$  be a minimizer for problem (NP) that satisfies the LICQ and strict complementarity and assume that  $\sigma_k > \|y^*\|_{\infty}$ , that Assumption 4.1 holds, and  $\|B_k\|_2 \leq b_B$  for some  $b_B > 0$ . It follows that there exist positive numbers  $c_2$  and  $\varepsilon_4$  such that if iterate  $k - 1$  is successful,  $w_k \in \mathcal{B}_{\varepsilon_4}(w^*)$  and  $M \in S_{\varepsilon_4}$ , then problem (4.38) is well-defined and the solution  $s_{\mathcal{T}}$  satisfies*

$$(g_k + M s_k^{\text{CP}} + \sigma_k J_k^T z_k)^T s_{\mathcal{T}} < -c_2 \|s_{\mathcal{T}}\|_2^2 \quad \text{for} \quad [z_k]_i = \begin{cases} 0 & \text{if } i \in \mathcal{V}_k, \\ -1 & \text{if } i \in \mathcal{S}_k, \end{cases} \quad (4.39)$$

where  $\mathcal{V}_k = \{i : [c_k + J_k s_k^{\text{CP}}]_i < 0\}$  and  $\mathcal{S}_k = \{i : [c_k + J_k s_k^{\text{CP}}]_i \geq 0\}$ .

*Proof.* Since the proof is very similar to Lemma 4.5, we only point out the differences. First, by choosing  $\lambda_{\min} \leq \lambda_{\min}^B$ , we have that  $B_k \in S_{\varepsilon}$  for all  $\varepsilon > 0$ . Second, since  $\sigma_k > \|y^*\|_{\infty}$  the predictor subproblem (1.4) is equivalent to problem (4.13) for the choice  $M = B_k$  provided that the trust-region constraint is inactive. Third, Lemma 4.2 shows that the solution to problem (4.13) with  $M = B_k$  correctly identifies the optimal active set if  $w_k$  is sufficiently close to  $w^*$ , so that the solution satisfies system (4.6). Equation (4.5) then shows that we can make the solution to problem (4.13) arbitrarily small by choosing  $w_k$  sufficiently close to  $w^*$ . Fourth, since iteration  $k - 1$  is successful by assumption, we know that the predictor trust-region radius is at least as large as  $\Delta_{\text{R}}$  for iteration  $k$  (see Algorithm 4.1). Combining all of this together, we know that there exists a positive number  $\varepsilon_4 < \mu/2$  ( $\mu$  is defined in Lemma 4.2) such that if  $w_k \in \mathcal{B}_{\varepsilon_4}(w^*)$  then the trust-region in the predictor step will be inactive,  $s_k^{\text{P}}$  correctly identifies the optimal active set (see Figure 2), and  $\|s_k^{\text{P}}\|_2$  is as small as we wish. The system that arises in place of (4.26) is

$$\begin{pmatrix} M & J_{\mathcal{A}}(x)^T \\ J_{\mathcal{A}}(x) & 0 \end{pmatrix} \begin{pmatrix} s \\ -q \end{pmatrix} = - \begin{pmatrix} g(x) - J_{\mathcal{A}}(x)^T y_{\mathcal{A}} + M s_k^{\text{CP}} \\ c_{\mathcal{A}}(x) + J_{\mathcal{A}}(x_k) s_k^{\text{CP}} \end{pmatrix}, \quad (4.40)$$

but since  $\|s_k^{\text{CP}}\|_2 \leq \|s_k^{\text{P}}\|_2$  we can ensure – by possibly decreasing  $\varepsilon_4$  – that parts 3a and 3b of Lemma 4.5 are once again satisfied. The rest of the proof is identical.  $\square$

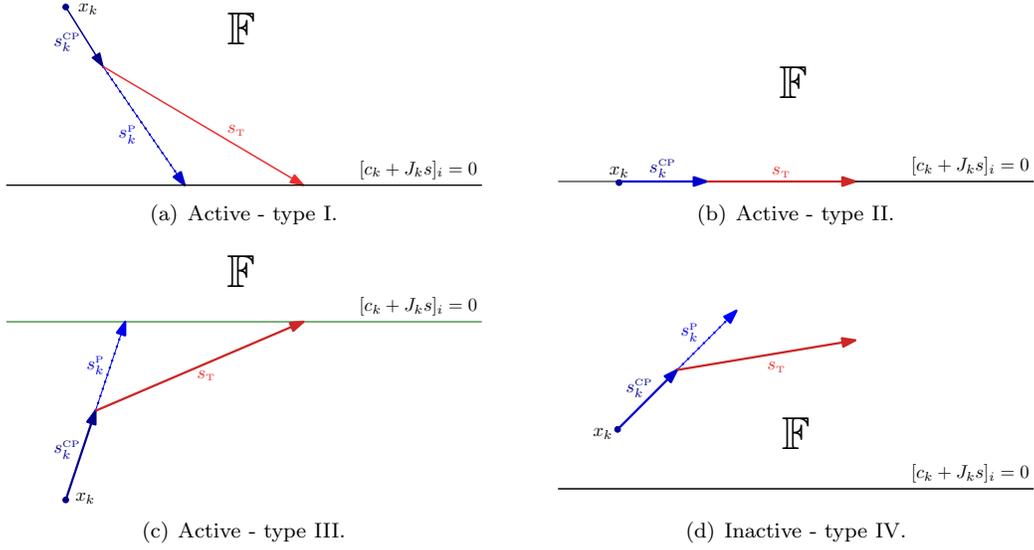


FIG. 2. *The only four possibilities in a small enough neighborhood of the solution  $w^*$ . (a) For type I, we have  $c_i(x_k) > 0$ ,  $c_i^* = 0$ , and  $\nabla c_i(x_k)^T s_T < 0$ . (b) For type II, we have  $c_i(x_k) = 0$ ,  $c_i^* = 0$ , and  $\nabla c_i(x_k)^T s_T = 0$ . (c) For type III, we have  $c_i(x_k) < 0$ ,  $c_i^* = 0$ , and  $\nabla c_i(x_k)^T s_T > 0$ . (d) For type IV, we have  $c_i^* > 0$ .*

**4.3. Local convergence with an SEQP step.** Our first rate of convergence result for Algorithm 4.1 assumes that the SQP step is computed from subproblem (SEQP) as discussed in [19, Section 2.3.2]. We restate this subproblem for convenience:

$$\begin{aligned}
 \text{(SEQP)} \quad & \underset{s \in \mathbb{R}^n}{\text{minimize}} && \bar{f}_k + (g_k + H_k s_k^P)^T s + \frac{1}{2} s^T H_k s \\
 & \text{subject to} && [J_k s]_{\mathcal{A}(s_k^P)} = 0, \quad \|s\|_2 \leq \Delta_k^S,
 \end{aligned}$$

where  $\mathcal{A}(s_k^P) = \{i : [c_k + J_k s_k^P]_i \leq 0\}$  and  $\bar{f}_k = f_k + g_k^T s_k^P + \frac{1}{2} s_k^{P,T} H_k s_k^P$ . Since this subproblem only defines multipliers for the constraints whose indices are in the set  $\mathcal{A}(s_k^P)$ , we form  $y_k^S$  by "scattering" the multipliers from subproblem (SEQP) into the appropriate locations of a zero-vector of length  $m$ .

**THEOREM 4.7 (SEQP local convergence result).** *Let  $w^*$  be a minimizer for problem (NP) that satisfies the strong second-order sufficient conditions as given by Definition 1.5. Let Assumption 4.1 hold and suppose that  $\sigma_k \equiv \sigma_b > \|y^*\|_\infty$  and  $\|B_k\|_2 \leq b_B$  for some  $b_B > 0$  and  $\sigma_b > 0$  and for all  $k \geq 0$ , the SQP step is computed from subproblem (SEQP) with the choice  $H_k \equiv H(x_k, y_k^P)$ , and  $\max_{\text{fails}} \geq 1$  in Algorithm 4.1. It follows that there exists a positive number  $\delta$ , such that if the SQP step is computed for every iteration once the first successful iterate of Algorithm 4.1 is contained in  $\mathcal{B}_\delta(w^*)$  then the sequences of iterates  $\{x_k\}$  and  $\{y_k\}$  generated by Algorithm 4.1 converge to  $x^*$  and  $y^*$  at a  $Q$ -superlinear and  $R$ -superlinear rate, respectively. Moreover, if  $H(w)$  is Lipschitz continuous in a neighborhood of  $w^*$ , then they convergence at a  $Q$ -quadratic and  $R$ -quadratic rate, respectively.*

*Proof.* We begin by setting  $\lambda_{\min} = \min(\lambda_{\min}^H/2, \lambda_{\min}^B)$  and  $\beta_{\max} = \max(b_B, \|H^*\|_2 + 1)$  in the definition of  $S(x; x^*)$  as given by (4.2), and by letting  $\beta$ ,  $\varepsilon_1$ ,  $\varepsilon_2$ , and  $\mu$  be the positive constants guaranteed by Lemmas 4.1 and 4.2; note that they satisfy  $0 < \mu \leq \varepsilon_2 \leq \varepsilon_1$  by construction, so that part (ii) of Lemma 4.1 implies

$$\mathcal{B}_{\mu/2}(w^*) \subseteq \mathcal{B}_\mu(w^*) \subseteq \mathcal{B}_{\varepsilon_2}(w^*) \subseteq \mathcal{B}_{\varepsilon_1}(w^*) \quad \text{and} \quad S_{\mu/2} \subseteq S_\mu \subseteq S_{\varepsilon_2} \subseteq S_{\varepsilon_1}, \quad (4.41)$$

where  $S_\varepsilon$  is defined by (4.2) and (4.3). By possibly decreasing  $\mu$ , we can also guarantee that if  $w$  and  $\bar{w}$  are contained in  $\mathcal{B}_\mu(w^*)$ , then the following conditions are satisfied:

$$\text{C1. } \|y - y^*\|_\infty < \sigma_b - \|y^*\|_\infty;$$

- C2.  $\|H(x, y^F(x))\|_2 \leq \|H^*\|_2 + 1$ , where  $y^F(x)$  is any estimate satisfying  $y^F(x) - y^* = O(\|x - x^*\|_2)$ ;  
 C3.  $s^T H(x, y^F(x)) s \geq (\lambda_{\min}^H/2) s^T s$  for all  $s$  satisfying  $J_A^* s = 0$ ;  
 C4. Newton's Method applied to the function  $F_N$  in equation (4.1) converges from the point  $w$  to  $w^*$ ; moreover, the Newton update  $w_+$  to  $w$  satisfies  $\|w_+ - w^*\|_2 \leq \|w - w^*\|_2$  (see Dennis and Schnabel [11, Theorem 5.2.1]).

With  $\mu$  defined, we now pick  $\delta_\Delta > 0$  so that

- C5.  $\delta_\Delta \leq \min(\mu/2, \varepsilon_4)$ , where  $\varepsilon_4$  is defined in Lemma 4.6; and  
 C6.  $\delta_\Delta \leq \eta_c \Delta_R/2$ , where  $0 < \Delta_R \leq \Delta_U$  and  $\eta_c$  are used in Algorithm 4.1.

Finally, we choose  $\delta > 0$  so that

- C7.  $\delta \leq \min(\mu/2, \varepsilon_4)$ , where  $\varepsilon_4$  is defined in Lemma 4.6; and  
 C8. if  $w \in \mathcal{B}_\delta(w^*)$ , then the following bound on the KKT equality conditions is satisfied:

$$\|f_{\text{KKT}}(w)\|_2 = \left\| \begin{pmatrix} g(x) - J(x)^T y \\ c(x) \cdot y \end{pmatrix} \right\|_2 < \frac{1}{4\beta} \min(\delta_\Delta, \eta_c \Delta_R).$$

Now let  $k-1$  be the first successful iterate generated by Algorithm 4.1 such that  $w_k \in \mathcal{B}_\delta(w^*)$ . By construction of Algorithm 4.1 and the fact that the SQP trust-region scale factor satisfies  $\tau_f \geq 1$ , we have

$$\Delta_k^P \geq \Delta_R > 0 \quad \text{and} \quad \Delta_k^S \geq \tau_f \Delta_R \geq \Delta_R > 0. \quad (4.42)$$

Since equation (4.41) and C7 imply that  $w_k \in \mathcal{B}_{\mu/2}(w^*)$ , it follows from C8, Lemma 4.2, and (4.42) that  $J_A(x_k)$  has full row rank and if  $M \in S_{\mu/2}$  then  $x_k(M)$  correctly identifies the optimal active set and satisfies

$$\|x_k(M) - x_k\|_\infty \leq 2\beta \|F_M(w_k; w_k)\|_2 \leq \frac{1}{2} \min(\delta_\Delta, \eta_c \Delta_R) \leq \frac{1}{2} \eta_c \min(\Delta_k^P, \Delta_k^S). \quad (4.43)$$

We now observe that  $B_k \in S_{\mu/2}$  by construction and is, in fact, positive definite. Furthermore, since C1 implies  $\sigma_k = \sigma_b > \|y_k(B_k)\|_\infty$  and equation (4.43) implies  $\|x_k(B_k) - x_k\|_\infty \leq (\eta_c/2) \Delta_k^P < \Delta_k^P$ , we must have  $w_k^P = w_k(B_k)$ . Thus the solution to the predictor subproblem satisfies  $s_k^P = x_k(B_k) - x_k$ , correctly identifies the optimal active set, and is not restricted by the trust-region constraint, i.e.,  $s_k^P$  is the solution to (4.23) with  $M = B_k$ .

Next we observe that C2 and C3 imply that  $H_k \in S(x^*; x^*) \subset S_{\mu/2}$ . Therefore, the point  $w_k(H_k)$  is well-defined, identifies the optimal active set, and is the unique minimizer of problem (4.13) in a neighborhood of  $w_k$  for  $M = H_k$ . Since  $J_A(x_k)$  has full row rank, it follows from (4.41), part (iii) of Lemma 4.1, and [1, Lemma 1.27] that subproblem (SEQP) has  $s_k^S$  as a unique solution. It follows that if  $\|x_k(H_k) - (x_k + s_k^P)\|_2 \leq \Delta_k^S$ , then  $s_k^S = x_k(H_k) - (x_k + s_k^P)$  (see Figure 3). Using the triangle inequality, the definition of  $w_k(B_k)$ , and equation (4.43), we have

$$\|x_k(H_k) - (x_k + s_k^P)\|_2 \leq \|x_k(H_k) - x_k\|_2 + \|s_k^P\|_2 = \|x_k(H_k) - x_k\|_2 + \|x_k(B_k) - x_k\|_2 \leq \eta_c \Delta_k^S \leq \Delta_k^S. \quad (4.44)$$

Thus, if  $s_k^P + s_k^S$  satisfies condition (1.7) then  $s_k = s_k^P + s_k^S$  and it follows that  $x_k + s_k = x_k(H_k)$  and  $y_k^S = y_k(H_k)$ . We now show that this is the case. If  $s_T \neq 0$  then C5 and Lemma 4.6 show that the vector  $s_T$ , which satisfies  $s_k^{\text{CP}} + s_T = x_k(H_k)$ , is a descent direction for the model  $M_k^H$ . Therefore,  $M_k^H(s_k^P + s_k^S) < M_k^H(s_k^{\text{CP}})$  so that condition (1.7) is satisfied by  $s_k^P + s_k^S$ . On the other hand, if  $s_T = 0$  then it follows that  $s_k^P = s_k^{\text{CP}}$  and  $s_k^S = 0$ , so that  $s_k^P + s_k^S = s_k^{\text{CP}}$  trivially satisfies condition (1.7).

If  $x_k + s_k$  is a successful step, then  $x_{k+1} \leftarrow x_k + s_k$ ; otherwise, the update  $x_{k+1} \leftarrow x_k + s_k$  is still made since  $\text{max\_fails} \geq 1$ , but a nonmonotone phase is entered. In either case, the vector  $w_{k+1}$  is the same vector that is obtained by performing one step of Newton's Method on the function  $F_N$  (see equation (4.1)) from the point  $(x_k, y_k^F)$  with the understanding that  $y_{k+1}$  is formed by "scattering"  $y_k^S$  into a zero-vector of length  $m$ . Since Algorithm 4.1 makes the assignment  $w_{k+1} \leftarrow w_k^S$ , it follows from C4 that  $w_{k+1} \in \mathcal{B}_\delta(w^*)$  and so the same argument may be repeated starting from the point  $w_{k+1}$ ; this results in a vector  $w_{k+2}$  that has the same properties as  $w_{k+1}$  and is, in fact, equivalent to performing one step of Newton's Method on the



restate this subproblem for convenience:

$$\begin{aligned}
 \text{(SIQP-E)} \quad & \underset{s \in \mathbb{R}^n}{\text{minimize}} && \bar{f}_k + (g_k + H_k s_k^{\text{CP}})^T s + \frac{1}{2} s^T H_k s + \sigma_k \| [c_k + J_k(s_k^{\text{CP}} + s)]_{\bar{V}_k} \|_1 \\
 & \text{subject to} && [c_k + J_k(s_k^{\text{CP}} + s)]_{S_k} \geq 0, \\
 & && (g_k + H_k s_k^{\text{CP}} + \sigma_k J_k^T z_k)^T s \leq 0, \quad \|s\|_\infty \leq \Delta_k^s,
 \end{aligned}$$

where  $z_k \in \mathbb{R}^m$  is defined by (4.39),  $\bar{f}_k = f_k + g_k^T s_k^{\text{CP}} + \frac{1}{2} s_k^{\text{CP}T} H_k s_k^{\text{CP}}$ , and  $(g_k + H_k s_k^{\text{CP}} + \sigma_k J_k^T z_k)^T s \leq 0$  is the so-called “descent-constraint”.

We begin by making two observations. First, since problem (SIQP-E) is generally a nonconvex *inequality* constrained QP, we will need to assume that the solution  $s_k^s$  is one of minimal norm; a similar assumption is made by Robinson in [36, Section 3]. Although this assumption is not ideal, it is not too offensive within our setting; if we use an active-set QP solver with a hot start based on the active set obtained from the predictor step, then the solution to subproblem (SIQP-E) will ultimately be the same as the solution to subproblem (SEQP). Theorem 4.7 validates that this is a good step and, therefore, if this strategy is used then the “minimum-norm solution” assumption is not necessary. The second observation is that if the SQP step is chosen to be one of minimal norm, then the proof of Theorem 4.7 carries over since 1) the Cauchy step  $s_k^{\text{CP}}$  satisfies  $\|s_k^{\text{CP}}\|_\infty \leq \|s_k^p\|_\infty$ ; 2) the vector  $x_k(H_k) - s_k^{\text{CP}}$  is a solution to subproblem (SIQP-E); and 3) Lemma 4.6 guarantees that the descent-constraint does not interfere with the step from  $s_k^{\text{CP}}$  to  $x_k(H_k)$ .

**THEOREM 4.9** (SIQP-E local convergence result). *Let  $w^*$  be a minimizer for problem (NP) that satisfies the strong second-order sufficient conditions as given by Definition 1.5. Let Assumption 4.1 hold and assume that  $\sigma_k \equiv \sigma_b > \|y^*\|_\infty$  and  $\|B_k\|_2 \leq b_B$  for some  $b_B > 0$  and  $\sigma_b > 0$  and all  $k \geq 0$ , the SQP step is computed from subproblem (SIQP-E) with the choice  $H_k \equiv H(x_k, y_k^r)$ , and  $\text{max\_fails} \geq 1$  in Algorithm 4.1. It follows that there exists a positive number  $\delta$ , such that if the SQP step is a solution of minimal-norm and is computed for every iteration once the first successful iterate of Algorithm 4.1 is contained in  $\mathcal{B}_\delta(w^*)$ , then the sequences of iterates  $\{x_k\}$  and  $\{y_k\}$  converge to  $x^*$  and  $y^*$  at a  $Q$ -superlinear and  $R$ -superlinear rate, respectively. Moreover, if  $H(w)$  is Lipschitz continuous in a neighborhood of  $w^*$ , then they converge at a  $Q$ -quadratic and  $R$ -quadratic rate, respectively.*

*Proof.* Follows from the proof of Theorem 4.7, the discussion above, and Lemma 4.5.  $\square$

**5. Numerical results.** We tested Algorithm 4.1 on the Hock-Schittkowski (HS) [23] test problems. The HS test suite is comprised of generally small and dense problems that are very useful during early stages of code development; the small size of the problems allows for relatively careful inspection of each problem. We note that problem *HS87* has been removed from the test set since the objective function is not continuous.

To be precise, we tested three variants of Algorithm 4.1—they differ in how we compute the SQP step and update the penalty parameter. In the first variant we computed the SQP step from the inequality constrained subproblem (SIQP-E) on page 25 and updated the penalty parameter by using “steering” as briefly described at the end of Section 3. In the second variant we computed the SQP step from the equality constrained subproblem (SEQP) on page 22 and again used steering to update the penalty parameter. Finally, in the third variant we computed the SQP step from the inequality constrained subproblem (SIQP-E) and updated the penalty parameter by using Algorithm 3.1.

Since the problems in the test set are of small dimension, we chose to update the positive-definite matrix  $B_k$  in the predictor subproblem (1.5) by using the BFGS update. To perform this update, we used the vectors  $s_k$  and  $d_k = \nabla_x \mathcal{L}(x_k + s_k, y_{k+1}) - \nabla_x \mathcal{L}(x_k, y_{k+1})$ . If these vectors did not result in a sufficiently positive-definite update, then we used the damped vectors given by equations (2.11) and (2.12). For simplicity, we chose  $B_0 = I$ .

In all cases for the SQP and Cauchy step calculations we chose  $H_k \equiv H(x_k, y_k^p)$ , where  $y_k^p$  is the multiplier vector from the predictor subproblem. We solved both the (convex) quadratic program (1.5) and the (generally indefinite) quadratic program (SIQP-E) using the GALAHAD [18] package QPC, which is a “cross-over” QP solver. In the first phase, QPC calls the GALAHAD interior-point QP solver QPB [8] to

compute an approximate solution and an estimate of the optimal active set. In the second phase, QPC calls the GALAHAD active-set QP solver QPA [20] to “refine” the approximate solution from the first phase. To solve the equality constrained QP (SEQP) we used the GALAHAD package EQP, which has been designed to solve problems of precisely this form. We should mention that most of the GALAHAD packages, including the QP solvers mentioned above, use the sparse solvers MA48 and MA57 from [24] to handle the required systems. The modular design of all the GALAHAD packages makes it easy to call these subroutines as needed.

The following parameters were used in all cases: primal/dual/complementarity slackness tolerances  $\tau_p = \tau_d = \tau_c = 1.0e^{-5}$ , successful/very successful tolerances  $\eta_s = 0.01$  and  $\eta_{vs} = 0.7$ , maximum predictor trust-region radius  $\Delta_U = 1000$ , trust-region “reset” radius  $\Delta_R = 1.0e^{-4}$ , SQP trust-region scale factor  $\tau_f = 4.0$ , number of nonmonotone steps allowed  $max\_fails = 1$ , and trust-region contraction and expansion factors  $\eta_c = 0.1$  and  $\eta_e = 5.0$ . We used an initial penalty parameter of  $\sigma = 1.0$  for the first and second strategies, and an initial penalty parameter of  $\sigma = 9.0$  for the third strategy. The larger (seemingly arbitrary) initial penalty parameter for the third variant was chosen based on performance and seems to be related to the less dynamic nature of the update as compared with steering.

Tables 1 and 2 on pages 27 and 28 give our preliminary numerical results for these three strategies; column SIQP-steer corresponds to the first strategy, column SEQP-steer corresponds to the second strategy, and column SIQP-seq corresponds to the third strategy. For each strategy we have recorded the number of function evaluations #fc and the number of gradient evaluations #gJ. Note that if the quantity #fc - #gJ is positive, then its value represents the number of unsuccessful iterations, i.e., the number of times that the trust-region radii were necessarily decreased in order to obtain good agreement between the faithful model  $M_k^H$  and the merit function  $\phi$ . An F indicates that more than 500 evaluations were required and an FQP indicates that the QP solver failed.

We believe that these preliminary results show that our method works quite well. The strictly convex predictor subproblem combined with either SQP subproblem (SIQP-E) or (SEQP) typically generates iterates that rapidly converge to a solution; the nonmonotone approach avoids the Maratos effect. The results also indicate that using steering to update the penalty parameter generally performs better than the method discussed in Section 3; this agrees with [4, 5]. In particular, steering was essential in solving *HS93* for otherwise the merit function converged to minus infinity, the constraints blew up, and the method failed. In a less clear manner, the update to the penalty parameter is important in solving *HS56*. When the update in Section 3 was used, the merit function again converged to minus infinity; the same occurred when the SQP step was computed from subproblem (SEQP) and steering was used.

**6. Conclusions and future work.** In [19], we proved global convergence of a second-derivative SQP method for minimizing the  $\ell_1$ -penalty function for a fixed value of the penalty parameter. This algorithm requires the definition of a positive-definite matrix that approximates the Hessian of the Lagrangian; in Section 2 we considered two possibilities. The first was a simple diagonal approximation that attempted to crudely estimate the size of the Hessian of Lagrangian. The second approach used a *limited*-memory BFGS update. We proceeded to show how the resultant dense QP could be transformed into an equivalent QP whose sparsity is essentially the same as the initial approximation (which will be sparse in practice). In Section 3 we gave details on a simple strategy for updating the penalty parameter based on minimizing the  $\ell_1$ -penalty function over a sequence of increasing values of the penalty parameter. Although the basic idea is certainly not new [3, 25, 26, 30, 34, 38, 42], the details of our very simple strategy have not been published to our knowledge. The main result of that section is that the penalty parameter will stay uniformly bounded and that an *approximate* solution to the nonlinear programming problem will be computed in a finite number of iterations. However, the primary purpose of this paper was to study the local convergence properties of a *nonmonotone* variant of the algorithm given in [19]. In Section 4 we gave two local convergence results—the first applies when the SQP step is computed from an *equality* constrained subproblem (the so-called SEQP approach) and the second applies when the SQP step is computed from an *inequality* constrained subproblem (the so-called SIQP approach). Both results show superlinear convergence of the iterates to a solution satisfying the strong second-order sufficiency conditions; under slightly stronger assumptions on

TABLE 1

We record the number of function and gradient evaluations required, and note that the difference between these values indicates the number of unsuccessful steps attempted; an F indicates that more than 500 evaluations were required.

	SIQP-steer		SEQP-steer		SIQP-seq	
Prob	#fc	#gJ	#fc	#gJ	#fc	#gJ
HS1	46	25	39	24	46	25
HS2	11	8	11	8	11	8
HS3	3	3	3	3	3	3
HS4	3	2	3	2	3	2
HS5	10	6	9	6	10	6
HS6	3	3	5	5	3	3
HS7	10	9	9	8	12	10
HS8	6	6	6	6	6	6
HS9	3	3	4	4	3	3
HS10	10	10	10	10	10	10
HS11	6	6	6	6	6	6
HS12	7	7	6	6	14	11
HS13	15	12	29	27	56	54
HS14	5	5	5	5	5	5
HS15	7	7	7	7	20	20
HS16	4	4	4	4	5	5
HS17	7	7	7	7	10	10
HS18	9	8	7	7	9	8
HS19	6	6	6	6	13	13
HS20	8	8	4	4	16	15
HS21	2	2	2	2	2	2
HS22	2	2	2	2	2	2
HS23	6	6	6	6	6	6
HS24	3	3	4	4	3	3
HS25	1	1	1	1	1	1
HS26	17	17	16	16	17	17
HS27	12	11	14	13	13	12
HS28	2	2	3	3	2	2
HS29	6	6	6	6	31	23
	SIQP-steer		SEQP-steer		SIQP-seq	
Prob	#fc	#gJ	#fc	#gJ	#fc	#gJ
HS30	10	10	10	10	10	10
HS31	8	8	9	7	5	5
HS32	3	3	4	4	3	3
HS33	5	5	14	11	5	5
HS34	9	8	9	8	34	28
HS35	2	2	3	3	2	2
HS36	3	3	4	4	3	3
HS37	5	5	5	5	5	5
HS38	84	38	65	46	84	38
HS39	12	11	13	11	12	11
HS40	4	4	4	4	4	4
HS41	2	2	2	2	2	2
HS42	6	6	4	4	4	4
HS43	7	7	8	8	12	10
HS44	2	2	3	3	2	2
HS45	3	3	5	4	3	3
HS46	16	16	16	16	16	16
HS47	15	15	20	18	19	17
HS48	2	2	3	3	2	2
HS49	16	16	16	16	16	16
HS50	9	9	6	6	9	9
HS51	2	2	2	2	2	2
HS52	2	2	2	2	2	2
HS53	2	2	2	2	2	2
HS54	9	9	14	14	22	22
HS55	2	2	2	2	2	2
HS56	110	82	F	F	F	F
HS57	8	6	6	6	8	6
HS59	10	9	8	8	18	15

the second-derivatives, the convergence is quadratic. We note that the second convergence result required that the so-called “descent-constraint” be inactive at a “minimum-norm” solution to problem (SIQP-E) on page 25; this result was presented as Lemma 4.6. In Section 5 we gave preliminary numerical results for the Hock-Schittkowski test problems. Our first set of results were based on using “steering” [4, 5] to update the penalty parameter and computing the SQP step from the inequality constrained subproblem (SIQP-E). Although this requires “solving” a potentially indefinite quadratic program, which is generally perceived as a bad idea, the results are quite good. Our second set of results also used steering to update the penalty parameter, but instead computed the SQP step from the equality constrained subproblem (SEQP) on page 22. Our last set of results were based on solving subproblem (SIQP-E) for the SQP step, but updating the penalty parameter by monitoring the norm of the constraint violation over a sequence of approximate minimizers of the merit function (see Section 3). We stress that these results are preliminary and that they are not intended to compare the (SIQP) approach with the (SEQP) approach, but rather to show that both approaches have the potential to be successful in practice. We must also mention that in

TABLE 2

We record the number of function and gradient evaluations required, and note that the difference between these values indicates the number of unsuccessful steps attempted; an F indicates that more than 500 evaluations were required and an FQP indicates that the QP solver QPC failed.

	SIQP-steer		SEQP-steer		SIQP-seq	
Prob	#fc	#gJ	#fc	#gJ	#fc	#gJ
HS60	7	7	7	7	7	7
HS61	5	5	5	5	14	11
HS62	12	6	8	7	12	6
HS63	7	7	7	7	7	7
HS64	15	15	20	17	23	23
HS65	6	6	6	6	22	16
HS66	4	4	4	4	4	4
HS67	7	7	10	10	14	14
HS68	27	20	38	27	20	16
HS69	24	19	42	30	45	34
HS70	21	17	41	33	21	17
HS71	5	5	5	5	5	5
HS72	16	15	15	14	46	45
HS73	3	3	3	3	3	3
HS74	8	8	8	8	8	8
HS75	8	8	8	8	8	8
HS76	2	2	4	4	2	2
HS77	12	12	12	12	12	12
HS78	4	4	4	4	4	4
HS79	5	5	5	5	5	5
HS80	7	7	8	8	7	7
HS81	6	6	6	6	6	6
HS83	6	6	6	6	16	16
HS84	4	4	4	4	5	5
HS85	8	8	13	12	21	20
HS86	4	4	4	4	4	4
HS88	20	18	25	21	43	41
HS89	23	21	64	46	45	43

	SIQP-steer		SEQP-steer		SIQP-seq	
Prob	#fc	#gJ	#fc	#gJ	#fc	#gJ
HS90	44	33	40	30	42	34
HS91	43	32	32	25	73	61
HS92	35	25	34	27	64	37
HS93	18	15	6	6	F	F
HS95	3	3	2	2	3	3
HS96	3	3	2	2	3	3
HS97	4	4	5	5	4	4
HS98	4	4	5	5	4	4
HS99	5	5	FQP	FQP	75	34
HS100	10	9	13	10	10	9
HS101	34	27	68	40	67	52
HS102	28	21	55	34	44	39
HS103	26	20	27	20	79	62
HS104	17	14	19	14	14	11
HS105	21	14	31	23	21	14
HS106	103	101	122	66	103	101
HS107	6	6	6	6	10	10
HS108	12	9	12	10	242	177
HS109	9	9	9	9	10	10
HS110	9	5	8	6	9	5
HS111	28	23	41	31	23	19
HS112	11	11	50	50	11	11
HS113	5	5	6	6	5	5
HS114	142	142	13	13	136	134
HS116	F	F	F	F	F	F
HS117	10	10	11	11	10	10
HS118	3	3	12	12	3	3
HS119	7	7	8	8	7	7

essentially simultaneous work, Morales, Nocedal, and Wu [27] have been developing an  $\ell_1$ -SQP *line-search* algorithm based on subproblem (SEQP) in which the optimal active set is predicted by solving a convex QP. During initial testing, the authors have reported that the use of the convex QP to predict the active set combined with the additional equality constrained subproblem is effective.

There are still many ideas to be explored and options to be added to our evolving and (soon to be) freely available Fortran 95 GALAHAD [18] package S2QP. First, we want to explore whether solving an SQP subproblem with perturbed constraints (a “correction step”) is sufficient for avoiding the Maratos effect. This deserves further investigation since if this is the case, then fast convergence would be guaranteed without the need for a nonmonotone approach. Second, we plan on experimenting with modern approaches for adjusting/defining the trust-region radius [13, 14, 39] as well as exploring new possibilities. Third, we want to examine the consequences of removing the trust-region constraint from the predictor step computation; the problem is strictly convex and, therefore, well-defined without this constraint. Fourth, we will investigate the notion of “steering”, as it pertains to our setting, as a reliable strategy for dynamically updating the penalty

parameter. Finally, we plan on extensively testing the package on larger problems from the CUTer [17] test set.

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