

# Generic identifiability and second-order sufficiency in tame convex optimization

J. Bolte <sup>\*</sup>; A. Daniilidis <sup>†</sup> and A. S. Lewis <sup>‡</sup>

Date: 19th January 2009

## Abstract

We consider linear optimization over a fixed compact convex feasible region that is semi-algebraic (or, more generally, “tame”). Generically, we prove that the optimal solution is unique and lies on a unique manifold, around which the feasible region is “partly smooth”, ensuring finite identification of the manifold by many optimization algorithms. Furthermore, second-order optimality conditions hold, guaranteeing smooth behavior of the optimal solution under small perturbations to the objective.

**Key words:** Tame optimization, partial smoothness, strong maximizer, o-minimal structure.

**AMS Subject Classification** *Primary* 90C31 ; *Secondary* 32B20, 90C05, 49K40

## 1 Introduction

“Identification” in constrained optimization signifies an important idea both in theory and for algorithms. Sensitivity analysis, the theory of how optimal solutions behave under data perturbations, depends on identifying active constraints and verifying associated optimality conditions. A variety of

---

<sup>\*</sup>This author acknowledges the support of the grant ANR-05-BLAN-0248-01 (France).

<sup>†</sup>This author’s research was partially supported by the grants ANR-05-BLAN-0248-01 (France) and MTM2008-06695-C03-03 (Spain)

<sup>‡</sup>This author’s research was partially supported by the ANR grant ANR-05-BLAN-0248-01 (France) and by the National Science Foundation Grant DMS-0806057 (USA)

practical algorithms for inequality-constrained problems aim to identify the active constraints: once the identification is successful, we have essentially converted to the easier, equality-constrained case.

An early survey of identification techniques, for optimization over polyhedra and generalizations, appears in [3]. For general convex feasible regions, a more abstract approach appeals, in part for its theoretical elegance, and in part because for constraints more complex than simple inequalities, such as the semidefinite inequalities common in modern optimization, simply deciding whether a constraint is active or not fails to capture crucial finer details. Such an abstract approach, based on the idea of an “identifiable surface”, appeared in [17]. As shown in [12], this idea has an equivalent but more geometric description: the surface turns out to be a manifold contained in the feasible region satisfying a property called “partial smoothness”. As well as its geometric transparency, the notion of partial smoothness has the merit of extending naturally to the nonconvex case. In this work, however, we confine ourselves to convex feasible regions.

Our goal here is to show that partial smoothness is a common phenomenon. Certainly the property can fail, either because the feasible region is somehow pathological, or because of the failure of the typical regularity conditions needed for standard sensitivity analysis. A good illustration is the convex optimization problem over  $\mathbb{R}^3$ ,

$$\inf\{w : w \geq (|u| + |v|)^2\}. \tag{1}$$

As we perturb the linear objective function slightly, the corresponding optimal solutions describe not one but two distinct manifolds.

Nonetheless, very generally, partial smoothness is indeed typical for linear optimization over a fixed compact convex feasible region  $F \subset \mathbb{R}^n$ . Specifically, we prove that if  $F$  is *semi-algebraic*—a finite union of sets defined by finitely many polynomial inequalities—or, more generally, “tame”, then, except for objectives lying in some exceptional set of dimension strictly smaller than  $n$ , the corresponding optimal solution is unique, and  $F$  is partly smooth around a corresponding unique manifold. Furthermore, a second-order sufficient optimality condition holds. A variety of algorithms will therefore identify the manifold and converge well, and standard sensitivity analysis applies.

Various authors have shown that, for some suitably structured convex optimization problem, the set of instances for which the optimal solution

has some beneficial property (such as “well-posedness” [5]) is generic. An interesting recent example is [8]. By contrast, we assume nothing about our feasible region, beyond its semi-algebraic or tame nature. Remarkably, nonetheless, generic problems are very well behaved.

## 2 Preliminaries and notation

Throughout the manuscript we deal with a finite-dimensional Euclidean space  $\mathbb{R}^n$  equipped with the usual scalar product  $\langle \cdot, \cdot \rangle$  and the corresponding Euclidean norm  $\|\cdot\|$ . We denote by  $B(x, r)$  the closed ball with center  $x \in \mathbb{R}^n$  and radius  $r > 0$ . We simply denote by  $\mathbb{B}$  the closed unit ball  $B(0, 1)$  and by  $S^{n-1}$  its boundary, that is, the unit sphere of  $\mathbb{R}^n$ . Given any  $E \subset \mathbb{R}^n$ , we denote by  $\text{ri } E$  its relative interior and by  $\overline{E}$  its closure.

### Preliminaries on variational analysis

We refer to [14] and [15] for basic facts about convex and variational analysis that we use.

Let  $X, Y$  be metric spaces and  $T : X \rightrightarrows Y$  be a set-valued mapping. We say that  $T$  is *outer semicontinuous* at a point  $\bar{x} \in X$  if, for any sequence of points  $x_r \in X$  converging to  $\bar{x}$  and any sequence of points  $y_r \in T(x_r)$  converging to  $\bar{y}$ , we must have  $\bar{y} \in T(\bar{x})$ . On the other hand, we say that  $T$  is *inner semicontinuous* at  $\bar{x}$  if, for any sequence of points  $x_r \in X$  converging to  $\bar{x}$  and any point  $\bar{y} \in Y$ , there exists a sequence  $y_r \in Y$  converging to  $\bar{y}$  such that  $y_r \in T(x_r)$  for all large  $r$ . If both properties hold, we call  $T$  *continuous* at  $\bar{x}$ .

Consider a nonempty closed convex set  $F \subset \mathbb{R}^n$ . The *normal cone*  $N_F(x)$  at a point  $x \in F$  is defined as follows:

$$N_F(x) = \left\{ c \in \mathbb{R}^n : \langle c, x' - x \rangle \leq 0, \forall x' \in F \right\}. \quad (2)$$

It is standard and easy to check that the mapping  $x \mapsto N_F(x)$  is outer semicontinuous on  $F$ . In a slightly different context, for a point  $x$  in a smooth submanifold  $\mathcal{M}$  of  $\mathbb{R}^n$ , we denote by  $N_{\mathcal{M}}(x)$  the normal space in the usual sense of elementary differential geometry, that is, the orthogonal complement in  $\mathbb{R}^n$  of the tangent space  $T_{\mathcal{M}}(x)$  of  $\mathcal{M}$  at  $x$ .

## Preliminaries on partial smoothness

We recall from [12] the definition of partial smoothness, specialized to the convex case.

*Definition 1.* A closed convex set  $F \subset \mathbb{R}^n$  is called *partly smooth* at a point  $\bar{x} \in F$  relative to a set  $\mathcal{M} \subset F$  if the following properties hold:

- (i)  $\mathcal{M}$  is a  $C^2$  submanifold of  $\mathbb{R}^n$  (called the *active manifold*) containing  $\bar{x}$ .
- (ii) The set-valued mapping  $x \mapsto N_F(x)$ , restricted to the domain  $\mathcal{M}$ , is continuous at  $\bar{x}$ .
- (iii)  $N_{\mathcal{M}}(\bar{x}) = N_F(\bar{x}) - N_F(\bar{x})$ .

While not obvious from the above definition, the active manifold for a partly smooth convex set is locally unique around the point of interest: see [13, Cor. 4.2].

Geometrically, condition (iii) guarantees “sharpness” around a kind of “ridge” in the set  $F$  defined by the active manifold, as illustrated in the following simple example.

*Example 1.* In  $\mathbb{R}^3$ , define

$$\begin{aligned} F &= \{(u, v, w) : w \geq u^2 + |v|\}, \\ \mathcal{M} &= \{(t, 0, t^2) : t \in (-1, 1)\}. \end{aligned}$$

Then the set  $F$  is partly smooth at the point  $\bar{x} = (0, 0, 0)$  relative to the one-dimensional manifold  $\mathcal{M}$ .

The following example illustrates the importance of normal cone continuity.

*Example 2 (failure of normal cone continuity).* In  $\mathbb{R}^3$ , consider the set and manifold

$$\begin{aligned} F &= \{(u, v, w) : v \geq 0, w \geq 0, v + w \geq u^2\} \\ \mathcal{M} &= \{(t, t^2, 0) : t \in (-1, 1)\}. \end{aligned}$$

Then  $F$  is convex and conditions (i) and (iii) of Definition 1 are satisfied at the point  $\bar{x} = (0, 0, 0)$ . But condition (ii) fails, since the normal cone mapping is discontinuous there, relative to  $\mathcal{M}$ .

For the purposes of sensitivity analysis, partial smoothness is most useful when combined with a second-order sufficiency condition, captured by the following definition.

*Definition 2.* Consider a vector  $\bar{c} \in \mathbb{R}^n$  and a closed convex set  $F \subset \mathbb{R}^n$  that is partly smooth at a point  $\bar{x} \in \operatorname{argmax}_F \langle \bar{c}, \cdot \rangle$  relative to a manifold  $\mathcal{M}$ . We say that  $\bar{x}$  is *strongly critical* if the following properties hold:

- (i)  $\bar{c} \in \operatorname{ri} N_F(\bar{x})$ .
- (ii) There exists  $\delta > 0$  such that

$$\langle \bar{c}, \bar{x} \rangle \geq \langle \bar{c}, x \rangle + \delta \|x - \bar{x}\|^2, \quad \text{for all } x \in \mathcal{M} \text{ near } \bar{x}.$$

Condition (i) can be interpreted as a kind of “strict complementarity” condition, while condition (ii) concerns quadratic decay. Notice that the above definition yields uniqueness of the maximizer  $\bar{x}$  of  $\bar{c}$ , as well as good sensitivity properties, as the following result shows: see [12].

**Theorem 3 (second-order sufficiency).** *Consider a closed convex set  $F \subset \mathbb{R}^n$  and assume that  $F$  is partly smooth at some point  $\bar{x} \in F$  and that  $\bar{x}$  is strongly critical point for the problem  $\max_F \langle \bar{c}, \cdot \rangle$ , relative to a manifold  $\mathcal{M}$ . Then for all vectors  $c \in \mathbb{R}^n$  sufficiently near  $\bar{c}$ , the perturbed problem  $\max_F \langle c, \cdot \rangle$  has a unique optimal solution  $x_c \in \mathcal{M}$ . The map  $c \mapsto x_c$  is  $C^1$  around  $\bar{c}$ .*

### Preliminaries on tame geometry

Let us first recall the definitions of an “o-minimal structure” (see for instance [4], [6] or [9] and references therein).

*Definition 4.* An *o-minimal structure* on  $(\mathbb{R}, +, \cdot)$  is a sequence of Boolean algebras  $\mathcal{O} = \{\mathcal{O}_n\}$ , where each algebra  $\mathcal{O}_n$  consists of subsets of  $\mathbb{R}^n$ , called *definable* (in  $\mathcal{O}$ ), and such that for every dimension  $n \in \mathbb{N}$  the following properties hold.

- (i) For any set  $A$  belonging to  $\mathcal{O}_n$ , both  $A \times \mathbb{R}$  and  $\mathbb{R} \times A$  belong to  $\mathcal{O}_{n+1}$ .
- (ii) If  $\Pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$  denotes the canonical projection, then for any set  $A$  belonging to  $\mathcal{O}_{n+1}$ , the set  $\Pi(A)$  belongs to  $\mathcal{O}_n$ .

- (iii)  $\mathcal{O}_n$  contains every set of the form  $\{x \in \mathbb{R}^n : p(x) = 0\}$ , for polynomials  $p : \mathbb{R}^n \rightarrow \mathbb{R}$ .
- (iv) The elements of  $\mathcal{O}_1$  are exactly the finite unions of intervals and points.

When  $\mathcal{O}$  is a given o-minimal structure, a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  (or a set-valued mapping  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ ) is called *definable* (in  $\mathcal{O}$ ) if its graph is definable as a subset of  $\mathbb{R}^n \times \mathbb{R}^m$ .

If a subset  $A$  of  $\mathbb{R}^n$  has the property that its intersection with every ball is definable in some o-minimal structure, then it is sometimes called *tame*. In this work we are concerned primarily with bounded sets: in that context, we use the terms “tame” and “definable” interchangeably.

Semi-algebraic sets constitute an o-minimal structure, as a consequence of the Tarski-Seidenberg principle, but richer structures also exist. In particular, the Gabrielov theorem implies that “subanalytic” sets are tame. These two structures in particular provide rich practical tools, because checking semi-algebraicity or subanalyticity of sets in concrete problems of variational analysis is often easy. We refer to [1], [2], and [7] for more details.

Definable sets and functions enjoy many structural properties. In particular, every definable set can be written as a finite disjoint union of manifolds (or “strata”) that fit together in a regular “stratification”: see [6, §4.2]. In particular, the *dimension* of the set is the maximum of the dimensions of the strata, a number independent of the stratification: see [4, Definition 9.14] for more details. We call a definable subset of a definable set *generic* if its complement has strictly smaller dimension.

In this paper we make fundamental use of a stratification result. We present a particular case—adapted to our needs—of a more general result: see [6, p. 502, §1.19 (2)] or [16] for the statement in its full generality. The result describes a decomposition of the domain of a definable function into subdomains on which the function has “constant rank”: a smooth function has *constant rank* if its derivative has constant rank throughout its domain. Such functions have a simple canonical form: they are locally equivalent to projections, as described by the following result from basic differential geometry (see [10, Thm 7.8]).

**Proposition 5 (Constant Rank Theorem).** *Let  $M_1$  and  $M_2$  be two differentiable manifolds, of dimensions  $m_1$  and  $m_2$  respectively, and let  $g : M_1 \rightarrow M_2$  be a differentiable mapping of constant rank  $r$ . Then for every point*

$x \in M_1$ , there exist neighborhoods  $O_i$  of zero in  $\mathbb{R}^{m_i}$  and local diffeomorphisms  $\psi_i : O_i \rightarrow M_i$  (for  $i = 1, 2$ ) with  $\psi_1(0) = x$  and  $\psi_2(0) = g(x)$ , such that mapping  $\psi_2^{-1} \circ g \circ \psi_1$  is just the projection  $\pi : O_1 \rightarrow O_2$  defined by

$$\pi(y_1, y_2, \dots, y_{m_1}) = (y_1, y_2, \dots, y_r, 0, \dots, 0) \in \mathbb{R}^{m_2}, \quad (y \in O_1). \quad (3)$$

The stratification result we use follows.

**Proposition 6 (Constant rank stratification).** *Let  $f : M \rightarrow \mathbb{R}^n$  be a definable function, where  $M$  is a submanifold of  $\mathbb{R}^n$ . Then there exists a  $C^2$ -stratification  $\mathcal{S} = \{S_i\}_i$  of  $M$  and a  $C^2$ -stratification  $\mathcal{T}$  of  $\mathbb{R}^n$  such that the restriction  $f_i$  of  $f$  onto each stratum  $S_i \in \mathcal{S}$  is a  $C^2$ -function,  $f_i(S_i) \in \mathcal{T}$  and  $f_i$  is of constant rank in  $S_i$ .*

The above statement yields that each restriction  $f_i : S_i \rightarrow f_i(S_i)$  is surjective,  $C^2$ , and of constant rank  $r_i$ . Thus  $r_i$  is also equal to the dimension of the manifold  $f_i(S_i)$ :

$$r_i = \text{rank } f_i = \dim \text{Im}(df_i(x)) = \dim(T_{f_i(S_i)}(f_i(x))), \quad \text{for all } x \in S_i.$$

### 3 Introductory results

We always consider a fixed nonempty compact convex set  $F \subset \mathbb{R}^n$ , and study the set of optimal solutions of the problem

$$\sup_F \langle c, \cdot \rangle$$

for vectors  $c \in \mathbb{R}^n$ . The optimal value of this problem, as a function of  $c$ , is called the *support function*, denoted  $\sigma_F$ . We denote by  $\text{argmax}_F \langle c, \cdot \rangle$  the set of optimal solutions. By scaling, we may as well assume  $c$  lies in the unit sphere  $S^{n-1}$ . We aim to show good behavior for objective vectors  $c$  lying in some large subset of the sphere. Classically, “large” might mean, for example, “full-measure”, or perhaps “generic”: a *generic* subset of a topological space is one containing a countable intersection of dense open sets. Clearly, these distinctions vanish for definable sets.

We begin our development with an easy and standard argument.

**Proposition 7 (Generic uniqueness).** *Consider a nonempty compact convex set  $F \subset \mathbb{R}^n$ . For all vectors  $c$  lying in a generic and full-measure subset of the sphere  $S^{n-1}$ , the linear functional  $\langle c, \cdot \rangle$  has a unique maximizer over  $F$ .*

**Proof.** We use various standard techniques from convex analysis [14, 15]. Note first

$$\operatorname{argmax}_F \langle c, \cdot \rangle = \partial \sigma_F(c),$$

where  $\partial$  denotes the convex subdifferential. This set is therefore a singleton if and only if the support function  $\sigma_F$  is differentiable at  $c$ . Being a finite, positively homogeneous, convex function, the set of points of differentiability is both generic and full-measure in  $\mathbb{R}^n$ , and is closed under strictly positive scalar multiplication. The result now follows.  $\square$

As a next step towards our main result, we prove stronger properties for at least a dense set of objectives. Density will suffice for our purposes once we move to a tame setting.

**Proposition 8 (Almost all linear functionals have strong maximizers).** *Let  $F$  be a nonempty compact convex subset of  $\mathbb{R}^n$ . Then corresponding to any vector  $c$  lying in some subset of  $S^{n-1}$  of full measure, there exist a vector  $x_c \in F$  and a constant  $\delta_c > 0$  such that*

$$\langle c, x_c \rangle \geq \langle c, x \rangle + \delta_c \|x - x_c\|^2, \quad \text{for all } x \in F, \quad (4)$$

that is,  $x_c$  is a strong (unique) maximizer of the linear functional  $\langle c, \cdot \rangle$  over  $F$ .

**Proof.** Let us denote by  $\sigma_F$  the support function of  $F$  and  $i_F$  the corresponding indicator function

$$i_F(c) := \begin{cases} 0 & \text{if } x \in F \\ +\infty & \text{otherwise.} \end{cases}$$

Notice that  $\sigma_F$ , a finite convex function, coincides with the Fenchel conjugate of  $i_F$  and  $\partial \sigma_F(c)$  is the set of maximizers of  $c$  on  $F$ . Applying Alexandrov's Theorem ([15, Theorem 13.51, p. 626]), we deduce that there exists a full measure subset  $A$  of  $\mathbb{R}^n$  on which  $\sigma_F$  has a quadratic expansion. (Thus in particular  $\sigma_F$  is differentiable there and  $\nabla \sigma_F(c) = x_c$ , where  $x_c$  denotes the unique maximizer of  $c$  at  $F$ .) In view of [15, Definition 13.1(c), p. 580], we have, for any fixed  $\bar{c} \in A$ , there exists a positive semidefinite matrix  $S$  such that for all  $c \in \mathbb{R}^n$ ,

$$\sigma_F(c) = \sigma_F(\bar{c}) + \langle \nabla \sigma_F(\bar{c}), c - \bar{c} \rangle + \frac{1}{2} \langle S(c - \bar{c}), c - \bar{c} \rangle + o(\|c - \bar{c}\|^2).$$



Hence there exists  $\varepsilon > 0$ ,  $\rho > 0$  such that for all  $c \in B(\bar{c}, \varepsilon)$  we have

$$\sigma_F(c) \leq \sigma_F(\bar{c}) + \langle x_{\bar{c}}, c - \bar{c} \rangle + \frac{\rho}{2} \|c - \bar{c}\|^2.$$

Further, we can clearly assume

$$\varepsilon^{-1} \text{diam}(F) < \rho. \quad (5)$$

Now consider  $x \in F$ . Recalling  $\sigma_F(\bar{c}) = \langle \bar{x}_c, \bar{c} \rangle$  we deduce successively

$$\begin{aligned} 0 = i_F(x) &= \sigma_F^*(x) = \sup_{c \in \mathbb{R}^n} \{ \langle x, c \rangle - \sigma_F(c) \} \\ &\geq \sup_{c \in B(\bar{c}, \varepsilon)} \{ \langle x, c \rangle - \sigma_F(c) \} \\ &\geq \sup_{c \in B(\bar{c}, \varepsilon)} \left\{ \langle x, c \rangle - \sigma_F(\bar{c}) - \langle x_{\bar{c}}, c - \bar{c} \rangle - \frac{\rho}{2} \|c - \bar{c}\|^2 \right\} \\ &= \sup_{c \in B(\bar{c}, \varepsilon)} \left\{ \langle x - x_{\bar{c}}, c \rangle - \frac{\rho}{2} \|c - \bar{c}\|^2 \right\} \\ &= \langle x - x_{\bar{c}}, \bar{c} \rangle + \sup_{u \in B(0, \varepsilon)} \left\{ \langle x - x_{\bar{c}}, u \rangle - \frac{\rho}{2} \|u\|^2 \right\}. \end{aligned}$$

In view of (5) it is easy to notice that the above supremum is realized at  $u = \rho^{-1}(x - x_{\bar{c}}) \in B(0, \varepsilon)$ . Replacing this value in the above inequality we deduce

$$0 \geq \langle x - x_{\bar{c}}, \bar{c} \rangle + \frac{1}{2\rho} \|x - x_{\bar{c}}\|^2, \quad \text{for all } x \in F.$$

which yields the asserted equation for  $\delta_c = (2\rho)^{-1}$ . The restriction of the result to  $S^{n-1}$  is straightforward.  $\square$

**Corollary 9 (Density of functionals with strong maximizer).** *The set of vectors  $c \in S^{n-1}$  such that there exist  $x_c \in F$ ,  $c \in \text{ri} N_F(x_c)$  and a constant  $\delta_c > 0$  such that*

$$\langle c, x_c \rangle \geq \langle c, x \rangle + \delta_c \|x - x_c\|^2, \quad \text{for all } x \in F,$$

*is a dense subset of the sphere  $S^{n-1}$ .*

**Proof** Given  $c \in A$ , take  $c'$  in  $\text{ri} N_F(x_c)$  and choose  $\eta > 0$  such that  $c' - \eta c \in \text{ri} N_F(x_c)$ . From the definition of the normal cone we deduce

$$\langle c', x_c - x \rangle = \langle \eta c, x_c - x \rangle + \langle c' - \eta c, x_c - x \rangle \geq \eta \delta_c \|x - x_c\|^2 = \eta \delta_c \|x - x_{c'}\|^2,$$

for all  $x$  in  $F$ . In other words: if  $\mathcal{C}$  denotes the set of linear functionals  $c \in \mathbb{R}^n$  satisfying (4) with  $c \in \text{ri} N_F(x_c)$ , then  $\bar{\mathcal{C}} \supset A$  and thus  $\bar{\mathcal{C}} = \bar{A} = \mathbb{R}^n$ . The density result on  $S^{n-1}$  follows easily.  $\square$

## 4 Main result

From now on we shall assume that the nonempty compact convex set  $F \subset \mathbb{R}^n$  is also definable in some o-minimal structure (see Definition 4). We are ready to state and prove the main result of this work. This result asserts that a generic linear optimization problem over  $F$  has a unique optimal solution, that  $F$  is partly smooth there, and strong criticality holds. As we see in the proof below, the active manifold arises naturally, by means of Proposition 6 (constant rank stratification) applied to an appropriately defined function.

To obtain the semi-algebraic version of the result below, simply replace the term “definable” by “semi-algebraic”.

**Theorem 10 (Main result).** *Let  $F$  be a nonempty compact convex subset of  $\mathbb{R}^n$  that is definable in some o-minimal structure. Then there exists a definable generic subset  $U$  of the unit sphere  $S^{n-1}$  with the following property: for each unit vector  $c \in U$ , there exists a unique vector  $x_c \in F$  and a definable set  $\mathcal{M}_c \subset F$  (unique in a neighborhood of  $x_c$ ) satisfying:*

- (i)  $\operatorname{argmax}_F \langle c, \cdot \rangle = \{x_c\}$ ;
- (ii)  $F$  is partly smooth at  $x_c$  relative to  $\mathcal{M}_c$ ;
- (iii)  $x_c$  is strongly critical.

**Proof** Let us consider the definable set-valued mapping  $\tilde{\Phi} : S^{n-1} \rightrightarrows F$  defined by

$$\tilde{\Phi}(c) = \operatorname{argmax}_F \langle c, \cdot \rangle, \quad (6)$$

and let us note, by the definition of the normal cone,

$$\tilde{\Phi}^{-1}(x) = N_F(x) \cap S^{n-1}.$$

Let  $D$  denote the dense subset of  $S^{n-1}$  asserted in Corollary 9 (Density of functionals with strong maximizer). Since  $F$  is a definable set, we deduce easily that  $D$  is also definable (see [1, Section 2.2], for example), and hence generic. In particular, the set

$$N_* = S^{n-1} \setminus D \quad (7)$$

has dimension strictly less than  $n - 1$ .

Let  $\Phi : D \rightarrow F$  denote the restriction of the mapping  $\tilde{\Phi}$  to  $D$ . Observe that  $\Phi$  is single-valued and, by the definition of the set  $D$ , satisfies the strict complementarity and quadratic decay conditions:

(i)  $c \in \text{ri } N_F(\Phi(c))$ ;

(ii)  $\langle c, \Phi(c) \rangle \geq \langle c, x \rangle + \delta \|x - \Phi(c)\|^2$ , for some  $\delta > 0$  and for all  $x \in F$ .

Applying Proposition 6 (Constant rank stratification) to the definable function

$$\begin{cases} \Phi : D \rightarrow F \\ c \mapsto \Phi(c), \end{cases} \quad (8)$$

we arrive at a stratification  $\mathcal{S} = \{S_j\}_{j \in J}$  of  $D$  such that for every index  $j \in J$ ,

- $\Phi_j := \Phi|_{S_j}$  is a  $C^2$  function of constant rank;
- $\Phi_j(S_j)$  is a manifold of dimension equal to the rank of  $\Phi_j$ ;
- the image strata  $\{\Phi(S_j)\}_j$  belong to a Whitney stratification of  $\mathbb{R}^n$ .

In particular,

$$D = \bigcup_{j \in J} S_j \quad (9)$$

and

$$j_1 \neq j_2 \Rightarrow \Phi(S_{j_1}) = \Phi(S_{j_2}) \quad \text{or} \quad \Phi(S_{j_1}) \cap \Phi(S_{j_2}) = \emptyset. \quad (10)$$

Denote the set of strata of full dimension by  $\{S_{j_1}, \dots, S_{j_\ell}\}$ . Set

$$U = \bigcup_{i=1}^{\ell} S_{j_i}$$

and observe that the above set is open and dense in  $D$ , and hence generic in  $S^{n-1}$ .

Our immediate objective is to show that for every vector  $c \in U$  there exists a manifold  $\mathcal{M} \subset F$  containing  $\Phi(c)$  such that  $F$  is partly smooth at  $\Phi(c)$  with respect to  $\mathcal{M}$ .

To this end, fix  $\bar{x} \in \Phi(U)$  and consider the set of “active” indices

$$I(x) := \{j \in J : \bar{x} \in \Phi(S_j)\}. \quad (11)$$

We aim to show that the set  $F$  is partly smooth at  $\bar{x}$  relative to the manifold

$$\mathcal{M} = \Phi_j(S_j), \quad (12)$$

for any  $j \in I(x)$ . Note that in view of property (10) the definition of  $\mathcal{M}$  is in fact independent of the choice of  $j$  in  $I(x)$ , and for the same reason the set of active indices  $I(x)$  is invariant for all  $x \in \mathcal{M}$ . In the sequel, this set will be simply denoted by  $I$ .

Clearly, property (i) of the definition of partial smoothness (Definition 1) holds. If we can prove properties (ii) and (iii), then our result will follow: since  $U \subset D$ , Corollary 9 (Density of functionals with strong maximizers) implies strong criticality for any objective  $c \in U$ .

### Step 1: normal cone continuity

We establish the continuity at  $\bar{x}$  of the normal cone mapping  $x \mapsto N_F(x)$  as  $x$  moves along the manifold  $\mathcal{M}$ .

The normal cone mapping is always outer semicontinuous (even in  $F$ ). To establish that the truncated normal cone mapping

$$x \mapsto \tilde{\Phi}^{-1}(x) = N_F(x) \cap S^{n-1} \quad (x \in \mathcal{M}) \quad (13)$$

is inner semicontinuous (which clearly suffices for our purposes), we decompose the above mapping with respect to the active strata. We set

$$N_j(x) = N_F(x) \cap S_j, \quad \text{for every } j \in J. \quad (14)$$

Note that for each  $x \in \mathcal{M}$  we have

$$N_j(x) \neq \emptyset \Leftrightarrow j \in I \Leftrightarrow \mathcal{M} = \Phi(S_j). \quad (15)$$

We can therefore decompose the truncated normal cone mapping (13) as follows:

$$N_F(x) \cap S^{n-1} = N_*(x) \cup \bigcup_{j \in I} N_j(x) \quad (16)$$

where

$$N_*(x) = N_F(\bar{x}) \cap N_*,$$

and the set  $N_*$  is defined by equation (7).

**Claim A.** For every  $x \in \mathcal{M}$  the set  $\cup_{j \in I} N_j(x)$  is dense in  $N_F(x) \cap S^{n-1}$ .

**Proof of Claim A.** Since we are assuming  $\bar{x} \in \Phi(U)$ , there exists an active index  $j_p$  with  $p \in \{1, \dots, \ell\}$  corresponding to a full-dimensional stratum

$S_{j_p}$  such that  $\mathcal{M} = \Phi_{j_p}(S_{j_p})$  (see property (15)). This yields that for every  $x \in \mathcal{M}$  there exists  $c \in S_{j_p}$  with  $x = \Phi(c)$ . Hence

$$c \in N_F(x) \cap S_{j_p} = N_{j_p}(x) \subset \bigcup_{j \in I} N_j(x).$$

Fix now any vector  $c_* \in N_F(x) \cap S^{n-1}$ , and consider the spherical path

$$c_t := \frac{c + t(c_* - c)}{\|c + t(c_* - c)\|}, \quad \text{for } t \in [0, 1].$$

It follows that  $c_t \in \text{ri } N_F(x)$ , for all  $t \in [0, 1)$ . Since  $c \in S_{j_p} \subset D$ , there exists a constant  $\delta_c > 0$  such that  $\langle c, x \rangle \geq \langle c, x' \rangle + \delta_c \|x - x'\|^2$ , for all  $x' \in F$ . By the definition of the normal cone, we also have  $\langle c_*, x \rangle \geq \langle c_*, x' \rangle$  for all  $x' \in F$ . Multiplying the aforementioned inequalities by  $(1-t)$  and  $t$  respectively, and adding, we infer that  $x$  is a strong maximizer of  $\langle c_t, \cdot \rangle$  over the set  $F$  for all  $0 \leq t < 1$ . In other words,  $c_t \in N_F(x) \cap D$ , which in view of equation (9) yields  $c_t \in \bigcup_{j \in I} N_j(x)$ , for  $t \in [0, 1)$ . Since  $c_t \rightarrow c_*$  as  $t \uparrow 1$ , Claim A follows.

In view of Claim A, it is sufficient to establish the inner continuity of the mapping

$$x \mapsto \bigcup_{j \in I} N_j(x) \quad x \in \mathcal{M}. \quad (17)$$

To see this, we use the following simple exercise.

**Lemma 11.** *Let  $X$  and  $Y$  be metric spaces, and consider two set-valued mappings  $G, T : X \rightrightarrows Y$  such that  $\text{cl}(G(x)) = T(x)$  for all points  $x \in X$ . If  $G$  is inner semicontinuous at a point  $\bar{x} \in X$ , then so is  $T$ .*

**Proof of Lemma 11.** Assume (towards a contradiction) that there exists a constant  $\rho > 0$ , a sequence  $\{x^k\} \subset X$  with  $x^k \rightarrow \bar{x}$  and a point  $\bar{y} \in T(\bar{x})$ , such that

$$\text{dist}(\bar{y}, T(x^k)) > \rho > 0.$$

Then pick any point  $\hat{y} \in B(\bar{y}, \rho/2) \cap G(\bar{x})$  and use the inner semicontinuity of  $G$  to get a sequence  $y^k \in G(x^k) \subset T(x^k)$  for  $k \in \mathbb{N}$  such that  $y^k \rightarrow \hat{y}$ . This gives a contradiction, proving the lemma.  $\square$

Applying this lemma to the set-valued mappings

$$G(x) = \bigcup_{j \in I} N_j(x) \quad \text{and} \quad T(x) = N_F(x) \cap S^{n-1}$$

accomplishes the reduction we seek.

Let us, therefore, prove the inner semicontinuity of the mapping defined in (17) at the point  $\bar{x}$ . To this end, fix any vector  $\bar{c} \in \cup_{j \in I} N_j(\bar{x})$  and consider any sequence  $\{x_k\}_k \subset \mathcal{M}$  approaching  $\bar{x}$ . For some index  $j \in I$  we have  $\bar{c} \in S_j$ . Let us restrict our attention to the constant-rank surjective mapping  $\Phi_j : S_j \rightarrow \mathcal{M}$  and let us recall that

$$\Phi_j(S_j) = \mathcal{M} \quad \text{and} \quad \Phi_j(\bar{c}) = \bar{x}.$$

Let  $d$  be the dimension of the stratum  $S_j$ , so

$$\text{rank}(d\Phi_j) = \dim \mathcal{M} := r \leq d \leq n - 1.$$

Denote by  $0_d$  (respectively  $0_r$ ) the zero vector of the space  $\mathbb{R}^d$  (respectively  $\mathbb{R}^r$ ). Then applying the Rank Theorem (Proposition 5), we infer that for some constants  $\delta, \varepsilon > 0$  there exist diffeomorphisms

$$\psi_1 : B(0_d, \delta) \rightarrow S_{j_0} \cap B(\bar{c}, \varepsilon) \quad \text{and} \quad \psi_2 : B(0_r, \delta) \rightarrow \mathcal{M} \cap B(\bar{x}, \varepsilon) \quad (18)$$

such that

$$\psi_1(0_d) = \bar{c} \quad \text{and} \quad \psi_2(0_r) = \bar{x}, \quad (19)$$

and such that all vectors  $y \in B(0_d, \delta)$  satisfy

$$(\psi_2^{-1} \circ \Phi_j \circ \psi_1)(y) = \pi(y), \quad (20)$$

where for  $y = (y_1, \dots, y_d) \in \mathbb{R}^d$  we have

$$\pi(y_1, \dots, y_r, y_{r+1}, \dots, y_d) = (y_1, \dots, y_r) \in B(0_r, \delta) \subset \mathbb{R}^r. \quad (21)$$

We may assume  $\{x_k\}_k \subset \mathcal{M} \cap B(\bar{x}, \varepsilon)$ . Thus, in view of definition (18), for every integer  $k \in \mathbb{N}$  there exists a vector  $z^k = (z_1^k, \dots, z_r^k) \in B(0_r, \delta)$  with

$$\psi_2(z^k) = x_k. \quad (22)$$

Note  $z^k \rightarrow 0_r = (\psi_2)^{-1}(\bar{x})$ . Define vectors

$$y^k := (z_1^k, \dots, z_r^k, 0, \dots, 0) \in \mathbb{R}^d$$

for every  $k \in \mathbb{N}$ . Since  $z^k \in B(0_r, \delta)$ , we know  $y^k \in B(0_d, \delta)$ , and clearly

$$y^k \rightarrow 0_d. \quad (23)$$

We now define vectors  $c_k := \psi_1(y^k)$  for each  $k$ . In view of definition (18) we see that  $c_k \in S_j \cap B(\bar{c}, \varepsilon)$ , and in view of properties (23) and (19),

$$c_k \rightarrow \psi_1(0_d) = \bar{c} \text{ as } k \rightarrow \infty.$$

To complete the proof of inner semicontinuity, it remains to show  $c_k \in N_F(x_k)$ . Since  $\Phi_j(c_k) = \Phi_j(\psi_1(y^k))$  we infer by properties (20) and (23) that

$$\psi_2^{-1}(\Phi_j(c_k)) = (\psi_2^{-1} \circ \Phi_j \circ \psi_1)(y^k) = \pi(y^k) = z^k.$$

Using now the fact that  $\psi_2$  is a diffeomorphism we deduce from equation (22) that  $\Phi_j(c_k) = \psi_2(z^k) = x_k$ . Thus  $c_k \in \Phi_j^{-1}(x_k) \subset N_F(x_k)$  which completes the proof of inner semicontinuity and hence of Step 1.

## Step 2: sharpness

It remains to verify that condition (iii) of Definition 1, namely

$$N_{\mathcal{M}}(\bar{x}) = N_F(\bar{x}) - N_F(\bar{x}), \quad (24)$$

is also fulfilled.

To this end, as in the proof of Claim A, we can choose an index  $j \in I$  corresponding thus to a stratum  $S_j$  of full dimension (that is, equal to  $n - 1$ ) such that  $\mathcal{M} = \Phi_j(S_j)$ . Recall that the definable  $C^2$ -mapping  $\Phi_j : S_j \rightarrow \mathcal{M}$  is surjective and has constant rank  $r = \dim \mathcal{M}$ , so

$$\dim N_{\mathcal{M}}(\bar{x}) = n - r.$$

It follows directly from the inclusion  $\mathcal{M} \subset F$  that  $N_F(\bar{x}) \subset N_{\mathcal{M}}(\bar{x})$ . Since the right-hand side is a subspace, we deduce

$$N_F(\bar{x}) - N_F(\bar{x}) \subset N_{\mathcal{M}}(\bar{x}). \quad (25)$$

Since  $\Phi_j$  is surjective and of maximal rank, we deduce easily that  $\Phi_j^{-1}(\bar{x})$  is a definable submanifold of  $S^{n-1}$  of dimension

$$\dim \Phi_j^{-1}(\bar{x}) = (n - 1) - r.$$

which, in view of definition (13) and equation (14) yields

$$\dim (N_F(\bar{x}) \cap S^{n-1}) \geq \dim N_j(\bar{x}) \geq (n - 1) - r.$$

Thus  $\dim N_F(\bar{x}) \geq n - r$ , which, along with inclusion (25), yields equation (24), as required.  $\square$

It is interesting to revisit the example in the introduction, problem (1). One can truncate the feasible region (by intersecting with the unit ball for example) to obtain a convex compact semi-algebraic set for which the functional  $\bar{c} = (0, 0, -1)$  has a unique maximizer (the origin) while the generic condition of Theorem 10 fails. In other words,  $\bar{c} \notin U$  according to notation of Theorem 10.

**Acknowledgment** A major part of this work was accomplished during a research visit of the second and third authors at the University Pierre et Marie Curie (Paris 6), in May 2008.

## References

- [1] BOLTE, J., DANILIDIS, A. & LEWIS, A.S., The Łojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems, *SIAM J. Optim.* **17** (2007), 1205–1223.
- [2] BOLTE, J., DANILIDIS, A. & LEWIS, A.S., Tame functions are semismooth, *Math. Program. (Series B)* **117** (2009), 5–19.
- [3] BURKE, J.V. & J.J. MORÉ, On the identification of active constraints, *SIAM J. Numerical Analysis.* **25** (1988), 1197–1211.
- [4] COSTE, M., *An Introduction to o-minimal Geometry*, RAAG Notes, 81 pages, Institut de Recherche Mathématiques de Rennes, November 1999.
- [5] DONTCHEV, A.L. & ZOLEZZI, T., *Well-Posed Optimization Problems*, Lecture Notes in Mathematics **1543** (Springer Verlag, Berlin 1993).
- [6] VAN DEN DRIES, L. & MILLER, C., Geometric categories and o-minimal structures, *Duke Math. J.* **84** (1996), 497–540.
- [7] IOFFE, A.D., An invitation to Tame Optimization, 23 p., preprint 2008.
- [8] IOFFE, A.D. & LUCCHETTI, R., Typical convex program is very well posed, *Math. Program.* **104** (2005), 483–499.



- [9] KURDYKA, K., On gradients of functions definable in o-minimal structures, *Ann. Inst. Fourier* **48** (1998), no. 3, 769–783.
- [10] LEE, J.M., *Introduction to Smooth Manifolds*, (Springer, New York 2003).
- [11] LEMARECHAL, C., OUSTRY, F. & SAGASTIZABAL, C., The  $\mathcal{U}$ -Lagrangian of a convex function, *Trans. Amer. Math. Soc.* **352** (2000), 711–729.
- [12] LEWIS, A.S., Active sets, nonsmoothness and sensitivity, *SIAM Journal on Optimization* **13** (2003), 702–725.
- [13] HARE, W.L. & LEWIS, A.S., Identifying active constraints via partial smoothness and prox-regularity, *Journal of Convex Analysis* **11** (2004), 251–266.
- [14] PHELPS, R.R., *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes in Mathematics **1364** (2nd ed.) (Springer Verlag, Berlin 1993).
- [15] ROCKAFELLAR, R.T. & WETS, R.J.-B., *Variational Analysis*, Grundlehren der Mathematischen Wissenschaften, Vol. **317**, (Springer, 1998).
- [16] LOI, TA LÊ, Thom stratifications for functions definable in o-minimal structures on  $(\mathbb{R}, +, \cdot)$ , *C. R. Acad. Sci. Paris Sér. I Math.* **324** (1997), no. 12, 1391–1394.
- [17] WRIGHT, S.J., Identifiable surfaces in constrained optimization, *SIAM Journal on Control and Optimization* **31** (1993), 1063–1079.

---

Jérôme BOLTE

UPMC Univ Paris 06, Équipe Combinatoire et Optimisation (UMR 7090),  
Case 189

Université Pierre et Marie Curie, 4 Place Jussieu, F-75252 Paris Cedex 05.

and

INRIA Saclay, CMAP, Ecole Polytechnique, F-91128 Palaiseau, France.

E-mail: [bolte@math.jussieu.fr](mailto:bolte@math.jussieu.fr) ;  
<http://www.ecp6.jussieu.fr/pageperso/bolte>  
Research supported by the ANR grant ANR-05-BLAN-0248-01 (France).

Aris DANIILIDIS

Departament de Matemàtiques, C1/320  
Universitat Autònoma de Barcelona  
E-08193 Bellaterra (Cerdanyola del Vallès), Spain.

*and*

Laboratoire de Mathématiques et Physique Théorique  
Université François Rabelais, Faculté des Sciences  
Parc de Grandmont, F-37200 Tours, France.

E-mail: [arisd@mat.uab.es](mailto:arisd@mat.uab.es) ; <http://mat.uab.es/~arisd>

Research supported by the grant MTM2008-06695-C03-03/MTM (Spain).

Adrian LEWIS

School of Operations Research and Information Engineering  
Cornell University  
Ithaca, NY 14853, USA.

E-mail: [aslewis@orie.cornell.edu](mailto:aslewis@orie.cornell.edu)

<http://www.orie.cornell.edu/~aslewis>

Research supported in part by National Science Foundation Grant DMS-0806057 (USA).