Nuclear norm minimization for the planted clique and biclique problems*

Brendan P.W. Ames[†] Stephen A. Vavasis[‡]

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Abstract

We consider the problems of finding a maximum clique in a graph and finding a maximum-edge biclique in a bipartite graph. Both problems are NP-hard. We write both problems as matrix-rank minimization and then relax them using the nuclear norm. This technique, which may be regarded as a generalization of compressive sensing, has recently been shown to be an effective way to solve rank optimization problems. In the special cases that the input graph has a planted clique or biclique (i.e., a single large clique or biclique plus diversionary edges), our algorithm successfully provides an exact solution to the original instance. For each problem, we provide two analyses of when our algorithm succeeds. In the first analysis, the diversionary edges are placed by an adversary. In the second, they are placed at random. In the case of random edges for the planted clique problem, we obtain the same bound as Alon, Krivelevich and Sudakov as well as Feige and Krauthgamer, but we use different techniques.

1 Introduction

Several recent papers including Recht et al. [17] and Candès and Recht [4] consider nuclear norm minimization as a convex relaxation of matrix rank minimization. *Matrix rank minimization* refers to the problem of finding a matrix $X \in \mathbf{R}^{m\times n}$ to minimize $\text{rank}(X)$ subject to linear constraints on X. As we shall show in Sections 3 and 4, the clique and biclique problems, both NP-hard, are easily expressed as matrix rank minimization, thus showing that matrix rank minimization is also NP-hard.

Each of the two papers mentioned in the previous paragraph has results of the following general form. Suppose an instance of matrix rank minimization is posed in which it is known *a priori* that a solution of very low rank exists. Suppose further that the constraints are

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[†]Department of Combinatorics and Optimization, University of Waterloo, 200 University Avenue W., Waterloo, Ontario N2L 3G1, Canada, bpames@math.uwaterloo.ca

[‡]Department of Combinatorics and Optimization, University of Waterloo, 200 University Avenue W., Waterloo, Ontario N2L 3G1, Canada, vavasis@math.uwaterloo.ca

random in some sense. Then the nuclear norm relaxation turns out to be exact, i.e., it recovers the (unique) solution of low rank. The *nuclear norm* of a matrix X, also called the *trace norm*, is defined to be the sum of the singular values of X.

These authors build upon recent breakthroughs in compressive sensing [10, 5, 3]. In compressive sensing, the problem is to recover a sparse vector that solves a set of linear equations. In the case that the equations are randomized and a very sparse solution exists, compressive sensing can be solved by relaxation to the l_1 norm. The correspondence between matrix rank minimization and compressive sensing is as follows: matrix rank (number of nonzero singular values) corresponds to vector sparsity (number of nonzero entries) and nuclear norm corresponds to l_1 norm.

Our results follow the spirit of Recht et al. but use different technical approaches. We establish results about two well known graph theoretic problems, namely maximum clique and maximum-edge biclique. The maximum clique problem takes as input an undirected graph and asks for the largest clique (i.e., induced subgraph of nodes that are completely interconnected). This problem is one of Karp's original NP-hard problems [8]. The maximum-edge biclique takes as input a bipartite graph (U, V, E) and asks for the subgraph that is a complete bipartite graph $K_{m,n}$ that maximizes the product mn. This problem was shown to be NP-hard by Peeters [16].

In Sections 3 and 4, we relax these problems to convex optimization using the nuclear norm. For each problem, we show that convex optimization can recover the exact solution in two cases. The first case, described in Section 3.2, is the adversarial case: the N-node graph under consideration consists of a single n-node clique plus a number of diversionary edges chosen by an adversary. We show that the algorithm can tolerate up to $O(n^2)$ diversionary edges provided that no non-clique vertex is adjacent to more than $O(n)$ clique vertices. We argue also that these two bounds, $O(n^2)$ and $O(n)$, are the best possible. We show analogous results for the biclique problem in Section 4.1.

Our second analysis, described in Sections 3.3 and 4.2, supposes that the graph contains a single clique or biclique, while the remaining nonclique edges are inserted independently at random with fixed probability p. This problem has been studied by Alon et al. [2] and by Feige and Krauthgamer [6]. In the case of clique, we obtain the same result as they do, namely, that as long as the clique has at least $O(N^{1/2})$ nodes, where N is the number of nodes in G, then our algorithm will find it. Like Feige and Krauthgamer, our algorithm also certifies that the maximum clique has been found due to a uniqueness result for convex optimization, which we present in Section 3.1. We believe that our technique is more general than Feige and Krauthgamer; for example, ours extends essentially without alteration to the biclique problem, whereas Feige and Krauthgamer rely on some special properties of the clique problem. Furthemore, Feige and Krauthgamer use more sophisticated probabilistic tools (martingales), whereas our results use only Chernoff bounds and classical theorems about the norms of random matrices. The random matrix results needed for our main theorems are presented in Section 2.

Our interest in the planted clique and biclique problems arises from applications in data mining. In data mining, one seeks a pattern hidden in an apparently unstructured set of data. A natural question to ask is whether a data mining algorithm is able to find the hidden pattern in the case that it is actually present but obscured by noise. For example, in the realm of clustering, Ben-David [1] has shown that if the data is actually clustered, then a clustering algorithm can find the clusters. The clique and biclique problems are both simple model problems for data mining. For example, Pardalos [13] reduces a data mining problem in epilepsy prediction to a maximum clique problem. Gillis and Glineur [11] use the biclique problem as a model problem for nonnegative matrix factorization and finding features in images.

2 Results on norms of random matrices

In this section we provide a few results concerning random matrices with independently identically distributed (i.i.d.) entries of mean 0. In particular, the probability distribution Ω for an entry A_{ij} will be as follows:

$$A_{ij} = \begin{cases} 1 & \text{with probability } p, \\ -p/(1-p) & \text{with probability } 1-p. \end{cases}$$

It is easy to check that the variance of A_{ij} is $\sigma^2 = p/(1-p)$.

We start by recalling a theorem of Füredi and Komlós [7]:

Theorem 2.1 *For all integers i, j, $1 \le j \le i \le n$, let A_{ij} be distributed according to Ω. Define symmetrically $A_{ij} = A_{ji}$ for all $i < j$.*

Then the random symmetric matrix $A = [A_{ij}]$ satisfies

$$\|A\| \le 3\sigma\sqrt{n}$$

with probability at least to $1 - \exp(-cn^{1/6})$ for some $c > 0$ that depends on σ.

Remark 1. In this theorem and for the rest of the paper, $\|A\|$ denotes $\|A\|_2$, often called the spectral norm. It is equal to the maximum singular value of A or equivalently to the square root of the maximum eigenvalue of $A^T A$.

Remark 2. The theorem is not stated exactly in this way in [7]; the stated form of the theorem can be deduced by taking $k = (\sigma/K)^{1/3} n^{1/6}$ and $v = \sigma\sqrt{n}$ in the inequality

$$P(\max|\lambda| > 2\sigma\sqrt{n} + v) < \sqrt{n}\exp(-kv/(2\sqrt{n} + v))$$

on p. 237.

Remark 3. As mentioned above, the mean value of entries of A is 0. This is crucial for the theorem; a distribution with any other mean value would lead to $\|A\| = O(n)$.

A similar theorem due to Geman [9] is available for unsymmetric matrices.

Theorem 2.2 *Let A be a $\lceil yn \rceil \times n$ matrix whose entries are chosen according to Ω for fixed $y \in \mathbf{R}_+$. Then, with probability at least $1 - c_1 \exp(-c_2 n^{c_3})$ where $c_1 > 0$, $c_2 > 0$, and $c_3 > 0$ depend on p and y,*

$$\|A\| \le c_4\sqrt{n}$$

for some $c_4 > 0$ also depending on p, y.

As in the case of [7], this theorem is not stated exactly this way in Geman's paper, but can be deduced from the equations on pp. 255–256 by taking $k = n^q$ for a q satisfying $(2\alpha + 4)q < 1$.

The last theorem about random matrices requires a version of the well known Chernoff bounds, which is as follows (see [15, Theorem 4.4]).

Theorem 2.3 (Chernoff Bounds) *Let $X_1, \ldots, X_k$ be a sequence of k independent Bernoulli trials, each succeeding with probability p so that $E(X_i) = p$. Let $S = \sum_{i=1}^{k} X_i$ be the binomially distributed variable describing the total number of successes. Then for $\delta > 0$*

$$P\Big(S > (1+\delta)pk\Big) \le \left(\frac{e^{\delta}}{(1+\delta)^{(1+\delta)}}\right)^{pk}. \tag{1}$$

It follows that for all $a \in (0, p\sqrt{k})$,

$$P(|S - pk| > a\sqrt{k}) \le 2\exp(-a^2/p). \tag{2}$$

The final theorem of this section is as follows.

Theorem 2.4 *Let A be an $n \times N$ matrix whose entries are chosen according to Ω. Let $\tilde{A}$ be defined as follows. For (i,j) such that $A_{ij} = 1$, we define $\tilde{A}_{ij} = 1$. For entries (i,j) such that $A_{ij} = -p/(1-p)$, we take $\tilde{A}_{ij} = -n_j/(n-n_j)$, where n_j is the number of 1's in column j of A. Then there exist $c_1 > 0$ and $c_2 \in (0,1)$ depending on p such that*

$$P(\|A - \tilde{A}\|_F^2 \le c_1 N) \ge 1 - (2/3)^N - N c_2^n. \tag{3}$$

Remark 1. The notation $\|A\|_F$ denotes the Frobenius norm of A, that is, $\left(\sum_i \sum_j A_{ij}^2\right)^{1/2}$. It is well known that $\|A\|_F \ge \|A\|$ for any A.

Remark 2. Note that $\tilde{A}$ is undefined if there is a j such that $n_j = n$. In this case we assume that $\|A - \tilde{A}\| = \infty$, i.e., the event considered in (3) fails.

Remark 3. Observe that the column sums of A are random variables with mean zero since the mean of the entries is 0. On the other hand, the column sums of $\tilde{A}$ are identically zero deterministically; this is the rationale for the choice of $\tilde{A} = -n_j/(n-n_j)$.

Proof: From the definition of $\tilde{A}$, for column j, there are exactly $n - n_j$ entries of $\tilde{A}$ that differ from those of A. Furthermore, the difference of these entries is exactly $(n_j - pn)/((1-p)(n-n_j))$. Therefore, for each $j = 1, \ldots, N$, the contribution of column j to the square norm difference $\|A - \tilde{A}\|_F^2$ is given by

$$\|A(:,j) - \tilde{A}(:,j)\|_F^2 = \frac{(n_j - pn)^2}{(1-p)^2(n-n_j)}.$$

Recall that the numbers $n_1, \ldots, n_N$ are independent, and each is the result of n Bernoulli trials done with probability p.

We now define Ψ to be the event that at least one n_j is very far from the mean. In particular, Ψ is the event that there exists a $j \in \{1, \ldots, N\}$ such that $n_j > qn$, where

$q = \min(\sqrt{p}, 2p)$. Let $\tilde{\Psi}$ be its complement, and let $\tilde{\psi}(j)$ be the indicator of this complement (i.e., $\tilde{\psi}(j) = 1$ if $n_j \leq qn$ else $\tilde{\psi}(j) = 0$). Let c be a positive scalar depending on p to be determined later. Observe that

$$\begin{aligned} P(\|A - \tilde{A}\|_F^2 \geq cN) &= P(\|A - \tilde{A}\|_F^2 \geq cN \ \wedge \ \tilde{\Psi}) + P(\|A - \tilde{A}\|_F^2 \geq cN \ \wedge \ \Psi) \\ &\leq P(\|A - \tilde{A}\|_F^2 \geq cN \ \wedge \ \tilde{\Psi}) + P(\Psi). \end{aligned} \tag{4}$$

We now analyze the two terms separately. For the first term we use a technique attributed to S. Bernstein (see Hoeffding [12]). Let ϕ be the indicator function of nonnegative reals, i.e., $\phi(x) = 1$ for $x \geq 0$ while $\phi(x) = 0$ for $x < 0$. Then, in general, $P(u \geq 0) \equiv E(\phi(u))$. Thus,

$$\begin{aligned} P(\|A - \tilde{A}\|_F^2 \geq cN \ \wedge \ \tilde{\Psi}) &= P(\|A - \tilde{A}\|_F^2 - cN \geq 0 \ \wedge \ \tilde{\psi}(n_1) = 1 \ \wedge \ \cdots \ \wedge \ \tilde{\psi}(n_N) = 1) \\ &= E(\phi(\|A - \tilde{A}\|_F^2 - cN) \cdot \tilde{\psi}(n_1) \cdots \tilde{\psi}(n_N)). \end{aligned}$$

Let h be a positive scalar depending on p to be determined later. Observe that for any such h and for all $x \in \mathbf{R}$, $\phi(x) \leq \exp(hx)$. Thus,

$$\begin{aligned} P(\|A - \tilde{A}\|_F^2 \geq cN \ \wedge \ \tilde{\Psi}) &\leq E(\exp(h\|A - \tilde{A}\|_F^2 - hcN) \cdot \tilde{\psi}(n_1) \cdots \tilde{\psi}(n_N)) \\ &= E\left(\exp\left(h \sum_{j=1}^{N} \left(\|A(:,j) - \tilde{A}(:,j)\|_F^2 - c\right)\right) \cdot \tilde{\psi}(n_1) \cdots \tilde{\psi}(n_N)\right) \\ &= E\left(\exp\left(h \sum_{j=1}^{N} \left(\frac{(n_j - pn)^2}{(1-p)^2(n - n_j)} - c\right)\right) \cdot \tilde{\psi}(n_1) \cdots \tilde{\psi}(n_N)\right) \\ &= E\left(\prod_{j=1}^{N} \exp\left(h\left(\frac{(n_j - pn)^2}{(1-p)^2(n - n_j)} - c\right)\right) \tilde{\psi}(n_j)\right) \\ &= \prod_{j=1}^{N} E\left(\exp\left(h\left(\frac{(n_j - pn)^2}{(1-p)^2(n - n_j)} - c\right)\right) \tilde{\psi}(n_j)\right) \qquad (5) \\ &= f_1 \cdots f_N, \qquad (6) \end{aligned}$$

where

$$f_j = E\left(\exp\left(h\left(\frac{(n_j - pn)^2}{(1-p)^2(n - n_j)} - c\right)\right) \tilde{\psi}(n_j)\right).$$

To obtain (5), we used the independence of the n_j's. Let us now analyze f_j in isolation.

$$\begin{aligned} f_j &= \sum_{i=0}^{n} \exp\left(h\left(\frac{(i - pn)^2}{(1-p)^2(n - i)} - c\right)\right) \tilde{\psi}(n_j) P(n_j = i) \\ &= \sum_{i=0}^{\lfloor qn \rfloor} \exp\left(h\left(\frac{(i - pn)^2}{(1-p)^2(n - i)} - c\right)\right) P(n_j = i) \\ &\leq \sum_{i=0}^{\lfloor qn \rfloor} \exp\left(h\left(\frac{(i - pn)^2}{(1-p)^2(n - \sqrt{p}n)} - c\right)\right) P(n_j = i). \end{aligned}$$

To derive the last line, we used the fact that $i \le \sqrt{p}n$ since $i \le qn$. Now let us reorganize this summation by considering first i such that $|i-pn| < \sqrt{n}$, and next i such that $|i-pn| \in [\sqrt{n}, 2\sqrt{n})$, etc. Notice that, since $i \le qn \le 2pn$, we need consider intervals only until $|i-pn|$ reaches pn.

$$\begin{aligned}
f_j &\le \sum_{k=0}^{\lfloor p\sqrt{n}\rfloor} \sum_{i:|i-pn|\in[k\sqrt{n},(k+1)\sqrt{n})} \exp\left(h\left(\frac{(i-pn)^2}{(1-p)^2(n-\sqrt{p}n)} - c\right)\right) P(n_j = i) \\
&\le \sum_{k=0}^{\lfloor p\sqrt{n}\rfloor} \sum_{i:|i-pn|\in[k\sqrt{n},(k+1)\sqrt{n})} \exp\left(h\left(\frac{(k+1)^2 n}{(1-p)^2(n-\sqrt{p}n)} - c\right)\right) P(n_j = i) \\
&= \sum_{k=0}^{\lfloor p\sqrt{n}\rfloor} \sum_{i:|i-pn|\in[k\sqrt{n},(k+1)\sqrt{n})} \exp\left(h\left(\frac{(k+1)^2}{(1-p)^2(1-\sqrt{p})} - c\right)\right) P(n_j = i) \\
&= \sum_{k=0}^{\lfloor p\sqrt{n}\rfloor} \exp\left(h\left(\frac{(k+1)^2}{(1-p)^2(1-\sqrt{p})} - c\right)\right) \sum_{i:|i-pn|\in[k\sqrt{n},(k+1)\sqrt{n})} P(n_j = i) \\
&\le 2\sum_{k=0}^{\lfloor p\sqrt{n}\rfloor} \exp\left(h\left(\frac{(k+1)^2}{(1-p)^2(1-\sqrt{p})} - c\right)\right) \exp(-k^2/p),
\end{aligned}$$

where, for the last line, we have applied (2). The theorem is valid since $k \le p\sqrt{n}$.

Continuing this derivation and overestimating the finite sum with an infinite sum,

$$\begin{aligned}
f_j &\le 2\exp(-hc)\cdot\sum_{k=0}^{\infty}\exp\left(\frac{h(k+1)^2}{(1-p)^2(1-\sqrt{p})} - k^2/p\right) \\
&= 2\exp\left(\frac{h}{(1-p)^2(1-\sqrt{p})} - hc\right) \\
&\qquad + 2\exp(-hc)\cdot\sum_{k=1}^{\infty}\exp\left[\frac{h(k+1)^2}{(1-p)^2(1-\sqrt{p})} - k^2/p\right].
\end{aligned}$$

Choose h so that $h/((1-p)^2)(1-\sqrt{p}) < 1/(8p)$, i.e., $h < (1-p)^2(1-\sqrt{p})/(8p)$. Then the second term in the square-bracket expression at least twice the first term for all $k \ge 1$, hence

$$f_j \le 2\exp\left(\frac{h}{(1-p)^2(1-\sqrt{p})} - hc\right) + 2\exp(-hc)\cdot\sum_{k=1}^{\infty}\exp\left(-k^2/(2p)\right). \tag{7}$$

Observe that $\sum_{k=1}^{\infty}\exp(-k^2/(2p))$ is dominated by a geometric series and hence is a finite number depending on p. Thus, once h is selected, it is possible to choose c sufficiently large so that each of the two terms in (7) is at most 1/3. Thus, with appropriate choices of h and c, we conclude that $f_j \le 2/3$. Thus, substituting this into (6) shows that

$$P(\|A-\tilde{A}\|_F^2 \ge cN \wedge \tilde{\Psi}) \le (2/3)^N. \tag{8}$$

We now turn to the second term in (4). For a particular j, the probability that $n_j > qn$ is bounded using (1) by v_p^n where $v_p = (e^\delta/(1+\delta)^{(1+\delta)})^p$, where $\delta = q/p - 1$, i.e., $\delta =$

$\min(p, \sqrt{p} - p)$. Then the union bound asserts that the probability that any j satisfies $n_j > qn$ is at most Nv_p^n. Thus,

$$P(\|A - \tilde{A}\|_F^2 \geq cN) \leq (2/3)^N + Nv_p^n.$$

This concludes the proof.

3 Maximum Clique

Let $G = (V, E)$ be a simple graph. The **maximum clique problem** focuses on finding the largest clique of graph G, i.e., the largest complete subgraph of G. For any clique K of G, the adjacency matrix of the graph K' obtained by taking the union of K and the set of loops for each $v \in V(K)$ is a rank-one matrix with 1's in the entries indexed by $V(K) \times V(K)$ and 0's everywhere else. Therefore, a clique K of G containing n vertices can be found by solving the rank minimization problem

$$\begin{aligned} &\min \operatorname{rank}(X) \\ &\text{s.t.} \sum_{i \in V} \sum_{j \in V} X_{ij} \geq n^2, && (9) \\ &\quad X_{ij} = 0 \ \text{ if } (i,j) \notin E \text{ and } i \neq j, && (10) \\ &\quad X \in [0,1]^{V \times V}. && (11) \end{aligned}$$

Unfortunately, this rank minimization problem is also NP-hard. We consider the relaxation obtained by replacing the objective function with the nuclear norm, the sum of the singular values of the matrix: $\|X\|_* = \sigma_1(X) + \cdots + \sigma_N(X)$.

Underestimating $\operatorname{rank}(X)$ with $\|X\|_*$, we obtain the following convex optimization problem:

$$\begin{array}{ll} \min & \|X\|_* \\ \text{s.t.} & \sum_{i \in V} \sum_{j \in V} X_{ij} \geq n^2, \\ & X_{ij} = 0 \ \text{ if } (i,j) \notin E \text{ and } i \neq j. \end{array} \qquad (12)$$

Notice that the relaxation has dropped the constraint $X_{ij} \leq 1$ that was present in the original formulation. This constraint turns out to be superfluous (and, in fact, unhelpful—see the remark following (20)) for our approach. Using the Karush-Kuhn-Tucker conditions, we derive conditions for which the adjacency matrix of a graph comprising a clique of G of size n together with n loops for each vertex in the clique is optimal for this convex relaxation.

3.1 Optimality Conditions

In this section, we prove a theorem that gives sufficient conditions for optimality and uniqueness of a solution to (12). These conditions involve multipliers λ_{ij} and μ and a matrix W. In subsequent subsections we explain how to select λ_{ij}, μ and W based on the underlying graph to satisfy the conditions.

Recall that if $f : \mathbf{R}^n \to \mathbf{R}$ is a convex function, then a *subgradient* of f at a point $\mathbf{x}$ is defined to be a vector $\mathbf{g} \in \mathbf{R}^n$ such that for all $\mathbf{y} \in \mathbf{R}^n$, $f(\mathbf{y}) - f(\mathbf{x}) \geq \mathbf{g}^T(\mathbf{y} - \mathbf{x})$. It is a well-known theorem that for a convex f and for every $\mathbf{x} \in \mathbf{R}^n$, the set of subgradients forms a nonempty closed convex set. This set of subgradients, called the *subdifferential*, is denoted as $\partial f(\mathbf{x})$.

In this section we consider the following generalization of (12) because it will also arise in our discussion of biclique below:

$$\begin{array}{ll} \min & \|X\|_* \\ \text{s.t.} & \sum_{i=1}^{M}\sum_{j=1}^{N} X_{i,j} \geq mn, \\ & X_{i,j} = 0 \text{ for } (i,j) \in \tilde{E} \end{array} \tag{13}$$

Here, $X \in \mathbf{R}^{M\times N}$, E is a subset of $\{1,\ldots,M\} \times \{1,\ldots,N\}$, and the complement of E is denoted $\tilde{E}$.

The following lemma characterizes the subdifferential of $\|\cdot\|_*$ (see [4, Equation 3.4] and also [18]).

Lemma 3.1 *Suppose $A \in \mathbf{R}^{m\times n}$ has rank r with singular value decomposition $A = \sum_{k=1}^{r} \sigma_k \mathbf{u}_k \mathbf{v}_k^T$. Then ϕ is a subgradient of $\|\cdot\|_*$ at A if and only if ϕ is of the form*

$$\phi = \sum_{k=1}^{r} \mathbf{u}_k \mathbf{v}_k^T + W$$

where W satisfies $\|W\| \leq 1$ such that the column space of W is orthogonal to $\mathbf{u}_k$ and the row space of W is orthogonal to $\mathbf{v}_k$ for all $k = 1, 2, \ldots, r$.

Let I be a subset of $\{1,\ldots,N\}$. We say that $\mathbf{u} \in \mathbf{R}^N$ is the *characteristic vector* of I if $u_i = 1$ for $i \in I$ while $u_i = 0$ for $i \in \{1,\ldots,N\} - I$.

Let U^* be a subset of $\{1,\ldots,M\}$ and V^* a subset of $\{1,\ldots,N\}$, and let $\bar{\mathbf{u}}$, $\bar{\mathbf{v}}$ be their characteristic vectors respectively. Suppose $|U^*| = m$ and $|V^*| = n$ with $m > 0$, $n > 0$. Let $X^* = \bar{\mathbf{u}}\bar{\mathbf{v}}^T$, an $M \times N$ matrix. Clearly X^* has rank 1. Note that Lemma 3.1 implies that

$$\partial\|\cdot\|_*(X^*) = \{\bar{\mathbf{u}}\bar{\mathbf{v}}^T/\sqrt{mn} + W : W\bar{\mathbf{v}} = \mathbf{0},\ W^T\bar{\mathbf{u}} = \mathbf{0},\ \|W\| \leq 1\}. \tag{14}$$

This leads to the main theorem for this section.

Theorem 3.1 *Let U^* be a subset of $\{1,\ldots,M\}$ of cardinality m, and let V^* be a subset of $\{1,\ldots,N\}$ of cardinality n. Let $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}$ be the characteristic vectors of U^*, V^* respectively. Let $X^* = \bar{\mathbf{u}}\bar{\mathbf{v}}^T$. Suppose X^* is feasible for* (13). *Suppose also that there exist $W \in \mathbf{R}^{M\times N}$, $\lambda \in \mathbf{R}^{M\times N}$ and $\mu \in \mathbf{R}_+$ such that $W\bar{\mathbf{v}} = \mathbf{0}$, $\bar{\mathbf{u}}^T W = \mathbf{0}$, $\|W\| \leq 1$ and*

$$\frac{\bar{\mathbf{u}}\bar{\mathbf{v}}^T}{\sqrt{mn}} + W = \mu\mathbf{e}\mathbf{e}^T + \sum_{(i,j)\in\tilde{E}} \lambda_{ij}\mathbf{e}_i\mathbf{e}_j^T. \tag{15}$$

Here, $\mathbf{e}$ denotes the vector of all 1's while $\mathbf{e}_i$ denotes the ith column of the identity matrix (either in $\mathbf{R}^M$ or $\mathbf{R}^N$). Then X^* is an optimal solution to (13)*. Moreover, for any $I \subset \{1,\ldots,M\}$ and $J \subset \{1,\ldots,N\}$ such that $I \times J \subset E$, $|I| \cdot |J| \leq mn$.*

Furthermore, if $\|W\| < 1$ and $\mu > 0$, then X^* is the unique optimizer of (13) *(and hence will be found if a solver is applied to* (13)*).*

Proof: The fact that X^* is optimal is a straightforward application of the well-known KKT conditions. Nonetheless, we now explicitly prove optimality because the inequalities in the proof are useful for the uniqueness proof below.

Suppose X is another matrix feasible for (13). We wish to show that $\|X\|_* \geq \|X^*\|_*$. To prove this, we use the definition of subgradient followed by (15). The notation $A \bullet B$ is used to denote the elementwise inner product of two matrices A, B.

$$\begin{aligned} \|X\|_* - \|X^*\|_* &\geq (\bar{\mathbf{u}}\bar{\mathbf{v}}^T/\sqrt{mn} + W) \bullet (X - X^*) & (16) \\ &= \mu(\mathbf{e}\mathbf{e}^T) \bullet (X - X^*) + \sum_{(i,j)\in\tilde{E}} \lambda_{ij}(\mathbf{e}_i\mathbf{e}_j^T) \bullet (X - X^*) & (17) \\ &= \mu\left((\mathbf{e}\mathbf{e}^T) \bullet X - mn\right) & (18) \\ &\geq 0. & (19) \end{aligned}$$

Equation (16) follows by the definition of subgradient and (14); (17) follows from (15); and (18) follows from the fact that $(\mathbf{e}\mathbf{e}^T) \bullet X^* = mn$ by definition of X^* and $(\mathbf{e}_i\mathbf{e}_j^T) \bullet X = (\mathbf{e}_i\mathbf{e}_j^T) \bullet X^* = 0$ for $(i,j) \in \tilde{E}$ by feasibility. Finally, (19) follows since $\mu \geq 0$ and $(\mathbf{e}\mathbf{e}^T) \bullet X \geq mn$ by feasibility. This proves that X^* is an optimal solution to (13).

Now consider (I, J) such that $I \times J \subset E$. Then $X' = \bar{\mathbf{u}}'(\bar{\mathbf{v}}')^T \cdot mn/(|I|\cdot|J|)$, where $\bar{\mathbf{u}}'$ is the characteristic vector of I and $\bar{\mathbf{v}}'$ is the characteristic vector of J, is also a feasible solution to (13). Recall that for a matrix of the form $\mathbf{u}\mathbf{v}^T$, the unique nonzero singular value (and hence the nuclear norm) equals $\|\mathbf{u}\|\cdot\|\mathbf{v}\|$. Thus, $\|X'\|_* = mn/(|I|\cdot|J|)^{1/2}$ and $\|X^*\|_* = \sqrt{mn}$. Since X^* is optimal, $\|X'\|_* \geq \|X^*\|$, i.e., $\sqrt{mn} \leq mn/(|I|\cdot|J|)^{1/2}$. Simplifying yields $|I|\cdot|J| \leq mn$.

Now finally we turn to the uniqueness of X^*, which is the most complicated part of the proof. This argument requires a preliminary claim. Let S_1 denote the subspace of $M \times N$ matrices Z_1 such that $\bar{\mathbf{u}}^T Z_1 = \mathbf{0}$ and $Z_1\bar{\mathbf{v}} = \mathbf{0}$. Let S_2 denote the subspace of $M \times N$ matrices that can be written in the form $\mathbf{x}\bar{\mathbf{v}}^T$, where $\mathbf{x} \in \mathbf{R}^M$ has all zeros in positions indexed by U^*. Let S_3 denote the subspace of $M \times N$ matrices that can be written in the form $\bar{\mathbf{u}}\mathbf{y}^T$, where $\mathbf{y} \in \mathbf{R}^N$ has all zeros in positions indexed by V^*. Let S_4 denote the subspace of all $M \times N$ matrices that can be written in the form $\bar{\mathbf{u}}\mathbf{y}^T + \mathbf{x}\bar{\mathbf{v}}^T$, where $\mathbf{x}$ has nonzeros only in positions indexed by U^*, $\mathbf{y}$ has nonzeros only in positions indexed by V^*, and the sum of entries of $\bar{\mathbf{u}}\mathbf{y}^T + \mathbf{x}\bar{\mathbf{v}}^T$ is zero. Finally, let S_5 be the subspace of $M \times N$ matrices of the form $\alpha\bar{\mathbf{u}}\bar{\mathbf{v}}^T$, where α is a scalar.

The preliminary claim is that $S_1, \ldots, S_5$ are mutually orthogonal and that $S_1 \oplus \cdots \oplus S_5 = \mathbf{R}^{M\times N}$. To check orthogonality, we proceed case by case. For example, if $Z_1 \in S_1$ and $Z_2 \in S_2$, then $Z_2 = \mathbf{x}\bar{\mathbf{v}}^T$ so $Z_1 \bullet Z_2 = Z_1 \bullet (\mathbf{x}\bar{\mathbf{v}}^T) = \mathbf{x}^T Z_1 \bar{\mathbf{v}} = 0$ since $Z_1\bar{\mathbf{v}} = \mathbf{0}$. The identity $Z \bullet (\mathbf{x}\mathbf{y}^T) = \mathbf{x}^T Z \mathbf{y}$ similarly shows that Z_1 is orthogonal to all of $S_2, \ldots, S_5$. Next, observe that $Z_2 \in S_2$ has nonzero entries only in positions indexed by $U^* \times \tilde{V}^*$, where $\tilde{V}^*$ denotes $\{1, \ldots, N\} - V^*$. Similarly, $Z_3 \in S_3$ has nonzero entries only in positions indexed by $\tilde{U}^* \times V^*$, and $Z_4 \in S_4$ and $Z_5 \in S_5$ have nonzero entries only in positions indexed by $U^* \times V^*$. Thus, the nonzero entries of S_2, S_3 and $S_4 \oplus S_5$ are disjoint, and hence these spaces are mutually orthogonal. The only remaining case is to show that S_4 and S_5 are orthogonal; this follows because a matrix in S_5 is a multiple of the all 1's matrix in positions indexed by $U^* \times V^*$, while the entries of a matrix in S_4, also only in positions indexed by $U^* \times V^*$, sum to 0.

Now we must show that $S_1 \oplus \cdots \oplus S_5 = \mathbf{R}^{M\times N}$. Select a $Z \in \mathbf{R}^{M\times N}$. We first split off an S_5 component: let $\alpha = \bar{\mathbf{u}}^T Z \bar{\mathbf{v}}/((\bar{\mathbf{u}}^T\bar{\mathbf{u}})(\bar{\mathbf{v}}^T\bar{\mathbf{v}}))$ and define $Z_5 = \alpha\bar{\mathbf{u}}\bar{\mathbf{v}}^T$. Then $Z_5 \in S_5$. Let

$\dot{Z} = Z - Z_5$. One checks from the definition of α that $\bar{\mathbf{u}}^T\dot{Z}\bar{\mathbf{v}} = 0$. It remains to write $\dot{Z}$ as a matrix in $S_1 \oplus \cdots \oplus S_4$.

Next we split off an S_1 component. Let $\mathbf{x} = \dot{Z}\bar{\mathbf{v}}/\bar{\mathbf{v}}^T\bar{\mathbf{v}}$ and $\mathbf{y} = \dot{Z}^T\bar{\mathbf{u}}/\bar{\mathbf{u}}^T\bar{\mathbf{u}}$. Observe that $\bar{\mathbf{u}}^T\mathbf{x} = \bar{\mathbf{u}}^T\dot{Z}\bar{\mathbf{v}}/\bar{\mathbf{v}}^T\bar{\mathbf{v}} = 0$. Similarly, $\bar{\mathbf{v}}^T\mathbf{y} = 0$. Let $\ddot{Z} = \mathbf{x}\bar{\mathbf{v}}^T + \bar{\mathbf{u}}\mathbf{y}^T$ and $Z_1 = \dot{Z} - \ddot{Z}$. Then

$$\begin{aligned} Z_1\bar{\mathbf{v}} &= \dot{Z}\bar{\mathbf{v}} - \ddot{Z}\bar{\mathbf{v}} \\ &= \dot{Z}\bar{\mathbf{v}} - \mathbf{x}\bar{\mathbf{v}}^T\bar{\mathbf{v}} - \bar{\mathbf{u}}\mathbf{y}^T\bar{\mathbf{v}} \\ &= \dot{Z}\bar{\mathbf{v}} - \mathbf{x}\bar{\mathbf{v}}^T\bar{\mathbf{v}} \\ &= \mathbf{0}, \end{aligned}$$

where the third line follows because $\bar{\mathbf{v}}^T\mathbf{y} = 0$ and the fourth by definition of $\mathbf{x}$. Similarly, $Z_1^T\bar{\mathbf{u}} = \mathbf{0}$. Thus, $Z_1 \in S_1$.

It remains to split $\ddot{Z}$ among S_2, S_3 and S_4. Write $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, where $\mathbf{x}_1$ is nonzero only in entries indexed by U^* while $\mathbf{x}_2$ is nonzero only in entries indexed by $\tilde{U}^*$. Similarly, split $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$ using V^* and $\tilde{V}^*$. Then $\ddot{Z} = \mathbf{x}_1\bar{\mathbf{v}}^T + \mathbf{x}_2\bar{\mathbf{v}}^T + \bar{\mathbf{u}}\mathbf{y}_1^T + \bar{\mathbf{u}}\mathbf{y}_2^T$. Then $\mathbf{x}_2\bar{\mathbf{v}}^T \in S_2$ and $\bar{\mathbf{u}}\mathbf{y}_2^T \in S_3$, so define $Z_2 = \mathbf{x}_2\bar{\mathbf{v}}^T$ and $Z_3 = \bar{\mathbf{u}}\mathbf{y}_2^T$. Finally, we must consider the remaining term $Z_4 = \ddot{Z} - \mathbf{x}_2\bar{\mathbf{v}}^T - \bar{\mathbf{u}}\mathbf{y}_2^T = \mathbf{x}_1\bar{\mathbf{v}}^T + \bar{\mathbf{u}}\mathbf{y}_1^T$. This has the form required for membership in S_4, but it remains to verify that the sum of entries of Z_4 add to zero. This is shown as follows:

$$\begin{aligned} Z_4 \bullet (\mathbf{e}\mathbf{e}^T) &= Z_4 \bullet (\bar{\mathbf{u}}\bar{\mathbf{v}}^T) \\ &= \bar{\mathbf{u}}^T Z_4\bar{\mathbf{v}} \\ &= (\bar{\mathbf{u}}^T\mathbf{x}_1)(\bar{\mathbf{v}}^T\bar{\mathbf{v}}) + (\bar{\mathbf{u}}^T\bar{\mathbf{u}})(\mathbf{y}_1^T\bar{\mathbf{v}}) \\ &= (\bar{\mathbf{u}}^T\mathbf{x})(\bar{\mathbf{v}}^T\bar{\mathbf{v}}) + (\bar{\mathbf{u}}^T\bar{\mathbf{u}})(\mathbf{y}^T\bar{\mathbf{v}}) \\ &= 0 + 0. \end{aligned}$$

The second line follows because Z_4 is all zeros outside entries indexed by $U^* \times V^*$. The fourth line follows because $\bar{\mathbf{u}}$ is zero outside U^* and similarly for $\bar{\mathbf{v}}$. The last line follows from equalities derived in the previous paragraph.

This concludes the proof of the claim that $S_1, \cdots, S_5$ split $\mathbf{R}^{M\times N}$ into mutually orthogonal subspaces.

Now we prove the uniqueness of X^* under the assumption that $\mu > 0$ and $\|W\| < 1$. Let X be a feasible solution different from X^*. Write $X - X^* = Z_1 + \cdots + Z_5$, where $Z_1, \ldots, Z_5$ lie in $S_1, \ldots, S_5$ respectively. Now we consider several cases.

The first case is that $Z_1 \neq 0$. Then since $\|W\| < 1$ and $Z_1\bar{\mathbf{v}} = \mathbf{0}$, $Z_1^T\bar{\mathbf{u}} = \mathbf{0}$, it follows from Lemma 3.1 that $W + \epsilon Z_1$ lies in $\partial\|\cdot\|_*(X^*)$ for $\epsilon > 0$ sufficiently small. This means that 'W' appearing in (16) above may be replaced by $W + \epsilon Z_1$ without harming the validity of the inequality. This adds the term $\epsilon Z_1 \bullet (X - X^*)$ to the right-hand sides of the inequalities following (16). Observe that $Z_1 \bullet (X - X^*) = Z_1 \bullet (Z_1 + \cdots + Z_5) = Z_1 \bullet Z_1 > 0$. Thus, a positive quantity is added to all these right-hand sides, so we conclude $\|X\|_* - \|X^*\|_* > 0$.

For the remaining cases, we assume $Z_1 = 0$. We claim that $Z_2 = Z_3 = 0$ as well. For example, suppose $Z_2 = \mathbf{x}\bar{\mathbf{v}}^T$. Recall that Z_2 is nonzero only for entries indexed by $\tilde{U}^* \times V^*$ (and in particular, $\mathbf{x}$ must be zero on U^*). Since all of Z_3, Z_4 and Z_5 are zero in $\tilde{U}^* \times V^*$, $Z_2(i,j) = X_{ij} - X^*_{ij}$ for $(i,j) \in \tilde{U}^* \times V^*$. Select an $i \in \tilde{U}^*$; we claim that there exists a $j \in V^*$ such that $(i,j) \notin E$. If not, then $(U^* \cup \{i\}) \times V^*$ would define a solution to (13) with greater cardinality (and hence lower objective value) than $U^* \times V^*$, but we have already proven that $U^* \times V^*$ defines the optimal solution. Thus, there is a constraint in (13) of the form $X_{i,j} = 0$ that must be satisfied by both X and X^*. This means that the (i, j) entry of Z_2 is zero. On the other hand, this entry is $x_i \bar{v}_j = x_i$. Thus, we conclude $x_i = 0$. Therefore, $\mathbf{x} = \mathbf{0}$ so Z_2 vanishes. The same argument shows Z_3 vanishes.

The last case is thus that Z_1, Z_2 and Z_3 are all zero, so at least one of Z_4 or Z_5 must be nonzero. Since the sum of entries of Z_4 is zero and X is feasible (and, in particular, feasible for the constraint $X \bullet (\mathbf{e}\mathbf{e}^T) \geq mn$), it follows that the sum of entries of Z_5 must be nonnegative, i.e., $Z_5 = \alpha \bar{\mathbf{u}}\bar{\mathbf{v}}^T$ with $\alpha \geq 0$. If $\alpha > 0$ then we are finished with the proof: the assumption $\mu > 0$ and $\alpha > 0$ imply that both factors in (18) are positive, hence $\|X\|_* - \|X^*\|_* > 0$.

Thus, we may assume that $Z_5 = 0$ so $Z_4 \neq 0$. Recall that Z_4 is nonzero only in positions indexed by $U^* \times V^*$. We can now draw the following conclusions about the singular values of X versus those of X^*. Recall that the rank of X^* is one, and its sole nonzero singular value is $\sqrt{mn}$ and hence $\|X^*\|_F = \|X^*\| = \|X^*\|_* = \sqrt{mn}$. Observe that the sum of entries of X, namely, $\bar{\mathbf{u}}^T X \bar{\mathbf{v}}$, is also mn. But $\bar{\mathbf{u}}^T X \bar{\mathbf{v}} \leq \|\bar{\mathbf{u}}\| \cdot \|X\| \cdot \|\bar{\mathbf{v}}\| = \|X\|\sqrt{mn}$. Thus, $\|X\| \geq \sqrt{mn}$, i.e., $\sigma_1(X) \geq \sigma_1(X^*)$, where $\sigma_k(A)$ is notation for the kth singular value of matrix A.

Next, note that $\|X\|_F > \|X^*\|_F$ for the following reason. Recall that the Frobenius norm is equivalent to the Euclidean vector norm applied to the matrix when regarded as a vector. Furthermore, when regarded as a vector, X is the sum of two orthogonal components, namely X^* and Z_4. Therefore, by the Pythagorean theorem, $\|X\|_F = (\|X^*\|_F^2 + \|Z_4\|^2)^{1/2}$. Since $Z_4 \neq 0$, $\|X\|_F > \|X^*\|_F$.

Thus, we know that $\sigma_1(X) \geq \sigma_1(X^*)$ and that $\sigma_1(X)^2 + \sigma_2(X)^2 > \sigma_1(X^*)^2$. These two inequalities imply that $\sigma_1(X) + \sigma_2(X) > \sigma_1(X^*)$, and therefore $\|X\|_* > \|X^*\|_*$.

Thus, we have shown that in all cases, if $\|W\| < 1$, $\mu > 0$ and X is a feasible point distinct from X^*, then $\|X\|_* > \|X^*\|_*$. This proves that X^* is the unique optimizer.

This theorem immediately specializes to the following theorem if we take the case that G is an N-node undirected graph, that $M = N$, $m = n$, and $E = E(G) \cup \{(i, i) : i \in V(G)\}$.

Theorem 3.2 *Let V^* be the nodes of an n-node clique contained in an N-node undirected graph $G = (V, E)$. Let $\bar{\mathbf{v}} \in \mathbf{R}^V$ be the characteristic vector of V^*. Let $X^* = \bar{\mathbf{v}}\bar{\mathbf{v}}^T$. (Clearly X^* is feasible for (12)). Suppose also that there exist $W \in \mathbf{R}^{V \times V}$, $\lambda \in \mathbf{R}^{V \times V}$ and $\mu \in \mathbf{R}_+$ such that $W\bar{\mathbf{v}} = \mathbf{0}$, $\bar{\mathbf{v}}^T W = \mathbf{0}$, $\|W\| \leq 1$ and*

$$\frac{\bar{\mathbf{v}}\bar{\mathbf{v}}^T}{n} + W = \mu \mathbf{e}\mathbf{e}^T + \sum_{(i,j) \in \tilde{E}} \lambda_{ij} \mathbf{e}_i \mathbf{e}_j^T. \tag{20}$$

Then X^* is an optimal solution to (12). Moreover, V^* is a maximum clique of G. Furthermore, if $\|W\| < 1$ and $\mu > 0$, then X^* is the unique optimizer of (12), and V^* is the unique maximum clique of G.

Remark: It may appear that we need to know the value of n prior to applying the theorem since n is present in the statement of (12). In fact, this is not the case: we observe that the factor n^2 appearing in (12) is the sole inhomogeneity in the problem. This means that we obtain the same solution, rescaled in the appropriate way, if we replace n^2 by 1 in (12). Thus, n does not need to be known in advance to apply this theorem.

For the next two subsections, we consider two scenarios for constructing G and try to find X^*, W and values for the multipliers to satisfy the conditions of the previous theorem. For both subsections, we use the following choices. We take $\mu = 1/n$ where $n = |V^*|$. We define W and λ by considering the following cases:

(ω_1) If $(i,j) \in V^* \times V^*$, we choose $W_{ij} = 0$ and $\lambda_{ij} = 0$. In this case, the entries on other side of (20) corresponding to this case become $1/n + 0 = 1/n + 0$.

(ω_2) If $(i,j) \in E - (V^* \times V^*)$ such that $i \neq j$, then we choose $W_{ij} = 1/n$ and $\lambda_{ij} = 0$. Then the two sides of (20) become $0 + 1/n = 1/n + 0$.

(ω_3) If $i \notin V^*$, we set $W_{ii} = 1/n$. Again the two sides of (20) become $0 + 1/n = 1/n + 0$.

(ω_4) If $(i,j) \notin E$, $i \notin V^*$, $j \notin V^*$, then we choose $W_{ij} = -\gamma/n$ and $\lambda_{ij} = -(1+\gamma)/n$ for some constant $\gamma \in \mathbf{R}$. The two sides of (20) become $0 - \gamma/n = 1/n - (1+\gamma)/n$. The value of γ is specified below.

(ω_5) If $(i,j) \notin E$, $i \in V^*$, $j \notin V^*$, then we choose

$$W_{ij} = -\frac{p_j}{n(n-p_j)}, \quad \lambda_{ij} = -\frac{1}{n} - \frac{p_j}{n(n-p_j)}$$

where p_j is equal to the number of edges in E from j to V^*.

(ω_6) If $(i,j) \notin E$, $i \notin V^*$, $j \in V^*$ then choose W_{ij}, λ_{ij} symmetrically with the previous case.

First, observe that $W\bar{\mathbf{v}} = \mathbf{0}$. Indeed, for entries $i \in V^*$, $W(i,:)\bar{\mathbf{v}} = 0$ since $W(i, V^*) = 0$ for such entries. For entries $i \in V - V^*$,

$$W(i,:)\bar{\mathbf{v}} = p_i \frac{1}{n} - (n - p_i)\frac{p_i}{n(n-p_i)} = 0$$

by our special choice of $W(i,j)$ in cases 5 and 6.

It remains to determine which graphs G yield W as defined by (ω_1)–(ω_6) such that $\|W\| < 1$. We present two different analyses.

3.2 The Adversarial Case

Suppose that the edge set of the graph $G = (V, E)$ is generated as follows. We first add a clique K_{V^*} with vertex set V^* of size n. Then, an adversary is allowed to add a number of the remaining $|V|(|V|-1)/2 - n(n-1)/2$ potential edges to the graph. We will show that, under certain conditions, our adversary can add up to $O(n^2)$ edges to the graph and K_V^* will still be the unique maximum clique of G.

We first introduce the following notation. Let $W^D \in \mathbf{R}^{V \times V}$ denote the matrix with diagonal entries equal to the diagonal entries of W and all other entries equal to 0. Let W^{ND} be the matrix whose nondiagonal entries are equal to the corresponding nondiagonal entries of W and whose diagonal entries are equal to 0. So $W = W^D + W^{ND}$.

Now suppose $G = (V, E)$ contains a clique K_{V^*} of size n with vertices indexed by $V^* \in \mathbf{R}^V$. Moreover, suppose that G contains at most r edges not in K_{V^*} and each vertex in $V - V^*$ is adjacent to at most δn vertices in V^* for some $\delta \in (0, 1)$. Consider W as defined by (ω_1)–(ω_6) with $\gamma = 0$. By the triangle inequality,

$$\|W\|^2 \leq (\|W^D\| + \|W^{ND}\|)^2 \leq 2(\|W^D\|^2 + \|W^{ND}\|^2) = 2(1/n^2 + \|W^{ND}\|^2)$$

since $\|W^D\| = 1/n$. Applying the bound $\|W\| \leq \|W\|_F$, it suffices to determine which values of r yield

$$\|W^{ND}\|_F^2 = 2\|W(V^*, V - V^*)\|_F^2 + \|W^{ND}(V - V^*, V - V^*)\|_F^2 < (n^2 - 2)/(2n^2)$$

since, by the symmetry of W,

$$W^{ND}(V^*, V - V^*) = W(V^*, V - V^*) = W(V - V^*, V^*).$$

The diagonal entries of $W^{ND}(V - V^*, V - V^*)$ are equal to 0 and at most $2r$ of the remaining entries are equal to $1/n$. Therefore,

$$\|W^{ND}(V - V^*, V - V^*)\|_F^2 \leq 2r/n^2.$$

Moreover, since $n - p_j \geq (1 - \delta)n$,

$$\begin{aligned}
\|W(V^*, V - V^*)\|_F^2 &= \sum_{j \in V - V^*} \left(p_j \cdot \frac{1}{n^2} + (n - p_j) \cdot \frac{p_j^2}{(n - p_j)^2 n^2} \right) \\
&= \sum_{j \in V - V^*} \left(\frac{p_j}{n^2} + \frac{p_j^2}{(n - p_j) n^2} \right) \\
&\leq \sum_{j \in V - V^*} \left(\frac{p_j}{n^2} + \frac{\delta n p_j}{(1 - \delta) n^3} \right) \\
&= \left(\frac{1}{1 - \delta} \right) \sum_{j \in V - V^*} \frac{p_j}{n^2} \\
&\leq \left(\frac{1}{1 - \delta} \right) \frac{r}{n^2}.
\end{aligned}$$

Thus, the optimality and uniqueness conditions given by Theorem 3.1 are satisfied by X^* if

$$\left(1 + \frac{1}{1 - \delta} \right) r < (n^2 - 2)/4.$$

Equivalently,

$$r < \frac{1 - \delta}{4(2 - \delta)} (n^2 - 2).$$

Therefore, G can contain up to $O(n^2)$ edges other than those in $V^* \times V^*$, and yet V^* will remain the unique maximum clique of G.

Note that these bounds are the best possible up to the constant factors. In particular, if the adversary were able to insert $(n+1)(n+2)/2$ edges, then a new clique could be created larger than the planted clique. Thus, the adversary must be limited to const $\cdot\, n^2$ edges for const $< 1/2$. Similarly, if the adversary could join a nonclique vertex to n clique vertices, then the adversary would have enlarged the clique. Thus, the restriction that a nonclique vertex is adjacent to at most const $\cdot\, n$ clique vertices is the best possible.

3.3 The Randomized Case

Let V be a set of vertices with $|V| = N$ and consider a subset $V^* \subseteq V$ such that $|V^*| = n$. We construct the edge set E of the graph $G = (V, E)$ as follows:

(Γ_1) For all $(i, j) \in V^* \times V^*$, $(i, j) \in E$.

(Γ_2) Each of the remaining $N(N-1)/2 - n(n-1)/2$ possible edges is added to E independently at random with probability $p \in [0, 1)$.

Notice that, by our construction of E, G contains a clique of size n with vertices indexed by V^*. We wish to determine which n, N yield G as constructed by (Γ_1) and (Γ_2) such that with high probability $X^* = \bar{\mathbf{v}}\bar{\mathbf{v}}^T$ is optimal for the convex relaxation of the clique problem given by (12). The following theorem states the desired result.

Theorem 3.3 *There exists an $\alpha > 0$ depending on p such that for all G constructed via (Γ_1), (Γ_2) with $n \geq \alpha\sqrt{N}$, the clique defined by $V^* \times V^*$ is the unique maximum clique of G and will correspond to the unique solution of* (12) *with probability tending exponentially to* 1 *as $N \to \infty$.*

Proof: Consider the matrix W constructed as in (ω_1)–(ω_6) with $\gamma = -p/(1-p)$. By Theorem 3.2, X^* is the unique optimum if

$$\|W\| < 1 \quad \text{and} \quad p_j < n \text{ for all } j \in V - V^*$$

We first show that $\|W\| < 1$ with probability tending exponentially to 1 as $N \to \infty$ in the case that $n = \Omega(\sqrt{N})$. We write $W = W_1 + W_2 + W_3 + W_4 + W_5$, where each of the five terms is defined as follows.

We first define W_1. For cases (ω_2) and (ω_4), choose $W_1(i, j) = W(i, j)$. For cases (ω_5) and (ω_6), take $W_1(i, j) = -p/((1-p)n)$. For case (ω_1), choose $W_1(i, j)$ randomly such that $W_1(i, j)$ is equal to $1/n$ with probability p and equal to $-p/((1-p)n)$ otherwise. Similarly, in case (ω_3), take $W_1(i, i)$ to be equal to $1/n$ with probability p and equal to $-p/((1-p)n)$ otherwise. By construction, each entry of W_1 is an independent random variable with the distribution

$$W_1(i, j) = \begin{cases} 1/n & \text{with probability } p, \\ -p/((1-p)n) & \text{with probability } 1-p. \end{cases}$$

Therefore, applying Lemma 2.1 shows that there exists constant $c_1 > 0$ such that

$$\|W_1\| \leq 3\left(\frac{p}{1-p}\right)^{1/2} \frac{\sqrt{N}}{n} \tag{21}$$

with probability at least to $1 - \exp(c_1 N^{1/6})$ for some constant $c_1 > 0$.

Next, W_2 is the correction matrix to W_1 in case (ω_1). That is, $W_2(i, j)$ is chosen such that

$$W_2(i, j) + W_1(i, j) = W(i, j) = 0$$

for all $(i,j) \in V^* \times V^*$ and is zero everywhere else. As before, applying Lemma 2.1 shows that

$$\|W_2\| \leq 3\left(\frac{p}{1-p}\right)^{1/2}\frac{1}{\sqrt{n}} \tag{22}$$

with probability at least $1-\exp(c_1 n^{1/6})$. Similarly, W_3 is the correction to W_3 in case (ω_3), that is

$$W_3(i,i) = W(i,i) - W_1(i,i)$$

for all $i \in V - V^*$ and all other entries are equal 0. Therefore, W_3 is a diagonal matrix with diagonal entries bounded by $2/n$. It follows that

$$\|W_3\| \leq \frac{2}{n}. \tag{23}$$

Finally, W_4 and W_5 are the corrections for cases (ω_5) and (ω_6) respectively. These are exactly of the form $(A-\tilde{A})/n$ as in Theorem 2.4, in which N in the theorem stands for $N-n$ in the present context. Examining each term of (3) shows that in the case $n = \Omega(N^{1/2})$, the probability on the right-hand side is the form $1 - c\exp(-kN^{c_2})$. It follows that there exists constant $\alpha_4 > 0$ such that

$$\|W_4\|^2 \leq \|W_4\|_F^2 < \alpha_4^2 N n^{-2}$$

with probability tending exponentially to 1 as $N \to \infty$. Moreover, since Condition F is satisfied in this case, $p_j < n$ for all $j \in V - V^*$. Notice that, by symmetry, $W_4 = W_5^T$. Thus, since each of $W_1, W_2, \ldots, W_5$ is bounded by an arbitrarily small constant if $n = \Omega(\sqrt{N})$, there exists constant $\alpha > 0$ such that $\|W\| < 1$ with probability tending exponentially to 1 as $N \to \infty$ as required.

4 Maximum Edge Biclique

Consider a bipartite graph $G = ((U,V),E)$ where $|U| = M$, $|V| = N$. The adjacency matrix of a biclique H of G is rank-one matrix $X \in \mathbf{R}^{M\times N}$. This matrix X has the property that $X_{ij} = 0$ for all $i \in U, j \in V$ such that $(i,j) \notin E$. It follows that a biclique of G of size mn can be found (if one exists) by solving the rank minimization problem

$$\begin{aligned} &\min \operatorname{rank}(X) \\ &\text{s.t.} \quad \sum_{i\in U}\sum_{j\in V} X_{ij} \geq mn, && (24)\\ &\qquad X_{ij} = 0 \ \ \forall\, (i,j) \in (U\times V) - E, && (25)\\ &\qquad X \in [0,1]^{U\times V}. && (26) \end{aligned}$$

A rank-one solution X^* to this problem corresponds to the adjacency matrix of a biclique of G containing at least mn edges. As with the maximum clique problem, this rank minimization problem is still NP-hard. As before, we underestimate $\operatorname{rank}(X)$ with $\|X\|_*$. We obtain the following convex optimization problem:

$$\begin{array}{ll} \min & \|X\|_* \\ \text{s.t.} & \sum_{i\in V}\sum_{j\in V} X_{ij} \geq mn, \\ & X_{ij} = 0 \ \text{ if } (i,j) \notin E. \end{array} \tag{27}$$

Using the Karush-Kuhn-Tucker conditions, we derive conditions for which the adjacency matrix of a graph comprising a biclique of G is optimal for this relaxation. Indeed, the following is an immediate consequence (essentially a restatement) of Theorem 3.1.

Theorem 4.1 *Let $U^* \times V^*$ be the vertex set of a biclique in G in which $|U^*| = m$ and $|V^*| = n$. Let $\bar{\mathbf{u}} \in \mathbf{R}^M$ be the characteristic vector of U^*, and let $\bar{\mathbf{v}} \in \mathbf{R}^N$ be the characteristic vector of V^*. Let $X^* = \bar{\mathbf{u}}\bar{\mathbf{v}}^T$. (Clearly X^* is feasible for (27)). Let $E = E(G)$ and let $\tilde{E}$ be its complement. Suppose also that there exist $W \in \mathbf{R}^{M\times N}$, $\lambda \in \mathbf{R}^{M\times N}$ and $\mu \in \mathbf{R}_+$ such that $W\bar{\mathbf{v}} = \mathbf{0}$, $\bar{\mathbf{u}}^T W = \mathbf{0}$, $\|W\| \le 1$ and*

$$\frac{\bar{\mathbf{u}}\bar{\mathbf{v}}^T}{\sqrt{mn}} + W = \mu \mathbf{e}\mathbf{e}^T + \sum_{(i,j)\in\tilde{E}} \lambda_{ij} e_i e_j^T. \tag{28}$$

Then X^* is an optimal solution to (27). Moreover, G does not contain any biclique with more than mn edges. Furthermore, if $\|W\| < 1$ and $\mu > 0$, then X^* is the unique optimizer of (27) and $U^* \times V^*$ is the unique optimal biclique.

In the next two subsections, we consider two scenarios for how to construct a bipartite graph G and biclique that satisfy the conditions of the theorem.

In both scenarios, we will take $\mu = 1/\sqrt{mn}$ and consider W and λ defined according to the following cases.

(ψ_1) For $(i,j) \in U^* \times V^*$, taking $W_{ij} = 0$ and $\lambda_{ij} = 0$ ensures the ij-entries of both sides of (28) are equal to $1/\sqrt{mn}$.

(ψ_2) For $(i,j) \in E - (U^* \times V^*)$, we take $W_{ij} = 1/\sqrt{mn}$ and $\lambda_{ij} = 0$. Again, the ij-entries of both sides of (28) are equal to $1/\sqrt{mn}$.

(ψ_3) For $(i,j) \notin E$ such that $i \notin U^*$ and $j \notin V^*$, we select $W_{ij} = -\gamma/\sqrt{mn}$ and $\lambda_{ij} = -(1+\gamma)/\sqrt{mn}$ where γ will be defined below. In this case, the ij-entries of each side of (28) are 0.

(ψ_4) For $(i,j) \notin E$ such that $i \notin U^*$ and $j \in V^*$, we choose

$$W_{ij} = -\frac{p_i}{(n-p_i)\sqrt{mn}} \text{ and } \lambda_{ij} = \frac{1}{\sqrt{mn}}\left(\frac{-p_i}{n-p_i} - 1\right)$$

where p_i is equal to the number of edges with left endpoint equal to i and right endpoint in V^*. Note that if $n = p_i$ then i is connected to every vertex of V^* and thus the KKT condition cannot possibly be satisfied. If $p_i < n$, both sides of (28) are equal to $-p_i/((n-p_i)\sqrt{mn})$.

(ψ_5) For $(i,j) \notin E$ such that $i \in U^*$ and $j \notin V^*$, we choose

$$W_{ij} = -\frac{q_j}{(m-q_j)\sqrt{mn}} \text{ and } \lambda_{ij} = \frac{1}{\sqrt{mn}}\left(\frac{-q_j}{m-q_j} - 1\right)$$

where q_j is equal to the number of edges with right endpoint equal to j and left endpoint in U^*. As before, this is appropriate only if $q_j < m$.

We next check that W satisfies the requirements for ϕ to be a subgradient of $\bar{\mathbf{u}}\bar{\mathbf{v}}^T$: $W\bar{\mathbf{v}} = \mathbf{0}$, $W^T\bar{\mathbf{u}} = \mathbf{0}$, and $\|W\| \le 1$. To show that $W\bar{\mathbf{v}} = \mathbf{0}$, choose row i of W and consider $W(i,:)\bar{\mathbf{v}} = \sum_{j\in V^*} W_{ij}$. If $i \in U^*$ then $W_{ij} = 0$ for all $j \in V^*$, so $W(i,:)\bar{\mathbf{v}} = 0$. In the case $i \notin U^*$, consider each $j \in V^*$. If $(i,j) \in E$ then, by Case 2, $W_{ij} = 1/\sqrt{mn}$. There are p_i such entries, with sum $p_i/\sqrt{mn}$. If $(i,j) \notin E$, then $W_{ij} = -p_i/((n-p_i)\sqrt{mn})$. There are $n - p_i$ such entries, with sum $-p_i/\sqrt{mn}$. It follows that $W(i,:)\bar{\mathbf{v}} = 0$ as required.

The proof that $W^T\bar{\mathbf{u}} = \mathbf{0}$ follows is symmetric. It remains to determine which bipartite graphs G yield W as defined above such that $\|W\| < 1$. As in the maximum clique case, we present two different analyses.

4.1 The Adversarial Case

Suppose that the edge set of the bipartite graph $G = ((U,V),E)$ is generated as follows. We first add a biclique $U^* \times V^*$ with $|U^*| = m$, $|V^*| = n$. Then, as in the adversarial case for the maximum clique problem, an adversary is allowed to add a number of the remaining $|U||V| - mn$ potential edges to the graph. We will show that, under certain conditions, our adversary can add up to $O(mn)$ edges to the graph and $U^* \times V^*$ will still be a maximum edge biclique of G.

We make the following assumptions on the structure of G:

1. G contains at most r edges aside from those of the optimal biclique.
2. Each vertex of $V - V^*$ is adjacent to at most αm vertices of U^* for some $\alpha \in (0,1)$.
3. Each vertex of $U - U^*$ is adjacent to at most βn vertices of V^* for some $\beta \in (0,1)$.

Consider W as defined by (ψ_1)-(ψ_5) with $\gamma = 0$. As before, we use the bound $\|W\| \le \|W\|_F$. Notice that at most r entries of $W(U - U^*, V - V^*)$ are equal to $1/\sqrt{mn}$ and the remainder are equal to 0. Therefore,

$$\|W(U - U^*, V - V^*)\|_F^2 \le \frac{r}{mn}.$$

Moreover, for each $j \in V - V^*$, $q_j \le \alpha m$. It follows that

$$\begin{aligned}
\|W(U^*, V - V^*)\|_F^2 &= \sum_{v\in V-V*} \left(\frac{q_v}{mn} + (m - q_v)\frac{q_v^2}{mn(m-q_v)^2} \right) \\
&= \sum_{v\in V^*} \frac{q_v}{mn}\left(1 + \frac{q_v}{m-q_v}\right) \\
&\le \sum_{v\in V^*} \frac{q_v}{mn}\left(1 + \frac{\alpha}{1-\alpha}\right) \\
&= \sum_{v\in V^*} \frac{q_v}{mn(1-\alpha)} \le \frac{r}{mn(1-\alpha)}.
\end{aligned}$$

Similarly,

$$\|W(U - U^*, V^*)\|_F^2 \le \frac{r}{(1-\beta)mn}.$$

Therefore, $\|W\| < 1$ if

$$r\left(1 + \frac{1}{1-\alpha} + \frac{1}{1-\beta}\right) < mn.$$

Thus, the graph can contain up to $O(mn)$ diversionary edges, yet the optimality and uniqueness conditions given by Theorem 4.1 are still satisfied. This result is the best possible up to constants for the same reasons explained at the end of Section 3.2.

4.2 The Random Case

Let y, z be fixed positive scalars. Let U, V be two disjoint vertex sets with $|V| = N$ and $|U| = \lceil yN \rceil$. Consider $U^* \subseteq U$ and $V^* \subseteq V$ such that $|V^*| = n$ and $|U^*| = m = \lceil zn \rceil$. Suppose the edges of the bipartite graph $G = ((U, V), E)$ are determined as follows:

(β_1) For all $(i, j) \in U^* \times V^*$, $(i, j) \in E$.

(β_2) For each of the remaining potential edges $(i, j) \in U \times V$, we add edge (i, j) to E with probability p (independently).

Notice G contains the biclique (U^*, V^*). As in the maximum clique problem, if $n = \Omega(\sqrt{N})$ and G is constructed as in (β_1), (β_2) then $U^* \times V^*$ is optimal for the convex problem (27). We have the following theorem.

Theorem 4.2 *There exists $\alpha > 0$ depending on p, y, z such that for each bipartite graph G constructed via (β_1), (β_2) with $n \geq \alpha\sqrt{N}$ the biclique defined by $U^* \times V^*$ is maximum edge biclique of G with probability tending exponentially to 1 as $N \to \infty$ and is found as the unique solution to the convex relaxation* (27).

Let W be constructed as in (ψ_1)–(ψ_5) with $\gamma = -p/(1-p)$. Then $X^* = \bar{\mathbf{u}}\bar{\mathbf{v}}^T$ is the unique optimal solution of (27) if

$$\|W\| < 1, \quad q_j < \lceil zn \rceil \ \forall\, j \in V - V^*, \ \text{ and } p_j < n \ \forall\, j \in U - U^*.$$

To prove that $\|W\| < 1$ with high probability as $N \to \infty$ in the case that $n = \Omega(\sqrt{N})$, we write

$$W = W_1 + W_2 + W_3 + W_4$$

where each of the summands is defined as follows. We first define W_1. If $(i, j) \in U^* \times V^*$, then we set $W_1(i, j) = 1/\sqrt{mn}$ with probability p and equal to $\gamma/\sqrt{mn}$ with probability $(1-p)$. For $(i, j) \in (U \times V) - (U^* \times V^*)$, we set $W_1(i, j) = 1/\sqrt{mn}$ if $(i, j) \in E$ and set $W_1(i, j) = \gamma/\sqrt{mn}$ otherwise. In order to bound $\|W_1\|$, we will use the following Theorem 2.2 to conclude that $\|W_1\| \leq \alpha\sqrt{N}/\sqrt{mn}$. Since $\sqrt{mn}$ equals $\sqrt{\lceil zn \rceil n}$ and hence is proportional to n, we see that $\|W_1\| \leq$ const with probability exponentially close to 1 provided $n = \Omega(\sqrt{N})$.

Next, set W_2 to be the correction matrix for W_1 for $U^* \times V^*$, that is,

$$W_2(i, j) = \begin{cases} -W_1(i, j) & \text{if } (i, j) \in U^* \times V^* \\ 0 & \text{otherwise,} \end{cases}$$

Again, by Theorem 2.2 we conclude that

$$\|W_2\| \le \alpha \frac{1}{\sqrt{n}}$$

with probability at least $1 - c_1' \exp(-c_2' n^{c_3'})$ for some $c_1', c_2', c_3' > 0$.

It remains to derive bounds for $\|W_3\|$ and $\|W_4\|$. Notice that the construction of $W(U^*, V - V^*)$ and $W(U - U^*, V^*)$ is identical to that in Case (ω_5) for the maximum clique problem. Thus, we can again apply Theorem 2.4, first to W_3 (in which case (n, N) in the theorem stand for $(\lceil zn \rceil, N - n)$) and second to W_4^T (in which case (n, N) in the theorem stand for $(n, \lceil yN \rceil - \lceil zn \rceil)$) to conclude that $\|W_3\|$ and $\|W_4\|$ are both strictly bounded above by constants provided $n = \Omega(\sqrt{N})$ with probability tending to 1 exponentially fast. Moreover, as before, Condition F is satisfied in this case and thus $q_j < \lceil zn \rceil$ for all $j \in V - V^*$ and $p_j < n$ for all $j \in U - U^*$ as required. ■

5 Conclusions

We have shown that the maximum clique and maximum biclique problems can be solved in polynomial time using nuclear norm minimization, a technique recently proposed in the compressive sensing literature, provided that the input graph consists of a single clique or biclique plus diversionary edges. The spectral technique used by Alon et al. [2] for the planted clique problem has been extended to other problems; see, e.g., McSherry [14]. It would be interesting to extend the nuclear norm approach to other NP-hard problems as well.

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