Free Material Optimization with Fundamental Eigenfrequency Constraints*

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Abstract

The goal of this paper is to formulate and solve free material optimization problems with constraints on the smallest eigenfrequency of the optimal structure. A natural formulation of this problem as linear semidefinite program turns out to be numerically intractable. As an alternative, we propose a new approach, which is based on a nonlinear semidefinite low-rank approximation of the semidefinite dual. We introduce an algorithm based on this approach and analyze its convergence properties. The article is concluded by numerical experiments proving the effectiveness of the new approach.

1 Introduction

Free material optimization (FMO) is a branch of structural optimization that gains interest in the recent years. The underlying FMO model was introduced in [4] and has been studied in several further articles as, for example, [2, 30]. The design variable in FMO is the full elastic stiffness tensor that can vary from point to point. The method is supported by powerful optimization and numerical techniques which allow for scenarios with complex bodies, fine finite-element meshes and several load cases. FMO has been successfully used for conceptual design of aircraft components; the most prominent example is the design of ribs in the leading edge of Airbus A380 [13].

As mentioned, the optimal elastic stiffness tensor can vary from point to point; it should be physically available but is otherwise not restricted. Given this freedom, we have to face the question of interpretation of the optimal result. A general anisotropic

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material that changes its properties at any point is certainly not easy to manufacture. The most natural interpretation is the use of fibre-reinforced composite materials, though other interpretations are possible, too. The problem gives the best physically attainable material and its result thus serves as a benchmark to any realized design. The question of practical interpretation of FMO results is intensively studied in the recent EU FP6 project PLATO-N, whose consortium includes several industrial partners.

The basic FMO model has other drawbacks, though. For example, structures may fail due to high stresses, or due to lack of stability of the optimal structure (compare [16, 17] for further discussion). In order to prevent this undesirable behavior, additional requirements have to be taken into account in the FMO model. Typically, such modifications lead to additional constraints on the set of admissible materials and/or the set of admissible displacements. These constraints usually destroy the favorable mathematical structure of the original problem (see [16, 17]). The particular cause of structural failure we want to investigate in this article is vibration resonance. Structural optimization problems with eigenvalue constraints have been intensively studied in the context of truss and topology optimization, see, e.g., [9, 10, 20, 22, 23, 24] and many others. We will give an appropriate formulation of the FMO problem, which takes care of this phenomenon, derive various discretized formulations and propose an efficient algorithm for the solution of the problem.

In contrast to the original FMO model which is based on a static PDE system, vibration of a body is a dynamic process. In the first part of the paper we demonstrate how we can bypass this additional challenge by using a reformulation as (time-independent) generalized eigenvalue problem. As a result, we obtain an extended FMO problem with an eigenfrequency condition. For this problem we are able to prove the existence of an optimal solution. In the second section we explain how an existing discretization scheme (proposed in [28]) can be extended to cover the additional eigenfrequency condition. In the third section we give a first formulation of the discretized FMO problem with vibration constraint as a linear semidefinite program. We further explain why this formulation is not suited to serve as a basis for efficient numerical calculations. In the framework of the fourth section we develop an algorithm which is based on a low-rank approximation of the semidefinite dual. The low-rank approach is motivated by ideas recently introduced by Burer and Monteiro (see [6]) for the solution of linear semidefinite programs. The article is concluded by numerical studies.

Throughout this article we use the following notation: We denote by $\mathbb{S}^N$ the space of symmetric $N \times N$ matrices equipped with the standard inner product $\langle \cdot, \cdot \rangle$ defined by $\langle A, B \rangle := \mathrm{Tr}(AB)$ for any pair of matrices $A, B \in \mathbb{S}^N$. We further denote by $\mathbb{S}^N_+$ the cone of all positive semidefinite matrices in $\mathbb{S}^N$ and use the abbreviation $A \succcurlyeq 0$ for matrices $A \in \mathbb{S}^N_+$. Moreover, for $A, B \in \mathbb{S}^N$, we say that $A \succcurlyeq B$ if and only if $A - B \succcurlyeq 0$, and similarly for $A \preccurlyeq B$.

2 The mathematical model

Material optimization deals with optimal design of elastic structures, where the design variables are material properties. The material can even vanish in certain areas, thus the so-called topology optimization (see, e.g., [3]) can be considered a special case of material optimization.

Let $\Omega \subset \mathbb{R}^2$ be a two-dimensional bounded domain[1] with a Lipschitz boundary.

[1]The entire presentation is given for two-dimensional bodies, to keep the notation simple. Extension to the three-dimensional space is straightforward.

By $u(x) = (u_1(x), u_2(x))$ we denote the displacement vector at a point x of the body under load f, and by

$$e_{ij}(u(x)) = \frac{1}{2}\left(\frac{\partial u_i(x)}{\partial x_j} + \frac{\partial u_j(x)}{\partial x_i}\right) \quad \text{for } i,j = 1,2$$

the (small-)strain tensor. We assume that our system is governed by linear Hooke's law, i.e., the stress is a linear function of the strain

$$\sigma_{ij}(x) = E_{ijk\ell}(x) e_{k\ell}(u(x)) \qquad \text{(in tensor notation)},$$

where E is the elastic (plane-stress) stiffness tensor. The symmetries of E allow us to write the 2^{nd} order tensors e and σ as vectors

$$e = (e_{11}, e_{22}, \sqrt{2}e_{12})^\top \in \mathbb{R}^3,\ \sigma = (\sigma_{11}, \sigma_{22}, \sqrt{2}\sigma_{12})^\top \in \mathbb{R}^3\,.$$

Correspondingly, the 4^{th} order tensor E can be written as a symmetric 3×3 matrix

$$E = \begin{pmatrix} E_{1111} & E_{1122} & \sqrt{2}E_{1112} \\ & E_{2222} & \sqrt{2}E_{2212} \\ \text{sym.} & & 2E_{1212} \end{pmatrix}. \tag{1}$$

In this notation, Hooke's law reads as $\sigma(x) = E(x)e(u(x))$.

For a given external load function $f \in [L_2(\Gamma)]^2$ we obtain the following basic boundary value problem of linear elasticity:

$$\begin{aligned} &\text{Find } u \in [H^1(\Omega)]^2 \text{ such that} \\ &\begin{array}{rcll} \operatorname{div}(\sigma) & = & 0 & \text{in } \Omega \\ \sigma \cdot n & = & f & \text{on } \Gamma \\ u & = & 0 & \text{on } \Gamma_0 \\ \sigma & = & E \cdot e(u) & \text{in } \Omega \end{array} \end{aligned} \tag{2}$$

Here Γ and Γ_0 are open disjunctive subsets of $\partial\Omega$. The corresponding weak form, the so called weak equilibrium equation, reads as:

$$\begin{aligned} &\text{Find } u \in \mathcal{V}, \text{ such that} \\ &\int_\Omega \langle E(x)e(u(x)), e(v(x))\rangle \mathrm{d}x = \int_\Gamma f(x) \cdot v(x) \mathrm{d}x, \quad \forall v \in \mathcal{V}, \end{aligned} \tag{3}$$

where $\mathcal{V} = \{u \in [H^1(\Omega)]^2 \,|\, u = 0 \text{ on } \Gamma_0\} \supset [H_0^1(\Omega)]^2$ reflects the Dirichlet boundary conditions. Below we will use the abbreviation

$$a_E(w, v) = \int_\Omega \langle E(x)e(w(x)), e(v(x))\rangle \mathrm{d}x \tag{4}$$

for the bilinear form on the left hand side of (3). In *free material optimization* (FMO), the design variable is the elastic stiffness tensor E which is a function of the space variable x (see [4]). The only constraint on E is that it is physically reasonable, i.e., that E is symmetric and positive semidefinite. This gives rise to the following definition

$$\mathcal{E}_0 := \left\{E \in L^\infty(\Omega)^{3\times 3} \mid E = E^\top, E \succcurlyeq 0 \text{ a.e. in } \Omega\right\}.$$

The choice of L^∞ is due to the fact that we want to allow for material/no-material situations. A frequently used measure of the stiffness of the material tensor is its trace.

In order to avoid arbitrarily stiff material, we add pointwise stiffness restrictions of the form $\mathrm{Tr}(E) \leq \overline{\rho}$, where $\overline{\rho}$ is a finite real number. We also allow for pointwise lower trace bounds $\mathrm{Tr}(E) \geq \underline{\rho} \geq 0$. Moreover we prescribe the total stiffness/volume by the constraint $v(E) = \bar{v}$. Here the volume $v(E)$ is defined as $\int_\Omega \mathrm{Tr}(E)\mathrm{d}x$ and $\bar{v} \in \mathbb{R}$ is an upper bound on overall resources. Accordingly, we define the *set of admissible materials* as

$$\mathcal{E} := \left\{E \in \mathcal{E}_0 \mid \underline{\rho} \leq \mathrm{Tr}(E) \leq \overline{\rho} \text{ a.e. in } \Omega,\ v(E) = \bar{v}\right\}.$$

We are now able to present the *minimum compliance single-load FMO problem*

$$\begin{aligned}
&\inf_{E\in\mathcal{E}} \int_\Gamma f(x) \cdot u_E(x)\mathrm{d}x \\
&\text{subject to} \\
&\qquad u_E \text{ solves (3).}
\end{aligned} \tag{5}$$

The objective, the so called compliance functional, measures how well the structure can carry the load f. In [28] it is shown that problem (5) can be given equivalently as

$$\inf_{E\in\mathcal{E}} c(E)$$

where $c(E)$ is a closed formula for the compliance given by

$$c(E) = \sup_{u\in\mathcal{V}} \left\{-a_E(u,u) + 2\int_\Gamma f \cdot u \,\mathrm{d}x\right\}.$$

Problem (5) has been extensively studied in [18, 28]. The most successful method for the solution of problem (5), based on dualization of the original problem [2, 28] gave rise to a software package MOPED, which was recently applied to real-world applications and lead to significant improvements of the classic design. On the other hand the underlying FMO model has certain limitations (other than the interpretation of the result discussed in the introduction). One of the drawbacks of problem (5) is that it does not count with possible instability of the structure (compare [16]). One possible source of such instability is vibration resonance. In the sequel we develop a generalized FMO model, which is more robust with respect to this phenomenon. As we will see, the modification in the model results in an additional constraint on the set of admissible materials $\mathcal{E}$.

Vibration of a body—as a dynamic process—can be modeled by the following time-dependent PDE:

$$\mathrm{div}(\sigma(x,t)) = \rho_E(x)\ddot{u}(x,t), \quad (x,t) \in \Omega \times [0,T], \tag{6}$$

with boundary conditions

$$\begin{aligned}
\frac{\partial}{\partial n}\sigma(x,t) &= 0 \quad \text{on } \Gamma \times [0,T] \\
u(x,t) &= 0 \quad \text{on } \Gamma_0 \times [0,T].
\end{aligned}$$

Here the material density term ρ is defined by $\rho_E(x) = \mathrm{tr}(E(x))$ and $E(x)$ is the material tensor introduced earlier in this section. As in this case there is no external force applied to the system, we call the solutions of (6) *free vibrations*. Using the assumption of Hooke's law and introducing the differential operator $S_E(\cdot) := \mathrm{div}(Ee(\cdot))$ we obtain from (6):

$$\begin{aligned} S_E(u(x,t)) &= \rho_E(x)\ddot{u}(x,t), \quad (x,t) \in \Omega \times [0,T], \\ \frac{\partial}{\partial n} S_E(u(x,t)) &= 0 \quad \text{on } \Gamma \times [0,T], \\ u(x,t) &= 0 \quad \text{on } \Gamma_0 \times [0,T]. \end{aligned} \tag{7}$$

Using Fourier transform we derive the following characterization of solutions of system (7):

Proposition 2.1. *The solutions of system (7) are of the form*

$$u(x,t) = \sum_{j=1}^{\infty} \left[a_j \cos(\sqrt{\lambda_j} t) + b_j \sin(\sqrt{\lambda_j} t) \right] w_j(x), \tag{8}$$

where a_j, b_j, are free real parameters and λ_j, w_j are the solutions (eigenvalues and eigenvectors, respectively) of the generalized eigenvalue problem

$$\begin{aligned} -S_E(w_j(x)) &= \lambda_j \rho_E(x) w_j(x), \quad x \in \Omega \\ \frac{\partial}{\partial n} \sigma(x) &= 0 \quad \text{on } \Gamma \\ u(x) &= 0 \quad \text{on } \Gamma_0. \end{aligned} \tag{9}$$

Introducing the bilinear form

$$b(w,v) = \int_\Omega m(x) w(x) v(x) \mathrm{d}x,$$

the weak form associated with (9) is:

$$\text{Find } \lambda \in \mathbb{R}, u \in \mathcal{V}, u \neq 0, \text{ such that : } a_E(u,v) = \lambda b(u,v) \quad \forall v \in \mathcal{V}. \tag{10}$$

In the standard dynamic analysis of a structure with a given isotropic material, the multiplier $m(x)$ in the definition of the bilinear form b has a meaning of the mass. In the following, we will relate it to the trace of the elasticity matrix. We follow the discussion in [3] and assume that, for a given E, the mass belongs to $\mathcal{M}(E)$, a set of all materials with the same elastic properties but different mass, i.e., $m \in \mathcal{M}(E) = \{m(\tau) \mid E(\tau) = E\}$. The tensor E is uniquely characterized by its principal invariants (up to a rotation that does not affect the mass), so we may write $m \in \mathcal{M}(I_1, I_2, I_3)$. It was shown in [3] that when we maximize the smallest eigenvalue subject to the volume constraint only (with no compliance constraint), the optimal material satisfies $I_2 = I_3 = 0$. In our case, we will *assume* this, so we will have that $m \in \mathcal{M}\rho_E(x)$. Further, motivated by the isotropic case, we will assume that m is in fact a linear function of the first invariant, i.e., $m(x) = c(x)\rho_E(x)$, $\underline{c}_{x,E} \leq c(x) \leq \overline{c}_{x,E}$. Finally, we will assume that the constant c is independent of x, which corresponds to the assumption that the optimal structure is made of the "same kind of material". This, certainly, is a simplification that needs to be taken into account in the interpretation of the results. In the rest of the paper we will thus use the following form of b:

$$b(w,v) := b_E(w,v) = \int_\Omega c\rho_E(x) w(x) v(x) \mathrm{d}x,$$

with certain $c > 0$.

We use the following definition of the smallest well defined eigenvalue.

Definition 2.2. *For each $E \in \mathcal{E}_0$, let $\lambda_{\min}(E)$ denote the* smallest well defined eigenvalue *of the system (10), i.e.*

$$\lambda_{\min} = \min \{\lambda \mid \exists u \in \mathcal{V} : \text{ Equation (10) holds for } (\lambda, u) \text{ and } u \notin \ker(b_E)\},$$

where

$$\ker(b_E) = \{z \mid b_E(z, v) = 0 \quad \forall v \in \mathcal{V}\}.$$

The square root of the smallest well-defined eigenvalue will be called fundamental eigenfrequency.

It is well known from engineering literature that the dynamic stiffness of a structure can be improved by raising its fundamental eigenfrequency. This is our motivation for considering the following problem: We search for a material distribution E such that the smallest well defined eigenvalue of the system (10) is larger than a prescribed positive lower bound. Denoting this value by $\hat{\lambda}$, we obtain the constraint

$$\lambda_{\min}(E) \geq \hat{\lambda}. \tag{11}$$

In Appendix A, Corollary A.3 it is shown that inequality (11) can be equivalently reformulated as

$$\inf_{u \in \mathcal{V}, \|u\|=1} \left\{ a_E(u, u) - \hat{\lambda} b_E(u, u) \right\} \geq 0. \tag{12}$$

Introducing the function

$$\begin{aligned} \mu_{\hat{\lambda}} : L^\infty(\Omega)^{3\times 3} &\to \mathbb{R} \cup \{\infty\} \\ E &\mapsto \inf_{u \in \mathcal{V}, \|u\|=1} \left\{ a_E(u, u) - \hat{\lambda} b_E(u, u) \right\}, \end{aligned}$$

we are able to state the *minimum compliance single-load FMO problem with vibration constraint*

$$\begin{aligned} &\inf_{E \in \mathcal{E}} \int_\Gamma f(x) \cdot u_E(x) \mathrm{d}x \\ &\text{subject to} \\ &\qquad u_E \text{ solves (3)}, \\ &\qquad \mu_{\hat{\lambda}}(E) \geq 0\,. \end{aligned} \tag{13}$$

Next we want to investigate the well-posedness of problem (13). We start with the following lemma:

Lemma 2.3. *The function $\mu_{\hat{\lambda}}$ is upper semicontinuous and concave.*

Proof. We first note that for fixed $u \in \mathcal{V}$ the mapping $E \mapsto a_E(u, u) - \hat{\lambda} b_E(u, u)$ is affine and consequently continuous w. r. t. E. Consequently, $\mu_{\hat{\lambda}}$ is the infimum of affine functionals and thus concave. Moreover, $\mu_{\hat{\lambda}}$ is upper semicontinuous, as it is the infimum of (infinitely many) continuous functionals [11, Proposition III.1.2]. □

Using Lemma 2.3 we are able to give details about the structure of the feasible set of problem (13):

Lemma 2.4. *The set $\mathcal{E}^{\hat{\lambda}} = \left\{ E \in \mathcal{E} \mid \mu_{\hat{\lambda}} \geq 0 \right\}$ is weakly-$*$ compact.*

Proof. The weakly-$*$ compactness of $\mathcal{E}$ has been shown in the proof of Theorem 2.1 in [2]. Thus the only thing we have to show is that $\mathcal{E}^{\hat{\lambda}}$ is closed. But this follows immediately from the closedness of $\mathcal{E}$ and Lemma 2.3. □

The following theorem can be proved exactly in the same way as Theorem 2.1 in [2]:

Theorem 2.5. *If the set $\mathcal{E}^{\hat{\lambda}}$ is non-empty, then problem (13) has at least one solution.*

We conclude this section by two remarks.

Remark 2.6. From the equivalence of (11) and (12) and Lemma 2.4 we immediately conclude that the function $\lambda_{\min} : L^\infty(\Omega)^{3\times 3} \to \mathbb{R} \cup \{\infty\}$ is upper semicontinuous and quasiconcave. Using this and the fact that the compliance functional, given by the formula

$$c(E) = \sup_{u \in \mathcal{V}} \left\{ -\frac{1}{2} a_E(u,u) + \int_\Gamma f \cdot u \, \mathrm{d}x \right\}$$

is convex and lower semicontinuous w. r. t. E (see [28]), we may repeat the arguments above in order to verify existence of at least one optimal solution for the following problems:

$$\inf_{E \in \mathcal{E}} v(E) \tag{14}$$
subject to
$$c(E) \le \delta,$$
$$\lambda_{\min} \ge \hat{\lambda} \, .$$

and

$$\inf_{E \in \mathcal{E}} -\lambda_{\min}(E) \tag{15}$$
subject to
$$c(E) \le \delta,$$
$$v(E) = \bar{v} \, .$$

Here $\delta \in \mathbb{R}$ is an upper bound on the compliance of the structure.

Remark 2.7. All results presented above remain true when we consider more general Dirichlet boundary conditions of the form

$$u_i = 0 \text{ on } \Gamma_0 \text{ for } i = 1 \text{ and/or } 2.$$

3 Discretization

In order to solve the infinite-dimensional problem (13) numerically, we have to use an appropriate discretization scheme. For the discretization, we use the standard isoparametric concept (see, e.g., [8]), using a piecewise constant approximation of the matrix function $E(x)$ and a piece-wise linear approximation of the displacements $u(x)$. Rather than presenting the full convergence analysis, we just note that the finite element approach and convergence analysis presented in [28] applies to our problem without changes and which generalizes the analysis presented in [25] for the variable thickness problems.

To keep the notation simple, we use the same symbols for the discrete objects (vectors) as for the "continuum" ones (functions). Suppose that Ω is approximated by a partitioning of M quadrilaterals called Ω_i. Let us denote by n the number of nodes (vertices of the elements). We approximate the matrix function $E(x)$ by a function that is constant on each element, i.e., characterized by a tuple of matrices $E = (E_1, \dots, E_M)$ of its element values. Hence the discrete counterpart of the set of admissible materials in algebraic form is

$$\widetilde{\mathcal{E}} = \left\{ E \in (\mathbb{S}^3)^M \mid E_i \succcurlyeq 0,\ \underline{\rho} \leq \mathrm{Tr}(E_i) \leq \overline{\rho},\ \ i = 1, \dots, M,\ \sum_{i=1}^{M} \mathrm{Tr}(E_i) = \hat{v} \right\}. \tag{16}$$

Here $\hat{v}$ is derived from the upper bound on resources introduced in (5) and the measure of a single element $\bar{\omega}$. Further we assume that the displacement vector $u(x)$ is approximated by a continuous function that is bilinear in each coordinate on every element. Such a function can be written as $u(x) = \sum_{i=1}^{n} u_i \vartheta_i(x)$ where u_i is the value of u at i^{th} node and ϑ_i is the basis function associated with i^{th} node (for details, see [8]). Recall that the displacement function is vector valued with 2 components. Consequently any function in the discrete set of admissible displacements can be identified with a vector in $\mathbb{R}^N$, where $N = 2n - \#(\text{components of } u \text{ fixed by Dirichlet b. c.})$ and we obtain

$$\widetilde{\mathcal{V}} = \mathbb{R}^N. \tag{17}$$

With the (reduced) family of basis functions $\vartheta_k, k = 1, 2, \dots, N$, we define the 3×2 matrix

$$\widehat{B}_k^\top = \begin{pmatrix} \frac{\partial \vartheta_k}{\partial x_1} & 0 & \frac{1}{2}\frac{\partial \vartheta_k}{\partial x_2} \\ 0 & \frac{\partial \vartheta_k}{\partial x_2} & \frac{1}{2}\frac{\partial \vartheta_k}{\partial x_1} \end{pmatrix}.$$

Now, for an element Ω_i, let $\mathcal{D}_i$ be an index set of nodes belonging to this element. Next we want to derive the discrete counter part of $a_E(\cdot,\cdot)$. We use a Gauss formula for the evaluation of the integral over each element Ω_i, assume that there are G Gauss integration points on each element and denote by $x_{i,k}^G$ the k-th integration point on the i-th element. Next we construct block matrices $B_{i,k} \in \mathbb{R}^{3\times N}$ composed of (3×2) blocks $\widehat{B}_j(x_{i,k}^G)$, at j-th position for all $j \in \mathcal{D}_i$ and zero blocks otherwise. Then the discrete counterpart of $a_E(\cdot,\cdot)$, the *stiffness matrix* is

$$A(E) = \sum_{i=1}^{M} A_i(E), \quad A_i(E) = \sum_{k=1}^{G} B_{i,k}^\top E_i B_{i,k}\,. \tag{18}$$

The matrices $A_i \in \mathbb{R}^{N\times N}$ are usually called *element stiffness matrices*. Now, assuming the load function f to be linear on each element and identifying such a function with a vector $f \in \mathbb{R}^N$, the discrete objective functional and equilibrium condition read as

$$f^\top u, \qquad A(E)u = f, \tag{19}$$

respectively. Similarly, we use the representation of the discrete displacement functions in the basis of $\vartheta_k, k = 1, 2, \dots, N$, to derive the discrete version of the bilinear form

$b_E(\cdot,\cdot)$: defining vectors $V_{i,k} \in \mathbb{R}^N,\ i = 1, 2, \ldots, M,\ k = 1, 2, \ldots, G$, with $\vartheta_j(x_i^k)$, $j \in \mathcal{D}_i$ at j-th position and zeros otherwise, the *mass matrix* is given by

$$M(E) = \sum_{i=1}^{M} M_i(E), \quad M_i(E) = \mathrm{Tr}(E) M_i, \quad M_i = \sum_{k=1}^{G} V_{i,k} V_{i,k}^\top . \tag{20}$$

As in (18), $M(E)$ is composed by a sum of matrices $M_i \in \mathbb{R}^{N\times N}$, the *element mass matrices*. Finally, the discrete counterpart of the condition on the lowest eigenfrequency of the structure (12) reads as

$$\inf_{u\in\mathbb{R}^n, \|u\|=1} u^\top \left(A(E) - \hat{\lambda} M(E) \right) u \geq 0. \tag{21}$$

After discretization, problem (13) becomes

$$\begin{aligned} &\min_{u\in\mathbb{R}^N, E\in\widetilde{\mathcal{E}}} f^\top u \\ &\text{subject to} \\ &\qquad A(E)u = f, \\ &\inf_{u\in\mathbb{R}^n, \|u\|=1} u^\top \left(A(E) - \hat{\lambda} M(E) \right) u \geq 0 . \end{aligned} \tag{22}$$

Problem (22) is a mathematical programming problem with linear matrix inequality constraints and standard nonlinear constraints; in the following section we will show how this problem can be turned into a standard linear semidefinite program.

4 The linear SDP approach

In the recent years excellent software packages, most of them based on the interior point idea, have been developed for the solution of linear SDP problems. For an overview, compare, for example, [29] or [19].

In the sequel we give an alternative formulation of the discrete FMO problem (22) as linear semidefinite program.

Proposition 4.1. *Problem (22) is equivalent to the following linear semidefinite program*

$$\begin{aligned} &\min_{\alpha\in\mathbb{R}, E\in\widetilde{\mathcal{E}}} \alpha \\ &\textit{subject to} \\ &\qquad \begin{pmatrix} \alpha & -f \\ -f & A(E) \end{pmatrix} \succcurlyeq 0 \\ &\qquad A(E) - \hat{\lambda} M(E) \succcurlyeq 0 . \end{aligned} \tag{23}$$

Proof. After introducing an auxiliary variable α, the assertion follows immediately from Proposition 3.1 in [1]. ☐

In the remainder of this section we will explain why the formulation as linear SDP is impractical for the efficient solution of FMO problems with vibration constraint, due to the large number of variables and the size of the matrix constraints. Our observations are based on practical experience and complexity estimates. We have solved Example 1 from Section 7 by state-of-the-art linear SDP solvers. The fastest solver needed about 50 hours on a high end computer with a processor speed of approximately 3 GHz. Using this number as a reference and taking into account that the computational complexity of all currently available linear SDP solvers depends at least quadratically (sometimes even cubically) on the size of the matrix constraints and typically cubically on the number of variables, it becomes obvious that formulation (23) is not suited to serve as a basis for the efficient solution of FMO problems of practical size.

Remark 4.2. Note that a formulation similar to (23) has been successfully applied to problems of Truss Topology design as well as variable thickness sheet problems in the past. The interested reader is referred to [22, 1, 16].

5 The dual problem and the low-rank approximation

The goal of this section is to find an alternative formulation to problem (23), which is numerically tractable. Our strategy is as follows: First, we derive the Lagrange dual to problem (23) and then present a low-rank approximation to the same, which is (under certain assumptions) equivalent to the original problem.

The following theorem allows us to identify problem (23) as the Lagrange dual of a convex semidefinite program:

Theorem 5.1. *Problem (23) is equivalent to the Lagrange dual of the problem*

$$\max_{\substack{u\in\mathbb{R}^N,\alpha\in\mathbb{R},W\succcurlyeq 0,\\ \beta^l\geq 0,\beta^u\geq 0}} 2f^\top u - \alpha V + \underline{\rho}\sum_{i=1}^{M}\beta_i^l - \overline{\rho}\sum_{i=1}^{M}\beta_i^u \tag{24}$$

subject to

$$g_i(u,\alpha,W,\beta^l,\beta^u) \preccurlyeq 0, \quad i=1,2,\ldots,M,$$

where $g_i(u,\alpha,W,\beta^l,\beta^u) : \mathbb{R}^{N+1}\times\mathbb{S}^N\times\mathbb{R}^{2M}\mapsto\mathbb{S}^3$ *is defined for all* $i=1,2,\ldots,M$ *as*

$$g_i(u,\alpha,W,\beta^l,\beta^u) = \sum_{j=1}^{G} B_{ij}^\top uu^\top B_{ij} + \sum_{j=1}^{G} B_{ij}^\top W B_{ij} - \hat{\lambda}\langle W, M_i\rangle I - \left(\alpha + \beta_i^l - \beta_i^u\right) I.$$

Moreover there is no duality gap and the optimal material matrices

$$E_i^*,\ i=1,2,\ldots,M$$

take the role of Lagrangian multipliers associated with the nonlinear inequality constraints in problem (24).

Proof. We will prove the theorem for formulation (22) which, by Proposition 4.1, is equivalent to (23). Rewriting the eigenfrequency constraints as in problem (23) and taking into account that $A(E)$ is positive definite for all $E\in\widetilde{\mathcal{E}}$, we observe that problem (22) can be written equivalently as

$$\min_{E\in\widetilde{\mathcal{E}}}\max_{u\in\mathbb{R}^N} 2f^\top u - u^\top A(E)u \tag{25}$$

subject to

$$A(E) - \hat{\lambda}M(E) \succcurlyeq 0\,.$$

The Lagrangian associated with problem (25) can be written in the form

$$\begin{aligned}\mathcal{L}(E,u,\alpha,W,\beta^l,\beta^u) := \max_{u\in\mathbb{R}^N} & \; 2f^\top u - u^\top A(E)u \\ &+ \sum_{i=1}^M \beta_i^l(\underline{\rho} - \mathrm{Tr}(E_i)) + \sum_{i=1}^M \beta_i^u(\mathrm{Tr}(E_i) - \overline{\rho}) \\ &+ \alpha(\sum_{i=1}^M \mathrm{Tr}(E_i) - V) + \langle W, \hat{\lambda}M(E) - A(E)\rangle,\end{aligned} \tag{26}$$

where

$$(E,u,\alpha,W,\beta^l,\beta^u) \in (\mathbb{S}_+^{d'})^M \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{S}_+^N \times \mathbb{R}_+^M \times \mathbb{R}_+^M.$$

Now problem (25) can be formulated as

$$\min_{E\succcurlyeq 0} \max_{\substack{u\in\mathbb{R}^N,\alpha\in\mathbb{R},W\succcurlyeq 0,\\ \beta^l\geq 0,\beta^u\geq 0}} \mathcal{L}(E,u,\alpha,W,\beta^l,\beta^u). \tag{27}$$

The Lagrange dual to (27) is

$$\max_{\substack{u\in\mathbb{R}^N,\alpha\in\mathbb{R},W\succcurlyeq 0,\\ \beta^l\geq 0,\beta^u\geq 0}} \min_{E\succcurlyeq 0} \mathcal{L}(E,u,\alpha,W,\beta^l,\beta^u). \tag{28}$$

Taking into account that

$$u^\top A(E)u = u^\top\Big(\sum_{i=1}^M\sum_{j=1}^G B_{ij}E_iB_{ij}^\top\Big)u = \sum_{i=1}^M \Big\langle E_i, \sum_{j=1}^G B_{ij}^\top uu^\top B_{ij}\Big\rangle$$

$$\langle W, A(E)\rangle = \Big\langle W, \sum_{i=1}^M\sum_{j=1}^G B_{ij}E_iB_{ij}^\top\Big\rangle = \sum_{i=1}^M \Big\langle E_i, \sum_{j=1}^G B_{ij}^\top W B_{ij}\Big\rangle$$

$$\langle W, M(E)\rangle = \Big\langle W, \sum_{i=1}^M \mathrm{Tr}(E_i)M_i\Big\rangle = \sum_{i=1}^M \langle E_i, I\rangle\langle W, M_i\rangle = \sum_{i=1}^M \Big\langle E_i, \langle W, M_i\rangle I\Big\rangle$$

and $\mathrm{Tr}(E_i) = \langle E_i, I\rangle$ for all $i = 1,2,\ldots,M$, we obtain

$$\begin{aligned}&\mathcal{L}(E,u,\alpha,W,\beta^l,\beta^u) = 2f^\top u - \alpha V + \underline{\rho}\sum_{i=1}^M \beta_i^l - \overline{\rho}\sum_{i=1}^M \beta_i^u \\ &- \sum_{i=1}^M \Big\langle E_i, \sum_{j=1}^G B_{ij}^\top uu^\top B_{ij} + \sum_{j=1}^G B_{ij}^\top W B_{ij} - \hat{\lambda}\langle W, M_i\rangle I - \big(\alpha + \beta_i^l - \beta_i^u\big)\, I\Big\rangle.\end{aligned}$$

Using this form and interpreting E_i $(i = 1,2,\ldots,M)$ as Lagrangian multipliers, problem (28) takes the form

$$\max_{\substack{u\in\mathbb{R}^N,\alpha\in\mathbb{R},W\succcurlyeq 0,\\ \beta^l\geq 0,\beta^u\geq 0}} 2f^\top u - \alpha V + \underline{\rho}\sum_{i=1}^M \beta_i^l - \overline{\rho}\sum_{i=1}^M \beta_i^u$$

subject to

$$\sum_{j=1}^G B_{ij}^\top(uu^\top + W)B_{ij} - \Big(\hat{\lambda}\langle W, M_i\rangle + \alpha + \beta_i^l - \beta_i^u\Big) I \preccurlyeq 0, \;\; i = 1,2,\ldots,M,$$

but this is problem (24). Finally, taking into account that problem (24) is convex and that the Slater condition holds (in order to construct a strictly feasible point, use arbitrary $W \succcurlyeq 0, u \in \mathbb{R}^N, \beta^l \geq 0, \beta^u \geq 0$ and choose α large enough, such that all inequalities in problem (24) are strictly feasible), the fact that the duality gap is zero follows from [5, Theorem 5.81]. □

Later we will make use of the following proposition. The proof is straightforward, but rather technical and therefore postponed to Appendix I of this article.

Proposition 5.2. *A tuple* $(u^*, \alpha^*, W^*, \beta^{l^*}, \beta^{u^*}; E^*) \in \mathbb{R}^{N+1} \times \mathbb{S}^N_+ \times \mathbb{R}^{2M}_+ \times (\mathbb{S}^3_+)^M$ *is a KKT point of (24) if and only if the conditions*

$$g_i(u^*, \alpha^*, W^*, \beta^{l^*}, \beta^{u^*}) \preccurlyeq 0 \quad (i = 1, 2, \ldots, M)$$

$$\underline{\rho} \leq \mathrm{Tr}(E_i^*) \leq \overline{\rho} \quad (i = 1, 2, \ldots, M)$$

$$\beta^{l^*}(\underline{\rho} - \mathrm{Tr}(E_i^*)) = 0, \quad \beta^{u^*}(\mathrm{Tr}(E_i^*) - \overline{\rho}) = 0, \quad (i = 1, 2, \ldots, M)$$

$$\sum\nolimits_{i=1}^{M} tr(E_i^*) = V, \quad A(E^*) - \lambda^* M(E^*) \succcurlyeq 0, \quad \langle A(E^*) - \lambda^* M(E^*), W^* \rangle = 0$$

$$A(E^*)u^* = f, \quad f^\top u^* = \alpha^* V - \underline{\rho} \sum\nolimits_{i=1}^{M} \beta_i^l + \overline{\rho} \sum\nolimits_{i=1}^{M} \beta_u^l$$

are satisfied.

Theorem 5.1 guarantees that we can retrieve the solution of (22) by calculating a primal-dual solution of (24). Consequently, we could apply any convex semidefinite programming solver which is able to generate primal-dual solutions. Note, however, that the computational complexity of problem (24) is not much better than that of the linear SDP problem (23). This is due to the large size of the positive semidefiniteness constraint $W \succcurlyeq 0$. For this reason, we follow the idea of Monteiro and Burer (see [6]), in order to construct a low-rank approximation of (24). Suppose for a moment that we know a primal solution E^* of problem (23). Then we define

$$R_0 := \dim \left(\ker \left(A(E^*) - \hat{\lambda} M(E^*) \right) \right), \tag{29}$$

which is equal to the dimension of the multiplicity of the smallest eigenvalue $\hat{\lambda}$ of the generalized eigenvalue problem $A(E^*)v = \hat{\lambda} M(E^*)v$. Now, assuming that Slater's condition holds for (23), we observe that the matrix W in problem (24) takes the role of the Lagrangian multiplier associated with the inequality constraint

$$A(E) - \hat{\lambda} M(E) \succcurlyeq 0.$$

Moreover it follows from the complementarity slackness condition that there exists an optimal multiplier W^*, with the property

$$\mathrm{rank}(W^*) \leq R_0.$$

This is our motivation to substitute W in problem (24) by $\sum_{\ell=1}^{L} w_\ell w_\ell^\top$ with $w_\ell \in \mathbb{R}^n$ and with some $L \in \mathbb{R}$. Doing this, we obtain

$$\max_{\substack{w_1, w_2, \ldots, w_L, u \in \mathbb{R}^N \\ \alpha \geq 0, \beta^l \geq 0, \beta^u \geq 0}} 2f^\top u - \alpha V + \underline{\rho} \sum_{i=1}^{M} \beta_i^l - \overline{\rho} \sum_{i=1}^{M} \beta_i^u \tag{30}$$

subject to

$$\tilde{g}_i(u, \alpha, w_1, w_2, \ldots, w_L, \beta^l, \beta^u) \preccurlyeq 0, \quad i = 1, \ldots, M,$$

where $\tilde{g}_i(u,\alpha,w_1,w_2,\ldots,w_L,\beta^l,\beta^u)$ is defined as

$$\sum_{j=1}^{G} B_{ij}^\top\Big(uu^\top + \sum_{\ell=1}^{L} w_\ell w_\ell^\top\Big)B_{ij} - \Big(\hat{\lambda}\Big\langle \sum_{\ell=1}^{L} w_\ell w_\ell^\top, M_i\Big\rangle + \alpha + \beta_i^l - \beta_i^u\Big)I$$

for all $i = 1,2,\ldots,M$. The following theorem provides a relation between problems (24) and (30).

Theorem 5.3. *Let E_0^* be a solution of (23) and R_0 be defined by (29). Then there exists $L \le R_0$ such that for all (global) solutions $\big(u^*,\alpha^*,w_1^*,\ldots,w_L^*,\beta^{l^*},\beta^{u^*}\big)$ of (30) the tuple $\big(u^*,\alpha^*,W^*,\beta^{l^*},\beta^{u^*}\big)$ with $W^* := \sum_{\ell=1}^{L} w_\ell^*(w_\ell^*)^\top$ is a solution of (24). Moreover each vector of Lagrangian multipliers E^* associated with the inequality constraints*

$$\tilde{g}_i(u,\alpha,w_1,w_2,\ldots,w_L,\beta^l,\beta^u) \preccurlyeq 0, \quad i = 1,2,\ldots,M,$$

forms an optimal solution of (23).

In order to proof Theorem 5.3, we make use of the following Lemmas:

Lemma 5.4. *Any local/global minimum of problem (30) is a local/global minimum of problem (24) with an additional rank constraint of the form*

$$\operatorname{rank}(W) \le L$$

and vice versa.

Proof. The assertion of Lemma 5.4 can be proven exactly in the same way as Proposition 2.3 in [7]. □

Lemma 5.5. *Robinson's constraint qualification (see [5]) is satisfied by problem (30) at any feasible point.*

Proof. Using [5, formula (5.195)] we can write Robinson's constraint qualification for an arbitrary feasible point $(\hat{u},\hat{\alpha},\hat{w_1},\ldots,\hat{w_L},\hat{\beta^l},\hat{\beta^u}) \in \mathbb{R}^{(L+1)N+2M+1}$ as follows: There exists a direction $h \in \mathbb{R}^{(L+1)N+2M+1}$ such that the inequality

$$\tilde{g}_i(\hat{u},\hat{\alpha},\hat{w_1},\ldots,\hat{w_L},\hat{\beta^l},\hat{\beta^u}) + \nabla\tilde{g}_i(\hat{u},\hat{\alpha},\hat{w_1},\ldots,\hat{w_L},\hat{\beta^l})\cdot h \prec 0 \qquad (31)$$

holds for all $i = 1,2,\ldots,M$. Obviously, the direction $(0,1,0,\ldots,0)$ with 1 in the position of the variable α satisfies (31). □

A simple consequence of Lemma 5.5 is that for each local minimum of problem (30) associated Lagrangian multipliers exist. Now we are able to prove Theorem 5.3:

Proof. Let $L = R_0$ and $\tilde{x}^* := \big(u^*,\alpha^*,w_1^*,\ldots,w_L^*,\beta^{l^*},\beta^{u^*}\big)$ be a (global) solution of (30). Then we conclude from Lemma 5.4 that $x^* := \big(u^*,\alpha^*,\sum w_\ell^* w_\ell^{*\top},\beta^{l^*},\beta^{u^*}\big)$ is a global solution of problem (24) with an additional rank constraint of the form $\operatorname{rank}(W) \le L$. But then we conclude from the definition of R_0 that x^* is a global solution of (24) (without any rank constraint). Moreover, Lemma 5.5 guarantees the existence of optimal Lagrangian multipliers $\tilde{E}^* \in (\mathbb{S}_+^3)^M$ associated with the inequality constraints

$$\tilde{g}_i(u,\alpha,w_1,w_2,\ldots,w_L,\beta^l,\beta^u) \preccurlyeq 0, \quad i = 1,2,\ldots,M.$$

Now we define a Lagrangian-type function for problem (24) as follows:

$$\widetilde{L}(x,E) = \begin{cases} f_0(x) + \sum_{i=1}^{M} \langle E_i, g_i(x)\rangle & x \in C,\ \ E_i \succcurlyeq 0\ \ (i=1,2,\dots,M) \\ -\infty & x \in C,\ \ E_i \not\succcurlyeq 0 \text{ for some } i \\ -\infty & x \notin C \end{cases}$$

where f_0 is the objective of (24) and C is the convex set

$$C := \mathbb{R}^{N+1} \times \mathbb{S}_+^M \times \mathbb{R}_+^{2M}.$$

As x^* is a global solution of (24) and

$$\widetilde{L}(x^*, \tilde{E^*}) = f_0(x^*),$$

we conclude that $(x^*, \tilde{E^*})$ is a saddle point of $\widetilde{L}$. Now we obtain for example from [26, Theorem 28.3] that $\tilde{E^*}$ is a solution of the dual problem to (24). $\square$

Theorem 5.3 allows us to replace problem (24) by a low-rank problem with an appropriate rank. The advantage of the low-rank problem is that the dimension of the optimization variable is significantly lower than in the original problem, as long as the multiplicity of the smallest generalized eigenvalue of the stencil $(A(E^*) \mid M(E^*))$ is not too large. Moreover, there is no large semidefinite constraint in the low-rank problem. On the other hand, problem (30) is a non-convex semidefinite program, which can still be considered large-scale. For such a problem it is generally difficult (if not impossible) to calculate a global solution. Even in the case a global solution has been found, it is not a trivial problem to detect globality. To cope with the first problem (finding a global optimum), from theoretical point of view, we cannot do much more than use an optimization algorithm with strong local convergence properties and provide a good start point. We will see in Section 7 that this is not a big problem in practice, as the local algorithm of our choice typically identifies the global optimum, provided our guess for the multiplicity of the smallest eigenvalue is large enough. This observation coincides with the experience reported by Burer and Monteiro in [6] for linear semidefinite programs approximated by low-rank problems. For the second problem (detecting that a local optimum is also a global one), we will present a practical globality test in the sequel. We start with the following proposition, which provides a characterization of KKT-points for problem (30). The proof uses almost exactly the same arguments as the proof of Proposition 5.2 and is therefore omitted.

Proposition 5.6. *A tuple* $(u^*, \alpha^*, w^*, \beta^{l^*}, \beta^{u^*}; E^*) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N\times L} \times \mathbb{R}_+^{2M} \times (\mathbb{S}_+^3)^M$ *is a KKT point of (30) if and only if the conditions*

$$g_i(u^*, \alpha^*, \textstyle\sum_{\ell=1}^{L} w_\ell^* w_\ell^{*\top}, \beta^{l^*}, \beta^{u^*}) \preccurlyeq 0 \quad (i=1,2,\dots,M)$$
$$\underline{\rho} \le \mathrm{Tr}(E_i^*) \le \overline{\rho} \quad (i=1,2,\dots,M), \qquad \textstyle\sum_{i=1}^{M} tr(E_i^*) = V$$
$$\beta^{l^*}(\underline{\rho} - \mathrm{Tr}(E_i^*)) = 0, \quad \beta^{u^*}(\mathrm{Tr}(E_i^*) - \overline{\rho}) = 0, \quad (i=1,2,\dots,M)$$
$$\langle A(E^*) - \lambda^* M(E^*), \textstyle\sum_{\ell=1}^{L} w_\ell^* w_\ell^{*\top}\rangle = 0$$
$$A(E^*)u^* = f, \quad f^\top u^* = \alpha^* V - \underline{\rho}\textstyle\sum_{i=1}^{M} \beta_i^l + \overline{\rho}\sum_{i=1}^{M} \beta_i^u$$

are satisfied.

The following corollary is a direct consequence of Proposition 5.2 and Proposition 5.2 and provides a globality test for an arbitrary KKT point of problem (30):

Corollary 5.7. *Suppose that the vector* $x^* = (u^*, \alpha^*, w_1^*, w_2^*, \dots w_L^*, \beta^{l*}, \beta^{u*}; E^*) \in \mathbb{R}^{(L+1)N+2M+1} \times (\mathbb{S}^3)^M$ *is a KKT point of problem (30) and*

$$A(E^*) - \hat{\lambda} M(E^*) \succcurlyeq 0.$$

Then $(u^*, \alpha^*, \sum_{\ell=1}^{L} w_L^* w_L^{*\top}, \beta^{l*}, \beta^{u*}; E^*)$ *is a KKT point of problem (24), and thus a primal-dual solution pair of problems (23) and (24).*

Moreover the following corollary can be derived directly from the KKT conditions in Proposition 5.2 and provides an interpretation of the solution vector.

Corollary 5.8. *Let* $(u^*, \alpha^*, \sum_{\ell=1}^{L} w_L^* w_L^{*\top}, \beta^{l*}, \beta^{u*}; E^*)$ *be a KKT point of problem (24). Then* u^* *is the optimal displacement field associated with the material* E^* *and the vectors* $w_1^*, w_2^*, \dots w_L^*$ *are eigenmodes associated with the generalized eigenvalue problem* $A(E^*)v = \hat{\lambda} M(E^*)v$*.*

Remark 5.9. As an alternative to the approximation strategy described above one may also try to use the 'standard' semidefinite dual of (23), which is itself a linear semidefinite program and to apply the non-linear reformulation by Monteiro-Burer (see [6]) directly. This approach has however an important disadvantage: the special structure of problem (23) is ignored. In particular, all semidefinite constraints (including the constraints on $E_i,\ i = 1, 2, \dots, m$) are merged into one and the low-rank character of the dual solution is lost. As a consequence the code SDPLR by Burer, implementing this 'direct approach' performs rather poor on this class of problems (see [19]).

6 The low-rank algorithm

Based on the results of Theorem 5.3 and Corollary 5.7, we present a low-rank algorithm for the free material optimization problem with control of the lowest eigenfrequency:

Algorithm 6.1.

Input: Problem (24), $L = 1$.

1. Solve (30) with rank L to get $(\tilde{u}, \tilde{\alpha}, \tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_L, \tilde{\beta}^l, \tilde{\beta}^u; \tilde{E})$.
2. Check optimality of $(\tilde{u}, \tilde{\alpha}, \tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_L, \tilde{\beta}^l, \tilde{\beta}^u; \tilde{E})$:
   If $A(\tilde{E}) - \lambda^* M(\tilde{E}) \succcurlyeq 0$ STOP; $(\tilde{u}, \tilde{\alpha}, \sum_{\ell=1}^{L} \tilde{w}_\ell \tilde{w}_\ell^\top, \tilde{\beta}^l, \tilde{\beta}^u; \tilde{E})$ is optimal.
3. Increase L and GOTO step 1.

Output: $(\tilde{u}, \tilde{\alpha}, \sum_{\ell=1}^{L} \tilde{w}_\ell \tilde{w}_\ell^\top, \tilde{\beta}^l, \tilde{\beta}^u; \tilde{E})$.

Theorem 6.2. *Let* E_0^* *be a solution of problem (23). Then Algorithm 6.1 converges in at most*

$$R_0 = \dim\left(\ker\left(A(E_0^*) - \hat{\lambda} M(E_0^*)\right)\right)$$

steps.

Proof. The assertion of Theorem 6.2 is a direct consequence of Theorem 5.3. □

Remark 6.3. With the globality test, the algorithm will never accept a "wrong" optimum. And in our numerical experiments we have never observed the case when, due to repeated failure of the test, the rank would be increased up to the theoretical bound by Pataki. Notice that even if it had happened it would have not been guaranteed that the computed solution of (30) was a global optimum and we would have had to adapt the penalization technique of [6]. Because the globality test had always been successful long before reaching the theoretical bound, we did not feel it was necessary to use this technique.

To solve low-rank problems of the form (30), we have chosen an algorithm based on a generalized augmented Lagrangian method. This algorithm solves general nonlinear semidefinite programs of the form

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} f(x) \\ &\text{subject to} \\ &\qquad \mathcal{G}_j(x) \preccurlyeq 0, \quad j \in \mathcal{J} = \{1, 2, \ldots, J\}\,; \end{aligned} \tag{32}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, and $\mathcal{G}_j(x) : \mathbb{R}^n \to \mathbb{S}^{m_j}$ $(j \in \mathcal{J})$ are twice continuously differentiable mappings. Local as well as global convergence properties under standard assumptions are discussed in detail in [27]. The algorithm, implemented in the code PENNON [15], has been recently applied to nonlinear SDP problems arising from various applications; compare, for example, [16, 17] and [14].

7 Numerical Experiments

The goals of the numerical experiments presented throughout this section are as follows:

- to study behavior of Algorithm 6.1, when applied to FMO problems of practical size;
- to study ability of the local algorithm applied in Step 1 of Algorithm 6.1 to find global optima;
- to compare the performance of the low-rank algorithm with the direct solution of the primal SDP problem (23).

All experiments have been performed on a Sun Opteron machine with 8 Gbyte of memory and processor speed of approximately 3 GHz.

Example 1 In our first example a rectangular two-dimensional body was clamped on its left boundary and subjected to a load from the right (see Figure 1). The design space was discretized by 5.000 finite elements. Without the eigenfrequency constraint the lowest eigenvalue in the optimal design was of order 10^{-9}. For the eigenfrequency constraint, we have used the eigenvalue threshold $\hat{\lambda} = 0.0125$ (notice that a too high threshold $\hat{\lambda}$ would lead to an infeasible problem). The lower and upper bound on the the material tensors $(\underline{\rho}, \overline{\rho})$ and the upper bound on overall resources $(\overline{v})$ have been chosen as $\underline{\rho} = 0$, $\overline{\rho} = 4$ and $\hat{v} = 5.000$, respectively.

Table 1: Example 1

rank	compliance	outer/inner iterations	$\lambda_{\min}$	time in sec.
0	5.81	29/137	$\approx 10^{-9}$	53
1	7.74	31/214	$2.5 \cdot 10^{-4}$	209
2	7.77	29/238	$1.25 \cdot 10^{-2}$	636

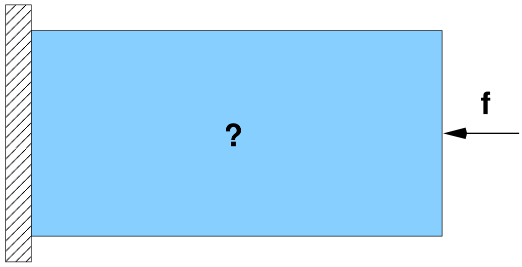

Figure 1: Basic test problem – boundary conditions

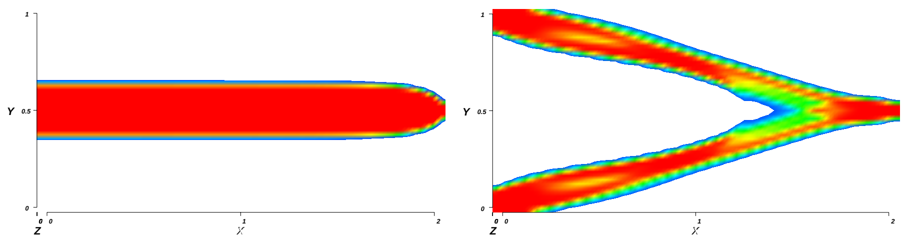

Figure 2: Optimal density plots without and with vibration constraint

We obtained the following results: The multiplicity of the lowest eigenvalue was two. The optimal compliance value was by 33 percent worse than in the pure compliance problem. The low-rank algorithm needed about 15 minutes to calculate the optimal result. The primal algorithm on the other hand needed already about 50 hours. Table 1 summarizes some more computational details: For each iteration of Algorithm 6.1 we report: the rank estimate, the compliance of the computed structure, the number of inner and outer iterations required by the augmented Lagrangian algorithm applied in step 1 of Algorithm 6.1, the minimal eigenvalue of the computed structure and the computation time in seconds. Optimal densities are visualized in Figure 2. In Figure 3, the displacement field is plotted along with the two eigenmodes.

Example 2 Our second example models a two-dimensional bridge which is clamped at the lower left and lower right corners. The bridge was subjected to vertical forces at the bottom (see Figure 4). The design space was discretized by 2.581 finite elements. Also in this example the smallest eigenfrequency in the pure compliance problem was of order 10^{-9}. We have chosen the eigenfrequency threshold $\hat{\lambda} = 0.02$. Furthermore we chose $\underline{\rho} = 0$, $\overline{\rho} = 4$ and $\hat{v} = 2.581$.

Our results can be summarized as follows: The multiplicity of the lowest eigenvalue

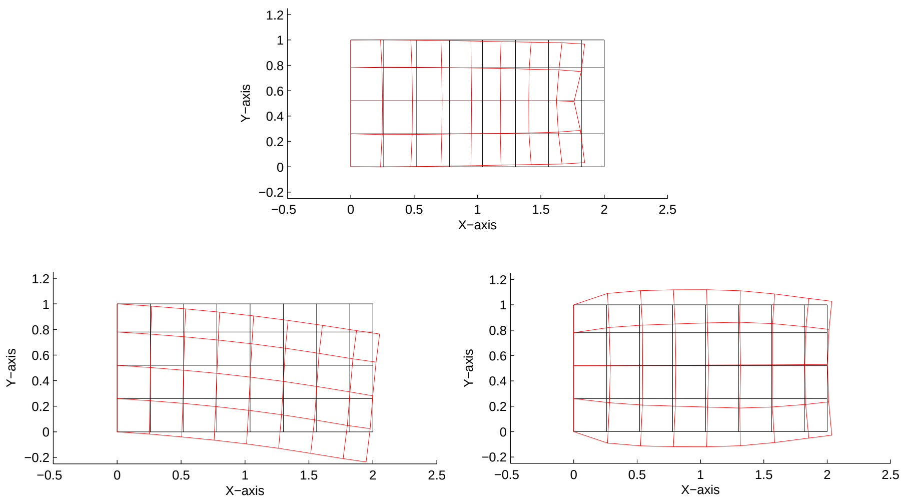

Figure 3: Displacement field (top) and eigenmodes (bottom)

Table 2: Example 2

rank	compliance	inner/outer iterations	$\lambda_{\min}$	time in sec.
0	168.29	13/68	$\approx 10^{-9}$	15
1	226.85	14/107	$5.5 \cdot 10^{-4}$	49
2	234.44	13/153	$2.0 \cdot 10^{-2}$	137

was again two. The optimal compliance value was by 25 percent worse than in the pure compliance problem. Computational details are provided in Table 2. Optimal densities are visualized in Figure 5. In Figure 6 we show the optimal displacement field along with the two eigenmodes corresponding to the lowest eigenfrequency.

Example 3 In our third example we considered a rectangular three-dimensional elastic body clamped on its left boundary and subjected to a load from the right (see Figure 7). The design space was discretized by 3.200 finite elements. The optimal design calculated for the pure compliance problem resulted in a fundamental eigenvalue of order 10^{-8}. For the problem with eigenfrequency constraint we put $\hat{\lambda} = 0.16$, $\underline{\rho} = 0, \overline{\rho} = 2.5$ and $\overline{v} = 3.200$.

This time we obtained the following results: The multiplicity of the lowest eigenvalue is three. The optimal compliance value is only by 5 percent worse than for the pure compliance problem. The low-rank algorithm needed about 5 hours to calculate the optimal result (compare Table 3 for details). Again we visualize optimal densities (see Figure 8) and the corresponding displacement field along with the two most significant eigenmodes of the optimal design (see Figure 9).

Remark 7.1. We calculated many more examples with different geometry, boundary conditions and values of $\hat{\lambda}$. In all these examples, the multiplicity of the smallest eigenvalue was never bigger than 8.

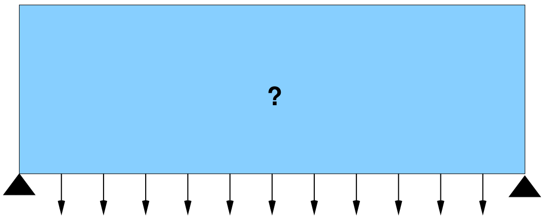

Figure 4: Basic test problem – boundary conditions

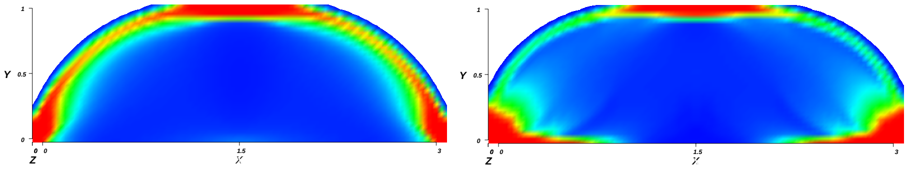

Figure 5: Optimal density plots without and with vibration constraint

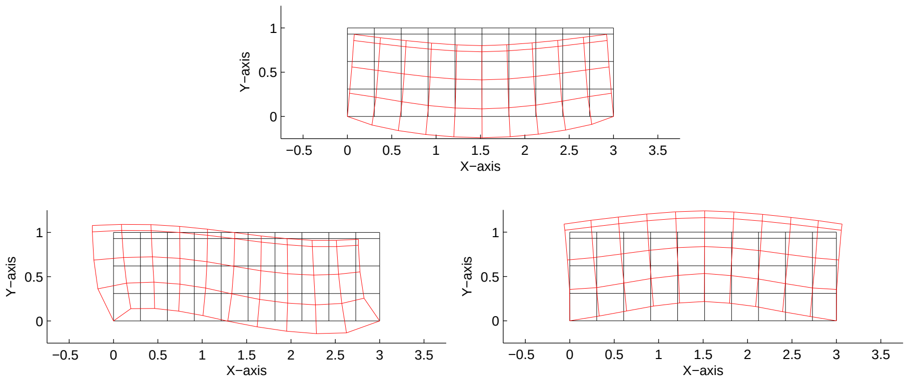

Figure 6: Displacement field (top) and eigenmodes (bottom)

Table 3: Example 3

rank	compliance	outer/inner iterations	$\lambda_{\min}$	time in sec.
0	1.67	30/122	$\approx 10^{-8}$	414
1	1.72	28/144	$8.0 \cdot 10^{-5}$	1670
2	1.73	29/181	$1.0 \cdot 10^{-2}$	5818
3	1.73	29/238	$1.6 \cdot 10^{-1}$	12776

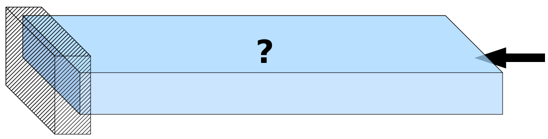

Figure 7: Basic test problem – boundary conditions

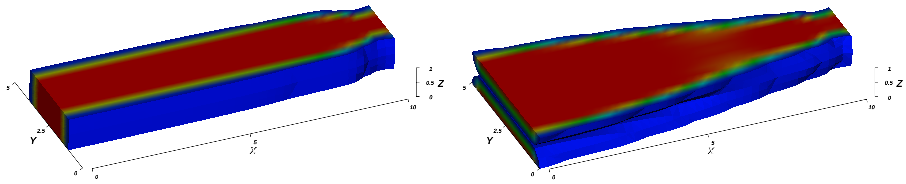

Figure 8: Optimal density plots without and with vibration constraint

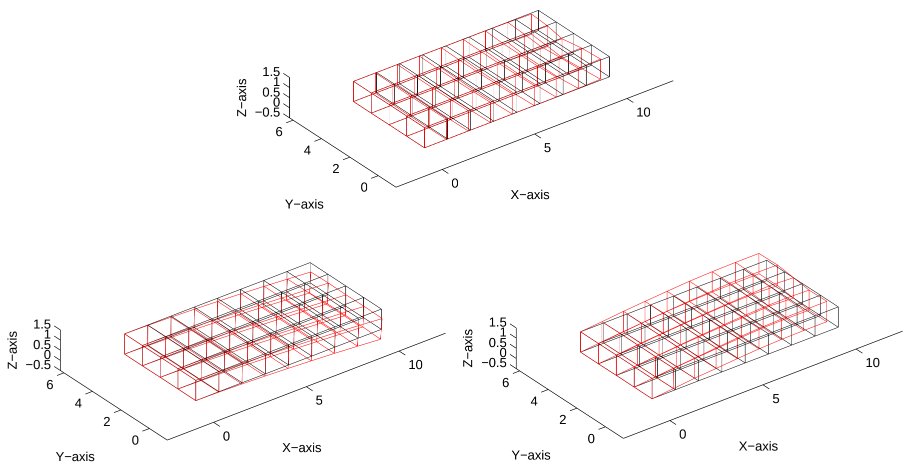

Figure 9: Displacement field (top) and 2 eigenmodes (bottom)

A Appendix

We consider the following situation: Let V be a Hilbert space, equipped with the inner product $(\cdot, \cdot)_V$ and the corresponding norm $\| \cdot \|_V$. Let further $a : V \times V \to \mathbb{R}$ be a bounded, symmetric and V-elliptic bilinear form. An abstract eigenvalue problem is defined as follows: Find $\lambda \in \mathbb{R}$ and $u \in V$, $u \neq 0$, such that

$$a(u, v) = \lambda(u, v)_V \quad \forall v \in V. \tag{33}$$

The following theorem deals with the existence of solutions of the latter problem (see, for example, [12] or [21]):

Theorem A.1. *There exists an increasing sequence of positive eigenvalues of problem (33) tending to* ∞*:*

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_k \to \infty \text{ for } k \to \infty$$

and an orthonormal basis $\{w_n\}$ *of* V *consisting of the normalized eigenfunctions associated with* λ_n*:*

$$a(w_n, v) = \lambda_n(w_n, v) \quad \forall v \in V, \ \|w_n\|_V = 1.$$

Furthermore the following formula holds true for λ_1*:*

$$\lambda_1 = \min_{v \in V, v \neq 0} \frac{a(v, v)}{\|v\|_V^2}.$$

Now we want to apply Theorem A.1 to the generalized eigenvalue problem (10). The following obvious inclusion holds true for the bilinear forms in (9):

$$\ker(b_E) \subset \ker(a_E). \tag{34}$$

Based on this observation, we define a Banach space $\overline{\mathcal{V}}$ as the factor space $\mathcal{V} \setminus \ker(b_E)$. On $\overline{\mathcal{V}} \times \overline{\mathcal{V}}$ we further define the inner product

$$(\overline{u}, \overline{v})_{\overline{\mathcal{V}}} := b_E(u, v), \tag{35}$$

where u, v are arbitrary representatives of the equivalence classes $\overline{u}$ and $\overline{v}$, respectively. The inner product $(\cdot, \cdot)_{\overline{\mathcal{V}}}$ induces the norm $\|\overline{u}\|_{\overline{\mathcal{V}}} := \sqrt{b_E(u, u)}$ on $\overline{\mathcal{V}}$. Consequently, $\overline{\mathcal{V}}$ is a Hilbert space. Next we define the bilinear form

$$\overline{a}_E(\overline{u}, \overline{v}) := a_E(u, v), \tag{36}$$

where again u, v are arbitrary representatives of the equivalence classes $\overline{u}$ and $\overline{v}$. Defining the eigenvalue problem: Find $\lambda \in \mathbb{R}$ and $\overline{u} \in \overline{\mathcal{V}}$, $\overline{u} \neq 0$, such that

$$\overline{a}_E(\overline{u}, \overline{v}) = \lambda(\overline{u}, \overline{v})_{\overline{\mathcal{V}}} \quad \forall \overline{v} \in \overline{\mathcal{V}}, \tag{37}$$

we are able to state the following corollary:

Corollary A.2. *There exists an increasing sequence of well defined eigenvalues of problem (10) tending to* ∞*:*

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_k \to \infty \text{ for } k \to \infty$$

and associated eigenfunctions $w_n, n = 1, 2, \ldots$, orthonormal w.r.t. the inner product $(\cdot,\cdot)_{\overline{\mathcal{V}}}$, such that

$$a_E(w_n, v) = \lambda_n b_E(w_n, v) \quad \forall v \in \mathcal{V}.$$

Furthermore the following formula holds true for λ_1:

$$\lambda_1 = \min_{v\in\mathcal{V}, v\notin\ker(b_E)} \frac{a_E(v,v)}{b_E(v,v)} = \min_{v\in\mathcal{V}, b_E(v,v)=1} a_E(v,v).$$

Proof. Suppose for a moment that $\overline{a}_E$ is $\overline{\mathcal{V}}$-elliptic. Then we are able to apply Theorem A.1 to problem (37) and all assertions of Corollary A.2 follow immediately from (34), (35) and (36). On the other hand, if $\overline{a}_E$ fails to be $\overline{\mathcal{V}}$-elliptic, we define a bilinear form $\overline{a}'_E$ by

$$\overline{a}'_E(\overline{u}, \overline{v}) := \overline{a}'_E(\overline{u}, \overline{v}) + \mu(u, v)_{\overline{\mathcal{V}}}.$$

Obviously $\overline{a}'_E$ is $\overline{\mathcal{V}}$-elliptic for any (arbitrary small) positive $\mu \in \mathbb{R}$. Applying Theorem A.1 to $\overline{\mathcal{V}}$ and $\overline{a}'_E$ we obtain the following estimates for the eigenvalues of problem (37)

$$-\mu < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lambda_k \to \infty \text{ for } k \to \infty.$$

As μ can be chosen arbitrarily small we conclude that $0 \leq \lambda_1$ and the proof is complete. □

Let us finally define the bilinear form $c_{E,\hat{\lambda}}(u, v) := a_E(u, v) - \hat{\lambda} b_E(u, v)$ and consider the eigenvalue problem:

$$\text{Find } \lambda \in \mathbb{R}, u \in \mathcal{V}, u \neq 0 \text{ such that } c_{E,\hat{\lambda}}(u, v) = \lambda(u, v) \quad \forall v \in \mathcal{V}. \tag{38}$$

Then the following result proves the equivalence of (11) and (12).

Corollary A.3. *The following assertions are equivalent:*

a) The smallest well defined eigenvalue of the generalized eigenvalue problem (10) is nonnegative.

b) The smallest eigenvalue of the eigenvalue problem (38) is nonnegative.

Proof. Using Corollary A.2 and equation (34), we have

$$\begin{aligned}
\text{a)} \quad &\Leftrightarrow \quad \frac{a_E(v,v)}{b_E(v,v)} \geq \hat{\lambda} \quad \forall v \in \mathcal{V},\ v \notin \ker(b_E) \\
&\Leftrightarrow \quad a_E(v,v) - \hat{\lambda} b_E(v,v) \geq 0 \quad \forall v \in \mathcal{V},\ v \notin \ker(b_E) \\
&\overset{(34)}{\Leftrightarrow} \quad a_E(v,v) - \hat{\lambda} b_E(v,v) \geq 0 \quad \forall v \in \mathcal{V},\ v \neq 0 \\
&\Leftrightarrow \quad \frac{a_E(v,v) - \hat{\lambda} b_E(v,v)}{\|v\|^2} \geq 0 \quad \forall v \in \mathcal{V},\ v \neq 0 \\
&\Leftrightarrow \quad a_E(v,v) - \hat{\lambda} b_E(v,v) \geq 0 \quad \forall v \in \mathcal{V},\ \|v\| = 1 \quad \Leftrightarrow \quad \text{b)}.
\end{aligned}$$

□

B Appendix

In the sequel we give a proof of Proposition 5.2:

Proof. We start with the KKT conditions of problem (24) in standard form: Let

$$\mathcal{F} := \big\{(u, \alpha, W, \beta^l, \beta^u) \in \mathbb{R}^{N+1} \times \mathbb{S}^N \times \mathbb{R}^{2M} \mid W \succcurlyeq 0, \beta^l \geq 0, \beta^u \geq 0\big\}$$

and $x^* := (u^*, \alpha^*, W^*, \beta^{l^*}, \beta^{u^*}) \in \mathcal{F}$. Then a pair (x^*, E^*) is a KKT point of (24) if and only if the conditions

$$g_i(x^*) \preccurlyeq 0, \quad E_i^* \succcurlyeq 0, \qquad i = 1, 2, \ldots, M, \tag{39}$$

$$\nabla_{u,\alpha} f(x^*) - \sum_{i=1}^{M} \langle E_i^*, \nabla_{u,\alpha} g_i(x^*) \rangle \;=\; 0, \tag{40}$$

$$\begin{aligned}(W^*, \beta^{l^*}, \beta^{u^*})^\top - \operatorname{Proj}_{\mathcal{F}}\Big((W^*, \beta^{l^*}, \beta^{u^*})^\top &- \nabla_{W,\beta^l,\beta^u} f(x^*) \\ &- \sum_{i=1}^{M} \langle E_i^*, \nabla_{W,\beta^l,\beta^u} g_i(x^*) \rangle\Big) \;=\; 0,\end{aligned} \tag{41}$$

$$\langle E_i^*, g_i(x^*) \rangle = 0, \quad i = 1, 2, \ldots, M, \tag{42}$$

with

$$f(u, \alpha, W, \beta^l, \beta^u) = 2f^\top u - \alpha V + \underline{\rho} \sum_{i=1}^{M} \beta_i^l - \overline{\rho} \sum_{i=1}^{M} \beta_i^u$$

are satisfied. Now we have

$$\begin{aligned}
&\frac{\partial}{\partial u} f(x^*) = 2f \\
&\frac{\partial}{\partial \alpha} f(x^*) = -V \\
&\frac{\partial}{\partial W} f(x^*) = 0 \\
&\frac{\partial}{\partial \beta_i^l} f(x^*) = \underline{\rho}, \quad (i = 1, 2, \ldots, M) \\
&\frac{\partial}{\partial \beta_i^u} f(x^*) = -\overline{\rho}, \quad (i = 1, 2, \ldots, M) \\
&\frac{\partial}{\partial u}\big(\sum_{i=1}^{M} \langle E_i^*, g_i(x^*) \rangle\big) = \sum_{i=1}^{M} \sum_{j=1}^{G} \frac{\partial}{\partial u} \langle E_i^*, B_{ij}^\top u u^\top B_{ij} \rangle|_{u=u^*} \\
&\qquad = \sum_{i=1}^{M} \sum_{j=1}^{G} 2 B_{ij} E_i^* B_{ij}^\top u^* \\
&\qquad = 2A(E^*) u^*
\end{aligned}$$

$$\frac{\partial}{\partial \alpha}\Big(\sum_{i=1}^{M}\langle E_i^*, \nabla g_i(x^*)\rangle\Big) = \sum_{i=1}^{M}\langle E_i^*, -I\rangle = -\sum_{i=1}^{M}\mathrm{Tr}(E_i^*)$$

$$\begin{aligned}
&\frac{\partial}{\partial W}\Big(\sum_{i=1}^{M}\langle E_i^*, \nabla g_i(x^*)\rangle\Big)\\
&\quad = \sum_{i=1}^{M}\langle E_i^*, \frac{\partial}{\partial W}(\sum_{j=1}^{G} B_{ij}^\top W B_{ij}) - \lambda^*\langle W, M_i\rangle I\rangle|_{W=W^*}\\
&\quad = \sum_{i=1}^{M}\sum_{j=1}^{G} B_{ij}E_i^* B_{ij}^\top - \lambda^* M_i\langle E_i^*, I\rangle = A(E^*) - \lambda^* M(E^*)
\end{aligned}$$

$$\frac{\partial}{\partial \beta^l}\Big(\sum_{i=1}^{M}\langle E_i^*, \nabla g_i(x^*)\rangle\Big) = \langle E_i^*, -I\rangle = -\mathrm{Tr}(E_i^*)$$

$$\frac{\partial}{\partial \beta^u}\Big(\sum_{i=1}^{M}\langle E_i^*, \nabla g_i(x^*)\rangle\Big) = \langle E_i^*, I\rangle = \mathrm{Tr}(E_i^*).$$

Hence (40) is equivalent to

$$A(E^*)u^* = f, \quad \sum_{i=1}^{M}\mathrm{Tr}(E_i^*) = V,$$

and (41) is equivalent to

$$\begin{gathered}
\underline{\rho} \le \mathrm{Tr}(E_i^*) \le \overline{\rho} \quad (i = 1, 2, \dots, M),\\
S\beta^{l^*}(\underline{\rho} - \mathrm{Tr}(E_i^*)) = 0, \quad \beta^{u^*}(\mathrm{Tr}(E_i^*) - \overline{\rho}) = 0, \quad (i = 1, 2, \dots, M),\\
A(E^*) - \lambda^* M(E^*) \succcurlyeq 0, \quad \langle A(E^*) - \lambda^* M(E^*), W^*\rangle = 0.
\end{gathered}$$

Next, we see from (39) that (42) is equivalent to

$$\sum_{i=1}^{M}\langle E_i^*, g_i(x^*)\rangle = 0.$$

We further calculate

$$\begin{aligned}
&\sum_{i=1}^{M}\Big\langle E_i^*, g_i(x^*)\Big\rangle\\
&\quad = \sum_{i=1}^{M}\Big\langle E_i^*, \sum_{j=1}^{G} B_{ij}^\top(u^* u^{*\top} + W^*)B_{ij} - (\hat{\lambda}\langle W^*, M_i\rangle + \alpha^* + \beta^{l^*} - \beta^{u^*})I\Big\rangle\\
&\quad = -\alpha^*\sum_{i=1}^{M}\mathrm{Tr}(E_i^*) + u^{*T}A(E^*)u^* + \langle W^*, A(E^*) - \hat{\lambda}M(E^*)\rangle\\
&\qquad\quad + \sum_{i=1}^{M}\beta_i^l\mathrm{Tr}(E_i^*) - \sum_{i=1}^{M}\beta_i^u\mathrm{Tr}(E_i^*)\\
&\quad = -\alpha^* V + f^\top u^* + \underline{\rho}\sum_{i=1}^{M}\beta^l - \overline{\rho}\sum_{i=1}^{M}\beta^u.
\end{aligned}$$

and the proof is complete. □

References

[1] W. Achtziger and M. Kočvara. Structural topology optimization with eigenvalues. Technical Report No. 315, Institute of Applied Mathematics, University of Dortmund, Germany, 2006.

[2] A. Ben-Tal, M. Kočvara, A. Nemirovski, and J. Zowe. Free material design via semidefinite programming. The multi-load case with contact conditions. *SIAM J. Optimization*, 9:813–832, 1997.

[3] M. Bendsøe and O. Sigmund. *Topology Optimization. Theory, Methods and Applications*. Springer-Verlag, Heidelberg, 2002.

[4] M. P. Bendsøe, J. M. Guades, R.B. Haber, P. Pedersen, and J. E. Taylor. An analytical model to predict optimal material properties in the context of optimal structural design. *J. Applied Mechanics*, 61:930–937, 1994.

[5] F. J. Bonnans and A. Shapiro. *Perturbation Analysis of Optimization Probelms*. Springer-Verlag New-York, 2000.

[6] S. Burer and R.D.C. Monteiro. A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization. *Mathematical Programming (series B)*, 95(2):329–357, 2003.

[7] S. Burer and R.D.C. Monteiro. Local minima and convergence in low-rank semidefinite programming. *Mathematical Programming (series A)*, 103:427–444, 2005.

[8] P. G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, New York, Oxford, 1978.

[9] A. Diaz and N. Kikuchi. Solution to shape and topology eigenvalue optimization problems using a homogenization method. *Int. J. Numer. Meth. Eng.*, 35:1487–1502, 1992.

[10] J. Du and N. Olhoff. Topological design of freely vibrating continuum structures for maximum values of simple and multiple eigenfrequencies and frequency gaps. *Structural and Multidisciplinary Optimization*, 34:91–110, 2007.

[11] I. Ekeland and Thomas Turnbull. *Infinite-Dimensional Optimization and Convexity*. University of Chicago Press, 1983.

[12] J. Haslinger and R. Mäkinen. *Introduction to shape optimization*. SIAM, Philadelphia, PA, 2002.

[13] M. Kočvara, M. Stingl, and J. Zowe. Free material optimization: recent progress. *Optimization*, 57:79–100, 2008.

[14] M. Kočvara, F. Leibfritz, M. Stingl, and D. Henrion. A nonlinear SDP algorithm for static output feedback problems in COMPlib. In Pavel Piztek, editor, *Proceedings of the 16th IFAC World Congress*. Elsevier, Amsterdam, 2005.

[15] M. Kočvara and M. Stingl. PENNON—a code for convex nonlinear and semidefinite programming. *Optimization Methods and Software*, 18(3):317–333, 2003.

[16] M. Kočvara and M. Stingl. Solving nonconvex SDP problems of structural optimization with stability control. *Optimization Methods and Software*, 19(5):595–609, 2004.

[17] M. Kočvara and M. Stingl. Free material optimization: Towards the stress constraints. *Structural and Multidisciplinary Optimization*, 33(4-5):323–335, 2007.

[18] M. Kočvara and J. Zowe. Free material optimization: An overview. In A.H. Siddiqi and M. Kočvara, editors, *Trends in Industrial and Applied Mathematics*, pages 181–215. Kluwer Academic Publishers, Dordrecht, 2002.

[19] H. D. Mittelmann. Several SDP codes on sparse and other SDP problems. *Department of Mathematics and Statistics, Arizona State University, Tempe, AZ*, September 14, 2003.

[20] T. Nakamura and M. Ohsaki. A natural generator of optimum topology of plane trusses for specified fundamental frequency. *Computer Methods in Applied Mechanics and Engineering*, 94:113–129, 1992.

[21] J. Nečas. *Monographie Les Methodes directes en théorie des equations elliptiques*. Masson et Cie, 1967.

[22] M. Ohsaki, K. Fujisawa, N. Katoh, and Y. Kanno. Semi-definite programming for topology optimization of truss under multiple eigenvalue constraints. *Comp. Meths. Appl. Mech. Engrg.*, 1999. To appear.

[23] N. Olhoff. Optimal design with respect to structural eigenvalues. In F.P.J. Rimrott and B. Tabarott, editors, *Theoretical and Applied Mechanics, Proc. XVth Int. IUTAM Congress*, pages 133–149. North-Holland, 1980.

[24] P. Pedersen. Maximization of eigenvalues using topology optimization. *Structural and Multidisciplinary Optimization*, 20:2–11, 2000.

[25] J. Petersson and J. Haslinger. An approximation theory for optimum sheet in unilateral contact. *Quarterly of Applied Mathematics*, 56:309–325, 1998.

[26] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, New Jersey, 1970.

[27] M. Stingl. *On the Solution of Nonlinear Semidefinite Programs by Augmented Lagrangian Methods*. PhD thesis, Institute of Applied Mathematics II, Friedrich-Alexander University of Erlangen-Nuremberg, 2006.

[28] R. Werner. *Free Material Optimization*. PhD thesis, Institute of Applied Mathematics II, Friedrich-Alexander University of Erlangen-Nuremberg, 2000.

[29] H. Wolkowicz, R. Saigal, and L. Vandenberghe. *Handbook on Semidefinite Programming*. Kluwer, 2000.

[30] J. Zowe, M. Kočvara, and M. Bendsøe. Free material optimization via mathematical programming. *Math. Prog., Series B*, 79:445–466, 1997.