# The advanced-step NMPC controller: optimality, stability and robustness ⋆

Victor M. Zavala [a], Lorenz T. Biegler [a]

[a] *Department of Chemical Engineering, Carnegie Mellon University 5000 Forbes Avenue, Pittsburgh PA, 15213, USA*

**Abstract**

Widespread application of dynamic optimization with fast optimization solvers leads to increased consideration of first-principles models for nonlinear model predictive control (NMPC). However, significant barriers to this optimization-based control strategy are feedback delays and consequent loss of performance and stability due to on-line computation. To overcome these barriers, recently proposed NMPC controllers based on nonlinear programming (NLP) sensitivity have reduced on-line computational costs and can lead to significantly improved performance. In this study, we extend this concept through a simple reformulation of the NMPC problem and propose the advanced-step NMPC controller. The main result of this extension is that the proposed controller enjoys the same nominal stability properties of the conventional NMPC controller without computational delay. In addition, we establish further robustness properties in a straightforward manner through input-to-state stability concepts. A case study example is presented to demonstrate the concepts.

*Key words:* nonlinear model predictive control; fast; large-scale; nonlinear programming; sensitivity; stability; Lyapunov functions.

## 1 Introduction

NMPC is a feedback control strategy based on the on-line solution of moving horizon optimal control problems (OCPs). As industrial NMPC applications demand the incorporation of increasingly larger and detailed dynamic process models [1–3], the development of efficient numerical methods for the solution of large-scale OCPs becomes essential. While advances in optimization strategies and algorithms have enabled the solution of increasingly larger OCPs, on-line implementations of NMPC still represent a challenge [4]. This is particularly true in large-scale applications where the solution of the OCP takes a non-negligible amount of time, giving rise to computational delays.

The effect of computational delays on the performance of NMPC has been noted by Santos et al. in a laboratory reactor [5] as well as in numerous industrial studies. Deterioration of stability has been studied in [6,7]. To address this issue, real-time NMPC strategies such as explicit NMPC, neighboring extremals, Newton-type controllers and NLP sensitivity-based controllers represent some alternatives. Explicit NMPC approaches compute off-line control actions based on a full enumeration of possible states. This approach is most suitable for systems with a few states where the effect of combinatorics is rather small [8,9]. For systems with large state spaces, on-line NMPC controllers represent a more efficient alternative. Among these, Newton-type controllers perform a single iteration (full Newton step) in the solution of the OCP at each time step. This requires the solution of a quadratic programming (QP) problem constructed around the solution of the QP at the previous time step [10–13]. This allows a fast disturbance rejection mechanism that has shown good practical performance in some applications. In addition, it can be shown that if the series of QPs is initialized around a sufficiently good reference solution, then the QP series can converge to the solution of the moving horizon OCPs. This result follows from the local convergence properties of Newton's method and the parametric properties of the moving horizon OCPs. With this, nominal stability of Newton-type controllers can be guaranteed in the face of this approximation by making use of the inherent robustness properties of NMPC [14].

The local convergence properties of Newton-type controllers might deteriorate in the face of nonlinear effects and strong perturbations, thus requiring extra safe-

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*Email addresses:* vzavala@andrew.cmu.edu (Victor M. Zavala), lb01@andrew.cmu.edu (Lorenz T. Biegler).

guards to promote global convergence [10,15]. A strategy able to ameliorate this drawback relies on the construction of approximate solutions around a continuously updated reference solution. Different variants of this are based on neighboring extremals [16–18] and NLP sensitivity concepts [19–22]. The nominal stability results of Newton-type controllers can also be used for these controller variants. However, a more general and constructive analysis of their stability properties can become cumbersome under these arguments, thus leading to unnecessarily conservative assumptions.

Motivated by these observations, we propose a new NMPC formulation, the advanced-step NMPC (asNMPC) controller. The main idea is to use the current control action to predict the future plant state in order to solve the future OCP in advance, while the current sampling period evolves. In the nominal case, the prediction matches the future plant state so that the current solution is already available, thus avoiding the computational delay. Similar ideas have been previously proposed in [6,7]. In this case, the asNMPC controller inherits the same nominal stability properties of the ideal NMPC (iNMPC) controller. An issue associated to this strategy is treatment of disturbances and model mismatch. To account for these, the proposed controller exploits the parametric property of the OCP and approximates the true optimal solution using NLP sensitivity concepts. A direct consequence of this is that a rigorous bound on the loss of optimality can be established and related to the bounds of the uncertainty description. With this, it is possible to analyze the inherent robustness properties of the asNMPC controller in a straightforward manner using input-to-state stability concepts. We show that the resulting closed-loop system is input-to-state stable and contrast its stability bounds with those of an iNMPC controller. We illustrate the concepts using a classical nonlinear continuous stirred tank reactor (CSTR) example and discuss computational issues associated to larger scale systems.

The paper is organized as follows. The optimality and stability properties of an ideal NMPC controller are presented in Section 2. These properties are contrasted against those of the asNMPC controller in Section 3. The simulation example is given in Section 4 while Section 5 closes the paper and presents directions for future work.

## 2 Ideal NMPC Formulation

We assume that the dynamics of an uncertain plant can be described by the following discrete-time dynamic model,

$$\begin{aligned} x(k+1) &= \hat{f}(x(k), u(k), w(k)) \\ &= f(x(k), u(k)) + g(x(k), u(k), w(k)) \end{aligned} \tag{1}$$

where $x(k) \in \Re^{n_x}$, $u(k) \in \Re^{n_u}$ and $w(k) \in \Re^{n_w}$ are the plant states, controls and disturbance signals, respectively, defined at time steps $t_k$ with integers $k > 0$. The mapping $f : \Re^{n_x+n_u} \mapsto \Re^{n_x}$ with $f(0,0) = 0$ represents the nominal model,

$$x(k+1) = f(x(k), u(k)) \tag{2}$$

while the term $g : \Re^{n_x+n_u+n_w} \mapsto \Re^{n_x}$ can be used to describe modeling errors, estimation errors and disturbances, among others. Having $x(k)$, the current plant state or its estimate, NMPC uses the nominal model,

$$z_{l+1} = f(z_l, v_l),\ z_0 = x(k),\ \ l = 0, \dots, N \tag{3}$$

to find a control sequence $\{v_0, v_1, ..., v_N\}$ and associated state sequence $\{z_0, z_1, ..., z_N\}$ that minimizes the cost function defined as,

$$J_N := F(z_N) + \sum_{l=0}^{N-1} \psi(z_l, v_l) \tag{4}$$

over a future prediction horizon containing $N$ time steps. Here, the computed controls $v_l \in \Re^{n_u}$ and predicted states $z_l \in \Re^{n_x}$ are enforced to satisfy the constraints $v_l \in \mathbb{U}$ and $z_l \in \mathbb{X}$ and the terminal constraints $z_N \in \mathbb{X}_f \subseteq \mathbb{X}, \forall\, l$. The cost function $J_N : \Re^{n_x+n_u} \mapsto \Re$ comprises the stage costs $\psi : \Re^{n_x+n_u} \mapsto \Re$ and a terminal penalty function $F : \Re^{n_x} \mapsto \Re$. This gives rise to a parametric NLP problem $\mathcal{P}_N(x(k))$ of the form,

$$\begin{aligned} \min_{v_l, z_l} \quad & J_N := F(z_N) + \sum_{l=0}^{N-1} \psi(z_l, v_l) \\ \text{s. t. } & z_{l+1} = f(z_l, v_l),\ z_0 = x(k) \quad l = 0, \dots N-1 \\ & z_l \in \mathbb{X},\ z_N \in \mathbb{X}_f,\ v_l \in \mathbb{U}. \end{aligned} \tag{5}$$

The solution of $\mathcal{P}_N(x(k))$, $(z_l^*, v_l^*)$, provides an *optimal cost value* $J_N(x(k))$ as well as the control $u(k) = v_0^*$ which is injected into the plant. In the nominal case, this drives the state of the plant towards $x(k+1) = z(k+1) = f(x(k), u(k))$ where $z(k+1) \in \Re^{n_x}$ is the nominal model prediction. In the face of uncertainty, the plant evolves as in (1) and generates the mismatch $x(k+1) - z(k+1) = g(x(k), u(k), w(k))$. Once $x(k+1)$ is known, the prediction horizon is shifted forward by one sampling instant and problem $\mathcal{P}_N(x(k+1))$ is solved to find $u(k+1)$. This recursive strategy gives rise to the feedback law,

$$u(k) = h^{id}(x(k)) \tag{6}$$

with $h^{id} : \Re^{n_x} \mapsto \Re^{n_u}$. Here, we will assume that the control action resulting from this conventional NMPC formulation can be computed instantaneously and term this the ideal NMPC (iNMPC) controller.

Many different choices of the penalty function $F(\cdot)$ and of the terminal set $\mathbb{X}_f$ have been proposed to guarantee stability of the nominal closed-loop system (6). Infinite horizon, quasi-infinite horizon, zero-state terminal constraint and dual-mode control represent some alternatives [23]. In addition, since the recursive solution of the nominal problem $\mathcal{P}_N(x(k))$ provides a mechanism to react to disturbances in (1), the nominal feedback law (6) provides some inherent robustness. This holds true in many important cases, except in the presence of state constraints for $\mathbb{X}, \mathbb{X}_f$ where a robust formulation of $\mathcal{P}_N(x(k))$ is required. In this work, we focus on nominal NMPC schemes and study their inherent robustness properties through input-to-state stability concepts. For a comprehensive summary of robustness analysis and general robust design of discrete-time NMPC algorithms please refer to [24].

### *2.1 Optimality Conditions and NLP Sensitivity*

The moving horizon OCP $\mathcal{P}_N(\cdot)$ is parametric in the initial state. Accordingly, it is possible to define the parameter vector $p := x(k)$. This study is based on the *post-optimal* analysis of solutions of the OCP. Here, we assume that the stage and terminal costs are chosen appropriately (e.g. through penalty terms) so that $x(k) \in \mathbb{X},\ z_N \in \mathbb{X}_f,\ u(k) \in \mathbb{U}$ are satisfied implicitly at a given solution. This simplifies the analysis below (see Remark 4).

The Lagrange function associated to $\mathcal{P}_N(p)$ is given by,

$$\mathcal{L} = F(z_N(p)) + \lambda_0(p)^T(z_0(p) - p) + \sum_{l=0}^{N-1} [\psi(z_l(p), v_l(p)) \\ + \lambda_{l+1}(p)^T(z_{l+1}(p) - f(z_l(p), v_l(p)))] \quad (7)$$

where $\lambda_l(p) \in \Re^{n_x}$ are vectors of Lagrange multipliers. Note that all the primal and dual variables become implicit functions of $p$. For simplicity in the presentation, we suppress this argument from the notation. The solution of $\mathcal{P}_N(p)$ needs to satisfy the first-order optimality or Karush-Kuhn-Tucker (KKT) conditions,

$$\begin{aligned} \nabla_{\lambda_0}\mathcal{L} &= z_0 - p \\ \left.\begin{aligned} \nabla_{z_l}\mathcal{L} &= \nabla_{z_l}\psi_l - A_l\lambda_{l+1} + \lambda_l = 0 \\ \nabla_{v_l}\mathcal{L} &= \nabla_{v_l}\psi_l - B_l\lambda_{l+1} = 0 \\ \nabla_{\lambda_{l+1}}\mathcal{L} &= z_{l+1} - f_l = 0 \end{aligned}\right\} & \ l = 0, \ldots, N-1 \\ \nabla_{z_N}\mathcal{L} &= \nabla_{z_N}F_N + \lambda_N = 0 \end{aligned} \quad (8)$$

where $f_l := f(z_l, v_l)$, $\psi_l := \psi(z_l, v_l)$, $F_N := F(z_N)$, $A_l = \nabla_{z_l} f_l$ and $B_l = \nabla_{v_l} f_l$. This set of nonlinear equations can be expressed in condensed form,

$$\varphi(s(p, N), p) = 0 \quad (9)$$

where the solution vector is defined as $s(p, N)^T = [\lambda_0^T, z_0^T, v_0^T, \lambda_1^T, z_1^T, v_1^T, ..., \lambda_N^T, z_N^T]$ and has optimal values $s^*(p, N)$. Newton-based NLP solvers search for a given solution $s^*(p_0, N)$ by successive linearization of (9) around the current point $s^j(p_0, N)$ with iteration counter $j$,

$$\mathbf{K}^j(p_0, N)\Delta s^j = -\varphi(s^j(p_0, N), p_0) \quad (10)$$

where, $\mathbf{K}^j(p_0, N) = \frac{\partial\varphi}{\partial s}|_{(s^j(p_0,N),p_0)}$ is the so-called KKT matrix. The above procedure, coupled to suitable safeguards to monitor the step size $\Delta s^j$, yields an optimal solution $s^*(p_0, N)$.

In large-scale applications, the formation and factorization of the KKT matrix is by far the most dominant expense in the solution of the OCP. The computational complexity of this step scales as $O(N(n_x + n_u))^\beta,\ \beta > 1$ and is directly linked to the on-line feedback delay introduced by a conventional NMPC scheme. To avoid this, we are interested in analyzing the effect of perturbations on $p$ around a given nominal solution in order to obtain fast approximate solutions to neighboring problems. For this, we make use of the following important result (adapted to the specifics of NMPC),

**Theorem 1** *(NLP Sensitivity) [25,26]. If $f(\cdot,\cdot)$, $\psi(\cdot,\cdot)$ and $F(\cdot)$ of the parametric problem $P_N(p)$ are twice continuously differentiable in a neighborhood of the nominal solution $s^*(p_0, N)$ and this solution satisfies the linear independence constraint qualifications (LICQ) and sufficient second order conditions (SSOC) then,*

- *$s^*(p_0, N)$ is an isolated local minimizer of $P_N(p_0)$ and the associated Lagrange multipliers are unique.*
- *For $p$ in a neighborhood of $p_0$ there exists a unique, continuous and differentiable vector function $s^*(p, N)$ which is a local minimizer satisfying SSOC and LICQ for $P_N(p)$.*
- *There exists a positive Lipschitz constant $\alpha$ such that $|s^*(p, N) - s^*(p_0, N)| \le \alpha|p - p_0|$ where $|\cdot|$ is the Euclidean norm.*
- *There exists a positive Lipschitz constant $L_J$ such that the optimal cost values $J_N(p)$ and $J_N(p_0)$ satisfy $|J_N(p) - J_N(p_0)| \le L_J|p - p_0|$.*

These results allow the application of the implicit function theorem to (9) at $s^*(p_0, N)$ to yield:

$$\mathbf{K}^*(p_0, N)\frac{\partial s^*}{\partial p} = -\left.\frac{\partial\varphi(s(p, N), p)}{\partial p}\right|_{s^*(p_0,N),p_0} \quad (11)$$

where $\mathbf{K}^*(p_0, N)$ is the KKT matrix of $\mathcal{P}_N(p_0)$ evaluated at $s^*(p_0, N)$ and, $\frac{\partial\varphi}{\partial p}|_{s^*(p_0,N),p_0} = [-\mathbb{I}_{n_x}, 0, \ldots, 0]^T$. If the nominal solution satisfies SSOC and LICQ, then the KKT matrix is non-singular [27] and can be used to compute the sensitivity matrix from (11). With this, first-order estimates of the solutions of neighboring problems can be obtained from the explicit form,

$$\tilde{s}(p, N) = s^*(p_0, N) + \frac{\partial s^*}{\partial p}(p - p_0) \quad (12)$$

where $\tilde{s}(p,N)$ is an approximate solution of $s^*(p,N)$. From continuity and differentiability of the optimal solution vector, there exists a positive Lipschitz constant $L_s$ such that,

$$|\tilde{s}(p,N) - s^*(p,N)| \leq L_s|p-p_0|^2. \quad (13)$$

**Remark 1.** The computation of the sensitivity matrix from (11) requires $n_x$ backsolves and becomes expensive for large-scale systems. To avoid this, we exploit the fact that the parameter vector $p$ enters linearly into (9). Consequently, the step $\Delta s(p,N) = \tilde{s}(p,N) - s^*(p_0,N)$ in (12) can also be found by linearization of the KKT conditions (9) around $s^*(p_0,N)$ to give,

$$\mathbf{K}^*(p_0,N)\Delta s(p,N) = -\varphi(s^*(p_0,N),p). \quad (14)$$

where the right-hand side corresponds to the KKT conditions (8) evaluated at the nominal solution. Therefore, $\varphi(s^*(p_0,N),p)^T = [(p_0-p)^T\ 0\ \ldots 0]$. Here, $\Delta s(p,N) = \tilde{s}(p,N) - s^*(p_0,N)$ is a Newton step taken from $s^*(p_0,N)$ towards the solution of a neighboring problem $\mathcal{P}_N(p)$ so that $\tilde{s}(p,N)$ satisfies (12)-(13). Furthermore, computing this step requires a single backsolve.

**Remark 2.** If $f(\cdot)$ is linear and $F(\cdot)$ and $\psi(\cdot)$ are convex quadratic functions, then $\tilde{s}(p,N) = s^*(p,N)$.

**Remark 3.** The exact structure of the sensitivity equations (14) is given by,

$$\begin{aligned} &\Delta z_0 = (p-p_0) \\ &\left.\begin{aligned} Q_l\Delta z_l + W_l\Delta v_l - A_l^T\Delta\lambda_{l+1} + \Delta\lambda_l = 0 \\ W_l^T\Delta z_l + R_l\Delta v_l - B_l^T\Delta\lambda_{l+1} = 0 \\ \Delta z_{l+1} - A_l\Delta z_l - B_l\Delta v_l = 0 \end{aligned}\right\} l = 0,\ldots,N-1 \\ &Q_N\Delta z_N + \Delta\lambda_N = 0 \end{aligned} \quad (15)$$

where $Q_N = \nabla_{z_N z_N}\mathcal{L} = \nabla_{z_N z_N}F$, $Q_l = \nabla_{z_l z_l}\mathcal{L}$, $W_l = \nabla_{z_l v_l}\mathcal{L}$ and $R_l = \nabla_{v_l v_l}\mathcal{L}$. In large-scale NLP algorithms, these linear equations are solved efficiently with some sort of sparse decomposition but for the analysis we can also apply a Riccati decomposition to give:

$$\begin{aligned} &\Delta z_0 = (p-p_0), \quad \Delta\lambda_0 = -\Pi_0\Delta z_0 \\ &\Delta v_l = K_l\Delta z_l, \quad \Delta z_{l+1} = A_l\Delta z_l + B_l\Delta v_l \\ &\Delta\lambda_{l+1} = -\Pi_{l+1}\Delta z_{l+1}, \quad l = 0,...,N \end{aligned} \quad (16)$$

with,

$$\begin{aligned} \Pi_N &= Q_N \\ \Pi_{l-1} &= Q_{l-1} + A_{l-1}^T\Pi_l A_{l-1} - (A_{l-1}^T\Pi_l B_{l-1} + W_{l-1}) \\ &\quad \times(R_{l-1} + B_{l-1}^T\Pi_l B_{l-1})^{-1}(B_{l-1}^T\Pi_l A_{l-1} + W_{l-1}^T) \\ K_{l-1} &= -(R_{l-1} + B_{l-1}^T\Pi_l B_{l-1})^{-1}(B_{l-1}^T\Pi_l A_{l-1} + W_{l-1}^T) \\ &\quad l = N,...,0 \end{aligned} \quad (17)$$

From the above recursion, it is clear that if we have an optimal control $v^*(p_0)$ from $\mathcal{P}(p_0)$, a fast approximate control action for a neighboring problem $\mathcal{P}(p)$ can be obtained from the correction,

$$\tilde{v}_0(p) = v_0^*(p_0) + K_0\cdot(p-p_0) \quad (18)$$

where $K_0$ is an analog of the Riccati gain matrix. Note that the above correction step is equivalent to obtain $\tilde{v}_0(p)$ from (12) with $\frac{\partial v_0^*}{\partial p} = K_0$.

**Remark 4.** The previous analysis accounts for inequality constraints implicitly. It is possible to extend this analysis to account for inequality constraints *explicitly* in the problem formulation but this would complicate the structure of the KKT conditions and the associated KKT matrix. In addition, note that if the perturbation $|p-p_0|$ induces an active-set change, the approximate solution $\tilde{s}(p)$ needs to be obtained through the solution of a quadratic programming problem. In this case, the factorization of the KKT matrix at the nominal solution can be re-used to correct the active-set on-line.

### *2.2 Stability Properties*

Ensuring stability of the closed-loop system (1) and (6) is a central problem in NMPC. Here, we establish well-known results on nominal and robust stability for the iNMPC controller. To start the discussion, we refer to [24,28] to make use of the following definitions and assumptions:

**Definition 1** *A continuous function $\alpha(\cdot):\Re\to\Re$ is a $\mathcal{K}$ function if $\alpha(0)=0, \alpha(s)>0, \forall s>0$ and it is strictly increasing. A continuous function $\beta:\Re\times\mathcal{Z}\to\Re$ is a $\mathcal{KL}$ function if $\beta(s,k)$ is a $\mathcal{K}$ function in $s$ for any $k>0$ and for each $s>0, \beta(s,\cdot)$ is decreasing and $\beta(s,k)\to 0$ as $k\to\infty$.*

**Definition 2** *(Lyapunov function) A function $V(\cdot)$ is called an Lyapunov function for system (2) if there exist an invariant set $\mathbb{X}$, $\mathcal{K}$ functions $\alpha_1,\alpha_2$ and $\alpha_3$ such that, $\forall x\in\mathbb{X}$,*

$$V(x) \geq \alpha_1(|x|) \quad (19a)$$
$$V(x) \leq \alpha_2(|x|) \quad (19b)$$
$$\Delta V(x) = V(f(x,h^{id}(x))) - V(x) \leq -\alpha_3(|x|) \quad (19c)$$

**Assumption 1** *(Nominal Stability Assumptions of iNMPC)*

- *The terminal penalty $F(\cdot)$, satisfies $F(z)>0, \forall z\in\mathbb{X}_f\backslash\{0\}$,*
- *There exists a local control law $u=h_f(z)$ defined on $\mathbb{X}_f$, such that $f(z,h_f(z))\in\mathbb{X}_f, \forall z\in\mathbb{X}_f$, and $F(f(z,h_f(z)))-F(z)\leq-\psi(z,h_f(z)), \forall z\in\mathbb{X}_f$.*

- *The optimal stage cost $\psi(x,u)$ satisfies $\alpha_p(|x|) \leq \psi(x,u) \leq \alpha_q(|x|)$ where $\alpha_p(\cdot)$ and $\alpha_q(\cdot)$ are $\mathcal{K}$ functions.*

**Assumption 2** *(Computational Delay of iNMPC) The control law $u = h^{id}(x)$ can be computed instantaneously.*

Nominal stability of iNMPC can be paraphrased by the following Theorem [23,29,30].

**Theorem 2** *(Nominal Stability of iNMPC) Consider the moving horizon problem $\mathcal{P}_N(x)$ defined in (5) and associated control law $u = h^{id}(x)$, that satisfies Assumptions 1 and 2. Then, $J_N(x)$ is a Lyapunov function and the closed-loop system is asymptotically stable.*

For the analysis of robust stability properties of the iNMPC controller, we apply definitions and properties of Input-to-State Stability (ISS) [24,31].

**Definition 3** *(Input-to-State Stability) The system,*

$$x(k+1) = \hat{f}(x(k), h^{id}(x(k)), w(k)),\ k \geq 0,\ x(0) = x_0 \tag{20}$$

*is said to be ISS in $\mathbb{X}$ if there exists a $\mathcal{KL}$ function $\beta$, and a $\mathcal{K}$ function $\gamma$ such that for all $w \in \mathcal{W}$,*

$$|x(k)| \leq \beta(|x_0|, k) + \gamma(|w|),\ \forall\, k \geq 0,\ \forall x_0 \in \mathbb{X} \tag{21}$$

**Definition 4** *(ISS-Lyapunov function) A function $V(\cdot)$ is called an ISS-Lyapunov function for system (20) if there exist a set $\mathbb{X}$, $\mathcal{K}$ functions $\alpha_1, \alpha_2, \alpha_3$ and $\sigma$ such that, $\forall x \in \mathbb{X}$ and $\forall w \in \mathcal{W}$,*

$$V(x) \geq \alpha_1(|x|) \tag{22a}$$
$$V(x) \leq \alpha_2(|x|) \tag{22b}$$
$$\Delta V(x,w) = V(\hat{f}(x, h^{id}(x), w)) - V(x) \leq -\alpha_3(|x|) + \sigma(|w|) \tag{22c}$$

**Lemma 3** *[24,31] Let $\mathbb{X}$ be a robustly invariant set for system (20) that contains the origin and let $V(\cdot)$ be an ISS-Lyapunov function for this system, then the resulting system is ISS in $\mathbb{X}$.*

To deal with robustness of the controller, we recognize that given $u(k)$ and the nominal model prediction $z(k+1) = f(x(k), u(k))$, there will exist a future mismatch $x(k+1) - z(k+1) = g(x(k), u(k), w(k))$ at the next time step, giving rise to two different problems $\mathcal{P}_N(z(k+1))$ and $\mathcal{P}_N(x(k+1))$, with optimal costs $J_N(z(k+1))$ and $J_N(x(k+1))$, respectively. To account for this, we define the mismatch term [32],

$$\epsilon(x(k+1)) := J_N(x(k+1)) - J_N(z(k+1)). \tag{23}$$

**Assumption 3** *Under Theorem 1, there exists a local positive Lipschitz constant $L_J$ such that $\forall x \in \mathbb{X}$,*

$$|\epsilon(x(k+1))| \leq L_J |g(x(k), u(k), w(k))|. \tag{24}$$

**Assumption 4** *(Robust Stability Assumptions) For $u = h^{id}(x)$,*
- *$g(x,u,w)$ can be described by $\mathcal{K}$ functions so that: $|g(x,u,w)| \leq \alpha_g(|x|) + \sigma(|w|)$.*
- *there exists a $\mathcal{K}$ function $\alpha_4$ and all $w \in \mathcal{W}$ and there exists a constant $M > 0$ such that:*

$$-\psi(x,u) + M(\alpha_g(|x|) + \sigma(|w|)) \leq -\alpha_4(|x|) + \sigma(|w|).$$

Robust stability of the iNMPC controller can be established from the following theorem.

**Theorem 4** *(Robust ISS Stability of iNMPC [24,31]) Under Assumptions 1 and 4, with $M \geq L_J$, the cost function $J_N(x)$ obtained from the solution of $\mathcal{P}_N(x)$, is an ISS-Lyapunov function and the resulting closed-loop system is ISS stable.*

**Proof:** We compare the costs of the neighboring problems $\mathcal{P}_N(x(k))$ and $\mathcal{P}_N(x(k+1))$ and introduce the effect of disturbances through $\epsilon(x(k+1))$,

$$\begin{aligned} & J_N(x(k+1)) - J_N(x(k)) \\ &= J_N(z(k+1)) - J_N(x(k)) + J_N(x(k+1)) - J_N(z(k+1)) \\ &\leq F(f(z_N^*, h_f(z_N^*))) - F(z_N^*) + \psi(z_N^*, h_f(z_N^*)) \\ &\quad -\psi(x(k), u(k)) + \epsilon(x(k+1)) \\ &\leq -\psi(x(k), u(k)) + \epsilon(x(k+1)) \end{aligned}$$

where the last two inequalities result from the fact that the solution of $\mathcal{P}_N(x(k))$ provides a feasible solution to $\mathcal{P}_N(z(k+1))$. With this, the existence of the terminal controller of Assumption 1 ensures that the stage cost $\psi(x(k), u(k))$ is the only accumulation point. Making use of Assumptions 3 and 4 to bound the mismatch term leads to,

$$J_N(x(k+1)) - J_N(x(k)) \leq -\alpha_4(|x(k)|) + \sigma(|w(k)|)$$

which fulfills (22c) with $M \geq L_J$. From Assumption 1 we also know that $J_N(x(k))$ satisfies (22a)-(22b). Therefore, the desired result follows from Lemma 3. In the nominal case, stability follows with $M = 0$. □

**Corollary 5** *(Asymptotic Robust Stability [24]) Assume that $\mathcal{W} = \{0\}$ and that Assumptions 1 -4 hold. Then the closed-loop system given by (1) and $u = h^{id}(x)$ is asymptotically stable.*

**Remark 5.** The above results assume Lipschitz continuity of the optimal cost function and of the control law. For the general nonlinear systems considered in this work, we guarantee Lipschitz continuity in a restricted neighborhood (possibly small) of an optimal solution satisfying the conditions of Theorem 1.

### *2.3 Computational Issues*

It is clear that Assumption 2 is too restrictive since in practical applications $\mathcal{P}_N(x)$ may be computationally expensive to solve. This implies that the control action $u = h^{id}(x)$ cannot be injected into the plant right after $x$ is obtained but only once the solution of $\mathcal{P}_N(x)$ has been obtained. As a consequence, the resulting delayed feedback action will be inconsistent with the current evolving state [5–7].

## 3 Advanced-Step NMPC Formulation

Consider that the state of the plant at $t_k$ is $x(k)$ and that we count with the control action $u(k)$. In the nominal case the system evolves as in (2). As a consequence, starting at $t_k$ we can predict the future state $z(k+1)$ and solve the predicted problem $\mathcal{P}_N(z(k+1))$ in advance. If this problem can be solved during the current sampling time, then $u(k+1) = h^{id}(x(k+1))$ will already be available at $t_{k+1}$. This simple strategy allows to remove the computational delay and preserves the iNMPC controller properties. In the presence of disturbances, the plant will evolve with uncertain dynamics towards the true state $x(k+1) = z(k+1) + g(x(k), u(k), w(k))$. In this case, the iNMPC control action cannot be computed in advance. In order to account for this, we exploit the parametric property of the OCP problem to compute a fast approximate solution of $\mathcal{P}_N(x(k+1))$ around the available nominal solution of $\mathcal{P}_N(z(k+1))$ to obtain fast feedback. We call the resulting algorithm the advanced-step NMPC controller (asNMPC):

**In background, between $t_k$ and $t_{k+1}$:**

- Having $x(k)$ and $u(k)$, predict the future state through forward simulation $z(k+1) = f(x(k), u(k))$. Set $p_0 = z(k+1)$ and solve the *predicted* problem $\mathcal{P}(p_0)$.
- At the solution $s^*(p_0, N)$, retain factors of $\mathbf{K}^*(p_0, N)$ or compute sensitivity matrix $\frac{\partial s^*}{\partial p}$ from (11).

**On-line, at $t_{k+1}$:**

- Obtain the true state $x(k+1)$ and set $p = x(k+1)$. Compute the fast approximate solution $\tilde{s}(p, N)$ from sensitivity (12) or as a perturbed Newton step (14), extract $u(k+1) = \tilde{v}_0(x(k+1))$ and return to background.

The above asNMPC algorithm yields the approximate control law, $u(k) = h^{as}(x(k))$.

**Theorem 6** *(Error Bound of asNMPC) From Theorem 1 with $p_0 = z(k+1)$ and $p = x(k+1) = z(k+1) + g(x(k), u(k), w(k))$, the approximation error between the asNMPC and iNMPC control laws satisfies $|h^{as}(x(k+1)) - h^{id}(x(k+1))| \le L_h^{as}|g(x(k), u(k), w(k))|^2$ with a local positive Lipschitz constant $L_h^{as}$.*

**Proof :** The asNMPC control action $\tilde{v}_0(x(k+1))$ is extracted from the approximate solution $\tilde{s}(x(k+1))$ obtained from the perturbation $p - p_0 = g(x(k), u(k), w(k))$. From (18) we have,

$$\begin{aligned} u(k+1) &= \tilde{v}_0(x(k+1)) \\ &= v_0^*(z(k+1)) + K_0 \cdot (x(k+1) - z(k+1)) \\ &= v_0^*(z(k+1)) + K_0 \cdot g(x(k), u(k), w(k)). \end{aligned} \quad (25)$$

The error bound follows from (13) and the equivalence between (18), (12) and (14) to give,

$$|u(k+1) - v_0^*(x(k+1))| \le L_h^{as}|g(x(k), u(k), w(k))|^2. \quad \square$$

For later reference, we note that solving the forward problem $\mathcal{P}_N(z(k+1)) = \mathcal{P}_N(f(x(k), h^{as}(x(k))))$ is equivalent to solve the following *extended* problem $\mathcal{P}_{N+1}(x(k), h^{as}(x(k)))$,

$$\begin{aligned} \min_{v_l, z_l} \quad & J_{N+1} := F(z_N) + \psi(x(k), h^{as}(x(k))) + \sum_{l=0}^{N-1} \psi(z_l, v_l) \\ \text{s. t. } & z_{l+1} = f(z_l, v_l), \quad l = 0, \dots N-1 \\ & z_0 = f(x(k), h^{as}(x(k))) \\ & z_l \in \mathbb{X},\ z_N \in \mathbb{X}_f,\ v_l \in \mathbb{U}. \end{aligned} \quad (26)$$

with fixed $h^{as}(x(k))$ computed from (25). In the following section, we will see that the cost function $J_{N+1}(x, h^{as}(x))$ associated to this problem can be used as a candidate Lyapunov function to derive sufficient stability conditions for the asNMPC control law.

### *3.1 Stability Properties*

To analyze the stability properties of the proposed controller we make use of the assumptions and definitions of Section 2 with a slight modification,

**Assumption 5** *(Computational Delay of asNMPC) The background calculations associated to the solution of the forward problem $\mathcal{P}_N(f(x, h^{as}(x)))$ can be obtained in one sampling time. Moreover, the sensitivity update can be obtained in a negligible amount of time.*

In the nominal case, the asNMPC and iNMPC controllers produce identical control actions. This follows from Theorem 6 with $g(x, u, w) = 0$. Under Assumption 5, Theorem 2 applies directly. For the analysis of the robustness properties of the asNMPC controller it is necessary to account for the effect of NLP sensitivity errors. As shown in Fig. 1, we recognize that the forward simulation $z(k+1) = f(x(k), u(k))$ will predict the future state at $t_{k+1}$. In the nominal case, this would give rise to the control action $h^{id}(z(k+1)) = h^{as}(z(k+1))$ that would be used to start the extended problem $\mathcal{P}_{N+1}(z(k+1), h^{id}(z(k+1)))$ with cost $J^{id}(z(k+1)) := J_{N+1}(z(k+1), h^{id}(z(k+1)))$. However, the plant will evolve with uncertain dynamics generating $x(k+1)$. Ideally, this would give rise to the optimal control action $h^{id}(x(k+1))$ that would be used to solve $\mathcal{P}_{N+1}(x(k+1), h^{id}(x(k+1)))$

at the next time step with cost $J^{id}(x(k+1)) := J_{N+1}(x(k+1), h^{id}(x(k+1)))$. In reality, we compute the approximate control $h^{as}(x(k+1))$ from (25) giving rise to problem $\mathcal{P}_{N+1}(x(k+1), h^{as}(x(k+1)))$ with cost $J^{as}(x(k+1)) := J_{N+1}(x(k+1), h^{as}(x(k+1)))$. Since this is a suboptimal cost that needs to be compared against the optimal cost $J^{id}(x(k+1))$. To account for this, we define the following mismatch terms,

$$\epsilon_s(x(k+1)) := J^{id}(x(k+1)) - J^{id}(z(k+1)) \quad (27a)$$

$$\epsilon_{as}(x(k+1)) := J^{as}(x(k+1)) - J^{id}(x(k+1)) \quad (27b)$$

where the first term accounts for the model mismatch as in (23) while the second term accounts for approximation errors introduced by NLP sensitivity.

**Assumption 6** *Under Theorems 1 and 6 there exist positive Lipschitz constants $L_J, L_h$ and $L_h^{as}$ such that $\forall x \in \mathbb{X}$,*

$$\begin{aligned} |\epsilon_s(x(k+1))| &\leq L_J(|x(k+1) - z(k+1)| \\ &\quad + |h^{id}(x(k+1)) - h^{id}(z(k+1))|) \\ &\leq L_J(1 + L_h)|g(x(k), u(k), w(k))| \quad (28a) \end{aligned}$$

$$\begin{aligned} |\epsilon_{as}(x(k+1))| &\leq L_J(|x(k+1) - x(k+1)| \\ &\quad + |h^{as}(x(k+1)) - h^{id}(x(k+1))|) \\ &\leq L_J L_h^{as} |g(x(k), u(k), w(k))|^2. \quad (28b) \end{aligned}$$

By comparing the successive costs $J^{as}(x(k))$ and $J^{as}(x(k+1))$, we can arrive at a similar ISS property as in Theorem 4.

**Theorem 7** (*Robust Stability of asNMPC) Under Assumptions 1, 4 and 5 with $M \geq L_J(1 + L_h + L_h^{as}|g(x, u, w)|) > 0$, the cost function $J^{as}(x)$ obtained from the solution of the extended problem $\mathcal{P}_{N+1}(x, u)$ with $u = h^{as}(x)$ is an ISS-Lyapunov function and the resulting closed-loop system is ISS stable.*

**Proof :** We compare the costs $J^{as}(x(k))$, $J^{as}(x(k+1))$ and use the mismatch terms in (28a)-(28b) to obtain,

$$\begin{aligned} & J^{as}(x(k+1)) - J^{as}(x(k)) \\ = \quad & J^{id}(z(k+1)) - J^{as}(x(k)) \\ & + J^{id}(x(k+1)) - J^{id}(z(k+1)) \\ & + J^{as}(x(k+1)) - J^{id}(x(k+1)) \\ \leq & -\psi(x(k), h^{as}(x(k))) + \epsilon_s(x(k+1)) + \epsilon_{as}(x(k+1)) \end{aligned}$$

the last inequality results from noting that the solution of $\mathcal{P}_{N+1}(x(k), h^{id}(x(k)))$ provides a feasible solution to $\mathcal{P}_{N+1}(z(k+1), h^{id}(z(k+1)))$. Applying the bounds (28b)-(28a), the result follows with $M \geq L_J(1 + L_h + L_h^{as}|g(x(k), u(k), w(k))|) > 0$. □

Note that if NLP sensitivity errors vanish (e.g. linear MPC) then $\epsilon_{as}(x) = 0$. Accordingly, $M \geq L_J(1 + L_h)$ is

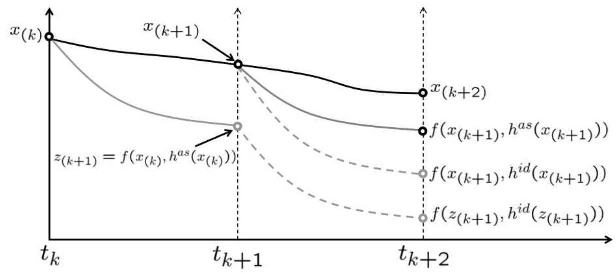


Fig. 1. Advanced step NMPC controller behavior.

sufficient and we recover *similar* (i.e. one step forward) robust stability properties of iNMPC as in Theorem 4. In the nominal case, $x(k+1) = z(k+1)$ and $M = 0$ is sufficient for nominal stability.

### *3.2 Computational Issues*

The proposed asNMPC algorithm is expected to reduce the on-line computational cost by two or three orders of magnitude. This results from the difference between the computational complexity of a single backsolve against that of the formation and factorization of the KKT matrix [22]. However, notice that Assumption 5 requires that the background solution can be obtained in one sampling time. Under mild conditions, if the sensitivity approximation is used to warm-start the next problem, the background problem can be converged in a few iterations [14]. Finally, note that it is possible to compute rigorous values of the Lipschitz constants that will implicitly take account the nonlinearity and disturbances of a particular system. Such bounds can be computed using NLP sensitivity concepts. This strategy has been applied in [32] to estimate $L_J$ as in Theorem 4 in order to characterize the robustness properties of iNMPC formulations.

## 4 Simulation Example

We consider a simulated NMPC scenario on a nonlinear CSTR model [33] represented by the following differential equations:

$$\frac{dz^c}{dt} = \frac{z^c - 1}{\theta} + k_0 z^c \exp\left[\frac{-E_a}{z^t}\right] \quad (29a)$$

$$\frac{dz^t}{dt} = \frac{z^t - z_f^t}{\theta} - k_0 z^c \exp\left[\frac{-E_a}{z^t}\right] + \alpha v(z^t - z_{cw}^t). \quad (29b)$$

The system involves two states $z = [z^c, z^t]$ corresponding to dimensionless concentration and temperature, and one control $v$ corresponding to the cooling water flowrate. The model parameters are $z_{cw}^t = 0.38$, $z_f^t = 0.395$, $E_a = 5$, $\alpha = 1.95 \times 10^4$, $\theta = 20$, and $k_0 = 300$. These differential equations are converted to the form of (2) through an implicit Runge-Kutta discretization, with each step representing a sampling time.

To simulate the plant evolution, we introduce off-set free plant-model mismatch by perturbing the nominal value of the reactor residence time $\theta$ from its nominal $\theta_{nom}$. In addition, we introduce Gaussian noise with $\sigma$ standard deviation, as a percentage on the initial states, to simulate the presence of measurement or estimation errors. The OCP is formulated using a quadratic function $\psi(z,v) = \hat{z}^T Q\hat{z} + R\hat{v}^2$ with $Q = \text{diag}\{1\times 10^6,\ 2\times 10^3\}$, $R = 1\times 10^{-3}$, terminal weight $F(z) = \hat{z}^T Q\hat{z}$ and $\hat{z} = z - z_{ss}, \hat{v} = v - v_{ss}$ where subscript $ss$ denotes steady-state value. The resulting NLP problems contain $18\times N$ variables and $6\times N$ constraints where $N$ is the number of time steps. While the resulting NLPs can be solved in a negligible amount of time, the performance characteristics apply to much larger examples as well [22].

We demonstrate the performance of the iNMPC and asNMPC controllers under different robust scenarios. Here, we choose $N = 10$ along with a zero terminal constraint $z_N = z_{ss}$. The controllers first perform the transition between two open-loop unstable steady states ($SS_1$ and $SS_2$) followed by a subsequent transition to a stable steady state, $SS_3$. The location of the three steady-states is illustrated in the $v - z^t$ bifurcation diagram in Fig. 2. The control is required to satisfy $250 \leq v \leq 450$ where the upper bound is set close to its corresponding value at $SS_2$. This tends to amplify approximation errors and thus illustrate the advantages and limitations of the proposed controller. In Fig. 3 we illustrate the effect of increasing model mismatch due to perturbations in the reactor residence time. From the top graph it is clear that for a perturbation of ($\theta = 0.75\theta_{nom}$) the performance of both iNMPC and asNMPC is nearly identical. Both controllers are able to handle relatively large perturbations. However, as the mismatch is increased ($\theta = 0.5\theta_{nom}$) the performance of asNMPC tends to drift away and the closed-loop system destabilizes due to the presence of approximation errors. This is particularly evident in the second transition. Interestingly, for a slightly larger mismatch ($\theta = 0.45\theta_{nom}$) the iNMPC controller is not able to reject the perturbation in the second transition either, and the close-loop becomes unstable. In other words, both controllers are able to tolerate similar levels of mismatch, suggesting that the effect of approximation errors in the asNMPC is not very strong. Similar behavior can be seen when the controllers are subjected to simultaneous noise and model mismatch ($\theta = 0.75\theta_{nom}$) as illustrated in Fig. 4. Again, for small levels of noise ($\sigma = 2.5\%$), the performance of the two controllers is almost identical. The asNMPC controller is able to tolerate large levels of noise (up to $\sigma = 7.5\%$) but its performance deteriorates due to approximation errors, specially in the transition from $SS_1$ to $SS_2$.

To illustrate the role of approximation errors on the stability of asNMPC, we perform a more detailed analysis on the second transition for scenario $\theta = 0.45\theta_{nom}$ from Fig. 3. The results are illustrated in Fig. 5. In the top graph, we present the profiles of the predicted $z(k)$ and the actual $x(k)$ temperatures. As can be seen, the perturbation in $\theta$ creates large deviations between both states. The mismatch is expected to generate a difference between the asNMPC control action $h^{as}(x)$ from that of the iNMPC $h^{id}(x)$ which is illustrated in the second graph. Interestingly, note that despite the relatively large mismatch, the asNMPC and iNMPC control actions are identical before the system destabilizes at time step 120. This would suggest that the system does not destabilize in the first place due to approximation errors. To validate this, we present profiles in the third graph of the left-hand side ($LHS$) and right-hand side ($RHS$) of (29), the sufficient stability condition from Theorem 7. Stability implies that $LHS \leq RHS$. As can be seen, this condition is fulfilled up to time step 95. However, note that even though the two control actions are identical at this point, there is a cross-over $LHS \geq RHS$ and the system destabilizes. To explain this, we present profiles for the mismatch terms in the bottom graph. As can be seen, the magnitude of the mismatch introduced by approximation errors $\epsilon_{as}(x)$ tends to be smaller compared to that introduced by the perturbations $\epsilon_s(x)$. However, at time step 95 the approximation errors become relevant and, *even though the injected control actions are identical*, the combined mismatch terms promote a cross-over in the stability condition (29) of Theorem 7. As predicted by (29), this destabilizes the system. It has been observed that, once the system becomes unstable, the oscillations become aggressive and the perturbations induce changes in the active-set for the perturbed problems (e.g. control profiles at time steps 120 and 140). These changes cannot be predicted by the NLP sensitivity calculation and requires the solution of a quadratic programming problem. While this might improve the quality of the approximations, understanding the implications of active-set changes in the stability of the asNMPC controller requires a deeper analysis of the Lipschitz continuity assumptions made in this work.

## 5 Conclusions and Future work

In this study, we derive and analyze the optimality and stability properties of the advanced-step NMPC (asNMPC) controller. This controller avoids feedback delays associated to the on-line solution of large-scale OCPs. Here, the moving horizon OCP is formulated with an advanced step control action and it is solved, in background, between sampling times. This study shows that the asNMPC controller has identical nominal stability properties of the ideal NMPC controller without computational delay. In the presence of disturbances, the controller exploits the parametric properties of the OCP through NLP sensitivity concepts to provide a fast on-line correction of the nominal solution. With this, a rigorous bound on the loss of optimality can be established and related to the bounds of the uncertainty description. This allows to characterize the robustness properties of

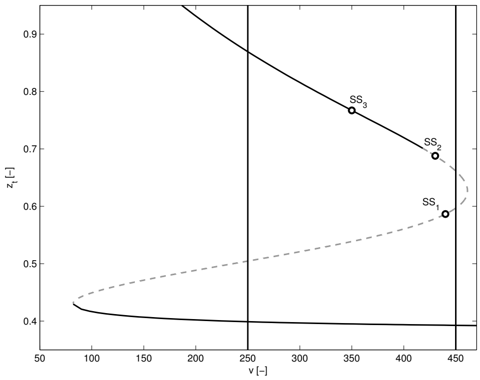


Fig. 2. Steady-state map between temperature $z^t$ and cooling water flow rate $v$. Solid vertical lines represents input and output constraints.

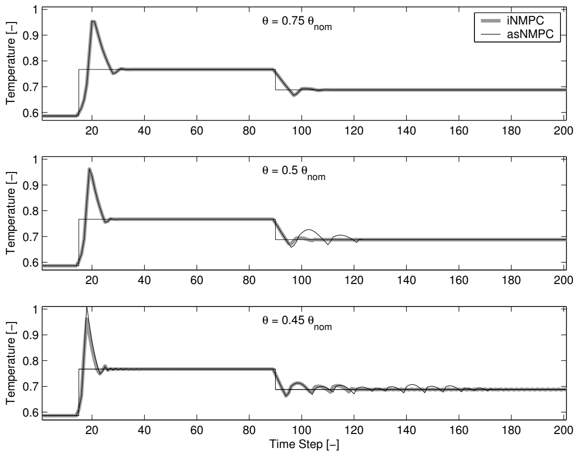


Fig. 3. Effect of plant-model mismatch on the performance of the controllers.

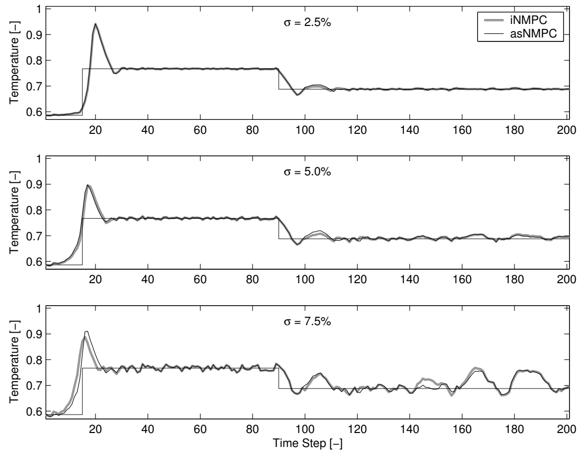


Fig. 4. Effect of noise on the performance of the controllers. Perturbation on residence time at 25% below nominal value.

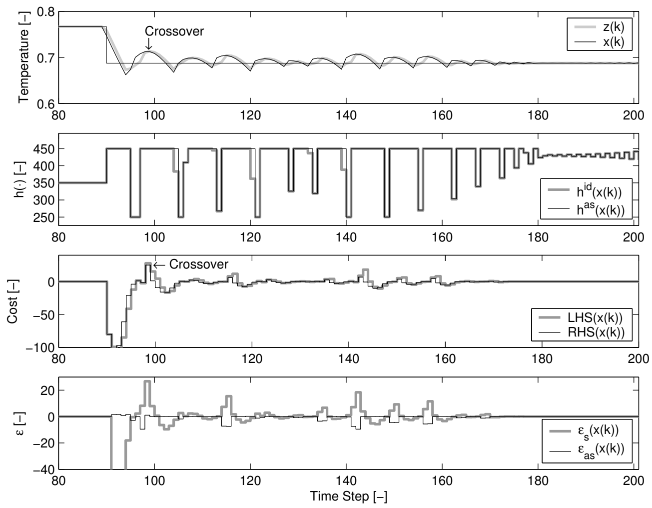


Fig. 5. Analysis of the effect of mismatch terms on the stability of the advanced step controller.

the controller through input-to-state stability concepts. As part of future work, we plan an extension of the asNMPC controller to more expensive background calculations that may take place over multiple time steps. Moreover, the effect of disturbances and model mismatch can be attenuated through on-line state estimation. For this, we plan to use fast NLP sensitivity-based moving horizon estimators developed in a recent study [34]. Finally, a deeper analysis of the impact of active-set changes on the stability of the approximate NMPC controllers is an important area of future research.

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