RISK-AVERSE FEASIBLE POLICIES FOR LARGE-SCALE MULTISTAGE STOCHASTIC LINEAR PROGRAMS

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Abstract. We consider risk-averse formulations of stochastic linear programs having a structure that is common in real-life applications. Specifically, the optimization problem corresponds to controlling over a certain horizon a system whose dynamics is given by a transition equation depending affinely on an interstage dependent stochastic process. We put in place a rolling-horizon time consistent policy. For each time step, a risk-averse problem with constraints that are deterministic for the current time step and uncertain for future times is solved. To each uncertain constraint corresponds both a chance and a Conditional Value-at-Risk constraint. We show that the resulting risk-averse problems are numerically tractable, being at worst conic quadratic programs. For the particular case in which uncertainty appears only on the right-hand side of the constraints, such risk-averse problems are linear programs. We show how to write dynamic programming equations for these problems and define robust recourse functions that can be approximated recursively by cutting planes. The methodology is assessed and favourably compared with Stochastic Dual Dynamic Programming on a real size water-resource planning problem.

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1. Introduction

Hedging risk is a challenging question in Stochastic Programming. Since uncertainty may appear in the objective function and/or in the constraints, risk-averse proposals can be gathered in three groups, depending on how uncertainty (and, hence, risk) is dealt with. The first two groups deal with uncertainty in both the objective function and in the constraints. More precisely, methods in the first group minimize some risk measure of the objective function, which depends on the decisions and on the underlying random variables. In this group, the so-called dynamic risk mappings [40], [34], including the different "perspectives" for polyhedral risk measures in [17], appear as powerful and useful tools. The second group, Robust Optimization, can use available data to define uncertainty sets and apply the worst case oriented robust optimization methodology [5]. Finally, general stochastic programs can always be reformulated as optimization problems with deterministic objective function and uncertain constraints. This makes possible the introduction of a last group of risk-averse models, especially designed to control risk on constraints. Among several proposals in this group, we mention the classical chance constraints [10], [37] and the more recent integrated chance constraints [26], [27], and stochastic ordering constraints [15].

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Our work belongs to the last group above, with some distinctive features. Specifically, most of the works in this group replace each uncertain constraint by a risk-averse counterpart, written as a single constraint. Instead, we use a couple of constraints in the risk-averse formulation: to each uncertain constraint corresponds both a chance and a Conditional Value-at-Risk (CVaR) constraint. Our goal is twofold; while a chance constraint gives a qualitative control for constraint violation, regardless of the amount of violation, a CVaR constraint gives a quantitative measure, by keeping control of such amount.

In addition, rather than defining a single risk-averse problem, we adopt a *rolling-horizon* point of view; there are as many risk-averse problems as time steps in the optimization period. At a given time step, the corresponding risk-averse problem considers constraints to be deterministic for the current time step and uncertain for future times. As a result, our approach builds *feasible* policies such that all the constraints over the optimization period are satisfied almost surely, an important requirement in applications with "hard" constraints that cannot be violated.

Another crucial matter when dealing with probabilistic constraints refers to numerical tractability. In this respect, since it is useful to take full advantage of the problem structure, we consider special stochastic linear programs arising when controlling a dynamical system with a transition equation depending affinely on an interstage dependent stochastic process. More precisely, given a multiperiod horizon, for each realization of uncertainty, we consider the problem:

$$
(1) \qquad \begin{cases} \min\limits_{x_{[1:T]},u_{[1:T]}} \sum\limits_{t=1}^{T} c_t^\top u_t & \\ x_t = A_{t-1}x_{t-1} + B_t u_t + C_t\tilde{\xi}_t + d_t & \text{for } t = 1,\dots,T, \qquad (\text{TRAN}) \\ E_t x_t + F_t(\tilde{\xi}_t)u_t \geq G_t\tilde{\xi}_t + h_t & \text{for } t = 1,\dots,T, \qquad (\text{INEQ}) \end{cases}
$$

where

- T is the number of time steps, possibly large;
- $\tilde{\xi}_t$ is a particular realization at time step t of an M-dimensional random process. Each process component $\xi_t(m)$ follows a generalized autoregressive model with time varying order; see Section 3.1 below. The realization $\tilde{\xi}_t$ becomes known at the beginning of time step t;
- $x_t \in \mathbb{R}^{N_x}$ is the state of the system at the end of time step t, with dynamics given by the transition equation (1)(TRAN) and known x_0;
- $u_t \in \mathbb{R}^{N_u}$ is the control variable, applied to the system at time step t; and
- $c_t^\top u_t$ is the cost at time step t.

Matrices A_t, B_t, C_t, E_t, G_t are $N_x \times N_x$, $N_x \times N_u$, $N_x \times M$, $q_t \times N_x$, $q_t \times M$, respectively. The "technology" matrix in the inequality constraints (1)(INEQ), i.e.,

$$
(2) \qquad F_t(\tilde{\xi}_t) = \mathring{F}_t + \begin{bmatrix} (F_{t,1}\tilde{\xi}_t)^\top \\ \vdots \\ (F_{t,q_t}\tilde{\xi}_t)^\top \end{bmatrix},
$$

is an affine function of the process realization. The respective sizes of matrices $\mathring{F}_t$ and $F_{t,i}$ are $q_t \times N_u$ and $N_u \times M$. Finally, vectors d_t and h_t have dimensions N_x and q_t, respectively. As stated,

(1) is just a problem instance, written for a given realization of the random process (ξ_t); for the moment, we do not explain how uncertainty is dealt with.

For many real-life planning problems, the evolution along the planning horizon is modelled by means of reservoirs (of a product, of water, of energy, of take-or-pay contracts, of oil). In this sense, the framework in (1) is rather comprehensive, and covers a variety of applications in inventory problems [3], [6], [29], electric energy, oil, and finance [39, Ch. 10], [19], [21], [46]; see also Section 2 below. For such applications, many of the constraints (INEQ) involve only the state x_t or only the control u_t. In this case, when certain rows in E_t are nonzero, the corresponding rows in $F_t(\tilde{\xi}_t)$ are null, and reciprocally.

When applying our rolling-horizon approach to problems like (1), the special underlying structure -of both the stochastic process and problem constraints- yields *fully tractable risk-averse* problems for each time step. Equivalent deterministic formulations of chance-constrained problems have already been proposed for a limited number of statistical frameworks and classes of optimization problems. Our result generalizes to a somewhat broader setting similar results in [10] and [12]. More precisely, we show in Theorem 4.4 that the risk-averse problems are deterministic conic quadratic programs, that become linear programs if the technology matrix in (2) is not random, that is, if $F_{t,1} = \ldots = F_{t,q_t} = 0$.

Our methodology is specially attractive for problems with large time horizon T. In such cases, there are often so many scenarios that only sampling methods can be employed, [33], [13], [36]. In this setting, many risk-averse formulations yield huge multistage stochastic linear programs which are difficult to solve, when not intractable. With our approach, we build a risk-averse feasible policy just by solving T deterministic linear or conic programs. Such programs are decomposable by stages (an issue of special algorithmic interest in a large-scale context) and can be solved efficiently by Dual Dynamic Programming. The question of how to measure risk in financial applications has been intensively developed over the past years [1], [38]. In the energy sector, similar ideas have been applied for hedging financial risk of price-taker companies operating in deregulated markets (we refer to [32], [44], [18], [17], and references therein). The problem of measuring *hydraulic risk* for the optimal operation problem (OOP) was first explored in [8], and more recently in [22], [20], [35], [43]. Our rolling horizon policy gives an alternative, computationally cheaper, risk-averse feasible policy that can be applied to the OOP.

Our paper is organized as follows. In Section 2 we start with an application example, the long-term planning of water reservoirs in hydro-thermal power systems. This application is used throughout the paper to explain and motivate the proposed approach. Section 3 gives the statistical model for the stochastic process as well as the main elements of our risk-averse rolling-horizon approach. Section 4 shows that the risk-averse problems are numerically tractable. In particular, when these problems are linear programs, in Section 5 we give a stagewise decomposition and show how to build approximate robust recourse functions if desired. Finally, in Section 6, the approach is assessed on the water-resource planning problem, by comparing its performance with the sampling method of Stochastic Dual Dynamic Programming from [33], set in a risk-neutral formulation.

We adopt the following notation and conventions. For $t_2 \geq t_1$, the short form $v_{(t_1,t_2]}$ (resp. $v_{[t_1,t_2]}$) stands for the concatenation $(v_{t_1+1}, \ldots, v_{t_2})$ (resp. $(v_{t_1}, \ldots, v_{t_2})$), with $v_{(t,t]}$ vacuous and knowing that the concatenated objects v_j can be vectors or matrices, depending on the context.

For sums and products, $\sum_{i=i_0}^{i_1} x_i = 0$ and $\prod_{i=i_0}^{i_1} x_i = 1$ whenever $i_0 > i_1$, knowing that for matrices X_i, if $i_0 > i_1$ then $\prod_{i=i_0}^{i_1} X_i = I$, the identity matrix. For a random variable X, its distribution function and its density are respectively denoted by $F_X(\cdot)$ and $f_X(\cdot)$, knowing that for $X \sim \mathcal{N}(0,1)$, we just write $F(\cdot) := F_X(\cdot)$. Finally, if X is continuous and larger values of the random variable are preferred, its Conditional Value-at-Risk of level $\varepsilon_{\mathtt{p}} \in [0,1]$ is denoted and defined by $CVaR_{\varepsilon_{\mathtt{p}}}(X) := -\mathbb{E}[X|X \leq F_X^{-1}(\varepsilon_{\mathtt{p}})]$, while the Value-at-Risk of level $\varepsilon_{\mathtt{p}}$ of X is $VaR_{\varepsilon_{\mathtt{p}}}(X) := -F_X^{-1}(\varepsilon_{\mathtt{p}})$; see [38].

2. Motivation

Our initial motivation is the long-term optimal management of water reservoirs in hydro-thermal power systems, [31]. For this problem, present operating decisions have future consequences that are difficult to quantify, because the water is a commodity of unknown value and uncertain availability. Moreover, it is important to set the problem in a risk-averse framework for economic indicators such as energy prices to better reflect the impact of extreme events.

In this setting, it is desirable for the computational model to provide two types of output:

`(RecFun)`: approximate recourse functions for time steps $t = 1, \ldots, T$; and

`(EcInd)`: economic indicators, such as mean marginal energy prices, expected load shedding, average supplied energy.

The approximate recourse functions from `(RecFun)` can be seen as a pricing mechanism, giving value to water, and defining a policy. Namely, with such prices it is possible to mimic the optimal operation of the hydro-thermal system. Recourse functions can also be used to couple planning models of different horizons and obtain a sound overall management of the system. Similar situations, arising when controlling in the long-term stochastic systems are described in [39, Ch. 10].

As for the indicators in `(EcInd)`, they measure the performance of the long-term management policy. They are computed by simulating the system operation over a high number of randomly generated scenarios, covering a large spectrum of foreseeable futures (including extreme droughts, floods, etc).

In Sections 3.2 and 5.2 we analyze how the robust rolling-horizon approach presented below compares to usual non-rolling-horizon methodologies to fulfill requirements `(RecFun)` and `(EcInd)`.

We now give the mathematical formulation for a simplified model of the water management application that will help in clarifying our rolling-horizon risk-averse methodology.

2.1. A simplified long-term energy planning problem.

The optimal management of a hydro-thermal power system in the long-term minimizes the operational cost throughout the period, subject to various technical constraints. Since the immediate hydro-cost is positive but negligible, operational costs are essentially related to the fuel burnt by thermal plants and penalties resulting from load shedding. In a predominantly hydro-electric system like Brazil's, the availability of (limited) amounts of hydro-power, in the form of water stored in reservoirs, makes the problem extremely complex. There are many reservoirs in cascade, some of them with a capacity of regularization that covers several years, which are spread over geographical regions with different seasonal rainfall.

For simplicity, we consider only one reservoir with large enough capacity and suppose there is neither spillage nor upper bound on the turbine's outflow. The system also has one run-of-river plant, but no thermal plants. In addition, the problem is formulated in energy variables, without entering into the issue of how to relate water to energy by explicit production functions.

In this simplified formulation, at each time step t we have a state variable x_t, the volume of the reservoir at the end of the time step; a control variable $u_t = (g_t, df_t)^\top \geq 0$, with g_t the turbines outflow, df_t the energy deficit; and $\tilde{\xi}_t$, the natural inflow of water arriving into the reservoir. Only a fraction $\gamma_t \in [0, 1)$ of this water can be stored; the remaining portion, $(1 - \gamma_t)\tilde{\xi}_t$, is immediately transformed into power by the run-of-river plant. We further assume that turbines of the run-of-river plant have enough capacity to generate power out of all of $(1 - \gamma_t)\tilde{\xi}_t$, without any losses (for a more general model, we refer to [25]).

In order to appropriately reflect seasonal variations, the stochastic process of water streamflows is usually represented by a periodic autoregressive model with a one year period, [30], [14]. If there are more reservoirs, the process is multivariate and there is a (nondiagonal) covariance matrix to express the dependence of inflows on neighboring geographical regions.

The objective function considers generation and shortage costs, and constraints are given below.

Water balance equation. If there is no evaporation,

$$x_t = x_{t-1} - g_t + \gamma_t \tilde{\xi}_t \tag{3}$$

corresponds to the transition equation (TRAN), with

$$A_{t-1} = 1,\ B_t = [-1 \quad 0],\ C_t = \gamma_t, \text{ and } d_t = 0.$$

This model can be generalized to more reservoirs. For example, if the system has no reservoirs in cascade then A_{t-1} is an identity matrix I, B_t a concatenation of $-I$ with a zero matrix, $C_t = \gamma_t I$ is a diagonal matrix, and d_t a null vector.

Demand satisfaction. Usually this constraint has the form $g_t + df_t = \max(\text{dem}_t - (1-\gamma_t)\tilde{\xi}_t, 0)$, where the energy deficit df_t is modelled as a (fictitious) thermal plant with large capacity and generation cost equal to the shortage cost. The demand dem_t at time step t is assumed deterministic for convenience.

Since our risk-averse formulation makes use of probabilistic constraints (that can be set only for inequality constraints), we rewrite the demand satisfaction constraint as an inequality:

$$g_t + df_t \geq \text{dem}_t - (1 - \gamma_t)\tilde{\xi}_t\,. \tag{4}$$

Although, in general, modifying the feasible set may alter the optimization problem, such is not the case here, when replacing the initial demand satisfaction constraint by inequality (4), thanks to the structure of the problem.

With respect to (INEQ), we see that the demand constraint corresponds to taking

$$E_t = 0,\ F_t = [1 \quad 1],\ G_t = -(1 - \gamma_t), \text{ and } h_t = \text{dem}_t,$$

with natural extensions to the multi-reservoir case.

Critical volume. Operators managing the system in real time are mostly concerned about keeping reservoirs above critical values, called *minzones* or reference trajectories, estimated empirically by

the operators or imposed by some regulatory rules. The corresponding constraints have the form

$$x_t \geq x_t^{crit}\,. \tag{5}$$

In the notation of (INEQ), constraint (5) sets

$$E_t = 1,\ F_t \text{ and } G_t \text{ null, and } h_t = x_t^{crit},$$

where the extension to more reservoirs is straightforward.

We assume that values $x_t^{crit} \geq 0$ are chosen so that

(A1) along time steps, the critical levels are nonincreasing: $x_0 \geq x_1^{crit} \geq x_2^{crit} \geq \ldots \geq x_T^{crit}$.

This assumption, together with the inflow condition stating that $\mathbb{P}(\xi_t \geq 0) = 1$ for every t, implies that recourse in problem (1) is *relatively complete*, [7, Ch. 3, p. 92]. If Assumption (A1) did not hold, or if we did not have $\xi_t \geq 0$ a.s., then the methodology described in the sequel would still be applicable. The reason is that relatively complete recourse can always be ensured for problem (1), by using slack variables $y_t \geq 0$ to constraints (5): $x_t + y_t \geq x_t^{crit}$, and penalizing y_t in the cost.

In this application the "technology" matrices F_t are deterministic and $F_{t,i} = 0, i = 1, \ldots, q_t$ ($q_t = 2$ for the simplified model considered in this section). If production functions were considered, and if such functions depended on uncertain factors, then the matrices F_t would be stochastic. Such is the case in the oil industry, for example [46], when dealing with refinery production planning with uncertain yields.

3. General setting

Consider the natural filtration $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_T$ induced by the process ξ_t, defining $\mathcal{F}_t$ as the sigma-algebra $\sigma(\xi_j,\ j \leq t)$. We are interested in finding a sequence of state and control mappings $x_t(\cdot)$ and $u_t(\cdot)$ for $t = 1, \ldots, T$, i.e., an implementable *policy* satisfying the following properties:

Nonanticipativity: state and control mappings $x_t(\cdot)$ and $u_t(\cdot)$ are $\mathcal{F}_t$-measurable and, hence, are functions of the available history $\tilde{\xi}_{[t]}$ of the process.

Feasibility: a non-anticipative policy satisfying (INEQ) and (TRAN) for $t = 1, \ldots, T$, with probability one.

The relatively complete recourse assumption ensures the existence of feasible policies for problem (1).

3.1. Statistical model.

Our multivariate discrete time stochastic process depends, in an affine manner, on previous values. In this context, for $m = 1, \ldots, M$, each component $\xi_t(m)$ is represented by a generalized autoregressive model, with varying orders $p_t(m) \geq 0$. Accordingly, for every integer t, there exist coefficients $\Phi_t^j(m)$ for $j = 1, \ldots, p_t(m)$, with non-null $\Phi_t^{p_t(m)}(m)$, such that

$$\xi_t(m) = \sum_{j=1}^{p_t(m)} \Phi_t^j(m)\xi_{t-j}(m) + \eta_t(m). \tag{6}$$

In this expression, $\eta_t, t = 1, \ldots, T$, are independent Gaussian vectors with $\mathbb{E}[\eta_t] = \mu_t$ and with an $M \times M$ covariance matrix $\Gamma_t := Cov(\eta_t)$. In particular, we will denote the standard deviation of $\eta_t(m)$ by $\sigma_t^\eta(m) > 0$.

In our development, we will need to express a given value of the process as a function of its past history. For this reason, it is convenient to introduce for $t = 1, \ldots, T-1$, $j = 1, \ldots, T-t$, and $m = 1, \ldots, M$, the integers

$$p_{t,j}^{\max}(m) = \max_{1 \le k \le j} \{p_{t+k}(m) - k\}, \tag{7}$$

as well as the useful past history of the process up to time step t

$$\tilde{\xi}_{[t]} = \left\{ \tilde{\xi}_{t-j}(m),\ m = 1, \ldots, M,\ j = 0, \ldots, \max(p_{t,T-t}^{\max}(m), t-1) \right\}. \tag{8}$$

The index $p_{t,j}^{\max}(m)$ specifies how much past information is needed at time step t to compute $\xi_{t+j}(m)$, for a process ξ_t modelled by (6). More precisely, as shown in [24], for $t = 1, \ldots, T-1$, $j = 1, \ldots, T-t$, and $m = 1, \ldots, M$, the relation

$$\xi_{t+j}(m) = \sum_{\ell=0}^{p_{t,j}^{\max}(m)} \alpha_{t,j}^{\ell}(m) \xi_{t-\ell}(m) + \sum_{\ell=1}^{j} \beta_{t+j}^{\ell}(m)\, \eta_{t+j-\ell+1}(m), \tag{9}$$

holds for certain coefficients $\alpha_{t,j}^{\ell}(m)$ and $\beta_{t+j}^{\ell}(m)$. Such coefficients can be derived from the model data in (6); we refer to [24] for the corresponding (recursive or explicit) formulæ, that are useful for the numerical implementation of the method.

3.2. **Rolling-horizon versus non-rolling-horizon policies.** We focus on models with recourse that, at time step t, make use of a recourse (or cost-to-go) function $\mathcal{Q}_{t+1}$. Usually, recourse functions depend only on the state variables, x_t. However, in our setting (6), the stochastic process is affinely dependent on previous values, and the state variable has to be augmented with the process history of realizations. As a result, the simulation phase, that is, the actual computation of the indicators from item `(EcInd)` in Section 2, involves solving (exactly or approximately) problems of the form

$$\begin{cases} \min\limits_{x_t, u_t} c_t^\top u_t + \mathcal{Q}_{t+1}(x_t, \tilde{\xi}_{[t]}) \\ x_t = A_{t-1} x_{t-1}(\tilde{\xi}_{[t-1]}) + B_t u_t + C_t \tilde{\xi}_t + d_t \\ E_t x_t + F_t(\tilde{\xi}_t) u_t \ge G_t \tilde{\xi}_t + h_t. \end{cases} \tag{10}$$

For these problems, $x_{t-1}(\tilde{\xi}_{[t-1]})$ is an optimal value of state x_{t-1} obtained solving a problem of form (10) for time step $t-1$ given the trajectory $\tilde{\xi}_{[t-1]}$ of process (ξ_t) up to stage $t-1$. A usual non-rolling-horizon approach approximates problem (10) by replacing $\mathcal{Q}_{t+1}(x_t, \tilde{\xi}_{[t]})$ with an approximate recourse function $\mathfrak{Q}_{t+1}(x_t, \tilde{\xi}_{[t]})$, depending on the augmented state $(x_t, \tilde{\xi}_{[t]})$. This approximation is built at the first stage, knowing $\tilde{\xi}_{[1]}$ and x_0 only.

By contrast, in a rolling-horizon setting, we can either determine an optimal solution to (10) or use an approximation $\mathfrak{Q}_{t+1}(\cdot, \cdot)$ of $\mathcal{Q}_{t+1}$ built at stage t, knowing the history $\tilde{\xi}_{[t]}$ and $x_{t-1}(\tilde{\xi}_{[t-1]})$.

3.3. **Building risk-averse rolling-horizon policies.** If the recourse functions in (10) are risk-neutral or risk-averse, so will be the corresponding policies. Since in our motivating application the Independent System Operator is interested in finding policies that not only keep the reservoir levels above some critical value with high probability, but also avoid too big shortfalls, we use two constraints to hedge risk. More precisely, each uncertain scalar constraint is replaced by a chance

constraint that prevents, with certain probability, constraint violation, and by a CVaR constraint, that prevents too big violations, if they occur.

For each $t = 1, \dots, T$, our risk-averse recourse function $\mathcal{Q}_{t+1}$ in (10) is defined considering certain t^{th} *risk-averse optimization problem*, written over the horizon $[t, T]$, considering that the past history $\tilde{\xi}_{[t]}$ is known. Accordingly, constraints at time step t are considered deterministic, while future constraints, for time steps $\tau = t+1, \dots, T$, are dealt with as uncertain.

In this setting, given a scenario $(\tilde{\xi}_1, \dots, \tilde{\xi}_T)$, and following [16, p. 16], given any random vector $Y(\xi_{t+1}, \dots, \xi_\tau)$ decomposed in the form

$$Y(\xi_{t+1}, \dots, \xi_\tau) = g_Y(\xi_{[t]}) + f_Y(\eta_{t+1}, \dots, \eta_\tau),$$

with $\xi_{[t]}$ and $(\eta_{t+1}, \dots, \eta_\tau)$ independent, we use the notation

$$Y(\xi_{t+1}, \dots, \xi_\tau)|\tilde{\xi}_{[t]} := g_Y(\tilde{\xi}_{[t]}) + f_Y(\eta_{t+1}, \dots, \eta_\tau)\,,$$

to refer to the corresponding conditional random variable.

For notational simplicity, and without loss of generality, from now on we suppose the inequality constraints in (1) are scalar, so matrices therein are row vectors (like each individual constraint in the water resource planning application from Section 2). In particular, in (2) we have that $q_t = 1$ and $F_t(\xi_t) = \mathring{F}_t + \xi_t^\top F_{t,1}^\top$.

The corresponding t^{th} risk-averse optimization problem, akin to problem (10), has the form

$$(11)\quad \begin{cases} \min\limits_{x_t, u_{[t,T]}} \sum\limits_{\tau=t}^{T} c_\tau^\top u_\tau \\ x_t = A_{t-1}x_{t-1}(\tilde{\xi}_{[t-1]}) + B_t u_t + C_t \tilde{\xi}_t + d_t, \\ E_t x_t + (\mathring{F}_t + \tilde{\xi}_t^\top F_{t,1}^\top)u_t \geq G_t \tilde{\xi}_t + h_t, \\ \text{and, for } \tau = t+1, \dots, T: \\ \mathbb{P}\left(E_\tau x_\tau(x_t, u_{(t:\tau]}, \xi_{(t:\tau]}) + (\mathring{F}_\tau + \xi_\tau^\top F_{\tau,1}^\top)u_\tau - G_\tau \xi_\tau \geq h_\tau \middle| \tilde{\xi}_{[t]}\right) \geq 1 - \varepsilon_{\mathtt{p}}, \\ -CVaR_{\varepsilon_{\mathtt{p}}}\left(E_\tau x_\tau(x_t, u_{(t:\tau]}, \xi_{(t:\tau]}) + (\mathring{F}_\tau + \xi_\tau^\top F_{\tau,1}^\top)u_\tau - G_\tau \xi_\tau \middle| \tilde{\xi}_{[t]}\right) \geq h_\tau - \varepsilon_{\mathtt{c}}(|h_\tau| + 1), \end{cases}$$

where $\varepsilon_{\mathtt{p}},\ \varepsilon_{\mathtt{c}} \in (0, 1)$ are given confidence levels and where $x_\tau(x_t, u_{(t:\tau]}, \xi_{(t:\tau]})$ is the expression of x_τ as a function of variables $x_t, u_{(t:\tau]}$ and of random vectors $\xi_{(t:\tau]}$. This expression is obtained by applying recursively transition equation (TRAN) between time steps $t+1$ and τ (see the next section).

Variables for problem (11) are of the here-and-now type, i.e., they are fixed deterministic vectors and not functions of past realizations. Having a solution $(x_t^{*t}, u_{[t,T]}^{*t})$ to (11) we only use its first components (x_t^{*t}, u_t^{*t}) to define the policy. More precisely, after solving for $t = 1, \dots, T$ all the risk-averse problems (11), the controls

$$u_{Rob} := (u_1^{*1}, \dots, u_t^{*t}, \dots, u_T^{*T})$$

give our rolling-horizon implementable policy, which is also time consistent [42]. The remaining optimal controls $u_{t+1}^{*t}, \dots, u_T^{*t}$ are not used. Since in (11) the solution depends on $\tilde{\xi}_{[t]}$, but not on future realizations $\tilde{\xi}_{(t,T]}$, the policy u_{Rob} is non-anticipative. As for feasibility, it results from the fact that a solution to the t^{th} risk-averse optimization problem (11) satisfies constraints (TRAN) and (INEQ) at time step t, by construction.

If uncertain inequality constraints were vectorial, instead of scalar, then each constraint component would have individual chance and CVaR constraints and, hence, different confidence levels $\varepsilon_{\mathtt{p}}^i, \varepsilon_{\mathtt{c}}^i$, for $i = 1, \ldots, q_t$, possibly varying with the time period $t = 1, \ldots, T-1$.

On confidence levels for our rolling-horizon policy. Confidence levels are parameters of our rolling-horizon methodology and, as such, they need some *calibration*, to ensure that the risk-averse problems (11) are feasible. Such parameters have to be taken large enough to ensure feasibility but also small enough to guarantee constraint satisfaction with a reasonably large probability ($\varepsilon_{\mathtt{p}}$), and/or to reasonably limit the amount of constraint violation ($\varepsilon_{\mathtt{c}}$). There is not a unique choice for confidence levels, since this is a problem-dependent issue. Furthermore, it is not possible to know in advance if, for a given choice of parameters, all problems (11) will be feasible on a given scenario $\tilde{\xi}_{[1:T]}$, for all time steps. As a result, instead of an a priori, static choice for these parameters, a *dynamic tuning* must be put in place. More precisely, on a given scenario and for a given time step, different instances of problem (11) can be solved, for a range of different confidence levels, increasing such levels as infeasibility arises. Once various values of $\varepsilon_{\mathtt{p}}$ and $\varepsilon_{\mathtt{c}}$ yielding feasible (11) are found, the combination with smaller $\varepsilon_{\mathtt{p}}$ should be preferred.

When uncertainty only appears in the right hand side as in our motivating application, the need of a dynamic calibration is not a real handicap of the approach, because each risk-averse problem (11) is a linear program (see the next section). The more general case, of uncertain matrices F_τ, may need a fine tuning for $\varepsilon_{\mathtt{p}} > \frac{1}{2}$. As shown in the next section, for smaller $\varepsilon_{\mathtt{p}}$, the risk-averse problems are convex programs, and the trial and error process for different confidence levels can still be employed.

4. Reformulating the chance and CVaR constraints

A usual concern when formulating robust or chance constraints is their tractability. We now develop some algebraic manipulations showing that our chance and CVaR constraints are conic quadratic at worst. Moreover, we show that when each $F_t(\xi_t) = \mathring{F}_t$ is deterministic (as in the water-resource planning application), problem (11) becomes a linear programming problem.

We start by applying the state transition equation (1)(tran) recursively to express a future state x_τ, for $\tau = t+1, \ldots, T$, as a function

- of the current state x_t,
- of future controls $u_{(t,\tau]} = (u_{t+1}, \ldots, u_\tau)$, and
- of random vectors $\xi_{(t,\tau]} = (\xi_{t+1}, \ldots, \xi_\tau)$ from time step $t+1$ to time step τ.

Accordingly, the random variables in the future inequality constraints of (11) become

$$X(\xi_{(t,\tau]}) := E_\tau x_\tau(x_t, u_{(t,\tau]}, \xi_{(t,\tau]}) + (\mathring{F}_\tau + \xi_\tau^\top F_{\tau,1}^\top)u_\tau - G_\tau \xi_\tau \tag{12}$$

for $\tau = t+1, \ldots, T$. The probability and CVaR in (11) are computed with respect to the distribution of random vectors $\xi_{t+1}, \ldots, \xi_\tau$ given $\tilde{\xi}_{[t]}$, i.e., with respect to the distribution of random vectors $\xi_{t+1}|\tilde{\xi}_{[t]}, \ldots, \xi_\tau|\tilde{\xi}_{[t]}$. Keeping in mind the relation (9), the argument in both the probability and CVaR depends on the random vectors $\xi_{[t]}$ and $(\eta_{t+1}, \ldots, \eta_\tau)$. Since the history $\tilde{\xi}_{[t]}$ is known,

both the conditional probability and the CVaR in (11) are finally computed with respect to the distribution of $(\eta_{t+1}, \dots, \eta_\tau)$.

We start with some technical relations that will be useful in the sequel.

Lemma 4.1 (Affine dependence on decision variables and random process)**.** *Consider the scalar random variable $X = X(\xi_{(t,\tau]})$ defined in* (12) *using the transition equation* (1)(TRAN)*. For all $\tau = t+1, \dots, T$ and each time step $t = 1, \dots, T-1$, X is an affine function of $\xi_{(t,\tau]}$:*

$$X = \sum_{j=t+1}^{\tau} \nu_{j,\tau} \xi_j + \nu_{t,\tau} \,,$$

where the coefficients have the expression

$$\nu_{j,\tau} = \begin{cases} E_\tau(\prod_{k=t}^{\tau-1} A_k)x_t + \sum_{i=t+1}^{\tau} E_\tau(\prod_{k=i}^{\tau-1} A_k)(B_i u_i + d_i) + \mathring{F}_\tau u_\tau & \in \mathbb{R} \qquad \text{for } j = t \\ E_\tau(\prod_{k=j}^{\tau-1} A_k) C_j & \in \mathbb{R}^{1,M} \quad \text{for } j = t+1, \dots, \tau-1 \\ E_\tau C_\tau + u_\tau^\top F_{\tau,1} - G_\tau & \in \mathbb{R}^{1,M} \quad \text{for } j = \tau \,. \end{cases}$$

Proof. Take fixed $t \in \{1, \dots, T-1\}$ and $\tau \in \{t+1, \dots, T\}$. A recursive application of the transition equation (1)(TRAN) yields for every $\ell \in \{1, \dots, \tau\}$:

$$x_\tau = \prod_{k=\ell-1}^{\tau-1} A_k x_{\ell-1} + \sum_{j=\ell}^{\tau} (\prod_{k=j}^{\tau-1} A_k) \left[B_j u_j + C_j \xi_j + d_j \right].$$

Now for $j = 0, \dots, \tau$, we let $\mathcal{A}_{j,\tau} = \prod_{k=j}^{\tau-1} A_k$ so that $\mathcal{A}_{\tau,\tau} = I$, the identity matrix. With this notation, the relation

$$x_\tau = \mathcal{A}_{\ell-1,\tau} x_{\ell-1} + \sum_{j=\ell}^{\tau} \mathcal{B}_{j,\tau} u_j + \sum_{j=\ell}^{\tau} \mathcal{C}_{j,\tau} \xi_j + \mathcal{d}_{\ell,\tau} \tag{13}$$

holds with

$$\mathcal{B}_{j,\tau} = \mathcal{A}_{j,\tau} B_j \,, \quad \mathcal{C}_{j,\tau} = \mathcal{A}_{j,\tau} C_j \,, \quad \text{and} \quad \mathcal{d}_{\ell,\tau} = \sum_{j=\ell}^{\tau} \mathcal{A}_{j,\tau} d_j \,.$$

Since $\xi_\tau^\top F_{\tau,1}^\top u_\tau = u_\tau^\top F_{\tau,1} \xi_\tau$, in (12) we have that

$$X = E_\tau x_\tau + \mathring{F}_\tau u_\tau + (u_\tau^\top F_{\tau,1} - G_\tau)\xi_\tau \,.$$

When using this identity and relation (13) for x_τ, written with $\ell = t+1$, we obtain that

$$\begin{aligned} X \;=\; & E_\tau \Big(\mathcal{A}_{t,\tau} x_t + \sum_{j=t+1}^{\tau} \mathcal{B}_{j,\tau} u_j + \mathcal{d}_{t+1,\tau} \Big) + \mathring{F}_\tau u_\tau \\ & + \sum_{j=t+1}^{\tau} E_\tau \mathcal{C}_{j,\tau} \xi_j + (u_\tau^\top F_{\tau,1} - G_\tau)\xi_\tau, \end{aligned}$$

where we gathered together all terms depending on $\xi_{(t,\tau]}$. The desired results follow, by associating $\nu_{t,\tau}$ with the first four terms on the right-hand side above, each $\nu_{j,\tau}$, $j = t+1, \dots, \tau-1$, equal to the corresponding term in the fifth summation, and $\nu_{\tau,\tau}$ as the remaining portion of the expression. □

From Lemma 4.1 we see that the dependence of coefficients ν on variables $(x_t, u_{(t,\tau]})$ is the following:

- $\nu_{t,\tau}$ is an affine scalar function of variables $(x_t, u_{(t,\tau]})$,
- $\nu_{j,\tau}$ is a constant M-dimensional row vector for all $j = t+1, \ldots, \tau-1$, and
- $\nu_{\tau,\tau}$ is an M-dimensional row vector, affine in u_τ if and only if $F_{\tau,1} \neq 0$, constant otherwise.

Since the relation in Lemma 4.1 is affine, the variable X from (12) is also affinely dependent on the decision variables in (11): $X = X(x_t, u_{(t,\tau]}, \xi_{(t,\tau]})$. In order to write down the chance and CVaR constraints in (11), we need to explicitly compute, for fixed $(x_t, u_{(t,\tau]})$, the mean and standard deviation of X, conditioned to $\tilde{\xi}_{[t]}$, i.e., knowing the history of realizations until time step t.

Lemma 4.2 (Conditional mean and standard deviation)**.** *Consider the scalar random variable X defined in* (12) *using the transition equation* (1)(TRAN)*, for a discrete time process ξ_t modelled by* (6)*. Consider the coefficients defined in Lemma 4.1, and let $\nu_{j,\tau}(m)$, $m = 1, \ldots, M$, denote the components of vectors $\nu_{j,\tau}$, for $j = t+1, \ldots, \tau$.*

Then for all $\tau = t+1, \ldots, T$, and $t = 1, \ldots, T-1$, the random variable $X|\tilde{\xi}_{[t]}$ is Gaussian with conditional mean

$$\mathbb{E}[X|\tilde{\xi}_{[t]}] = \nu_{t,\tau} + \sum_{j=1}^{\tau-t} \sum_{m=1}^{M} \nu_{t+j,\tau}(m) \mathbb{E}[\xi_{t+j}(m)|\tilde{\xi}_{[t]}]$$

where

$$\mathbb{E}[\xi_{t+j}(m)|\tilde{\xi}_{[t]}] = \sum_{\ell=0}^{p_{t,j}^{\max}(m)} \alpha_{t,j}^{\ell}(m) \tilde{\xi}_{t-\ell}(m) + \sum_{\ell=1}^{j} \beta_{t+j}^{\ell}(m)\, \mu_{t+j-\ell+1}(m)\,.$$

As for the conditional standard deviation, it is given by

$$\sigma(X|\tilde{\xi}_{[t]}) = \sigma(\sum_{j=1}^{\tau-t} \sum_{m=1}^{M} \nu_{t+j,\tau}(m)\xi_{t+j}(m)|\tilde{\xi}_{[t]}) = \sqrt{\sum_{\ell=1}^{\tau-t} \gamma_{t,\tau,\ell}^{\top} \Gamma_{t+\ell} \gamma_{t,\tau,\ell}}\,, \tag{14}$$

where $\Gamma_{t+\ell} = Cov(\eta_{t+\ell})$ and the M-dimensional vectors $\gamma_{t,\tau,\ell}$ have components

$$\gamma_{t,\tau,\ell}(m) := \sum_{j=\ell}^{\tau-t} \nu_{t+j,\tau}(m) \beta_{t+j}^{j-\ell+1}(m) \tag{15}$$

that are affine functions of u_τ if and only if $F_{\tau,1} \neq 0$, and are constant otherwise.
As a result, the conditional mean is an affine function of variables $(x_t, u_{(t,\tau]})$, while the conditional standard deviation is constant if and only if $F_{\tau,1} = 0$, otherwise it is a conic quadratic function of u_τ of the form $\|\mathcal{L}_{t,\tau} u_\tau + \ell_{t,\tau}\|_2$ for some matrix $\mathcal{L}_{t,\tau}$ and vector $\ell_{t,\tau}$ of appropriate dimensions.

Proof. Combining relation (9) with the expression for X in Lemma 4.1, we see that

$$X = \nu_{t,\tau} + \sum_{j=1}^{\tau-t} \sum_{m=1}^{M} \nu_{t+j,\tau}(m) \left(\sum_{\ell=0}^{p_{t,j}^{\max}(m)} \alpha_{t,j}^{\ell}(m) \xi_{t-\ell}(m) + \sum_{\ell=1}^{j} \beta_{t+j}^{\ell}(m) \eta_{t+j-\ell+1}(m) \right). \tag{16}$$

The desired result for $\mathbb{E}[X|\tilde{\xi}_{[t]}]$ follows, by taking the conditional expectation in both members above. Such expression depends affinely on $\nu_{t,\tau}, \ldots, \nu_{\tau,\tau}$ and, in turn, these coefficients depend affinely on $(x_t, u_{(t,\tau]})$, by Lemma 4.1.

Each component of $\gamma_{t,\tau,\ell}$ from (15) depends affinely on coefficients $\nu_{t+j,\tau}(m)$, and satisfies the relation

$$\sum_{j=1}^{\tau-t}\sum_{m=1}^{M}\nu_{t+j,\tau}(m)\sum_{\ell=1}^{j}\beta_{t+j}^{\ell}(m)\eta_{t+j-\ell+1}(m)=\sum_{\ell=1}^{\tau-t}\eta_{t+\ell}^{\top}\gamma_{t,\tau,\ell}; \tag{17}$$

with independent random vectors $\eta_{t+1},\ldots,\eta_\tau$. The identity (14) follows, by plugging (17) into (16). Note in particular, that the expression for the standard deviation is a conic quadratic function of $\nu_{t+1,\tau},\ldots,\nu_{\tau,\tau}$. By Lemma 4.1, coefficients $\nu_{t+1,\tau}(m),\ldots,\nu_{\tau-1,\tau}(m)$ are always constant, while $\nu_{\tau,\tau}(m)$ is either constant, or affinely depending on u_τ if and only if $F_{\tau,1}\neq 0$, as stated.

Finally, because the expression obtained for $X|\tilde{\xi}_{[t]}$ when using (17) in (16) is an affine combination of Gaussian random variables, the random variable $X|\tilde{\xi}_{[t]}$ is Gaussian too. □

Before showing that (11) is a tractable linear or conic quadratic program, we give an explicit equivalence between VaR and CVaR for Gaussian random variables; see [16, Ex. 6.3].

Lemma 4.3. *Let $X\sim\mathcal{N}(\mu,\sigma^2)$ be a Gaussian random variable. Then for any $\varepsilon_{\mathtt{p}}\in[0,1]$ we have $CVaR_{\varepsilon_{\mathtt{p}}}(X)=VaR_{\varphi(\varepsilon_{\mathtt{p}})}(X)$ where the (bijective) function $\varphi:[0,1]\to[0,\frac{1}{2}]$ is given by $\varphi(x)=1-F\left(\frac{\exp(-(F^{-1}(1-x))^2/2)}{\sqrt{2\pi}x}\right)$ for $x\in(0,1]$ and $\varphi(0)=0$.*

Proof. The relation is trivial if $\varepsilon_{\mathtt{p}}=1$ or $\varepsilon_{\mathtt{p}}=0$. When $\varepsilon_{\mathtt{p}}\in(0,1)$, first note that $VaR_{\varepsilon_{\mathtt{p}}}(X)=-F_X^{-1}(\varepsilon_{\mathtt{p}})=-\mu-F^{-1}(\varepsilon_{\mathtt{p}})\sigma=-\mu+F^{-1}(1-\varepsilon_{\mathtt{p}})\sigma$. Finally, the algebraic manipulations below

$$\begin{aligned}
CVaR_{\varepsilon_{\mathtt{p}}}(X) &= -\mathbb{E}[X|X\leq F_X^{-1}(\varepsilon_{\mathtt{p}})]=-\frac{1}{\varepsilon_{\mathtt{p}}}\int_{-\infty}^{\mu+F^{-1}(\varepsilon_{\mathtt{p}})\sigma}xf_X(x)dx\\
&= -\frac{1}{\varepsilon_{\mathtt{p}}\sqrt{2\pi}\sigma}\int_{-\infty}^{\mu+F^{-1}(\varepsilon_{\mathtt{p}})\sigma}x\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx\\
&= -\frac{\mu}{\varepsilon_{\mathtt{p}}}\int_{-\infty}^{F_X^{-1}(\varepsilon_{\mathtt{p}})}f_X(x)dx-\frac{\sigma}{\varepsilon_{\mathtt{p}}\sqrt{2\pi}}\int_{-\infty}^{\mu+F^{-1}(\varepsilon_{\mathtt{p}})\sigma}\left(\frac{x-\mu}{\sigma^2}\right)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx\\
&= -\mu+\frac{\sigma}{\varepsilon_{\mathtt{p}}\sqrt{2\pi}}\exp\left(-(F^{-1}(\varepsilon_{\mathtt{p}}))^2/2\right),
\end{aligned}$$

give the desired result, recalling that $F^{-1}(\varepsilon_{\mathtt{p}})=-F^{-1}(1-\varepsilon_{\mathtt{p}})$. □

The statement that a chance constraint for an affine relation of a Gaussian random variable is tractable and reduces to a conic quadratic constraint, is already known; see [10], [11], [12]. We now extend this result to a more general setting, and, more importantly, give a **completely explicit** formulation for problem (11), by means of (9), when each component $\xi_t(m),\ m=1,\ldots,M$, is a generalized autoregressive process with time varying order and possibly correlated noise components.

Theorem 4.4 (Tractability of problem (11))**.** *Let the scalar random variable X be defined in* (12) *using the transition equation* (1)(TRAN)*, for a discrete time process ξ_t modelled by* (6) *and let $\varphi(\cdot)$ be the bijection defined in Lemma 4.3. Then each pair of future constraints in* (11)

$$\begin{cases}\mathbb{P}\left(X\geq h_\tau\middle|\tilde{\xi}_{[t]}\right) &\geq 1-\varepsilon_{\mathtt{p}},\\ -CVaR_{\varepsilon_{\mathtt{p}}}\left(X\middle|\tilde{\xi}_{[t]}\right) &\geq h_\tau-\varepsilon_{\mathtt{c}}(|h_\tau|+1),\end{cases} \tag{18}$$

is equivalent to any of the three representations below:

Robust formulation
$$\begin{cases} \mathbb{E}[X|\tilde{\xi}_{[t]}] - F^{-1}(1-\varepsilon_{\mathtt{p}})\sigma(X|\tilde{\xi}_{[t]}) & \geq & h_\tau, \\ \mathbb{E}[X|\tilde{\xi}_{[t]}] - F^{-1}(1-\varphi(\varepsilon_{\mathtt{p}}))\sigma(X|\tilde{\xi}_{[t]}) & \geq & h_\tau - \varepsilon_{\mathtt{c}}(|h_\tau|+1), \end{cases}$$

where the explicit expressions for the conditional mean and standard deviation are given in Lemma 4.2;

VaR formulation
$$\begin{cases} -VaR_{\varepsilon_{\mathtt{p}}}(X|\tilde{\xi}_{[t]}) & \geq & h_\tau, \\ -VaR_{\varphi(\varepsilon_{\mathtt{p}})}(X|\tilde{\xi}_{[t]}) & \geq & h_\tau - \varepsilon_{\mathtt{c}}(|h_\tau|+1)\,; \end{cases}$$

and

CVaR formulation
$$\begin{cases} -CVaR_{\varphi^{-1}(\varepsilon_{\mathtt{p}})}(X|\tilde{\xi}_{[t]}) & \geq & h_\tau, \\ -CVaR_{\varepsilon_{\mathtt{p}}}(X|\tilde{\xi}_{[t]}) & \geq & h_\tau - \varepsilon_{\mathtt{c}}(|h_\tau|+1)\,. \end{cases}$$

Therefore, if $\varepsilon_{\mathtt{p}} \in (0, \frac{1}{2})$, future constraints in (11) *are conic quadratic and* (11) *is a conic quadratic program. Since future constraints* (18) *are affine in variables $(x_t, u_{(t,\tau]})$ if and only if $F_{\tau,1} = 0$; when $F_{\tau,1} = 0$ for every τ then* (11) *is a linear program for every $\varepsilon_{\mathtt{p}} \in (0,1)$.*

Proof. For a Gaussian random variable Z with expectation $\mathbb{E}[Z]$ and standard deviation $\sigma(Z)$, from the proof of Lemma 4.3 we obtain

$$\mathbb{P}(Z \geq \underline{\mathtt{Z}}) \geq 1-\varepsilon_{\mathtt{p}} \Leftrightarrow \underline{\mathtt{Z}} \leq \mathbb{E}[Z] - F^{-1}(1-\varepsilon_{\mathtt{p}})\sigma(Z) \Leftrightarrow \underline{\mathtt{Z}} \leq -VaR_{\varepsilon_{\mathtt{p}}}(Z).$$

Since the random variable $Z = X|\tilde{\xi}_{[t]}$ is Gaussian, the VaR and CVaR formulations follow from Lemma 4.3 and from the equivalences above with $\underline{\mathtt{Z}} = h_\tau$. In addition, the left-hand side equivalence yields the following expression:

$$\begin{array}{lcl} \mathbb{E}[X|\tilde{\xi}_{[t]}] - F^{-1}(1-\varepsilon_{\mathtt{p}})\sigma(X|\tilde{\xi}_{[t]}) & \geq & h_\tau, \\ \mathbb{E}[X|\tilde{\xi}_{[t]}] - F^{-1}(1-\varphi(\varepsilon_{\mathtt{p}}))\sigma(X|\tilde{\xi}_{[t]}) & \geq & h_\tau - \varepsilon_{\mathtt{c}}(|h_\tau|+1). \end{array}$$

When $\varepsilon_{\mathtt{p}} \in (0, \frac{1}{2})$ we have $\varphi(\varepsilon_{\mathtt{p}}) \in (0, \frac{1}{2})$, then $F^{-1}(1-\varepsilon_{\mathtt{p}})$ and $F^{-1}(1-\varphi(\varepsilon_{\mathtt{p}}))$ are both positive. Together with the last assertion in Lemma 4.2, it follows that problem (11) is a conic quadratic program when $\varepsilon_{\mathtt{p}} \in (0, \frac{1}{2})$ or a linear program if $F_{\tau,1} = 0$ for every τ. □

If in (11) some element of A_t, C_t, E_t, or G_t is no longer deterministic, but uncertain and, like F_t, depending affinely on ξ_t, then both Lemma 4.2 and Theorem 4.4 fail to hold because the product of two Gaussian random variables is not a Gaussian random variable.

In Theorem 4.4, the fact that the random variable X is Gaussian leads to the equivalent VaR and CVaR formulations, by Lemma 4.3. Explicit expressions for the chance and CVaR constraints would still be available for any distribution of η_t, as long as the distribution of any linear combination of noises $\eta_t, t = 1, \ldots, T$, is known. For some processes, log-normal distributions are preferred to normal ones. Such is the case of our initial motivation, for which streamflows must remain non-negative for all realizations. In this case, one needs to consider approximations of the generalized inverse of the cumulative distribution function of the (no longer normal) random variable $X|\tilde{\xi}_{[t]}$; see [45] and [2]. By suitably adapting Theorem 4.4, we would once more obtain a deterministic (linear or conic quadratic) program, that *approximates* problem (11).

5. Defining robust recourse functions

Robust optimization usually refers to worst-case oriented methodologies. Notwithstanding, our chance and CVaR constraints can also be considered as a robust way of dealing with uncertainty

(in the sense of [4]), because noises η_t are Gaussian. More precisely, the probabilistic constraint is nothing but its robust version, taking as uncertainty set for η_t the ellipsoid

$$\{x_t \,:\, (x_t-\mu_t)^\top \Gamma_t^{-1}(x_t-\mu_t) \leq (F^{-1}(1-\varepsilon_{\mathtt{p}}))^2\}\,. \tag{19}$$

Similarly, the CVaR constraint in (11) coincides with the robustified version of the constraint $X|\tilde{\xi}_{[t]} \geq h_\tau - \varepsilon_{\mathtt{c}}(|h_\tau|+1)$ for $\tau = t+1,\ldots,T$, using for η_ℓ ($\ell = t+1,\ldots,\tau$) the uncertainty ellipsoidal set

$$\{x_\ell \,:\, (x_\ell-\mu_\ell)^\top \Gamma_\ell^{-1}(x_\ell-\mu_\ell) \leq (F^{-1}(1-\varphi(\varepsilon_{\mathtt{p}})))^2\}\,. \tag{20}$$

These relations justify the naming "robust" for the recourse function $\mathcal{Q}_{t+1}(x_t,\tilde{\xi}_{[t]})$ given by

$$\left\{\begin{array}{l} \min\limits_{u_{[t+1,T]}} \sum\limits_{\tau=t+1}^{T} c_\tau^\top u_\tau \\ \text{s.t., for } \tau = t+1,\ldots,T: \\ \mathbb{P}\left(E_\tau x_\tau(x_t,u_{(t:\tau]},\xi_{(t:\tau]}) + (\mathring{F}_\tau + \xi_\tau^\top F_{\tau,1}^\top)u_\tau - G_\tau\xi_\tau \geq h_\tau \Big| \tilde{\xi}_{[t]}\right) \geq 1-\varepsilon_{\mathtt{p}}, \\ -CVaR_{\varepsilon_{\mathtt{p}}}\left(E_\tau x_\tau(x_t,u_{(t:\tau]},\xi_{(t:\tau]}) + (\mathring{F}_\tau + \xi_\tau^\top F_{\tau,1}^\top)u_\tau - G_\tau\xi_\tau \Big| \tilde{\xi}_{[t]}\right) \geq h_\tau - \varepsilon_{\mathtt{c}}(|h_\tau|+1). \end{array}\right. \tag{21}$$

With respect to (11), this function represents the cost-to-go function which depends on the choice of confidence levels. As a result, at a same time stage, the dynamic choice of confidence levels explained in Section 3.3 may end up taking *different* confidence levels for different sampled scenarios. In this case, cuts approximating the recourse functions can only be shared among scenarios having the *same* confidence level.

We start with a useful rewriting of problem (11).

5.1. **Stagewise decomposition of** (11)**.** Our goal is to write dynamic programming equations for (11) by decomposing its feasible set by stages. Recall our motivating application and consider in (11) a feasible set of the form

$$\mathcal{S} := \left\{\begin{array}{ll} (x_t,(g_\tau,df_\tau)_{\tau=t}^T): & \\ x_t = x_{t-1}(\tilde{\xi}_{[t-1]}) - g_t + \gamma_t\tilde{\xi}_t,\ g_t\geq 0,\ df_t\geq 0 & \\ g_t + df_t \geq \mathrm{dem}_t - (1-\gamma_t)\tilde{\xi}_t & \\ x_t \geq x_t^{crit} & \\ \text{and, for } \tau = t+1,\ldots,T: & \\ g_\tau\geq 0,\ df_\tau\geq 0 & \\ -CVaR_{\varepsilon_{\mathtt{p}}}\left(x_\tau(x_t,g_{(t:\tau]},\xi_{(t:\tau]})|\tilde{\xi}_{[t]}\right) \geq x_\tau^{crit}(1-\varepsilon_{\mathtt{c}}) - \varepsilon_{\mathtt{c}} & (a) \\ \mathbb{P}\left(g_\tau + df_\tau \geq \mathrm{dem}_\tau - (1-\gamma_\tau)\xi_\tau|\tilde{\xi}_{[t]}\right) \geq 1-\varepsilon_{\mathtt{p}} & (b) \end{array}\right\},$$

with only one type of risk-averse formulation for each future inequality constraint, and noting that the set depends on $x_{t-1}(\tilde{\xi}_{[t-1]})$ and on the realization history: $\mathcal{S} = \mathcal{S}(x_{t-1}(\tilde{\xi}_{[t-1]}),\tilde{\xi}_{[t]})$.

The first important observation is that by taking the conditional expectation in the (stochastic) flow balance transition equation for the reservoir, we obtain a (deterministic) transition equation for the mean reservoir volumes:

$$\bar{x}_\tau := \mathbb{E}[x_\tau|\tilde{\xi}_{[t]}] \quad \text{satisfies} \quad \bar{x}_\tau = \left[\begin{array}{ll} x_{t-1}(\tilde{\xi}_{[t-1]}) - g_t + \gamma_t\tilde{\xi}_t & \tau = t \\ \bar{x}_{\tau-1} - g_\tau + \gamma_\tau\mathbb{E}[\xi_\tau|\tilde{\xi}_{[t]}] & \tau = t+1,\ldots,T\,. \end{array}\right.$$

The expressions above use the fact that decision variables g_τ are of the here-and-now type for (11), i.e., they are deterministic.

With these new variables $\bar{x}_\tau$, the robust formulation in Theorem 4.4 for the critical volume (affine) constraints (a) is written down as follows:

$$\bar{x}_\tau \geq x_\tau^{R\ crit} := x_\tau^{crit}(1-\varepsilon_{\mathtt{c}}) - \varepsilon_{\mathtt{c}} + F^{-1}(1-\varphi(\varepsilon_{\mathtt{p}}))\sigma(x_\tau|\tilde{\xi}_{[t]})\,.$$

The explicit expression for the conditional standard deviation above is obtained from (12), written by taking all the matrices therein null, except for E_τ, set to the scalar 1. The corresponding affine parameters in Lemma 4.1 are $\nu_{j,\tau} = \gamma_j$ for $j = t+1,\ldots,\tau$, so that

$$x_\tau = x_t - \sum_{j=t+1}^{\tau} g_j + \sum_{j=t+1}^{\tau} \gamma_j \xi_j \text{ and, hence, } \sigma(x_\tau|\tilde{\xi}_{[t]}) = \sigma(\sum_{j=t+1}^{\tau} \gamma_j \xi_j|\tilde{\xi}_{[t]})\,,$$

which can be computed explicitly by using (9).

A similar reasoning, applying Theorem 4.4 to affine constraints (b), gives the following reformulation for future demand constraints:

$$g_\tau + df_\tau \geq \mathrm{dem}_\tau^R - (1-\gamma_\tau)\mathbb{E}[\xi_\tau|\tilde{\xi}_{[t]}] \text{ where } \mathrm{dem}_\tau^R := \mathrm{dem}_\tau + (1-\gamma_\tau)F^{-1}(1-\varepsilon_{\mathtt{p}})\sigma(\xi_\tau|\tilde{\xi}_{[t]})\,.$$

In order to obtain a stagewise decomposition of the feasible set in (11), we use the subset

$$\mathcal{S}_t^t(\bar{x}_{t-1}) := \left\{ (\bar{x}_t, g_t, df_t): \begin{array}{l} \bar{x}_t = \bar{x}_{t-1} - g_t + \gamma_t\tilde{\xi}_t,\ g_t \geq 0,\ df_t \geq 0 \\ g_t + df_t \geq \mathrm{dem}_t - (1-\gamma_t)\tilde{\xi}_t \\ \bar{x}_t \geq x_t^{crit} \end{array} \right\}$$

and, for $\tau = t+1,\ldots,T$, the subsets $\mathcal{S}_\tau^t(\bar{x}_{\tau-1})$, given by

$$\left\{ (\bar{x}_\tau, g_\tau, df_\tau): \begin{array}{l} \bar{x}_\tau = \bar{x}_{\tau-1} - g_\tau + \gamma_\tau\mathbb{E}[\xi_\tau|\tilde{\xi}_{[t]}],\ g_\tau \geq 0,\ df_\tau \geq 0 \\ g_\tau + df_\tau \geq \mathrm{dem}_\tau^R - (1-\gamma_\tau)\mathbb{E}[\xi_\tau|\tilde{\xi}_{[t]}] \\ \bar{x}_\tau \geq x_\tau^{R\ crit} \end{array} \right\}.$$

For our motivating application, the objective function is also separable by stages, making problem (11) solvable by dynamic programming, introducing the functions

$$Q_\tau^t(\bar{x}_{\tau-1}, \tilde{\xi}_{[t]}) = \min\left\{ c_\tau^\top u_\tau + Q_{\tau+1}^t(\bar{x}_\tau, \tilde{\xi}_{[t]}) : (\bar{x}_\tau, u_\tau) \in \mathcal{S}_\tau^t(\bar{x}_{\tau-1}) \right\}, \tag{22}$$

where we used the short notation $u_\tau := (g_\tau, df_\tau)$ for the controls and $\bar{x}_{t-1} := x_{t-1}(\tilde{\xi}_{[t-1]})$ for the input state. The functions above are defined recursively for $\tau = T, T-1,\ldots,t$, starting from $Q_{T+1}^t(\cdot) \equiv 0$. The super index t in the notation reflects the fact that functions Q_τ^t correspond to the stagewise decomposition of the t^{th} risk-averse optimization problem.

As a result, given a history $\tilde{\xi}_{[t]}^i$, risk-averse problem (11) has the stagewise decomposition:

$$\left\{ \begin{array}{l} \min\limits_{\bar{x}_{[t:T]}, u_{[t:T]}} \sum\limits_{\tau=t}^{T} c_\tau^\top u_\tau \\ \bar{x}_\tau = A_{\tau-1}\bar{x}_{\tau-1} + B_\tau u_\tau + C_\tau\mathbb{E}[\xi_\tau|\tilde{\xi}_{[t]}^i] + d_\tau \\ E_\tau\bar{x}_\tau + \mathring{F}_\tau^\top u_\tau \geq G_\tau\mathbb{E}[\xi_\tau|\tilde{\xi}_{[t]}^i] + h_\tau^t \\ \text{for } \tau = t, t+1,\ldots,T, \end{array} \right. \tag{23}$$

where

$$h_\tau^t := h_\tau + \max\left(F^{-1}(1-\varepsilon_{\mathtt{p}})\sigma(X|\tilde{\xi}^i_{[t]}), -\varepsilon_{\mathtt{c}}(|h_\tau|+1) + F^{-1}(1-\varphi(\varepsilon_{\mathtt{p}}))\sigma(X|\tilde{\xi}^i_{[t]})\right). \tag{24}$$

The representation (23) of the risk-averse problems shows that (11) can be derived from a problem instance (1) by replacing:

- future realizations $\tilde{\xi}_\tau$ by their conditional means $\mathbb{E}[\xi_\tau|\tilde{\xi}_{[t]}]$ and
- right-hand sides h_τ by some "robust" counterparts h_τ^t strictly greater than h_τ.

In the particular case of the hydro-thermal problem from Section 2, this means that each risk averse problem is obtained by replacing in a problem instance (1):

- the inflows by their means conditioned to the history $\tilde{\xi}_{[t]}$;
- the nominal demands dem_τ by "robust" demands dem_τ^R, which are strictly greater than the nominal demands; and
- the critical levels x_τ^{crit} by "robust" critical levels $x_\tau^{R\ crit}$ which, for $\varepsilon_{\mathtt{c}}$ sufficiently small, are strictly greater than the critical levels.

Since we can write (deterministic) dynamic programming equations for our risk-averse problems formulated as in (23), each one of these problems can be solved efficiently by Dual Dynamic Programming (DDP). This decomposition method can be interesting when T is large as in our motivating application. Moreover, as explained in the next section, a DDP approach has the advantage to define cutting planes for our recourse functions.

We mention that it is still possible to write down for (11) Dynamic Programming relations similar to the above even if $F_{\tau,i} \neq 0$ for some $\tau \in \{t+1,\dots,T\}$ and $i \leq q_\tau$.

5.2. **Approximate recourse functions.** Keeping in mind requirement `(RecFun)` in Section 2, we now explain how to approximate the recourse functions (21) for our model. The first step consists in sampling N scenarios $(\tilde{\xi}_2^i,\dots,\tilde{\xi}_T^i)$, $i=1,\dots,N$, for $(\xi_2,\dots,\xi_T)$ (recall that realization $\tilde{\xi}_1$ is available at $t=1$). Then our rolling-horizon policy is implemented along each scenario, as follows.

For scenario i and time step t, we solve risk-averse problem (11) written with $\tilde{\xi}_{[t]}$ replaced by $\tilde{\xi}^i_{[t]}$ by writing (11) in the form (23) and we denote a corresponding optimal value of $\bar{x}_t$ by $\bar{x}_t^i := x_t(\tilde{\xi}^i_{[t]})$. The recourse function $\mathcal{Q}_{t+1}$ from (21) has the expression

$$\mathcal{Q}_{t+1}(\bar{x}_t,\tilde{\xi}_{[t]}) := \left\{\begin{array}{l} \min\limits_{\bar{x}_{[t+1:T]},u_{[t+1:T]}} \sum\limits_{\tau=t+1}^{T} c_\tau^\top u_\tau \\ \bar{x}_\tau = A_{\tau-1}\bar{x}_{\tau-1} + B_\tau u_\tau + C_\tau \mathbb{E}[\xi_\tau|\tilde{\xi}_{[t]}] + d_\tau,\ \tau = t+1,\dots,T, \\ E_\tau \bar{x}_\tau + \mathring{F}_\tau^\top u_\tau \geq G_\tau \mathbb{E}[\xi_\tau|\tilde{\xi}_{[t]}] + h_\tau^t,\ \tau = t+1,\dots,T. \end{array}\right. \tag{25}$$

By Lemmas 4.1 and 4.2, this recourse function is convex. As a result, it is possible to define a cut (an hyperplane lying below the function) for $\mathcal{Q}_{t+1}$ at $(\bar{x}_t^i,\tilde{\xi}^i_{[t]})$. Collecting these cuts, after a simulation phase over a set of k scenarios, we have for $\mathcal{Q}_{t+1}$ the approximation

$$\mathfrak{Q}_{t+1}^k(\bar{x}_t,\tilde{\xi}_{[t]}) = \max_{1\leq i\leq k}\ \bar{x}_t^\top a_t^i + \tilde{\xi}_{[t]}^\top b_t^i + c_t^i$$

for each $t = 1, 2, \ldots, T-1$, with $\mathfrak{Q}^k_{T+1} = \mathcal{Q}_{T+1} \equiv 0$. The cut coefficients (a^k_t, b^k_t, c^k_t) are given in the following proposition. Their expressions depend on the optimal multipliers of the equality and inequality constraints in (23), denoted by $\pi^{t,i}_\tau$ and $\lambda^{t,i}_\tau$, respectively. Note that multipliers $\lambda^{t,i}_\tau$ are scalar because $q_t = 1$. The extension to the case $q_t > 1$ is straightforward.

Proposition 5.1 (Cuts computation for robust recourse functions)**.** *Consider risk averse problem* (23) *for some* $t \in \{1, \ldots, T-1\}$ *and let* $(\pi^{t,i}_t, \pi^{t,i}_{t+1}, \ldots, \pi^{t,i}_T, \lambda^{t,i}_t, \lambda^{t,i}_{t+1}, \ldots, \lambda^{t,i}_T)$ *be a dual solution. Let* $\mathcal{Q}_{t+1}$ *be the corresponding recourse function defined in* (25)*. Then valid cuts for* $\mathcal{Q}_{t+1}$ *are given by the expression* $\bar{x}_t^\top a^i_t + \tilde{\xi}^\top_{[t]} b^i_t + c^i_t$ *for*

$$
\begin{aligned}
&a^i_t = A_t^\top \pi^{t,i}_{t+1},\\
&c^i_t = \sum_{\tau=t+1}^{T} \left[d_\tau^\top \pi^{t,i}_\tau + \lambda^{t,i}_\tau h^t_\tau + \sum_{m=1}^{M} \sum_{\ell=1}^{\tau-t} \left(\sum_{p=1}^{N_x} \pi^{t,i}_\tau(p) C_\tau(p,m) + \lambda^{t,i}_\tau G_\tau(1,m) \right) \beta^\ell_\tau(m) \mu_{\tau-\ell+1}(m) \right],\\
&b^i_t = [b^{i,1}_t; b^{i,2}_t; \ldots; b^{i,M}_t] \text{ where } b^{i,m}_t \text{ is the } (p^{\max}_{t,T-t}(m)+1)\text{-vector given by}\\
&b^{i,m}_t(\ell+1) = \sum_{\tau \in I_{t,\ell}(m)} \left[\sum_{p=1}^{N_x} \pi^{t,i}_\tau(p) C_\tau(p,m) + \lambda^{t,i}_\tau G_\tau(1,m) \right] \alpha^\ell_{t,\tau-t}(m),\ \ell = 0, 1, \ldots, p^{\max}_{t,T-t}(m),
\end{aligned}
$$

with $I^\ell_t(m) := \left\{ \tau : t+1 \le \tau \le T,\ \ell \le p^{\max}_{t,\tau-t}(m) \right\}$ *and with* h^t_τ *given by* (24)*.*

With respect to the notation in (22)*, the identity* $\mathcal{Q}_{t+1} = Q^t_\tau$ *with* $\tau = t+1$ *holds.*

Proof. The problem dual to (25) gives $\mathcal{Q}_{t+1}(\bar{x}_t, \tilde{\xi}_{[t]})$ as the optimal value of the linear program

$$
\left\{
\begin{array}{l}
\max\limits_{\pi_{[t+1:T]}, \lambda_{[t+1:T]}} \pi^\top_{t+1} A_t \bar{x}_t + \sum\limits_{\tau=t+1}^{T} \left[\pi_\tau^\top (C_\tau \mathbb{E}[\xi_\tau | \tilde{\xi}_{[t]}] + d_\tau) + \lambda_\tau (G_\tau \mathbb{E}[\xi_\tau | \tilde{\xi}_{[t]}] + h^t_\tau) \right]\\
\pi_\tau - A_\tau^\top \pi_{\tau+1} + E_\tau^\top \lambda_\tau = 0,\ \ \tau = t+1, \ldots, T-1,\\
-B_\tau^\top \pi_\tau + \mathring{F}_\tau \lambda_\tau = c_\tau,\ \lambda_\tau \ge 0,\ \tau = t+1, \ldots, T,\\
\pi_T + E_T^\top \lambda_T = 0.
\end{array}
\right. \tag{26}
$$

Since $(\pi^{t,i}_{t+1}, \pi^{t,i}_{t+2}, \ldots, \pi^{t,i}_T, \lambda^{t,i}_{t+1}, \lambda^{t,i}_{t+2}, \ldots, \lambda^{t,i}_T)$ is feasible for (26), the optimal value $\mathcal{Q}_{t+1}(\bar{x}_t, \tilde{\xi}_{[t]})$ is bounded from below by the cut

$$
\bar{x}_t^\top A_t^\top \pi^{t,i}_{t+1} + \sum_{\tau=t+1}^{T} \left[(C_\tau \mathbb{E}[\xi_\tau | \tilde{\xi}_{[t]}] + d_\tau)^\top \pi^{t,i}_\tau + \lambda^{t,i}_\tau (G_\tau \mathbb{E}[\xi_\tau | \tilde{\xi}_{[t]}] + h^t_\tau) \right]. \tag{27}
$$

By Lemmas 4.1 and 4.2, the standard deviation $\sigma(X|\tilde{\xi}_{[t]})$ does not depend on $\tilde{\xi}_{[t]}$ but only on the problem data between time steps $t+1$ and τ. Therefore, recalling definition (24) of h^t_τ, the only terms in the lower bound (27) that are functions of $\tilde{\xi}_{[t]}$ are $\mathbb{E}[\xi_\tau | \tilde{\xi}_{[t]}]$, $\tau = t+1, \ldots, T$. By (9), these terms have the expression $\mathbb{E}[\xi_\tau(m) | \tilde{\xi}_{[t]}] = \sum_{\ell=0}^{p^{\max}_{t,\tau-t}(m)} \alpha^\ell_{t,\tau-t}(m) \tilde{\xi}_{t-\ell}(m) + \sum_{\ell=1}^{\tau-t} \beta^\ell_\tau(m) \mu_{\tau-\ell+1}(m)$, for $m = 1, \ldots, M$, and $\tau = t+1, \ldots, T$. Plugging this expression of $\mathbb{E}[\xi_\tau(m) | \tilde{\xi}_{[t]}]$ into the lower bound (27), we see that the latter can be split into a sum of constant terms, a term depending on $\bar{x}_t$, and terms depending on $\tilde{\xi}_{[t]}$. By identifying these terms with, respectively, c^i_t, $\bar{x}_t^\top a^i_t$, and $\tilde{\xi}^\top_{[t]} b^i_t$,

we obtain the desired expressions for a_t^i and c_t^i. Finally, the relations

$$\begin{aligned}\tilde{\xi}_{[t]}^\top b_t^i &= \sum_{m=1}^{M}\sum_{\tau=t+1}^{T}\sum_{\ell=0}^{p_{t,\tau-t}^{\max}(m)}\left[\sum_{p=1}^{N_x}\pi_\tau^{t,i}(p)C_\tau(p,m)+\lambda_\tau^{t,i}G_\tau(1,m)\right]\alpha_{t,\tau-t}^{\ell}(m)\tilde{\xi}_{t-\ell}(m)\\ &= \sum_{m=1}^{M}\sum_{\ell=0}^{p_{t,T-t}^{\max}(m)}\sum_{\tau\in I_t^\ell(m)}\left[\sum_{p=1}^{N_x}\pi_\tau^{t,i}(p)C_\tau(p,m)+\lambda_\tau^{t,i}G_\tau(1,m)\right]\alpha_{t,\tau-t}^{\ell}(m)\tilde{\xi}_{t-\ell}(m)\end{aligned}$$

give the desired formula for b_t^i (in the last equality, we used the fact that the sequence $(p_{t,k}^{\max}(m))_k$ from (7) is nondecreasing, by definition). □

As a result, after simulating the rolling-horizon policy on a set of N scenarios, we obtain N cuts for each recourse function.

At this point, it is important to underline the differences between the above algorithm and SDDP. In our rolling-horizon setting, since $\mathcal{Q}_t = Q_t^{t-1}$ is the recourse function for the $(t-1)^{th}$ risk-averse problem while $\mathcal{Q}_{t+1} = Q_{t+1}^t$ in (25) is the recourse function for the t^{th} risk-averse problem, there is no dynamic programming relation linking two consecutive recourse functions. By contrast, with SDDP, successive recourse functions are always linked by dynamic programming relations.

For both approaches the i-th forward pass computes a set of extended states $(\bar{x}_t^i, \tilde{\xi}_{[t]}^i)$, $t = 1,\dots,T-1$. With SDDP, the state $\bar{x}_t^i$ is computed using the available approximation of recourse function $\mathcal{Q}_{t+1}$. In our case, since each risk-averse problem is solved exactly, the state $\bar{x}_t^i$ corresponds to using implicitly an exact recourse function $\mathcal{Q}_{t+1}$.

Finally, SDDP needs a computationally heavy backward pass whereas our cuts are immediately available at the forward pass.

As stated in Proposition 5.1, $\mathcal{Q}_{t+1}(x_t, \tilde{\xi}_{[t]})$ from (10) is given by (21) and corresponds to $Q_{t+1}^t(x_t, \tilde{\xi}_{[t]})$, defined by (22) written for $\tau = t+1$.

By virtue of (11), Lemma 4.2, and Theorem 4.4, the functions Q_τ^t are convex polyhedral functions of $(\bar{x}_{\tau-1}, \tilde{\xi}_{[t]})$ for all $\tau = t,\dots,T$. Therefore, an alternative decomposition method can be obtained by computing cuts for Q_{t+1}^t, solving (11) by dual dynamic programming, using a Benders' method capable of handling infeasible states in the forward step; see [7] and also [9]. Along iterations, a set of cuts is built for $\mathcal{Q}_\tau^t$ at some points $(\bar{x}_{\tau-1}^{t,k}, \tilde{\xi}_{[t]}^i)$. In particular, when $\tau = t+1$, cuts are generated for the recourse function $\mathcal{Q}_{t+1} = \mathcal{Q}_{t+1}^t$.

To conclude, we emphasize the fact that, unlike SDDP, the rolling-horizon approach does not *need* to define approximate recourse functions to define a feasible risk-averse policy. However, if desired for some other use, it is *possible* to build such approximations in the rolling-horizon setting, using Proposition 5.1 or by solving the risk-averse problems (11) by nested decomposition, as explained above. For the sake of comparison, in our numerical experience we also implemented a risk-averse policy, denoted by `RA-NRH`, that makes use of these approximate recourse functions in a nonrolling horizon context, see Section 6.5. It is worth noting, however, that for such approximations to be useful, they need to be built for confidence levels fixed *a priori*, and not dynamically changed along scenarios (cf. our comments after (21)).

6. Numerical experience

We assess the rolling-horizon approach on a large-scale instance of the water-planning problem described in Section 2, similar to Brazil's power system, except for the standard deviations of noises η_t; see Section 6.1 below.

6.1. **Power system data.** We consider a hydro-thermal power system operating over an horizon of 4 years, discretized in $T = 48$ time steps, from January 2005 to December 2008. Most of the data was made available by CEPEL[1] and corresponds to part of Brazil's power system, represented by 4 different subsystems that can trade energy in the form of import-export exchanges. Each subsystem, South-East (SE), South (S), North-East (NE), and North (N), corresponds to a geographical region; some energy exchanges between the N, NE, and SE subsystems make use of a fifth, fictitious, node (F). In a specific subsystem, a single reservoir aggregates all the hydro-power, while thermal generation is considered individually: there are 24, 14, 6, and 0 thermal plants in the SE (the largest one), S, NE, and N subsystems, respectively.

With respect to the simplified model given in Section 2, there are additional control variables (spillage and import-export exchanges), as well as additional lower and upper bounds for hydro and thermal generation, for subsystem exchanges, and for reservoir volumes. The water balance equation (3) now considers spillage, while demand constraints (4) consider the subsystem thermal generation as well as import-export exchanges, with a total monthly demand of 54 804 MWMonth[2], taken constant over the horizon. The water balance relations (3) are started by setting the initial reservoir levels $x_0(m)$ at full capacity. Each reservoir critical level $x_t^{crit}(m)$ in (5) was set to 20% of the maximum level of the reservoir, for all time steps, so that assumption (A1) holds (recall that if (A1) did not hold, our methodology would still be applicable, by adding slack variables to the critical minimal volume constraints). With this assumption, relatively complete recourse is guaranteed if $\mathbb{P}(\xi_t(m) \geq 0) = 1$, and we checked numerically that this relation is satisfied beyond machine precision.

The objective function is given by the total thermal operating cost (ranging between R$ 6.27 per MWh and R$ 1047 per MWh) plus the cost of load shedding (the shortage cost is set at R$ 4170.44 per MWh). Unnecessary spillage and exchanges are discouraged by introducing small penalties and trading costs between subsystems.

Following the lines of [30], the inflows in each reservoir are modelled by a periodic autoregressive model of the form (6). The parameters of each model were estimated based on historical data from 1931 to 2005, with one important modification, relative to standard deviations. Namely, we reduced the estimated value of $\sigma_t^{\eta}(m)$ because, with the original estimations, the model generated too many negative water inflows that have no meaningful physical interpretation. We adopted the former approach not only for simplicity, but also for noises to remain Gaussian, so that we can use exact expressions for the inverse of the cumulative distribution functions in Theorem 4.4. As for the conditional expectations and standard deviations in Theorem 4.4, they are computed exactly, using the recursive expressions derived in [24] for the coefficients in (9) and model (6) parameters.

[1]The authors specially acknowledge the good will and availability of Débora Dias Jardim Penna.
[2]We adopt the convention 1 MWMonth$=\frac{365.25\times 24}{12}$ MWh$=$ 730.5 MWh

Due to the modified standard deviations, our results should be interpreted as an illustration of our methodology, rather than reflecting the real behavior of the Brazilian power system.

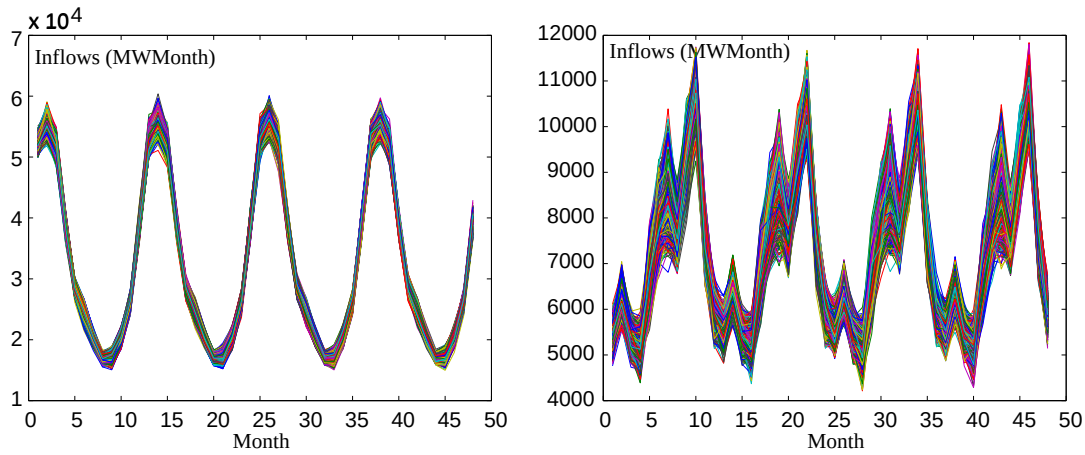

FIGURE 1. 10 000 inflow scenarios for the S and SE subsystems (left and right, respectively).

To obtain the indicators referred to in item `(EcInd)` of Section 2, the simulation phase uses $N = 10\,000$ streamflow scenarios (we displayed them in Figure 1 for the S and SE subsystems).

The implementation was done in Matlab, using Mosek's optimization library to solve linear programming problems (`http://www.matlab.com` and `http://www.mosek.com`).

6.2. **Impact of confidence levels.** To determine the influence of the confidence levels on the behavior of the rolling-horizon policy (denoted `RH` in the sequel), we first consider only CVaR constraints with $\varepsilon_{\mathtt{p}} = 0.2$ and different values for $\varepsilon_{\mathtt{c}}$, chosen a priori and taken constant along the different time steps and reservoirs: $\varepsilon_{\mathtt{c}} \in \{0.01, 0.05, 0.1, 0.2, 0.4, 0.6, 0.8\}$. The results for the corresponding mean and standard deviation of the total cost, as well as the evolution of load shedding (percentage of unsatisfied demand over all time steps and scenarios) are reported in Figure 2.

Taking higher values of the confidence levels naturally yields more load shedding. As a result, increasing the confidence level $\varepsilon_{\mathtt{c}}$ tends to increase the mean total cost. In this experiment, the increase in the confidence level also yielded higher standard deviations.

A similar behavior was observed when the risk-averse problems (11) have only chance constraints.

If fixed *a priori*, confidence levels should be chosen sufficiently small. However, for some applications, too small confidence levels can produce infeasible risk-averse problems (11). Such was not the case in our water management problem.

6.3. `RH` **policy with a dynamic choice of confidence levels.** We tested our robust rolling-horizon policy for a dynamic choice of confidence levels, as explained in Section 3.3. Each risk-averse problem (11) was solved directly as a linear program, without using the stagewise decomposition in Section 5.

We consider three variants, taking in (11) only CVaR constraints, only chance constraints, and both CVaR and chance constraints. For these variants, respectively referred to as `RH CVaR`, `RH CC`,

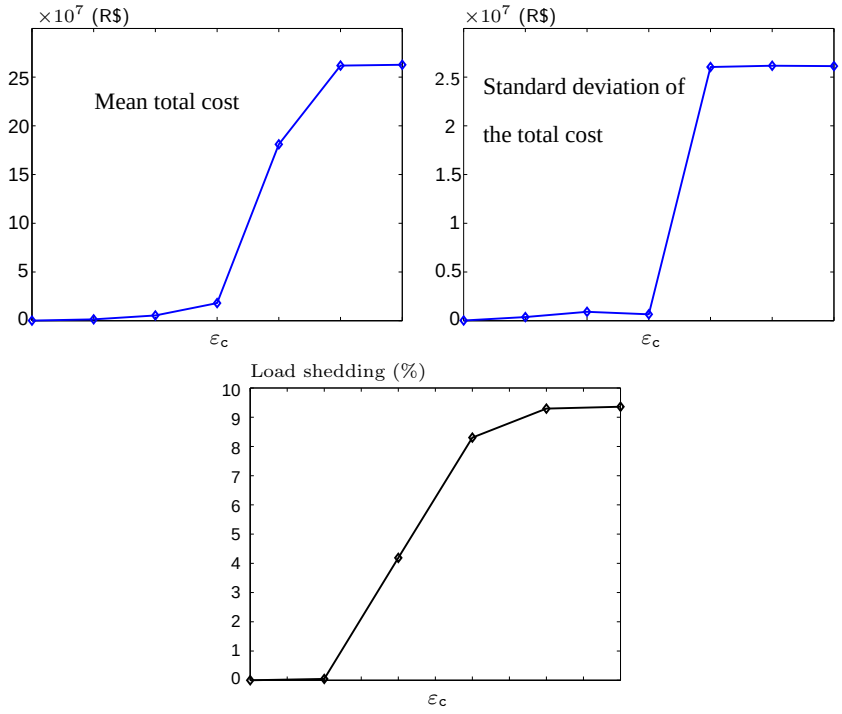

FIGURE 2. `RH` policy with only CVaR constraints: variation of the mean and s.d of the total cost as well as of load shedding as confidence level ε_c increases.

and `RH CC-CVaR`, we report on cost related values in Table 1. This table contains the empirical mean and standard deviation (s.d.) of the whole system cost (generation and exchange cost plus penalties paid for unsatisfied demand) over the 10 000 scenarios, as well as the corresponding VaR p%, for $p = 1, 5$, and 90, where VaR p% is the $(1 - p/100)$-quantile of the empirical distribution of the cost.[3] For variants `RH CVaR` and `RH CC-CVaR`, the parameter ε_c was fixed to two different

TABLE 1. Central and dispersion characteristics of the total cost (R$)

Output	`RH CVaR`	`RH CVaR`	`RH CC`	`RH CC-CVaR`	`RH CC-CVaR`
Parameter ε_c	$\varepsilon_c = 0.005$	$\varepsilon_c = 0.05$	-	$\varepsilon_c = 0.005$	$\varepsilon_c = 0.05$
Mean	5.447×10^7	7.624×10^7	6.942×10^7	7.091×10^7	7.779×10^7
s.d.	3.292×10^7	2.219×10^7	3.216×10^7	3.476×10^7	3.819×10^7
VaR 1%	1.638×10^8	1.399×10^8	1.742×10^8	1.814×10^8	1.975×10^8
VaR 5%	1.189×10^8	1.132×10^8	1.324×10^8	1.387×10^8	1.521×10^8
VaR 90%	2.072×10^7	4.966×10^7	3.649×10^7	3.361×10^7	3.679×10^7

values, with smaller values yielding smaller mean costs as with the a priori setting of confidence levels. At each stage, candidate values for ε_p are $\{0.01, 0.05, 0.1, 0.2, 0.4, 0.6, 0.8\}$.

We still observe lower mean costs for smaller values of ε_c, even if both the mean and the standard deviation of the cost is similar for the different experiments. The two exceptions are the mean cost (resp. s.d) for `RH CVaR` variant with $\varepsilon_c = 0.005$ (resp. $\varepsilon_c = 0.05$), which are significantly smaller.

[3]If X is a continuous random variable for which lower values are preferred, its Value-at-Risk of level ε_p is given by $VaR_{\varepsilon_p}(X) := F_X^{-1}(1 - \varepsilon_p)$ for any $\varepsilon_p \in [0, 1]$.

On this set of scenarios and for our data, the lowest mean cost was obtained with RH CVaR policy, for $\varepsilon_{\mathtt{c}} = 0.005$, while the highest one is given by RH CC-CVaR variant with $\varepsilon_{\mathtt{c}} = 0.05$. For these variants, no load shedding was observed in the simulation phase.

We also made a set of unreported tests to evaluate the sensitivity of RH to variations in the critical levels $x_t^{crit} \in \{0.1, 0.2, 0.5, 0.8\}x_t^{\max}$ and we observed a natural behavior for a hydro-dominated system: costs became prohibitive for the higher values of the critical levels, due to the appearance of load shedding for many time steps.

6.4. **Policies used in the comparison.** For our comparison, we implemented three policies:

- RH, a rolling-horizon policy using a dynamic choice for confidence levels and chance constraints only, noting that the chosen confidence levels were always inferior to 0.1.
- RA-NRH, a non-rolling risk-averse policy that, instead of solving (11) directly, uses the approximate recourse functions defined by the cuts in Proposition 5.1. The optimization phase, defining the approximate recourse functions uses 1 000 scenarios. Only chance constraints were considered in risk-averse problems (11) with confidence level $\varepsilon_{\mathtt{p}} = 0.1$ fixed *a priori*.
- SDDP: a non-rolling horizon risk-neutral policy that approximates the recourse functions by Stochastic Dual Dynamic Programming as in [33], using 200 scenarios for the forward pass and 20 discrete realizations for each noise η_t in the backward pass. Convergence was obtained after 10 iterations after observing a stabilization of the lower bound (increase of the lower bound inferior to 0.5%) with the following lower bounds along iterations: $10^7 \times (3.015, 4.571, 5.082, 5.362, 5.641, 5.809, 5.934, 6.016, 6.077, 6.104)$.

Both RA-NRH and SDDP are closed-loop policies that compute the indicators from (EcInd) in Section 2 in two steps, by first defining approximate recourse functions on a reduced set of scenarios, and then implementing the policy over a large set of scenarios, simulating the system operation. By contrast, being an open-loop policy, RH computes indicators in one shot, solving directly the risk-averse problems over a large number of scenarios and making a dynamic selection of confidence levels. We expect RH to be faster than SDDP and also less volatile. As for RA-NRH, it is also faster and more robust than SDDP but less accurate than RH. These expectations are confirmed by the results reported below.

6.5. **Comparison of different policies.** Table 2 reports the statistical indicators from Table 1 for these three policies. We observe that robust policies yield a higher mean cost than SDDP (11% higher for RH and 42% higher for RA-NRH) but a much smaller standard deviation of the cost. In fact, for SDDP, as can be seen considering the very low (resp. very high) value of quantile VaR 90%

TABLE 2. Central and dispersion characteristics of the total cost (R$)

Output	RH	RA-NRH	SDDP
Mean	6.942×10^7	8.860×10^7	6.251×10^7
s.d.	3.216×10^7	2.126×10^7	1.025×10^8
VaR 1%	1.742×10^8	1.509×10^8	4.204×10^8
VaR 5%	1.324×10^8	1.259×10^8	2.668×10^8
VaR 90%	3.649×10^7	6.399×10^7	4.754×10^5

(resp. VaR 10%), some scenarios are of very low cost (using nearly no thermal power) while others yield prohibitive costs.

More precisely, the differences between the distributions of the cost can be seen in Figure 3, that compares the empirical distribution of the total cost over the 10 000 scenarios for the three policies. This figure shows that in a large number of the simulated scenarios, `RA-NRH` cost is larger

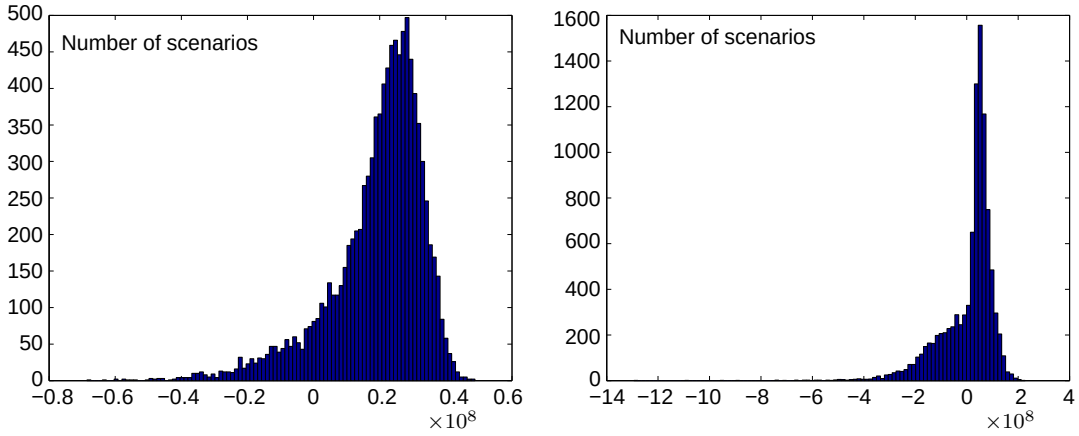

FIGURE 3. Empirical distribution of $C_{\texttt{RA-NRH}} - C_{\texttt{RH}}$ (on the left) and of $C_{\texttt{RH}} - C_{\texttt{SDDP}}$ (on the right) where $C_{\texttt{RA-NRH}}, C_{\texttt{RH}}$, and $C_{\texttt{SDDP}}$ respectively correspond to the total cost with `RA-NRH`, `RH`, and `SDDP` policies.

than `RH` cost (up to more than $R\$4\times10^7$), putting in evidence the superiority in terms of cost of `RH` over `RA-NRH`. Also, on a majority of scenarios, `SDDP` cost is lower than `RH` cost. However, there is quite a number of scenarios for which `SDDP` cost is superior to `RH` cost by more than $R\$10^8$. This is a natural result, because `RH` is risk-averse while `SDDP` uses a risk-neutral formulation. This is also consistent with Table 2, showing larger values for the quantiles VaR 1% and VaR 5% for `SDDP` policy.

It is also important to look at the portion of demand left unsatisfied by each policy. In our runs, only `SDDP` incurred load shedding. Even though the percentage of load shedding is very low for this policy and observed for some scenarios and one time step only, the corresponding deficit yields a significant increase in the mean cost (shortage cost is high: R$ 4170.44 per MWh). Figure 4 reports the average and 0.05-quantile of the *equivalent reservoir level* (sum of all the reservoir storages) for all policies. On the left plot, we observe that `SDDP` mean reservoir levels are well below the `RH` ones, themselves below the mean levels of `RA-NRH`. The right plot shows the same relations for the 0.05-quantiles.

For reservoir levels, the impact of modelling risk-aversion is an increased caution with respect to the use of water, keeping reservoir levels sufficiently above those obtained with a risk-neutral policy (`SDDP`). Over the first years, `RH` and `RA-NRH` are more conservative than `SDDP` because at the beginning of the process ($t \approx 1$), (11) needs to foresee a far away future ($\tau \in [2, 48]$), using robust uncertainty sets (19) and (20) that are overly pessimistic with respect to the final stages ($\tau \approx 48$). By contrast, over the last years, as the number of uncertain parameters in (11) decreases,

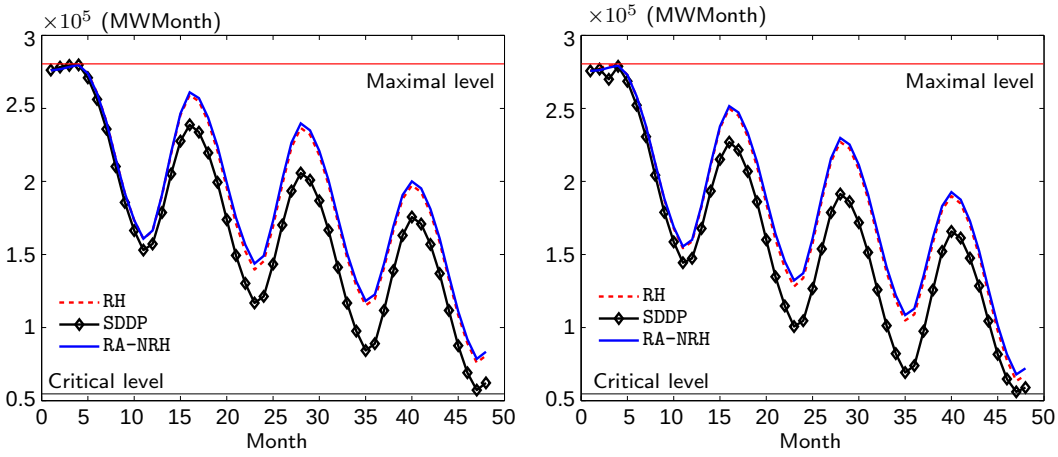

FIGURE 4. Mean and 0.05-quantiles for the equivalent reservoir level with `RH`, `RA-RH`, and `SDDP` policies

uncertainty sets become tighter, `RH` "realizes" that more water than necessary has been stored and it uses more water than `SDDP`.

Another useful indicator to compare the policies is the marginal cost (mean optimal Lagrange multipliers for the demand satisfaction constraint). In accordance with previous observations and with Figure 4, both the mean and s.d of the marginal cost are smaller with `SDDP`, but not significantly. In fact, mean marginal costs are low for all policies (considering the range of thermal unit costs), which is expected from a hydro-dominated system like the Brazilian one.

An additional comparison of both policies is given in Table 3, showing how severe is `SDDP` computational load when compared to `RH`.

TABLE 3. Computational load of the implemented policies for the test-case

`SDDP` optimization phase (one iteration)	Number of LPs	Each LP has 56 variables and box constraints, 8 equality/inequality constraints, and for $t < T$, a number of cuts depending on iteration k
Forward	9 600	200(k-1)+1
Backward	192 000	200k+1

Policy	Computational Effort
`SDDP` optimization phase (10 iterations)	96 000 Forward LPs 1.92×10^6 Backward LPs
`SDDP` simulation phase	480 000 LPs with 2001 cuts
`SDDP` total CPU time	One week
`RH`, for each $t = 1, \ldots, 48$	10 000 LPs with $56(T-t+1)$ variables and box constraints and $8(T-t+1)$ equality/inequality constraints
`RH` total CPU time	One day

It is worth mentioning that `SDDP` indicators could be improved by setting up an `SDDP` rolling-horizon methodology, but, in view of the figures in Table 3, it is clear that this alternative is computationally impossible.

Note also that implementing SDDP in a risk-averse formulation, for example along the lines of [23], [35], or [43], needs more variables and constraints, and would increase the already heavy computational effort. Moreover, even if distributing calculations in many different processors could make the computational load acceptable, approaches like [43] would still require to find at each stage suitable weights defining a trade-off between the mean and the CVaR of the future cost. At first sight, choosing sound values for these weights for a specific problem is not a simpler task than finding adequate confidence levels $\varepsilon_{\mathtt{p}}$ and $\varepsilon_{\mathtt{c}}$ for the rolling-horizon methodology.

Finally, the way of defining a sound statistical stopping test for risk-averse SDDP variants is a subject of current research, that has not been answered yet. Both [43] and [35] stop after having done as many iterations as required for the *risk-neutral* variant to stop with a statistical test (that is indeed well defined if there is no risk aversion).

Final considerations

The novelty of this paper is in the specific integration of topics from risk modelling, dynamic programming and multistage stochastic optimization to arrive at mathematically sound solution methods working efficiently for large-scale models. Another distinctive feature of this work is that we handle explicitly interstage dependent stochastic processes, giving recursive expressions that use a minimal amount of past information. Results of this type are heavy to write down, but are crucial for implementations. Like most of the multistage stochastic programs related papers, neither [43] nor [35] deal explicitly with such algebraic complications, leaving a gap for practitioners to fill, when implementing the approaches for interstage dependent uncertainty.

When the optimization horizon is large, (1) becomes intractable and very few policies can be implemented, even in a risk-neutral formulation. One of the rare exceptions is SDDP, which traverses a scenario tree by randomly choosing some paths. When T is large and when the underlying stochastic process is periodic autoregressive as in our application, such a scenario tree (generated by discretizing the continuous distribution of the stochastic process defining the uncertainty in (1)) has an astronomical number of paths and the size of the state vector can be large. In this setting, it takes considerable computational resources to obtain reasonable approximations of the recourse functions. On the contrary, for our test-case, the rolling-horizon policy yielded average results similar to SDDP, but obtained with much less computational effort: from Table 3, we see that RH's overall CPU time was about 7 times smaller than SDDP's.

With the rolling-horizon approach, there is also a reduction in volatility, not only because the optimal decision at time step t depends *only* on the information provided by the t^{th} risk-averse problem, but also because our SDDP implementation was risk-neutral and is not re-solved at each time stage (a risk-averse rolling-horizon formulation would have made SDDP computational times unacceptable). An interesting observation is that the gain in stability provided by RH did not increase much the mean marginal cost. Finally, RA-NRH can be seen as a closed-loop feedback approximation of RH. Naturally, the former policy is poorer than the latter, because RA-NRH uses only information available at the first stage to build the recourse functions.

Since we assume the model parameters in (6) to be known and since values of the inverse distribution F^{-1} in Theorem 4.4 can be obtained with arbitrary accuracy, the rolling-horizon approach can fully exploit the probabilistic structure of uncertainty, without resorting to discretizations or

approximations. If parameters in (6) are only estimations, or just approximate generalized inverse distributions are available, then the rolling-horizon policy would be an approximation. The question of how to control the quality of the approximate policy in this case is an interesting topic, combining statistical inference and sensitivity analysis.

At least from a theoretical point of view, we can always assume that the objective function of a stochastic optimization problem is uncertainty independent (problems $\min\{f(x,\tilde{\xi}) : x \in X\}$ and $\min\{t : t \geq f(x,\tilde{\xi}), x \in X\}$ are equivalent). For this reason, our analysis also applies to stochastic optimization problems with feasible set as in (1), but with objective function depending affinely on the uncertain parameters. Furthermore, the methodology would still be applicable if in (1) a linear term $H_t x_{t-1}$ was introduced in the right hand side of (INEQ), or for matrices B_t similar to the technology matrices F_t in (2). For the water-resource application, in particular, these extensions would make it possible to consider uncertain demand and generation costs, as in [28], [41], as long as the rolling-horizon risk-averse subproblems remain tractable. Moreover, it would be interesting to find theoretical upper bounds for the bias and variance of the total cost using `RH` policy.

Finally, instead of CVaR constraints, one could use integrated chance constraints (ICC), as in [27]:

$$\mathbb{E}[\min(X, h_\tau)] \geq h_\tau - \varepsilon_{\mathtt{c}}(|h_\tau| + 1),$$

because the set of ICC controls remains closed and convex. For comparison, our CVaR constraints can be written in the form $\mathbb{E}[X|X \leq \tilde{h}_\tau] \geq h_\tau - \varepsilon_{\mathtt{c}}(|h_\tau| + 1)$, by taking $\tilde{h}_\tau = -VaR_{\varepsilon_{\mathtt{p}}}(X)$ (the connection with ICC is given by the relation $\mathbb{E}[\min(X,x)] = x\mathbb{P}(X > x) + \mathbb{E}[X|X \leq x]\mathbb{P}(X \leq x)$). However, even for null technology matrices, ICC would result in convex non-linear optimization problems (still tractable, but no longer linear programs).

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