The master equality polyhedron with multiple rows

Sanjeeb Dash Ricardo Fukasawa Oktay Günlük
IBM Research

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Abstract

The master equality polyhedron (MEP) is a canonical set that generalizes the Master Cyclic Group Polyhedron (MCGP) of Gomory. We recently characterized a *nontrivial polar* for the MEP, i.e., a polyhedron T such that an inequality defines a nontrivial facet of the MEP if and only if its coefficient vector forms a vertex of T.

In this paper we study the MEP when it is defined by $m > 1$ rows. We define the notion of a *polaroid*, a set containing all nontrivial facet defining inequalities. We show how to use linear programming to efficiently solve the separation problem for the MEP when the polaroid has a compact polyhedral description, and obtain such descriptions via subadditivity conditions when $m = 2$ or $m = 3$. For the MCGP and the MEP defined by a single constraint, the notions of two-term subadditivity and valid inequalities for MEP are essentially equivalent. We show this is not true in the case of the MEP when $m \geq 3$; In fact, we prove that subadditivity conditions with a sub-exponential number of terms do not imply validity. In particular, when $m = 3$, we show that four-term subadditivity conditions are necessary and sufficient for validity.

1 Introduction

In [10], Gomory defined *master polyhedra*, and characterized the convex hull of their nontrivial facets, i.e., facets which do not have the form $x_i \geq 0$ for some variable x_i. He used these facets to define valid inequalities for general integer programs. Gomory and Johnson [11] later studied a mixed-integer extension of the *master cyclic group polyhedron* (MCGP) – a special case of the master polyhedron when it is defined by a single constraint – and also studied an "infinite group" variant of the MCGP. Over the last decade, the MCGP has been studied in great detail; see [3, 7, 15] for some facet classes, and [12, 9] for properties of the infinite group variant.

Recently, many authors have studied polyhedral properties of sets defined by two or more rows. Andersen, Louveaux, Weismantel, and Wolsey [1] derive valid inequalities for $\{x \in \mathbb{R}^t : Ax \equiv b (\text{mod } \mathbf{1}), x \geq 0\}$ where A has two rows, $\mathbf{1}$ is a vector with all components 1 and a vector equals another modulo a third vector if and only if the component-wise modular equations hold. Cornuéjols and Margot [5] study the infinite group version of the two-constraint problem above, while Borozan and Cornuéjols [4] study the m-constraint version, where $m \geq 2$. Dey and Richard [8] study the infinite group variant of the master polyhedron defined by two constraints.

In [6], we defined the master equality polyhedron (MEP), a generalization of the MCGP, and also of the *master knapsack polyhedron* of Araóz [2], and described a *nontrivial polar* for the MEP, i.e., a polyhedron T with the property that an inequality defines a nontrivial facet of MEP if and only if its coefficient vector defines a vertex of T. In this paper we study the m-row extension (for $m \geq 2$) of the MEP, which can be viewed as a generalization of the m-row master polyhedron. Let n be a strictly positive vector in $\mathbb{Z}^m$ for some $m \geq 1$. Let I^+ be the set of all non-zero integer vectors with kth component contained in $[0, n_k)$ for $k = 1, \ldots, m$. Then for $r \in I^+$, the m-row MGP, in its most general form, equals

$$\text{conv}\left\{x \in \mathbb{Z}^{I^+} : \sum_{i \in I^+} i x_i \equiv r (\text{mod } n), x \geq 0\right\}.$$

Let I be the set of vectors in $\mathbb{Z}^m$ with kth component contained in $[-n_k, n_k]$. If the values of m and n are not clear from the context, we refer to I as $I^m(n)$. For a nonzero integer vector r such that $n \geq r \geq 0$, we define the (m-row) MEP as

$$K^m(n, r) = \text{conv}\left\{x \in \mathbb{Z}^I : \sum_{i \in I} i x_i = r, x \geq 0\right\}. \tag{1}$$

where m is a positive integer. Let e_j stand for the unit vector in $\mathbb{R}^m$ with a one in the jth component and 0 in the remaining components, and let $I_N = \{-n_d e_d\}_{d=1}^m$; then an equivalent definition of the m-row master polyhedron is

$$P^m(n, r) = \text{conv}\left\{x \in \mathbb{Z}^{I^+ \cup I_N} : \sum_{i \in I^+} i x_i + \sum_{i \in I_N} i x_i = r, x \geq 0\right\}. \tag{2}$$

It is clear that $P^m(n, r)$ defines a face of $K^m(n, r)$ (fix x_i to zero for $i \in I \setminus (I^+ \cup I_N)$). In the rest of the paper, we assume the vector n in $K^m(n, r)$ has equal-valued components; our results proved under this restriction hold under the more general case. Therefore, we assume the parameter n in $K^m(n, r)$ is a number and not a vector.

Gomory characterized the convex hull of nontrivial facets of $P^m(n, r)$ as follows.

Theorem 1.1 *[10] The inequality $\bar{\pi} x \geq 1$ defines a nontrivial facet of $P^m(n, r)$ if and only if $\bar{\pi} \in \mathbb{R}^{I^+}$ is an extreme point of*

$$Q^m(n, r) = \begin{cases} \pi_i + \pi_j \geq \pi_{(i+j)\, mod n} & \forall i, j \in I^+, & \text{(SA)} \\ \pi_i + \pi_j = \pi_r & \forall i, j \in I^+ \text{ such that } r = (i+j)\, mod n, & \text{(CO)} \\ \pi_j \geq 0 & \forall j \in I^+, & \\ \pi_r = 1. & & \end{cases}$$

We call the property (SA) *two-term sub-additivity* and the property (CO) *complementarity.* Two-term subadditivity is a central concept in the study of the master polyhedron. It is essentially equivalent to validity: if $\bar{\pi}$ satisfies (SA), then $\bar{\pi} x \geq \bar{\pi}_r$ is a valid inequality for the master polyhedron, and any *minimal* valid inequality for the master polyhedron satisfies (SA). In [6], we described a nontrivial polar for $K^1(n, r)$ and proved a result similar to Theorem 1.1.

Theorem 1.2 *[6] The inequality $\bar{\pi}x \geq 1$ defines a nontrivial facet of $K^1(n,r)$ for $n \geq r > 0$ if and only if $\bar{\pi} \in \mathbb{R}^{2n+1}$ is an extreme point of*

$$T^1(n,r) = \begin{cases} \pi_i + \pi_j \geq \pi_{i+j}, & \forall i,j \in I, \quad 1 \leq i+j \leq n & \text{(RSA1)} \\ \pi_i + \pi_j + \pi_k \geq \pi_{i+j+k}, & \forall i \in \{-n,\ldots,-1\}, \quad j,k,i+j+k \in \{1,\ldots,n\} & \text{(RSA2)} \\ \pi_i + \pi_j = \pi_r, & \forall i,j \in I, \quad i+j=r & \text{(RCO)} \\ \pi_r = 1, & & \text{(NC1)} \\ \pi_{-n} = \pi_{\mathbf{0}} = 0. & & \text{(NC2)} \end{cases}$$

Theorem 1.2 implies that given a point $\bar{x} \notin K^1(n,r)$, a violated facet can be found in polynomial time by finding an extreme point solution of $\min\{\pi\bar{x} : \pi \in T^1(n,r)\}$. Notice that any set $Q = \text{conv}\{x \in \mathbb{Z}^t : \sum_{i=1}^t a_i x_i = b, x \geq 0\}$ defined by a single constraint with integral coefficients can be viewed as a face of $K^1(n,r)$ where $r = |b|$ and n is taken larger than all $|a_i|$ and $|b|$. This result yields a pseudo-polynomial time algorithm to solve the separation problem for the set Q.

In [6] we also show that two-term subadditivity conditions imply validity for $K^1(n,r)$ as they do for $P^m(n,r)$. In this paper we show that two-term subadditivity conditions imply validity for $K^2(n,r)$ as well. However, two and three-term subadditivity conditions do not imply validity for $K^3(n,r)$, but four-term subadditivity conditions are sufficient for validity. Further, we show that subadditivity conditions with exponentially many terms are necessary for validity for $K^m(n,r)$ for $m \geq 3$. Therefore, in a fundamental sense, $K^m(n,r)$ is a more complicated set than $P^m(n,r)$. Our second main result is a polynomial-time separation algorithm for $K^2(n,r)$ and $K^3(n,r)$ based on the notion of a polaroid. This is a polyhedral set, similar to a nontrivial polar, containing the coefficient vectors of all nontrivial facets. We present compact descriptions of polaroids for $K^2(n,r)$ and $K^3(n,r)$ and show that given a point not contained in $K^m(n,r)$ when $m = 2$ or 3, a violated facet can be obtained by optimizing *two* linear functions over the corresponding polaroid.

The paper is organized as follows. In Section 2, we discuss the difference between $K^m(n,r)$ and $P^m(n,r)$ as a source of cutting planes for integer programs and argue why it is hard to generalize Theorem 1.2 to $K^m(n,r)$ when $m > 1$. In Section 3, we describe *polaroids*, and show how to use them to solve the associated separation problem. In Section 4 we give polaroids for $m = 2$ and $m = 3$, and in Section 5, we study the relationship between validity and subadditivity for $K^m(n,r)$. In Section 6 we show that the size of certain natural polaroids grow exponentially with m.

2 Preliminaries

2.1 Group relaxation of MEP

As discussed earlier, $P^m(n,r)$ is a lower dimensional face of $K^m(n,r)$. In addition, it is possible to map points in $K^m(n,r)$ to points in $P^m(n,r)$ using a simple mapping which also gives a relaxation of $K^m(n,r)$ which we call the *group relaxation*. For simplicity, we next describe the group relaxation for $m = 1$ and present an example that shows that this relaxation can be arbitrarily bad. It is easy to extend the description and example to $m > 1$.

Let $n, r \in \mathbb{Z}_+ \setminus \{0\}$ be given and let $P = P^1(n, r)$ and $K = K^1(n, r)$. Furthermore, let K^C denote the continuous relaxation of K and let

$$P = \text{conv}\left\{y \in \mathbb{Z}_+^n : -ny_{-n} + \sum_{i=1}^{n-1} iy_i = r\right\} = \left\{y \in \mathbb{R}_+^n : Ay \geq b\right\}$$

where $Ay \geq b$ gives a linear description of P.

Note that for $x \in K$, we have $\sum_{i=-n}^{n} ix_i = \sum_{i=1}^{n-1} i(x_i + x_{i-n}) - n(\sum_{i=1}^{n} x_{-i} - x_n) = r$ and therefore $x \in K \Rightarrow y \in P$ where

$$y_i = \begin{cases} \sum_{i=1}^{n} x_{-i} - x_n & i = -n \\ x_i + x_{i-n} & i = 1, \ldots, n-1. \end{cases}$$

Based on this observation, it is possible to obtain the *group relaxation* K^G of K as the collection of points in K^C whose image satisfies $Ay \geq b$. More precisely,

$$K^G = \left\{x \in \mathbb{R}_+^{2n+1} : \sum_{i=-n}^{n} ix_i = r,\ Ay \geq b,\ y_{-n} = \sum_{i=1}^{n} x_{-i} - x_n,\ y_i = x_i + x_{i-n} \ \ i = 1, \ldots, n-1\right\}.$$

Clearly, $K^C \subseteq K^G \subseteq K$. We next give an example that shows that when optimizing a linear function, the group relaxation can be very weak.

Example 2.1 *Consider minimizing cx over $K^1(15, 3)$ where c_i is 1 if i is odd, and 0 otherwise. Let z^* denote the optimal value of this problem. Clearly any integer feasible solution has to have $x_i \geq 1$ for some i odd, and therefore $z^* \geq 1$. Furthermore, as $x^K \in K^1(15, 3)$ where x_i^K is 1 if $i = 3$ and 0 otherwise, we have $z^* = 1$.*

Let K^C and K^G denote the continuous and group relaxation of $K^1(15, 3)$ and let z^C and z^G be the optimal value of the corresponding relaxed problem. Clearly, $z^C, z^G \geq 0$ as $K^C, K^G \geq 0$ and $c \geq 0$. Now consider $\bar{x}$ where $\bar{x}_i$ is $1/2$ if $i = 8, 10, -12$ and 0 otherwise. Clearly, $c\bar{x} = 0$. As $\sum_{i=-n}^{n} i\bar{x}_i = r$, we have $\bar{x} \in K^C$ and $z^C = 0$. Furthermore, defining $\bar{y}_{-15} = \sum_{i=1}^{15} \bar{x}_{-i} - \bar{x}_{15}$ and $\bar{y}_i = \bar{x}_i + \bar{x}_{i-15}$ for $i = 1, \ldots, 14$, it is not hard to see that $\bar{y} = 1/2(y_1 + y_2)$ where

$$y_i^1 = \begin{cases} 1 & i = 3 \\ 0 & \textit{otherwise}, \end{cases} \qquad y_i^2 = \begin{cases} 1 & i = 8, 10, -15 \\ 0 & \textit{otherwise}, \end{cases}$$

Furthermore, notice that $y_1, y_2 \in P^1(15, 3)$ and therefore $\bar{y} \in P^1(15, 3)$ implying $A\bar{y} \geq b$. Therefore, $\bar{x} \in K^G$ and $z^G = 0$.

2.2 Basic properties of $K^m(n, r)$ and its facets

We now define some basic notation that will be used throughout the paper. As defined previously, $e_d \in \mathbb{R}_+^m$ will denote the d'th unit vector. Given a point $i \in I$, let $f_i \in \mathbb{R}_+^I$ denote the unit vector with a 1 in the i-th component and zero everywhere else. Finally, let $I_r = \{i \in I : r - i \in I\}$ and let $I_N = \{-ne_d\}_{d=1}^m \subset I$. These sets will be used later on to represent, respectively, vectors involved in complementarity equations and vectors involved in normalization equations.

We next make some basic observations regarding the polyhedral structure of $K^m(n, r)$. These observations are extensions of the results in [6]. We start with the dimension of $K^m(n, r) \subset \mathbb{R}^I$.

Lemma 2.2 *The dimension of $K^m(n,r)$ is $|I|-m$. In other words, there are no extra implied equations.*

Proof. All points in $K^m(n,r)$ satisfy the m linearly independent equations that define $K^m(n,r)$. Therefore, the dimension is at most $|I|-m$. Now suppose $\alpha x = \beta$ holds for all $x \in K^m(n,r)$. Then, we can assume that $\alpha_{-ne_d} = 0$ for all $d = 1,\ldots,m$. Now, since $f_r \in K^m(n,r)$, we have that $\alpha_r = \beta$. Moreover, since $nf_{e_d} + f_r + f_{-ne_d} \in K^m(n,r)$, then $n\alpha_{e_d} + \alpha_r + \alpha_{-ne_d} = \beta \Rightarrow \alpha_{e_d} = 0$. Similarly, $f_{e_d} + f_{-e_d} + f_r \in K^m(n,r)$ so $\alpha_{-e_d} = 0$. Finally, for any point $i \in I$, $f_i + \sum_{d:i_d<0} |i_d| f_{e_d} + \sum_{d:i_d\geq 0} i_d f_{-e_d} + f_r \in K^m(n,r)$, therefore $\alpha_i + \sum_{d:i_d<0}|i_d|\alpha_{e_d} + \sum_{d:i_d\geq 0} i_d \alpha_{-e_d} + \alpha_r = \beta \Rightarrow \alpha_i = 0$. Therefore, $\alpha = \beta = 0$. ∎

By using a very similar proof (which we omit for the sake of brevity), we can also show the following result:

Lemma 2.3 *If $n \geq 2$, then $x_i \geq 0$ defines a facet of $K^m(n,r)$ for all $i \in I$.*

We call any facet-defining inequality of $K^m(n,r)$ which is not a nonnegativity constraint *nontrivial*. We next show some basic properties of these inequalities.

Lemma 2.4 *The following conditions are satisfied by any nontrivial facet-defining inequality $\pi x \geq \eta$ of $K^m(n,r)$:*

$$\pi_i + \pi_{r-i} = \pi_r, \forall i \in I_r, \tag{RCO}$$

$$\pi_r = \eta, \tag{NC1}$$

In addition, for all $q \in \mathbb{Z}^I_+$ such that $\bar{q} := \sum_{i\in I} q_i \cdot i \in I$

$$\sum_{i\in I} q_i \pi_i \geq \pi_{\bar{q}}. \tag{KSA}$$

Proof. Let $q \in \mathbb{Z}^I_+$ and $t \in I$ such that $\sum_{i\in I} q_i \cdot i = t$. Since $\pi x \geq \eta$ is a nontrivial facet-defining inequality of $K^m(n,r)$, there is a point $x^* \in K^m(n,r)$ such that $\pi x^* = \eta$ and $x^*_t > 0$. But then define $x' = x^* - f_t + \sum_{i\in I} q_i f_i$. We have that $\eta \leq \pi x' = \pi x^* - \pi_t + \sum_{i\in I} q_i \pi_i$ and (KSA) follows.

To prove (RCO), let x' and x'' be points in $K^m(n,r)$ such that $\pi x' = \pi x'' = \eta$, $x'_i > 0$ and $x''_{r-i} > 0$. Then $\overline{x} = x' + x'' - f_i - f_{r-i} \in K^m(n,r) \Rightarrow \eta \leq \pi\overline{x} = \pi x' + \pi x'' - \pi_i - \pi_{r-i} \Rightarrow \pi_i + \pi_{r-i} \leq \eta$. But then, (KSA) implies (RCO).

Finally, $f_r \in K^m(n,r) \Rightarrow \pi_r \geq \eta$. But now pick any point $\overline{x} \in K^m(n,r)$ such that $\pi\overline{x} = \eta$ and note that $\overline{x} + \overline{x} - f_r \in K^m(n,r) \Rightarrow \eta \leq 2\eta - \pi_r \Rightarrow \pi_r \leq \eta$. ∎

Note that for $i = \mathbf{0}$, inequality (RCO) implies that $\pi_{\mathbf{0}} = 0$ for all nontrivial facet-defining inequalities. We will call inequalities (KSA) as *k-term subadditive inequalities* whenever $||q||_1 \leq k$.

As $K^m(n,r)$ is not a full-dimensional polyhedron, any valid inequality $\pi x \geq \eta$ for $K^m(n,r)$ has an equivalent representation with $\pi_i = 0$ for all $i \in I_N$. If a valid inequality does not satisfy $\pi_i = 0$

for some $i \in I_N$, one can add an appropriate multiple of the equation $\sum_{i\in I} i_d x_i = r_d$ to it. We next show that after this normalization, all facet-defining nontrivial inequalities have a strictly positive right-hand-side.

Lemma 2.5 *Let $\pi x \geq \eta$ be a nontrivial facet-defining inequality of $K^m(n,r)$ that satisfies $\pi_i = 0$ for all $i \in I_N$. Then $\eta > 0$.*

Proof. For any $i \in I$ such that $i \geq \mathbf{0}$ and $i \neq \mathbf{0}$, we have that $n \cdot i + \sum_{d=1}^m i_d(-ne_d) = \mathbf{0}$ so (KSA) implies that $n\pi_i + \sum_{d=1}^m i_d \pi_{-ne_d} \geq \pi_{\mathbf{0}} \Rightarrow \pi_i \geq 0$. Therefore, $\eta \geq 0$ as $\eta = \pi_r$.

Now assume, for the sake of contradiction that $\eta = 0$. By (NC1) we have that $\pi_r = 0$.

We also assume, without loss of generality, that there is a number $m' \geq 1$ such that $r_d > 0$ for all $d \in D^+ = \{1, \ldots, m'\}$ and $r_d = 0$ for all $d \in D^o = \{m'+1, \ldots, m\}$.

For all $0 \leq i \leq r$, (RCO) implies that $0 \leq \pi_i + \pi_{r-i} = \pi_r = 0$ and therefore $\pi_i = 0$. In particular, $\pi_{e_d} = 0$ for all $d \in D^+$. Furthermore, by inequality (KSA) $n\pi_{-e_d} \geq \pi_{-ne_d} = 0$, and thus $\pi_{-e_d} \geq 0$ for all $d = 1, \ldots, m$. In addition, for $d \in D^+$,we have $\pi_{-ne_d} + (n-1)\pi_{e_d} \geq \pi_{-e_d}$ implying $\pi_{-e_d} = 0$.

So far we have shown that $\pi_{e_d} = \pi_{-e_d} = 0$ for all $d \in D^+$. For $d \in D^o$, on the other hand, $\pi_{-ne_d} + \pi_{r+ne_d} = \pi_r$ and therefore $\pi_{r+ne_d} = 0$.

Now pick any point $i \in I$ and let $S_i = \{d : i_d < 0\}$ and define

$$i' \;=\; i + n\sum_{d\in S_i} e_d \;=\; i + \sum_{d\in S_i}(r + ne_d) + |S_i| \sum_{d\in D^+} r_d(-e_d).$$

Note that the last term above is indeed $-|S_i|r$ as $r_d = 0$ for $d \notin D^+$. Clearly, $\mathbf{0} \leq i' \in I$ and therefore $\pi_{i'} \geq 0$. Therefore, (KSA) implies that

$$\pi_i + \sum_{d\in D^+\cap S_i}(\pi_r + n\pi_{e_d}) + \sum_{d\in D^o\cap S_i}\pi_{r+ne_d} + |S_i|\sum_{d\in D^+} r_d\pi_{-e_d} \;\geq\; \pi_{i'} \;\geq\; 0.$$

Therefore $\pi_i \geq 0$ for all $i \in I$.

But then $\pi x \geq \eta$ is implied by all nonnegativity inequalities $x \geq 0$, which contradicts the fact that it is a facet-defining inequality. Thus $\eta > 0$. ∎

Therefore, without loss of generality, we can assume that if $\pi x \geq \eta$ is a nontrivial facet-defining inequality then it satisfies the following normalization conditions:

$$\pi_i = 0, \qquad \forall i \in I_N. \tag{NC2}$$

$$\eta = 1, \tag{NC3}$$

Throughout the rest of the paper we assume that all nontrivial facet-defining inequalities satisfy equations (NC3) and (NC2).

2.3 Extending the result from $K^1(n,r)$

An important property of $T^1(n,r)$ that we use in the proof of Theorem 1.2 is that it contains all unit vectors f_i for $i \notin I_r \cup I_N$ as rays. Using this property, we can argue that extreme points of $T^1(n,r)$ are in one-to-one correspondence with nontrivial facet-defining inequalities of $K^1(n,r)$. Notice that the conditions that guarantee that all $\pi \in T^1(n,r)$ yield valid inequalities are the relaxed sub-additivity conditions (RSA1) and (RSA2) which have the form

$$\sum_{i \in I} q_i \pi_i \geq \pi_{\bar{q}}, \qquad \forall q \in Q \tag{3}$$

where $\bar{q}$ denotes $\sum_{i \in I} q_i \cdot i$ and Q is a given subset of $\mathbb{Z}^I_+$. Since we wanted the property that f_i for $i \notin I_r \cup I_N$ are rays, we needed to exclude from Q the points q such that $\sum_{i \in I} q_i \cdot i \in I_r \cup I_N$. Moreover $||q||_1 \leq 3$ for all $q \in Q$ implying that $|Q|$ and therefore the number of inequalities defining $T^1(n,r)$ is small.

A natural generalization of this idea for $K^m(n,r)$ would require a set $Q^m \subset Z_+^{I^m(n)}$ with the following properties: (i) if π satisfies sub-additivity condition (3) for all $q \in Q^m$, then $\pi x \geq 1$ is valid for $K^m(n,r)$, and (ii) all $q \in Q^m$ have small norm and $\sum_{i \in I^m(n)} q_i \cdot i \notin I_r \cup I_N$. We next present an example that shows that such a set does not exist for $m \geq 2$ because for the first condition to hold, Q^m must contain an element q that has $||q||_1 = \Omega(n)$.

Example 2.6 (Swirszcz [16]) *Consider the following point $\bar{x} \in K^2(n,r)$ where $r = [n,n]^T$*

$$\bar{x}_i = \begin{cases} 2 & if\ i = [-n,n]^T \\ n, & if\ i = [3,-1]^T, \\ 0 & otherwise. \end{cases}$$

Clearly $\bar{x} \geq 0$ and $\sum_{i \in I} \bar{x}_i \cdot i = r$ implying $\bar{x} \in K^2(n,r)$. Now consider the following $\bar{\pi} \in \mathbb{Z}^I$

$$\bar{\pi}_i = \begin{cases} 0 & if\ i \in \{[-n,n]^T, [3,-1]^T, [-n,0]^T, [0,-n]^T, [0,0]^T\}, \\ 1 & if\ i = [n,n]^T, \\ 1/2 & otherwise \end{cases}$$

and note that $\bar{\pi}$ satisfies the normalization and complementarity conditions (NC1), (NC2) *and* (RCO). *Clearly $\bar{\pi}x \geq 1$ is not a valid inequality as $\bar{\pi}\bar{x} = 0$. However $\bar{\pi}$ satisfies $\sum_{i \in I} q_i \bar{\pi}_i \geq \bar{\pi}_{\bar{q}}$ for all $q \in \mathbb{Z}^I_+$ such that $\bar{q} \notin I_r \cup I_N$ and $1 < ||q||_1 < n/3 + 1$.*

Until this example was presented to us by our colleague Grzegorz Swirszcz, we made multiple attempts to characterize a set Q^m with properties (i) and (ii) described above. Based on this example we decided that, even for $m = 2$, it would be hard to find a compact characterization of a nontrivial polar for $K^m(n,r)$.

3 Separation using polaroids

In Section 2.3 we argued that it appears hard to obtain a compact description of a nontrivial polar for $K^m(n,r)$. In spite of this, we discuss in this section how to solve the separation problem efficiently.

Given a point $x^* \in \mathbb{R}^I$, the *separation problem* is to first decide if $x^* \in K^m(n,r)$, and if $x^* \notin K^m(n,r)$, then to find a valid inequality violated by x^*. The separation problem can be easily solved if an explicit and compact description of the polar or nontrivial polar is known. In this section, we present a polar-like set which also leads to an efficient separation algorithm. We next formally define the properties of this set.

Definition 3.1 *A* polaroid *T for $K^m(n,r)$ is a polyhedral set in $\mathbb{R}^I$ such that:*

1. *If $\pi \in T$, then π satisfies the normalization conditions (NC1) and (NC2).*
2. *If $\pi \in T$ then $\pi x \geq 1$ is valid for all $x \in K^m(n,r)$;*
3. *If $\pi x \geq 1$ is a nontrivial facet-defining inequality of $K^m(n,r)$ satisfying (NC1) and (NC2), then $\pi \in T$.*

Let $LP(K^m(n,r))$ denote the continuous relaxation of $K^m(n,r)$.

Theorem 3.2 *Given a point $x^* \in LP(K^m(n,r))$, and a polaroid T for $K^m(n,r)$. Then*

1. *x^* can be separated from $K^m(n,r)$ by solving a linear program (LP) over T, and,*
2. *if $x^* \notin K^m(n,r)$ then a violated facet-defining inequality can be obtained by solving a second LP over T provided that the solution to the second LP is an extreme point solution.*

Proof. Since $x^* \in LP(K^m(n,r))$, it satisfies all trivial inequalities $x \geq 0$. To solve the separation problem, solve:

$$\begin{array}{ll} \min & \pi x^* \\ \text{s.t.} & \pi \in T \end{array} \tag{LP1}$$

We start by establishing some properties of a solution $\hat{\pi} \in T$. As $\hat{\pi} x \geq 1$ is valid for $K^m(n,r)$ it can be obtained by relaxing the right hand side of a conic combination of facet-defining inequalities plus a linear combination of the equations $\sum_{i \in I} i_d x_i = r_d$ for $d = 1, \ldots, m$. In other words, there exists multipliers $\lambda \in \mathbb{R}^M_+$ and $\alpha \in \mathbb{R}^d$ such that

$$\hat{\pi}_i \;\geq\; \sum_{k=1}^{M} \lambda_k \pi_i^k + i\alpha, \qquad \forall\, i \in I \tag{4}$$

$$1 \;\leq \sum_{k=1}^{M} \lambda_k + r\alpha \tag{5}$$

hold. Here $\pi^k x \geq 1$ for $k = 1, \ldots, M$ are all nontrivial facet-defining inequalities normalized according to (NC1) and (NC2). The inequalities in (4) follow from the fact that $\hat{\pi}_i = \sum_{k=1}^{M} \lambda_k \pi_i^k + \sum_{d=1}^{m} i_d \alpha_d + \mu_i$, for $i \in I$ where the last term $\mu_i \geq 0$, corresponds to adding μ_i times the trivial inequality $x_i \geq 0$. Notice that, for each $d = 1, \ldots, m$, inequality (4) for $i = -ne_d$ becomes: $\hat{\pi}_i \geq \sum_{k=1}^{M} \lambda_k \pi_i^k - n\alpha_d$. But since $\hat{\pi}_i = \pi_i^k = 0$, we have that $\alpha_d \geq 0$. From the fact that $\pi'_r = \pi_r^k = 1$ we have that (4) for $i = r$ becomes $1 \;\geq\; \sum_{k=1}^{M} \lambda_k + \sum_{d=1}^{m} r_d \alpha_d$. Combining this with inequality (5) we obtain $\sum_{k=1}^{M} \lambda_k + r\alpha = 1$. We have already shown that $\alpha \geq 0$, so this also implies that $1 \geq \sum_{k=1}^{M} \lambda_k \geq 0$.

Now let $h \in \mathbb{R}$ be an arbitrary negative number less than $\min\{\pi^k x^*\}_{k=1}^{M}$. For any $\hat{\pi} \in T$, we have that $\hat{\pi} x^* \geq \sum_{k=1}^{M} \lambda_k \pi^k x^* + \sum_{i \in I} (x_i^* \cdot i)\alpha \geq \sum_{k=1}^{M} \lambda_k h + r\alpha \geq h$. Therefore, (LP1) is bounded and has an optimal solution π^*. If π^* is such that $\pi^* x^* < 1$, then $\pi^* x \geq 1$ is a valid inequality separating x^* from $K^m(n,r)$. If $\pi^* x^* \geq 1$, then all nontrivial facet-defining inequalities are satisfied by x^* and all trivial ones are also satisfied by it. Therefore $x^* \in K^m(n,r)$.

For the facet separation problem, note that the violated inequality $\pi^* x \geq 1$ obtained using (LP1) is not necessarily facet-defining. To handle this, let $z^* = \pi^* x^*$ and solve:

$$\begin{array}{ll} \min & \sum_{i \in I} \pi_i \\ \text{s.t.} & \pi \in T \\ & \pi x^* = z^* \end{array} \tag{LP2}$$

Clearly π^* is feasible for (LP2) and any (extreme) solution π' to (LP2) gives a violated inequality $\pi' x \geq 1$ that separates x^* from $K^m(n,r)$. We now need to argue that $\pi' x \geq 1$ defines a facet of $K^m(n,r)$.

Since $\pi' \in T$ we can still say that there exists multipliers $\lambda \in \mathbb{R}_+^M$ and $\alpha \in \mathbb{R}^d$ such that (4) and (5) hold. As π' gives a violated inequality, we have that

$$0 < 1 - \pi' x^* \leq \sum_{k=1}^{M} \lambda_k + r\alpha - \sum_{k=1}^{M} \lambda_k \pi^k x^* - \sum_{i \in I} i\alpha x_i^* = \sum_{k=1}^{M} \lambda_k - \sum_{k=1}^{M} \lambda_k \pi^k x^*. \tag{6}$$

But notice that if $\sum_{k=1}^{M} \lambda_k = 0$, then $\lambda_k = 0$ for all k which contradicts (6). Hence $\sum_{k=1}^{M} \lambda_k > 0$. Thus we can set $\gamma_k = \lambda_k / \sum_{k=1}^{M} \lambda_k$ and rewrite (6) as

$$0 < 1 - \pi' x^* \leq \left(\sum_{k=1}^{M} \lambda_k \right) \left(1 - \sum_{k=1}^{M} \gamma_k \pi^k x^* \right) \tag{7}$$

which implies that $1 - \sum_{k=1}^{M} \gamma_k \pi^k x^* > 0$.

Using $\sum_{k=1}^{M} \lambda_k \leq 1$, we can now write

$$0 < 1 - \pi^* x^* \leq 1 - \sum_{k=1}^{M} \gamma_k \pi^k x^* = \sum_{k=1}^{M} \gamma_k (1 - \pi^k x^*) \leq 1 - \pi' x^*. \tag{8}$$

The last inequality follows from the fact that π' is also an optimal solution to (LP1). Since equality has to hold throughout, we have that $\sum_{k=1}^{M} \lambda_k = 1$.

But now, notice that

$$\begin{aligned} \sum_{i \in I} \pi_i' \geq \sum_{i \in I} \sum_{k=1}^{M} \lambda_k \pi_i^k + \sum_{i \in I} i\alpha &= \sum_{i \in I} \sum_{k=1}^{M} \lambda_k \pi_i^k + \sum_{i \in I} \sum_{d=1}^{m} i_d \alpha_d \\ &= \sum_{i \in I} \sum_{k=1}^{M} \lambda_k \pi_i^k + \sum_{d=1}^{m} \alpha_d \sum_{i \in I} i_d = \sum_{k=1}^{M} \lambda_k \sum_{i \in I} \pi_i^k \geq \sum_{i \in I} \pi_i' \end{aligned}$$

where the first inequality follows from (4) and the last equation follows from $\sum_{i\in I} i_d = 0$ for all $d = 1,\dots,m$. Since equality has to hold throughout, all inequalities in (4) hold as equality. In particular, the inequalities for $i = -ne_d$ imply that $\alpha = 0$. Thus, $\pi' = \sum_{k=1}^{M} \lambda_k \pi_i^k$, where $\sum_{k=1}^{M} \lambda_k = 1$ and $\lambda_k \geq 0$. As π' is assumed to be an extreme point solution to (LP2), it can not be a convex combination of other optimal solutions and therefore π' is a nontrivial facet-defining inequality. ■

Notice that in the proof above we also showed the following.

Corollary 3.3 *If T is a polaroid and $\pi x \geq 1$ is a nontrivial facet-defining inequality of $K^m(n,r)$, then π is an extreme point of T.*

4 A polaroid for $K^m(n,r)$ for $m = 2, 3$

In Section 3, we showed that a compact description of a polaroid is useful for separation over $K^m(n,r)$. In this section, we obtain compact descriptions of polaroids when $m = 2$ and $m = 3$.

Lemmas 2.4 and 2.5 imply that a polyhedral set in $\mathbb{R}^{I^m(n)}$ defined by (NC1), (NC2) and any subset of the k-term subadditivity constraints (KSA) will automatically satisfy properties 1 and 3 in the definition of a polaroid. If for some fixed $l > 0$, the k-term subadditivity constraints with $k \leq l$ imply (along with (NC1)) validity for $K^m(n,r)$, then we can define a polaroid T via (NC1), (NC2) and a subset of the constraints (KSA). Clearly there are at most $|I^m(n)|^k$ k-term subadditivity inequalities. To obtain a polaroid defined by few inequalities, we would like to compute the smallest number $k^*(m)$ such that $k^*(m)$-term subadditivity and (NC1) imply validity for $K^m(n,r)$. We emphasize that Gomory [10] shows that the 2-term subadditivity constraints $\pi_i + \pi_k \geq \pi_{(i+k) \bmod n}$ are sufficient for validity for $P^m(n,r)$ for all $m \in \mathbb{Z}^+$ (for $P^m(n,r)$, these constraints are defined in a modular fashion). Therefore, if we show that $k^*(m) > 2$ for any $m \geq 2$, we will have shown that the set $K^m(n,r)$ is fundamentally a more complicated object than $P^m(n,r)$.

In the remainder of this Section, we obtain a constant upper bound on $k^*(m)$ for $m = 2, 3$ via some geometrical properties of vectors in $\mathbb{R}^m$. We start with the following notation.

A vector $b \in \mathbb{R}^2$ has *sign pattern* $(+-)$ (for example) if $b_1 \geq 0$ and $b_2 < 0$. We also denote this fact by $b \in (+-)$. We use similar notation for vectors in $\mathbb{R}^m$ for $m \geq 2$. Given a collection of vectors $(b_1, \dots, b_t)$, we say that a sign pattern p is absent from the collection if no vector in the collection has sign pattern p, otherwise we say that p is present. If two sign patterns have opposite signs in each co-ordinate, we say they are *opposing* sign patterns.

The following observation regarding vectors with opposing sign patterns is true for any m.

Observation 4.1 *If $b, c \in [-1,1]^m$ have opposing sign patterns, then $b + c \in [-1,1]^m$.*

Proof. A vector $v \in \mathbb{R}^m$ is in $[-1,1]^m$ if and only if $||v||_\infty \leq 1$. But since b and c have opposing sign patterns, for $j = 1,\dots,m$ the number $(b+c)_j$ has absolute value less than or equal to $\max_j\{|b_j|, |c_j|\}$, . Thus $||b+c||_\infty \leq \max\{||b||_\infty, ||c||_\infty\} \leq 1$. ■

4.1 The case $m = 2$

The following result regarding collections of vectors in $[-1,1]^2$ will imply that $k^*(2) \le 2$.

Lemma 4.2 *In any collection of vectors $S = (b_1, \ldots, b_t)$ such that $t \ge 2$, $b_i \in [-1,1]^2$ for $i = 1, \ldots, t$ and $\sum_{i=1}^t b_i \in [-1,1]^2$, there is a pair of vectors b_k, b_l with $b_k + b_l \in [-1,1]^2$.*

Proof. Without loss of generality we can assume that $\sum_{i=1}^t b_i = r = (r_1, r_2)$ with $(0,0) \le (r_1, r_2) \le (1,1)$. If S contains vectors with opposing sign patterns, then by Observation 4.1 the result holds for S. Assume that opposing sign patterns are not present in S; without loss of generality we can assume the sign pattern $(-\,+)$ is not present in S. Then the pattern $(+\,+)$ is present in S as $r_1 \ge 0$, and therefore $(-\,-)$ is not present in S. Let b_k be any vector with sign pattern $(+\,+)$. If S has a vector b_l with sign pattern $(+\,-)$, then $b_k + b_l \in [-1,1]^2$ as the first component of $b_k + b_l$ is contained in $[0, r_1] \subseteq [0,1]$. Otherwise, all vectors in S have sign pattern $(+\,+)$ and any vector $b_l \ne b_k$ satisfies $b_k + b_l \in [0,1]^2$. ∎

Note that, in Lemma 4.2, the choice of the interval $[-1,1]^2$ is arbitrary. The following remark formalizes this comment.

Remark 4.3 *Let $\alpha \in R^+$, $m \in \mathbb{Z}^+$ where $\alpha, m > 0$ and $b_i \in [-\alpha, \alpha]^m$ for all $i \in S$. Then $\sum_{i \in S} b_i \in [-\alpha, \alpha]^m$, if and only if $\sum_{i \in S} b_i/\alpha \in [-1,1]^m$.*

Let $I = I^2(n)$. Let I_{++} be the set of points in I with sign pattern $(+\,+)$, and let I_{--}, I_{+-} and I_{-+} be defined in a similar manner. Given a vector $i \in I$, let f_i be the unit vector in $\mathbb{R}^I$ with a 1 in component i and zero in all other components.

Theorem 4.4 *If $\pi \in \mathbb{R}^I$ satisfies 2-term subadditivity and $\pi_r \ge 1$, then $\pi x \ge 1$ is valid for $K^2(n,r)$. Moreover, the theorem is still true if π only satisfies $\pi_r \ge 1$ and the following 2-term subadditivity conditions:*

$$\pi_i + \pi_j \ge \pi_{i+j}, \qquad \forall i \in I_{-+}, j \in I_{+-}, \tag{9}$$

$$\pi_i + \pi_j \ge \pi_{i+j}, \qquad \forall i \in I_{++}, j \in I, \qquad i + j \in I, \tag{10}$$

Proof. Suppose $\pi x \ge 1$ is not valid for $K^2(n,r)$. Then let x^* be a point in $K^2(n,r)$ such that $\pi x^* < 1$ and $||x^*||_1$ is as small as possible. If $||x^*||_1 = 1$, then $\sum_{i \in I} i x_i^* = r$ implies that $x_r^* = 1$, and $x_i^* = 0$ for $i \ne r$; which contradicts the fact that $\pi_r \ge 1$. Therefore $||x^*||_1 \ge 2$. Define $t = ||x^*||_1$ and a collection $S = (b_1, \ldots, b_t)$ of vectors having x_i^* copies of vector i for all $i \in I$. Therefore $b_j \in [-n,n]^2$ for $j = 1, \ldots, t$. Using Lemma 4.2 and Remark 4.3, we know that there are vectors b and c in S such that $b + c \in [-n,n]^2$. Moreover, as $b, c \in I = [-n,n]^2 \cap \mathbb{Z}^2$, we know that $b + c \in I$. Define $x' := x^* - f_c - f_b + f_{b+c}$. Then $x' \in K^2(n,r)$ and yet $||x'||_1 < ||x^*||_1$ and $\pi x' = \pi x^* - \pi_b - \pi_c + \pi_{b+c} \le \pi x^*$. The previous inequality holds because π satisfies the 2-term subadditivity condition $\pi_b + \pi_c \ge \pi_{b+c}$. But this contradicts the choice of x^*.

The second part of the theorem follows from the proof of Lemma 4.2, which guarantees that either b and c can be chosen to have opposing sign patterns or b can be chosen to be in I_{++}. (notice that the case where $b \in I_{++}$ and $c \in I_{--}$ satisfies both conditions). ∎

Theorem 4.4 implies that $k^*(2) = 2$. Moreover, we can state the following corollary.

Corollary 4.5 *The following is a description of a polaroid for $K^2(n, r)$:*

$$\pi_i + \pi_j \geq \pi_{i+j}, \qquad \forall i \in I_{-+}, j \in I_{+-}, \tag{11}$$

$$\pi_i + \pi_j \geq \pi_{i+j}, \qquad \forall i \in I_{++}, j \in I, \qquad i + j \in I, \tag{12}$$

$$\pi_r = 1, \tag{13}$$

$$\pi_i = 0, \qquad \forall i \in I_N. \tag{14}$$

4.2 The case $m = 3$

We now prove an analogue of Lemma 4.2 for vectors in $\mathbb{R}^3$. As noted in Remark 4.3, such a result is equivalent to its variant obtained by replacing 1 by any $\alpha > 0$.

Lemma 4.6 *Let $S = (b_1, \ldots, b_t)$ be a collection of vectors with $t \geq 2$, $b_i \in [-1, 1]^3$ for $i = 1, \ldots, t$ and $\sum_{i=1}^t b_i \in [-1, 1]^3$. Then there is a set $T \subseteq \{1, \ldots, t\}$ with $2 \leq |T| \leq 4$ such that $\sum_{i \in T} b_i \in [-1, 1]^3$.*

Proof. First, note that the order of elements in the collection S does not matter. In addition we allow S to have multiple copies of the same element. In this proof we abuse the notation and use set operations on collections as follows: (i) $S' \subseteq S$ means that for every distinct element in S', the collection S has at least as many copies of that element as S' does, (ii) $S' = S$ means that $S' \subseteq S$ and $S \subseteq S'$, (iii) $S' \cup S$ denotes the collection obtained by concatenating S' and S, (iv) For $S' \subseteq S$, we define $S \setminus S'$ to denote their difference, that is $(S \setminus S') \cup S' = S$.

Without loss of generality we can assume that $\sum_{i=1}^t b_i = r$ has sign pattern $(+ + +)$. If the lemma is not true, then consider the collection of vectors S which forms the smallest counterexample to the claim, i.e., $\sum_{i=1}^t b_i = r$, where $0 \leq r \leq 1$, and no subcollection of four or fewer vectors in S has its sum contained in $[-1, 1]^3$. Clearly, $t \geq 5$. If any vector in S (say b_1) has the sign pattern $(+ + +)$, then $\sum_{i=2}^t b_i = r - b_1 \in [-1, 1]^3$, and there is a subcollection of four or fewer vectors from $b_2, \ldots, b_t$ with sum contained in $[-1, 1]^3$, a contradiction to the fact that S has no such subcollection.

We can therefore assume that the sign pattern $(+ + +)$ is not present in S. We now divide the remaining sign patterns into three groups.

$$A = \{(+ + -), (+ - +), (- + +)\}, B = \{(- - +), (- + -), (+ - -)\}, C = \{(- - -)\}.$$

Case 1: Assume none of the sign patterns from A are present in S. Let p be the sign pattern from B with fewest vectors in S (we may assume by symmetry that p is $(- - +)$). Let S' be a subcollection of vectors consisting of all vectors with sign pattern p, and an equal number of vectors with the other two sign patterns from B. If $S' \neq \emptyset$, let c_1, c_2, c_3 be vectors from S' where each has a distinct sign pattern from B (say c_1 has the first sign pattern and so on). Clearly $c_1 + c_2 \notin [-1,1]^3$, and as a consequence the first component of $c_1 + c_2$ is less than -1 (c_1 and c_2 have opposite signs in the last two co-ordinates, and therefore the sums in these co-ordinates are contained in $[-1,1]$). As $c_2 + c_3$ and $c_1 + c_3$ are also not contained in $[-1,1]^3$, we conclude that $c_1 + c_2 + c_3 < 0$. Since this holds for all triples of vectors in S' with distinct sign patterns, the sum of vectors in S' is negative if $S' \neq \emptyset$, or 0 if $S' = \emptyset$. But in either case the vectors in $S \setminus S'$ have a negative sign in the last co-ordinate, and thus $\sum_{i=1}^{t} b_i$ cannot equal $r \geq 0$.

Case 2: Assume exactly one of the sign patterns from A is present in S; without loss of generality, we can assume it is $(+\,+\,-)$. The first sign patterns of A and B are opposing patterns, and therefore the first sign pattern from B cannot be present in S. Then all sign patterns in S have a negative sign in the third co-ordinate, and $\sum_{i=1}^{t} b_i$ cannot equal $r \geq 0$.

Case 3: Assume exactly two sign patterns from A are present in S. Assume, without loss of generality, that the third sign pattern from A is absent from S. Note that this rules out the presence in S of the first two sign patterns from B.

Case 3a: Assume $(-\,-\,-)$ is not present in S. In this case all vectors in S have a positive sign in the first co-ordinate. By Lemma 4.2, there are two vectors in S, say b_i and b_j, such that the last two components of $b_i + b_j$ are contained in $[-1,1]^2$. But the first component of $b_i + b_j$ is contained in $[0, r_1]$, and therefore $b_i + b_j \in [-1,1]^3$, a contradiction.

Case 3b: Assume the pattern $(-\,-\,-)$ is present in S. As the sum of no pair of vectors from S is contained in $[-1,1]^3$, it follows that if c_1, c_2 and c_3 are any three vectors with the sign patterns $(+\,+\,-), (+\,-\,+), (-\,-\,-)$ respectively, then $c_1 + c_2 + c_3$ has the sign pattern $(+\,-\,-)$. Now let S' be a maximal subcollection of S with equal number of vectors from the above three sign patterns. Again $\sum_{i \in S'} b_i$ has sign pattern $(+\,-\,-)$. If $S \setminus S'$ does not contain either of the first two patterns, then $\sum_{i=1}^{t} b_i$ is negative in some co-ordinate. Therefore $S \setminus S'$ contains a vector $b_i \in (+\,+\,-)$ and $b_j \in (+\,-\,+)$, and no vector with sign pattern $(-\,-\,-)$. In this case, the first component of $b_i + b_j \in [0, r_1]$, and therefore $b_i + b_j \in [-1,1]^3$, a contradiction.

Case 4: Assume all three sign patterns from A are present in S. Then all sign patterns from B are absent from S. Let c_1, c_2, c_3 be any three vectors having the distinct sign patterns from A (in order). As $c_1 + c_2 \notin [-1,1]^3$, the first component of $c_1 + c_2$ lies in $(1,2]$, and therefore the first component of $c_1 + c_2 + c_3$ lies in $(0,2)$. Arguing in a similar fashion for the other components, we conclude that $c_1 + c_2 + c_3 \in (0,2)^3$, and at least one of its components is greater than 1.

Case 4a: The pattern $(-\,-\,-)$ is present in S. Let $c_4 \in (-\,-\,-)$. Then $\sum_{i=1}^{3} c_i > 0 \Rightarrow \sum_{i=1}^{4} c_i \geq -\mathbf{1}$. Furthermore, for $i = 1,2,3$, the ith component of $c_4 + c_{4-i}$ is contained in $[-2,-1)$ and hence $\sum_{i=1}^{4} c_i \leq \mathbf{1}$. Therefore $\sum_{i=1}^{4} c_i \in [-1,1]^3$.

Case 4b: The pattern $(-\,-\,-)$ is absent from S. Let $S' \supseteq (c_1, c_2, c_3)$ be a maximal subcollection of S with equal number of vectors from the three sign patterns in A. Clearly, $\sum_{b_i \in S'} b_i$ has sign pattern $(+\,+\,+)$, as the sum of every subset of three vectors with distinct sign patterns is strictly positive in each component.

Without loss of generality we can assume that the pattern $(-++)$ is absent from $S \setminus S'$; then all vectors in $S \setminus S'$ have a positive sign in the first co-ordinate. But note that S' can be partitioned into v collections of three vectors $S' = \bigcup_{u=1,\dots,v} (c_1^u, c_2^u, c_3^u)$ such that $c_1^u \in (++-)$, $c_2^u \in (+-+)$ and $c_3^u \in (-++)$ for all $u = 1,\dots,v$. Let $d^u = \sum_{j=1}^3 c_j^u$ for $u = 1,\dots,v$. We already argued that $d^u \in (+++)$ for all $u = 1,\dots,v$. Now define S^- as the collection of vectors from $(S \setminus S') \cup (d^u)_{u=1}^v$ and note that $\sum_{b\in S} b = \sum_{b \in S^-} b = r \in [0,1]^3$. But since the first component of all vectors in S^- are in $[0,1]$, then the sum of any subcollection of vectors from S^- will have its first component in $[0, r_1]$. From Lemma 4.2, we know that there is a pair of vectors b and c in S^- such that all their components lie in $[-1,1]$. If b or c are in $S \setminus S'$, we are done, since b and c will give us our desired subcollection of S.

Hence, assume $b = c_1^u + c_2^u + c_3^u$ and $c = c_1^{u'} + c_2^{u'} + c_3^{u'}$ for some $u, u' \in 1,\dots,v$. But then $b, c \in (+++)$ and hence $\mathbf{0} \le b \le b + c \le \mathbf{1}$. ■

Let $I = I^3(n)$. Given a vector $i \in I$, let f_i be the unit vector in $\mathbb{R}^I$ with a 1 in component i and zero in all other components.

Theorem 4.7 *If $\pi \in \mathbb{R}^I$ satisfies 4-term subadditivity and $\pi_r \ge 1$ then $\pi x \ge 1$ is valid for $K^3(n,r)$.*

Proof. Suppose $\pi x \ge 1$ is not valid for $K^2(n,r)$. Let x^* be a point in $K^3(n,r)$ such that $\pi x^* < 1$ and $||x^*||_1$ is as small as possible. As in the proof of Theorem 4.4, we can assume that $||x^*||_1 \ge 2$, and that there is a collection $S = (b_1,\dots,b_t)$ of integer vectors from $[-n,n]^3$ with $t = ||x^*||_1$ such that $\sum_{j=1}^t b_j = r$, and each b_j satisfies $x^*_{b_j} > 0$. By the previous theorem, there is a subset T of $\{1,\dots,t\}$ with $2 \le |T| \le 4$ such that $\sum_{j\in T} b_j = \lambda \in I$. If $x' = x^* - \sum_{j \in T} f_{b_j} + f_\lambda$, then $||x'||_1 < ||x^*||_1$. Further, $x' \in K^3(n,r)$ and $\pi x' = \pi x^* - \sum_{j\in T} \pi_{b_j} + \pi_\lambda \le \pi x^*$, a contradiction. ■

Therefore, $k^*(3) \le 4$ and we have the following corollary.

Corollary 4.8 *The following is a description of a polaroid for $K^3(n,r)$:*

$$\pi_i + \pi_j + \pi_k + \pi_l \ge \pi_{i+j+k+l} \qquad \forall i,j,k,l \in I, \text{ with } (i+j+k+l) \in I, \tag{15}$$

$$\pi_r = 1, \tag{16}$$

$$\pi_i = 0, \qquad \forall i \in I_N. \tag{17}$$

5 Validity via subadditivity

In the previous Section, we focused on obtaining an upper bound on $k^*(m)$, that is, the smallest number such that $k^*(m)$-term subadditivity and (NC1) guarantee validity for $K^m(n,r)$. In this Section, we focus on deriving lower bounds for $k^*(m)$. In particular, the results in this Section show that the upper bound of $k^*(3) \le 4$ obtained in Section 4 is tight.

We start with an example for $K^3(n,r)$ that shows that 3-term sub-additivity of the coefficients is not sufficient to guarantee validity of an inequality.

Example 5.1 *Let*

$$a = \begin{bmatrix} -10 \\ 10 \\ 10 \end{bmatrix} \quad b = \begin{bmatrix} 10 \\ -10 \\ 10 \end{bmatrix} \quad c = \begin{bmatrix} 1 \\ 1 \\ -9 \end{bmatrix}$$

and consider the inequality $\bar{\pi}x \geq 1$ *and the point* $\bar{x} \in K^3(10, \mathbf{2})$ *where* $\mathbf{2} \in \mathbb{R}^\mathbf{3}$ *is a vector of all twos and*

$$\bar{\pi}_i = \begin{cases} 0 & if\ i \in \{a,b,c\} \\ 1, & otherwise, \end{cases} \qquad and \qquad \bar{x}_i = \begin{cases} 1 & if\ i \in \{a,b\} \\ 2 & if\ i = c \\ 0, & otherwise. \end{cases}$$

Notice that even though $\bar{\pi}\bar{x} = 0 \not\geq 1$, $\bar{\pi}$ *satisfies all 2 and 3-term subadditivity conditions. To see this, first note that as* $\bar{\pi} \in \{0,1\}^I$, *if* $\bar{\pi}_i + \bar{\pi}_j \geq \bar{\pi}_{i+j}$ *is violated by some* $i, j \in I$ *with* $i + j \in I$, *then* $\bar{\pi}_i = \bar{\pi}_j = 0$ *and* $\bar{\pi}_{i+j} = 1$. *In other words* $i, j \in \{a,b,c\}$ *and* $i + j \in I$. *This, however, is not possible as* $i + j \notin I$ *if* $i, j \in \{a,b,c\}$.

Similarly, if $\bar{\pi}_i + \bar{\pi}_j + \bar{\pi}_k \geq \bar{\pi}_{i+j+k}$ *is violated by some* $i, j, k \in I$ *with* $i + j + k \in I$, *then* $\bar{\pi}_i = \bar{\pi}_j = \bar{\pi}_k = 0$ *and* $\bar{\pi}_{i+j+k} = 1$. *This, again, is not possible as* $i + j + k \notin I$ *if* $i, j, k \in \{a,b,c\}$.

Given a point x in $\mathbb{Z}^I$, we define its *support*, denoted as $supp(x)$, to be the collection of indices $i \in I$ for which $x_i > 0$. For the point $\bar{x}$ in Example 5.1, $supp(\bar{x}) = \{a,b,c\}$. We next define a function $\mathcal{F}$ which maps a matrix consisting of columns from I and a vector of column weights to a point in $\mathbb{Z}^I$. More formally, for some $l > 0$ let Q be a matrix in $\mathbb{Z}^{m \times l}$ with Q_j (the jth column of Q) contained in I for $j = 1, \ldots, l$. Furthermore, assume that all Q_j are distinct and let $w \in \mathbb{Z}^l_+$ be a vector with non-negative entries. Then $p = \mathcal{F}(Q, w)$ is a a point in $\mathbb{Z}^I_+$ defined as follows:

$$p_i = \begin{cases} w_j & if\ i = Q_j, \\ 0 & otherwise. \end{cases}$$

Therefore $||p||_1 = ||w||_1$ and $\sum_{i \in I} p_i \cdot i = Qw$ and $supp(p) = \{Q_j\}_{\{j=1,\ldots,l\ :\ w_j > 0\}}$. From this point on, whenever we say that $p = \mathcal{F}(Q, w)$, we are implicitly assuming that Q is a matrix in $\mathbb{Z}^{m \times l}$ with distinct columns contained in I and $w \in \mathbb{Z}^l_+$.

Conversely, if $y \in \mathbb{Z}^I$ is a vector where $supp(y) \subseteq \{Q_1, \ldots, Q_k\}$, then we define $v = \mathcal{F}^{-1}(Q, y)$ as a vector in $\mathbb{Z}^k_+$ such that for all $j = 1, \ldots, k$, $v_j = y_{Q_j}$. Note that $Qv = \sum_{i \in I} y_i \cdot i$ and $||v||_1 = ||y||_1$. Using the above definitions, $\bar{x}$ in Example 5.1 equals $\mathcal{F}([a,b,c], [1,1,2]^T)$ and $[1,1,2]^T = \mathcal{F}^{-1}([a,b,c], \bar{x})$.

We next formalize this example and produce increasing lower bounds on $k^*(m)$ for $m \geq 3$. We start with the following definition.

Definition 5.2 *Let* $p = \mathcal{F}(Q, w) \in \mathbb{Z}^I_+$. *The point* p *is called* a bad point *for* $K^m(n, r)$ *if it satisfies the following conditions: (i)* $r = Qw \in [1, n]^m$, *(ii)* $||w||_1 > 2$, *and, (iii)* $Qt \notin I$ *for all* $t \in \mathbb{Z}^l_+$ *with* $1 < ||t||_1 < ||w||_1$.

Further, we say p *is* an extremely bad point *if in addition to being a bad point,* Q_j *has a component of* $-n$ *for all* $j = 1, \ldots, l$.

Using this definition, the point $\bar{x}$ of Example 5.1 is a bad point. We next formally state that bad points lead to lower bounds on $k^*(m)$.

Lemma 5.3 *Let $p \in \mathbb{Z}_+^I$ be a bad point, for $K^m(n, r_p)$. Then $k^*(m) \geq ||p||_1$.*

Proof. The proof essentially follows from Example 5.1. Consider the inequality $\bar{\pi}x \geq 1$ where $\bar{\pi}_i$ takes the value 1 if $p_i = 0$ and 0 otherwise. Clearly $\bar{\pi}p = 0 \not\geq 1$ while $\bar{\pi}$ satisfies k-term subadditivity for all $k < \sum_{i \in I} p_i$. ■

Clearly, the point presented in Example 5.1 is not an extremely bad point. However, it is easy to see that the following point $q^3 = \mathcal{F}(Q^3, w^3) \in K^3(10, \mathbf{1})$ is an extremely bad point where

$$w^3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad Q^3 = \begin{bmatrix} -10 & 10 & 1 \\ 1 & -10 & 10 \\ 10 & 1 & -10 \end{bmatrix}.$$

Define $r^3 = \mathbf{1}$ and $n^3 = 10$. We next describe an iterative procedure that, for $m > 3$, constructs an extremely bad point $q^m \in K^m(n^m, r^m)$ using $q^{m-1} \in K^{m-1}(n^{m-1}, r^{m-1})$. The points $\{q^m\}$ constructed by this procedure will have the property that $||q^m||_1$ would grow exponentially with m.

Given $q^{m-1} = \mathcal{F}(Q^{m-1}, w^{m-1})$, we define *the scaling factor* $\alpha = 2||w^{m-1}||_1$ and let $n^m = \alpha n^{m-1}$. We define $Q^m \in \mathbb{Z}^{m \times m}$ and $w^m \in Z_+^m$ as follows:

$$w^m = \begin{bmatrix} w^{m-1} \\ ||w^{m-1}||_1 - 1 \end{bmatrix} \quad \text{and} \quad Q^m = \left[\begin{array}{c|c} & -1 \\ \alpha Q^{m-1} & \vdots \\ & -1 \\ \hline n^m \dots n^m & -n^m \end{array}\right]. \tag{18}$$

Finally, we define the new point q^m as $\mathcal{F}(Q^m, w^m)$. Let $r^m = Q^m w^m$; then

$$r^m = \begin{bmatrix} \alpha Q^{m-1} w^{m-1} - (||w^{m-1}||_1 - 1)\mathbf{1_{m-1}} \\ n^m ||w^{m-1}||_1 - n^m(||w^{m-1}||_1 - 1) \end{bmatrix} = \begin{bmatrix} \alpha \cdot r^{m-1} - (||w^{m-1}||_1 - 1) \cdot \mathbf{1}_{m-1} \\ n^m \end{bmatrix}.$$

As $r^{m-1} \in [1, n^{m-1}]^{m-1}$ and $\alpha = 2||w^{m-1}||_1$, one can verify that $r^m \in [1, n^m]^m$. Note that $||w^m||_1 = 2||w^{m-1}||_1 - 1$, and therefore $||q^m||_1 = ||w^m||_1 = 2^{m-2} + 1$ for all $m \geq 3$.

Also notice that if all diagonal entries of Q^{m-1} are equal to $-n^{m-1}$, then all diagonal entries of αQ^{m-1} are equal to $-\alpha n^{m-1}$ and therefore all diagonal entries of Q^m are equal to $-n^m$. As this condition holds for $m = 3$, it holds for all Q^m with $m \geq 3$.

Lemma 5.4 *The points $\{q^m\}$ generated by the procedure above are extremely bad for their corresponding $K^m(n^m, r^m)$, for $m \geq 3$.*

Proof. The case $m = 3$ can be easily checked to be true. When $m > 3$, due to the construction, we have (i) $r^m \in [1, n^m]^m$ and (ii) $||q^m||_1 > 2$. Furthermore, all columns of Q^m have a component of $-n^m$. Let $I = I^m(n^m)$ and $\alpha = n^m/n^{m-1} = 2||q^{m-1}||_1$. To see that q^m satisfies the remaining condition for being an extremely bad point, let $t \in \mathbb{Z}_+^m$ satisfy $1 < ||t||_1 < ||q^m||_1$. Let $s = Qt$, and $t^- = [t_1, \ldots, t_{m-1}]^T$.

We next show that $s = Qt \notin I$ by considering the following cases.

[*Case 1*] $||t^-||_1 = 0$. As $t_m = ||t||_1 > 1$, the last component of s is strictly less than n^m and $s \notin I$.

[*Case 2*] $||t^-||_1 = 1$. In this case let $t_j = 1$ for some $m-1 \geq j \geq 1$ and note that $t_m \geq 1$. As the j'th diagonal entry of Q^m is $-n^m$ and as the last column only has negative entries, we have $s \notin I$ as its j'th row is strictly less than $-n^m$.

[*Case 3*] $||q^{m-1}||_1 > ||t^-||_1 > 1$. Let $I^- = I^{m-1}(n^{m-1})$. As q^{m-1} is an extremely bad point for $K^{m-1}(n^{m-1}, r^{m-1})$ and $||q^{m-1}||_1 > ||t^-||_1 > 1$, we have $Q^{m-1}t^- \notin I^-$. In other words, $Q^{m-1}t^-$ has a component that is not contained in $[-n^{m-1}, n^{m-1}]$. Therefore, $\alpha Q^{m-1}t^-$ has a component with absolute value greater than or equal to $\alpha(n^{m-1}+1) = n^m + \alpha$. As the first $m-1$ rows of s are given by $\alpha Q^{m-1}t^- - t_m \mathbf{1_{m-1}}$ and $t_m < ||t||_1 < ||q^m||_1 < \alpha$, we have $s \notin I$.

[*Case 4*] $||t^-||_1 \geq ||q^{m-1}||_1$. In this case, $t_m \leq ||t||_1 - ||q^{m-1}||_1 < ||q^m||_1 - ||q^{m-1}||_1 = ||q^{m-1}||_1 - 1$, implying that $t_m \leq ||q^{m-1}||_1 - 2$. Therefore, the m'th component of s is at least $||q^{m-1}||_1 n^m - (||q^{m-1}||_1 - 2)n^m = 2n^m$ and $s \notin I$. ∎

As $||q^m||_1 = 2^{m-2}+1$, Lemma 5.4 together with Lemma 5.3 imply that $k^*(m) \geq 2^{m-2}+1$. We next present another family of points and show that this lower bound on $k^*(m)$ can be improved. Let $p^3 = \mathcal{F}(P^3, v^3)$ be the bad point for $K^3(\bar{n}^3, \bar{r}^3)$ presented in Example 5.1, where $P^3 = [a, b, c]$, $v^3 = [1,1,2]^T$, $\bar{n}^3 = 10$ and $\bar{r}^3 = \mathbf{2}$. For $m \geq 4$, let $q^{m-1} = \mathcal{F}(Q^{m-1}, w^{m-1})$ be the extremely bad point constructed above and define the scaling factor $\beta = 4||q^{m-1}||_1$ and let $\bar{n}^m = \beta n^{m-1}$. We next define the point $p^m = \mathcal{F}(P^m, v^m)$ where

$$v^m = \begin{bmatrix} w^{m-1} \\ 2||w^{m-1}||_1 - 2 \end{bmatrix} \quad \text{and} \quad P^m = \left[\begin{array}{c|c} \beta Q^{m-1} & \begin{array}{c} -1 \\ \vdots \\ -1 \end{array} \\ \hline \bar{n}^m \ldots \bar{n}^m & -\bar{n}^m/2 - 1 \end{array}\right].$$

and note that $\bar{n}^m$ is even and $-\bar{n}^m/2 - 1$ is indeed an integer. Also note that all entries of P^m are in $[-\bar{n}^m, \bar{n}^m]$. Let $\bar{r}^m = P^m v^m$, then,

$$\begin{aligned} \bar{r}^m &= \begin{bmatrix} \beta Q^{m-1} w^{m-1} - (2||w^{m-1}||_1 - 2)\mathbf{1_{m-1}} \\ \bar{n}^m ||w^{m-1}||_1 - \bar{n}^m(||w^{m-1}||_1 - 1) - (2||w^{m-1}||_1 - 2) \end{bmatrix} \\ &= \begin{bmatrix} \beta \cdot r^{m-1} - 2(||w^{m-1}||_1 - 1) \cdot \mathbf{1} \\ \bar{n}^m - 2||w^{m-1}||_1 + 2 \end{bmatrix}. \end{aligned}$$

Note that as $||w^{m-1}||_1 > 1$ we have $\bar{r}^m < \bar{n}^m \mathbf{1}$. In addition, $\bar{r}^m \geq \mathbf{1}$ as (i) $r^{m-1} \geq \mathbf{1}$ and therefore $\beta r^{m-1} \geq \beta = 4||q^{m-1}||_1 = 4||w^{m-1}||_1 > 2||w^{m-1}||_1 + 2$. Furthermore, (ii) $\bar{n}^m > 2||q^m||_1$.

Lemma 5.5 *The points $\{p^m\}$ generated by the procedure above are bad points for their corresponding $K^m(\bar{n}^m, \bar{r}^m)$, for all $m > 3$.*

Proof. The proof is very similar to that of Lemma 5.4. We again have (i) $\bar{r}^m \in [1, \bar{n}^m]^m$ and (ii) $||p^m||_1 > 2$. Let $I = I^m(\bar{n}^m)$ and $\beta = \bar{n}^m/n^{m-1} = 4||q^{m-1}||_1$. Assume $t \in \mathbb{Z}_+^m$ satisfy $1 < ||t||_1 < ||p^m||_1$ and let $s = Qt$, and $t^- = [t_1, \ldots, t_{m-1}]^T$.

If $||q^{m-1}||_1 > ||t^-||_1$, then the proof is the same as Lemma 5.4, substituting α by β, n^m by $\bar{n}^m$, Q^m by P^m and q^m by p^m.

Therefore we need to consider $||t^-||_1 \geq ||q^{m-1}||_1$. In this case, $t_m \leq ||t||_1 - ||q^{m-1}||_1 < ||p^m||_1 - ||q^{m-1}||_1 = 2||q^{m-1}||_1 - 2$, implying that $t_m \leq 2||q^{m-1}||_1 - 3$. Therefore, the m'th component of s is at least $||q^{m-1}||_1\bar{n}^m - (\bar{n}^m/2+1)(2||q^{m-1}||_1 - 3) = (\bar{n}^m+3) + (\bar{n}^m - 4||q^{m-1}||_1)/2$. As $\bar{n}^m = \beta n^{m-1} = 4||q^{m-1}||_1 n^{m-1}$, the m'th component of s is strictly greater than $\bar{n}^m$ and we have $s \notin I$. ∎

Remember that $||q^m||_1 = 2^{m-2}+1$ for all $m \geq 3$ and therefore $||p^m||_1 = 3\cdot 2^{m-3}+1$ for all $m > 3$. Combining this observation and the bad point $\bar{x} \in K^3(10, \mathbf{2})$ of Example 5.1 with Lemma 5.3, we have the following result:

Corollary 5.6 *For all $m \geq 3$, there exists a vector π of appropriate dimension that satisfies up to $(3/8)2^m$-term subadditivity conditions and yet the corresponding inequality $\pi^m x \geq 1$ is not valid for $K^m(n, r)$. In other words, $k^*(m) \geq (3/8)2^m + 1$.*

In particular, this shows that $k^*(3) \geq 4$ and hence from the results in Section 4, $k^*(3) = 4$.

6 Lower bounds on the size of the description of a polaroid

In Section 5 we derived lower bounds on the smallest number $k^*(m)$ that would guarantee that any π that satisfies $k^*(m)$-term subadditivity and (NC1) would yield a valid inequality $\pi x \geq 1$ for $K^m(n, r)$. Even though this lower bound grows exponentially in m, this does not necessarily imply that one can not obtain a compact description of a polaroid. In particular, recall from the discussion in Section 4 that all nontrivial facet-defining inequalities of $K^m(n, r)$ satisfy the complementarity conditions (RCO) and the normalization conditions (NC1) and (NC2). It is therefore, possible that, for some $k^{**}(m) < k^*(m)$, $k^{**}(m)$-term subadditivity conditions plus (NC1), (NC2) *and* (RCO) would give a description of a polaroid. In this section, we will show that, even in the presence of (NC1), (NC2) and (RCO), $k^{**}(m)$-term subadditivity does not guarantee validity if $k^{**}(m)$ is small. In fact, the same lower bounds for $k^*(m)$ derived from the bad points constructed in the previous section still hold for $k^{**}(m)$.

We next show that bad points that satisfy some additional conditions lead to lower bounds on $k^{**}(m)$. We will later show that the bad points constructed in the previous section satisfy such conditions.

Lemma 6.1 *Let $m \geq 3$. Let $p = \mathcal{F}(Q, w)$ be a bad point for $K^m(n, r)$ such that $0 < w \in \mathbb{Z}^l$. If $Q(w - t) \notin I$ for all $t \in \mathbb{Z}_+^l$ such that $1 \leq ||t||_1 < (||p||_1 + 1)/2$, then there exists $\pi \in \mathbb{R}^I$ that satisfies ($||p||_1$-1)-term subadditivity,* (RCO)*,* (NC1) *and* (NC2) *and such that $\pi x \geq 1$ is not valid for $K^m(n, r)$.*

Proof. Let $\gamma = ||p||_1 + 1$. Construct $\pi \in \mathbb{R}^I$ as follows:

$$\pi_i = \begin{cases} 1/\gamma & \text{if } i \in supp(p), \\ 1 & \text{if } i = r, \\ 0 & \text{if } i = 0, \\ 1/2 & \text{otherwise.} \end{cases}$$

The inequality $\pi x \geq 1$ is not valid for $K^m(n,r)$ as $p \in K^m(n,r)$, yet $\pi p = ||p||_1/(||p||_1 + 1) < 1$.

Claim: π satisfies (RCO):

Recall that $I_r = \{i \in I : r - i \in I\}$ and note that $I_r \cap supp(p) = \emptyset$. Indeed, if this were not true and $p_j > 0$ for some $j \in I_r$, then let $c(j)$ be the index of the column of Q that is equal to j and define $w' = w - e_{c(j)}$, where $e_{c(j)} \in \mathbb{Z}^l$ is the unit vector with a 1 in the $c(j)$-th component. But then $w' \in \mathbb{Z}^l_+$ and $r = Qw = Qw' + Q_{c(j)} = Qw' + j$ and thus $Qw' = r - j \in I$, which contradicts the fact that p is a a bad point. Therefore, for all $i \in I_r$, we have $\pi_i + \pi_{r-i} = 1$.

Claim: π satisfies ($||p||_1$-1)-term subadditivity:

If for some k with $2 \leq k \leq ||p||_1 - 1$, π violates a k-term subadditivity condition, then

$$\sum_{i \in I} q_i \pi_i < \pi_{\bar{q}} \leq 1, \tag{19}$$

for some $q \in \mathbb{Z}^I_+$ satisfying $\bar{q} := \sum_{i \in I} q_i \cdot i \in I$ and $k = ||q||_1 \leq ||p||_1 - 1$. We may assume that $q_0 = 0$, since otherwise another k'-term subadditivity condition is violated for $2 \leq k' < k$.

If $supp(q) \subseteq supp(p) = \{Q_1, \ldots, Q_l\}$ then let $t = \mathcal{F}^{-1}(Q, q)$ and note that $\sum_{i \in I} q_i \cdot i = Qt \notin I$ as p is a a bad point. Therefore there exists some $j \in I$ such that $j \notin supp(p)$ and $q_j \geq 1$. But (19) implies that $\sum_{i \notin supp(p)} q_i \leq 1$ and $j \neq r$. Thus $\pi_j = 1/2$ and $q_j = 1$, which also implies that $\pi_{\bar{q}} = 1 \iff \sum_{i \in I} q_i \cdot i = r$.

We have argued above that $\{i : i \neq j, q_i > 0\}$ is contained in $supp(p)$ and since $\pi_j = 1/2$, (19) implies that

$$\sum_{i \neq j} q_i \pi_i < 1/2 \Rightarrow \sum_{i \neq j} q_i < \gamma/2 = (||w||_1 + 1)/2.$$

But then, let $q' \in \mathbb{Z}^I_+$ be such that $q'_i = q_i$ for $i \in supp(p)$, $q'_i = 0$ for $i \notin supp(p)$. Note that $1 \leq ||q'||_1 = \sum_{i \neq j} q_i < (||w||_1 + 1)/2$. Now let $t' = \mathcal{F}^{-1}(Q, q')$. As $r = Qw = \sum_{i \in I} q_i \cdot i = j + Qt'$, we have $Qw - Qt' = j \in I$. But this contradicts the assumption on p given in the Lemma.

Modify π to satisfy (NC2):

We have shown so far that π satisfies $\pi_r = 1$, (RCO), (NC1) and k-term subadditivity for $k \leq ||p||_1 - 1$. However, it may not satisfy (NC2), i.e., π_{-ne_j} may not equal zero. In order to make π satisfy (NC2), we just need to add multiples of the equations $\sum_{i \in I} ix_i = r$ to $\pi x \geq 1$, with the multiple for the jth equation being π_{-ne_j}. More formally, let $v \in \mathbb{R}^m$ stand for the vector with $v_j = \pi_{-ne_j}$, for $j = 1, \ldots, m$. Let $\delta = 1/(1 + r^T v/n)$ and $\pi'_i = \delta(\pi_i + i^T v/n)$ for $i \in I$. By definition, $\pi'_j = 0$ for all $j \in I_N$, $\pi'_r = 1$ and for all $i \in I_r$ we have

$$\pi'_i + \pi'_{r-i} = \delta\left(\pi_i + i^T v/n + \pi_{r-i} + (r-i)^T v/n\right) = \delta\left(1 + r^T v/n\right) = 1.$$

Further, π' also satisfies k-term subadditivity for $2 \le k \le ||p||_1 - 1$. This is because for all $q \in \mathbb{R}^{|I|}$ such that $||q||_1 \le ||p||_1 - 1$ with $\sum_{i \in I} q_i \cdot i \in I$, we have

$$\frac{1}{\delta}\sum_{i\in I} q_i\pi'_i = \sum_{i\in I} q_i\pi_i + \sum_{i\in I} q_i \cdot i^T v/n \ge \pi_{\bar{q}} + (\sum_{i\in I} q_i \cdot i)^T v/n = \frac{1}{\delta}\pi'_{\bar{q}}.$$

Finally, as p satisfies the constraints $\sum_{i\in I} ix_i = r$,

$$\pi' p = \sum_{i\in I} \pi'_i p_i = \delta(\pi p + r^T v/n) < 1,$$

because $\pi p < 1$ and $r^T v > 0$. ■

We now show that the bad points $\{q^m\}$ and $\{p^m\}$ constructed in Section 5 satisfy the conditions in Lemma 6.1. We first start with an observation about bad points.

Observation 6.2 *Let $q = \mathcal{F}(Q, w)$ be a a bad point. Then $Q(w - t) \notin I$ for any $t \in \mathbb{Z}^l_+$ such that $1 \le ||t||_1 \le ||w||_1 - 2$ and $t \le w$.*

Proof. The proof follows trivially from the definition of a bad point and the fact that $(w - t)$ is a nonnegative vector with norm at least 2 and at most $||q||_1 - 1$. ■

Notice that, for the bad point q^3, the condition $1 \le ||t||_1 < (||w^3||_1 + 1)/2$ translates to $||t||_1 = 1$. Since $w^3 \ge \mathbf{1}$, by Observation 6.2, q^3 satisfies the condition that $Q^3(w^3 - t) \notin I$ for any $t \in \mathbb{Z}^l_+$ such that $1 \le ||t||_1 < (||w^3||_1 + 1)/2$. We now prove a slightly stronger statement for $m \ge 4$.

Lemma 6.3 *The points $\{q^m = \mathcal{F}(Q^m, w^m)\}$ generated in Section 5 have the property that, for all $m \ge 4$, if $t \in \mathbb{Z}^m_+$ satisfies $1 \le ||t||_1 \le (||w^m||_1 + 1)/2$, then $Q^m(w^m - t) \notin I^m(n^m)$.*

Proof. Let $I = I^m(n^m)$ and $\alpha = n^m/n^{m-1} = 2||w^{m-1}||_1$. Notice that $w^m \ge \mathbf{1}$, so by Observation 6.2, if $||t||_1 = 1$, then $Q^m(w^m - t) \notin I$. Thus, let $t \in \mathbb{Z}^m_+$ satisfy $2 \le ||t||_1 \le (||w^m||_1 + 1)/2$. Let $t^- = [t_1, \dots, t_{m-1}]^T$; therefore $||t||_1 = ||t^-||_1 + t_m$. We have that

$$Q^m w^m = \begin{bmatrix} \alpha \cdot Q^{m-1}w^{m-1} - (||w^{m-1}||_1 - 1)\cdot \mathbf{1}_{m-1} \\ n^m \end{bmatrix} \text{ and } Q^m t = \begin{bmatrix} \alpha Q^{m-1}t^- - t_m \mathbf{1}_{m-1} \\ (||t^-||_1 - t_m)n^m \end{bmatrix}.$$

Therefore, if $||t^-||_1 - t_m \le -1$ or $||t^-||_1 - t_m \ge 3$, then $Q^m(w^m - t) \notin I$, since its last component will not be in $[-n^m, n^m]$. We can thus assume $0 \le ||t^-||_1 - t_m \le 2$ for any value of $m \ge 4$.

Note that $0 \le ||t^-||_1 - t_m \le 2$ and $||t||_1 = ||t^-||_1 + t_m$ imply that $1 \le ||t^-||_1 \le (||t||_1 + 2)/2$ and $t_m \le ||t||_1/2$ for all $m \ge 4$. The fact that $||t||_1 \le (||w^m||_1 + 1)/2 = ||w^{m-1}||_1$ implies that $||t^-||_1 \le (||w^{m-1}||_1 + 2)/2$. But $||w^{m-1}||_1$ is odd and hence $||t^-||_1 \le (||w^{m-1}||_1 + 1)/2$ for all $m \ge 4$.

We prove the result for all m by induction.

For the base case $m = 4$, we have that $w = [1, 1, 1, 2]^T$ and $||w^4||_1 = 5$, so we need to check that for all $t \in \mathbb{Z}^4_+$ with $||t||_1 \le 3$, $Q^4(w^4 - t) \notin I$.

Note that $t_4 \le ||t||_1/2 \Rightarrow t_4 \le 1$ and $||t^-||_1 \le (||w^3||_1 + 1)/2 \Rightarrow ||t^-||_1 \le 2$. But by Observation 6.2, we may assume $t \not\le w^4$, and hence we must have that $t_j = 2$ for some $j = 1, \dots, 3$ and

$t_i = 0$ for all $i \in \{1,2,3\} \setminus \{j\}$. But, by construction, the j-th component of Q^4_j has value $-n^4$. Since the j-th component of Q^4_4 equals -1, then the j-th component $Q^4 t$ is strictly less than $-n^4$ and hence $Q^4(w^4 - t) \notin I$.

Now assume Lemma 6.3 is true for q^{m-1} for some $m \geq 5$.

Since $||t^-||_1 \leq (||w^{m-1}||_1 + 1)/2$, by the induction hypothesis, $Q^{m-1}(w^{m-1} - t^-) \notin I^{m-1}(n^{m-1})$, and hence $Q^{m-1}(w^{m-1} - t^-)$ has a component with absolute value strictly greater than n^{m-1}. Therefore, $\alpha Q^{m-1}(w^{m-1} - t^-)$ has a component with absolute value greater than or equal to $\alpha(n^{m-1} + 1) = n^m + \alpha$.

The first $m-1$ components of $Q^m(w^m - t)$ are equal to $\alpha \left(Q^{m-1}(w^{m-1} - t^-)\right) + (t_m - ||w^{m-1}||_1 + 1)\mathbf{1}_{m-1}$. As $0 \leq t_m \leq ||t||_1/2 < ||w^{m-1}||_1/2$ we have that $|t_m - ||w^{m-1}||_1 + 1| \leq ||w^{m-1}||_1/2 + ||w^{m-1}||_1 + 1 = 1.5||w^{m-1}||_1 + 1 < \alpha = 2||w^{m-1}||_1$. Hence, one of the first $m-1$ components of $Q^m(w^m - t)$ has absolute value strictly greater than n^m, therefore $Q^m(w^m - t) \notin I$. ∎

Using Lemma 6.3, we next show that the bad points $\{p^m\}$ also satisfy the desired property.

Lemma 6.4 *For all integers $m \geq 3$ the points $\{p^m = \mathcal{F}(P^m, v^m)\}$ constructed in Section 5 have the property that for any $t \in \mathbb{Z}^m_+$ with $1 \leq ||t||_1 < (||p^m||_1 + 1)/2$, $P^m(v^m - t) \notin I^m(\bar{n}^m)$.*

Proof. Let $I = I^m(\bar{n}^m)$. Recall from Section 5 that $p^m = \mathcal{F}(P^m, v^m)$ is a bad point for $K^m(\bar{n}^m, \bar{r}^m)$ for $m \geq 3$ where P^m has m columns and $v^m \in \mathbb{R}^m$, and $||p^m||_1 = ||v^m||_1$. Notice that $v^m \geq \mathbf{1}$, so by Observation 6.2, if $||t||_1 = 1$, then $P^m(v^m - t) \notin I$ for $m \geq 3$. Thus, let $t \in \mathbb{Z}^m_+$ satisfy $2 \leq ||t||_1 < (||v^m||_1 + 1)/2$.

For p^3, the only possible value for $||t||_1$ is 2, and since $v^3 = [1,1,2]^T$, we may assume by Observation 6.2 that $t_j = 2$ for some $j \in \{1,2\}$ and $t_i = 0$ for $i \neq j$. But then, the j-th component of $P^3 t$ equals $-2\bar{n}^3$ and hence $P^3(v^m - t) \notin I$.

For p^4, $2 \leq ||t||_1 \leq 3$. Since $v^4 = [1,1,1,4]^T$, we may assume by Observation 6.2 that $t_j \geq 2$ for some $j \in \{1,2,3\}$. Hence, $t_4 \leq 1$. If $t_4 = 0$, then the last component of $P^4 t$ is at least $2\bar{n}^4$ and since the last component of $P^4 v^4$ is strictly less than n^4, we have that $P^4(v^4 - t) \notin I$. Thus, $t_4 = 1$ and $t_j = 2$, but then the j-th component of $P^4 t$ is strictly less than $-\bar{n}^m$ and hence $P^4(v^4 - t) \notin I$.

Therefore, assume that $m \geq 5$. Recall that $\bar{n}^m = \beta n^{m-1}$, where $\beta = 4||q^{m-1}||_1$. Let $t \in \mathbb{Z}^m_+$ satisfy *(i)* $2 \leq ||t||_1 < (||p^m||_1 + 1)/2$. Recall that $||p^m||_1 = 3||q^{m-1}||_1 - 2$, and $||q^{m-1}||_1$ is an odd number, which means that $||p^m||_1$ is an odd number and $||t||_1 \leq (3||q^{m-1}||_1 - 3)/2$.

Let $t^- = [t_1, \ldots, t_{m-1}]^T$; therefore $||t||_1 = ||t^-||_1 + t_m$. We have

$$\bar{r}^m = P^m v^m = \left[\begin{array}{c} \beta \cdot r^{m-1} - 2(||q^{m-1}||_1 - 1) \cdot \mathbf{1} \\ \bar{n}^m - 2||q^{m-1}||_1 + 2 \end{array} \right] \text{ and } P^m t = \left[\begin{array}{c} \beta Q^{m-1} t^- - t_m \cdot \mathbf{1} \\ (||t^-||_1 - t_m/2)\bar{n}^m - t_m \end{array} \right]$$

Since $\bar{n}^m = 4||q^{m-1}||_1 n^{m-1}$, we have that $\bar{n}^m/2 < \bar{r}^m_m$. If $||t^-||_1 - t_m/2 \leq -1/2$, then the last component of $P^m(v^m - t)$ is greater than $\bar{n}^m$ and $P^m(v^m - t) \notin I$. Therefore $||t^-||_1 - t_m/2 \geq 0$. Further if $||t^-||_1 - t_m/2 \geq 2$, then the last component of $P^m(v^m - t)$ is less than or equal to $-\bar{n}^m - 2(||q^{m-1}||_1 - 1) + t_m$. However as $t_m \leq ||t||_1 < 3||q^{m-1}||_1/2$, the last component of $P^m(v^m - t)$ is less than $-\bar{n}^m$ and $P^m(v^m - t) \notin I$.

Therefore

$$||t^-||_1 - t_m/2 \le 3/2 \;\Rightarrow\; 2||t^-||_1 \le t_m + 3 \;\Rightarrow\; 3||t^-||_1 \le ||t||_1 + 3$$

$$\Rightarrow 3||t^-||_1 \le (3||q^{m-1}||_1 - 3)/2 + 3$$

$$\Rightarrow\; ||t^-||_1 \le (||q^{m-1}||_1 + 1)/2.$$

Using Lemma 6.3, we can assert that one of the first $m-1$ components of $\beta(r^{m-1} - Q^{m-1}t^-)$ has absolute value greater than or equal to $\beta(n^{m-1}+1) = \bar{n}^m + \beta$. Now the first $m-1$ components of $P^m(v^m - t)$ are equal to $\beta(r^{m-1} - Q^{m-1}t^-) + (t_m - 2||q^{m-1}||_1 + 2) \cdot \mathbf{1}_{m-1}$. As $t_m < 3||q^{m-1}||_1/2$ and $\beta = 4||q^{m-1}||_1$, we have that $|t_m - 2||q^{m-1}||_1 + 2| \le t_m + 2||q^{m-1}||_1 + 2 < 3.5||q^{m-1}||_1 + 2$ and hence one of the first $m-1$ components of $P^m(v^m - t)$ has absolute value strictly greater than $\bar{n}^m$.
■

We have therefore shown that bad points lead to an exponential lower bound on $k^{**}(m)$. More formally:

Corollary 6.5 *For all $m \ge 3$, there exists a vector π of appropriate dimension that satisfies up to $(3/8)2^m$-term subadditivity conditions,* (RCO), (NC1) *and* (NC2), *and yet the corresponding inequality $\pi^m x \ge 1$ is not valid for $K^m(n,r)$. In other words, $k^{**}(m) \ge (3/8)2^m + 1$.*

This also leads to the following observation about polaroids.

Corollary 6.6 *If the following set*

$$T_k^m(n,r) = \begin{cases} \pi_i + \pi_{r-i} &= \pi_r, & \forall i \in I_r, \\ \sum_{i\in I} q_i\pi_i &\ge \pi_{\bar{q}}, & \forall q \in \mathbb{Z}_+^I : ||q||_1 \le k, \quad \bar{q} := \sum_{i\in I} q_i \cdot i \in I \\ \pi_r &= 1, & \\ \pi_i &= 0, & \forall i \in I_N. \end{cases}$$

describes a polaroid for $K^m(n,r)$, for all $n \in \mathbb{Z}_+ \setminus \{0\}$ and $r \in [0,n]^m \cap \mathbb{Z}^m \setminus \{\mathbf{0}\}$, then $k \ge (3/8)2^m + 1$.

7 Conclusions

In this paper we analyzed $K^m(n,r)$, the m-row extension of the MEP defined in [6]. We were unable to give a compact description of a nontrivial polar for $K^m(n,r)$ for $m \ge 2$. A straightforward extension of Theorem 1.2 to $K^2(n,r)$ seems not to be possible as indicated by Example 2.6. However, by using the notion of a polaroid, which can be viewed as a weakening of the notion of a nontrivial polar, we show how to solve the separation problem for $K^2(n,r)$ and $K^3(n,r)$ by solving a linear program. This result yields simple, LP based, pseudo-polynomial time separation algorithms for the set $Q = \text{conv}(\{x \in \mathbb{Z}^t : Ax = b, x \ge 0\})$ where A has two or three rows. Alternative pseudo-polynomial algorithms for the above separation problem (A can have any fixed number of rows) can be obtained by combining Papadimitriou's [17] dynamic programming algorithm to optimize a linear function over Q with the "equivalence of separation and optimization" result of Grötschel, Lovász and Schrijver [13].

Gomory showed earlier that the notions of two-term subadditivity and validity are essentially equivalent for $P^m(n,r)$. This equivalence also holds for $K^1(n,r)$ [6], and for $K^2(n,r)$ (Theorem 4.4) but not for $K^m(n,r)$ for $m \geq 3$. We proved that subadditivity conditions with a sub-exponential number of terms do not imply validity when $m \geq 3$. The above fact suggests that nontrivial polars or polaroids for $K^m(n,r)$ have substantially more complicated descriptions when $m \geq 3$ than the corresponding objects in the case of $P^m(n,r)$, $K^1(n,r)$ and $K^2(n,r)$.

Our results providing compact descriptions of polaroids for $K^2(n,r)$ and $K^3(n,r)$ do not depend on the parameters n and r as they are based on the following simple property of vectors in $[-1,1]^m$ when $m = 2$ or 3: If S is a collection of vectors in $[-1,1]^m$ where $\sum_{b \in S} b \in [-1,1]^m$, then S contains a subset S' such that $|S'|$ is at most $\tau(m)$ and $\sum_{b \in S'} b \in [-1,1]^m$. For $m \geq 4$, we have obtained exponential lower bounds on $\tau(m)$ but we have not been able to obtain upper bounds. An upper bound on $\tau(m)$ would be an interesting result as it would imply, based on our results on polaroids, the existence of LP based separation algorithms for $K^m(n,r)$.

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