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Computational experience with modified conjugate gradient methods for unconstrained optimization

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Abstract:

In this report, several modifications of the nonlinear conjugate gradient method are described and investigated. Theoretical properties of these modifications are proved and their practical performance is demonstrated using extensive numerical experiments.

Keywords:
Numerical optimization, conjugate direction methods, conjugate gradient methods, global convergence, numerical experiments.

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1 Introduction

Conjugate gradient methods are widely studied and used for unconstrained minimization of function $F : R^n \to R$, see [1]–[75], [77]–[117], [120]–[133]. These methods are descent direction methods. It means that after introducing a starting approximation $x_1 \in R^n$ they generate a sequence of points $\{x_i\} \subset R^n$ by the rule

$$x_{i+1} = x_i + \alpha_i s_i, \quad i \in N, \tag{1}$$

where $s_i \in R^n$ is a direction vector satisfying a descent condition $s_i^T g(x_i) < 0$ ($g(x_i)$ is a gradient of the function F at a point x_i), and $\alpha_i > 0$ is a step-length chosen in such a way that the generalized Wolfe conditions

$$F(x_{i+1}) - F(x_i) \le \varepsilon_1 \alpha_i s_i^T g(x_i), \tag{2}$$

$$\varepsilon_2 s_i^T g(x_i) \le s_i^T g(x_{i+1}) \le \varepsilon_3 |s_i^T g(x_i)|, \tag{3}$$

with $0 < \varepsilon_1 < \varepsilon_2 < 1$ and $\varepsilon_3 \ge 0$, are satisfied, see [118]–[119]. If $\varepsilon_3 = \varepsilon_2$, we speak about the strong Wolfe conditions and if $\varepsilon_3 = \infty$, we speak about the weak Wolfe conditions. In case that $s_i^T g(x_{i+1}) = 0$, the step length is exact, otherwise is inexact.

We will use a shortened notation $F_i = F(x_i)$, $g_i = g(x_i)$, $G_i = G(x_i)$, $i \in N$, where $G(x_i)$ is a Hessian matrix of the function F at a point x_i. We will assume that function F is twice continuously differentiable and satisfies the conditions

$$F(x_i) \ge \underline{F} \qquad \forall x_i \in R^n, \tag{4}$$

$$\|G(x_i)\| \le \overline{G} \qquad \forall x_i \in R^n, \tag{5}$$

where $\underline{F}$ and $\overline{G}$ are suitable constants. We will investigate methods where vectors $s_i \in R^n$, $i \in N$, satisfy the condition

$$-s_i^T g_i \ge \varepsilon_0 \|g_i\|^2, \tag{6}$$

with $\varepsilon_0 > 0$. It can be shown (see e.g. [24]) that this condition implies the inequality

$$\sum_{i=1}^{\infty} \frac{\|g_i\|^4}{\|s_i\|^2} < \infty. \tag{7}$$

Definition 1 *We say that a descent direction method is a conjugate gradient method if*

$$s_1 = -g_1 \quad \text{and} \quad s_{i+1} = -g_{i+1} + \beta_i s_i \quad \text{for} \quad i \in N, \tag{8}$$

where a parameter β_i is chosen so that the direction vectors s_i, $1 \le i \le n$, were mutually G–orthogonal, i.e. $s_j^T G s_i = 0$, $1 \le j < i \le n$, if we apply this method to a strictly convex quadratic function

$$Q(x) = \frac{1}{2}(x - x^*)^T G (x - x^*)$$

with an exact choice of a step-length.

If we denote $d_i = x_{i+1} - x_i = \alpha_i s_i$ and $y_i = g_{i+1} - g_i$, then for the quadratic function Q we have $y_i = G d_i$ and the G–orthogonality condition of vectors s_i, s_{i+1} can be written as $\alpha_i s_i^T G s_{i+1} = y_i^T s_{i+1} = 0$ (we assume that $\alpha_i \neq 0$). This together with (8) lead to the equation $\beta_i y_i^T s_i - y_i^T g_{i+1} = 0$ or

$$\beta_i = \frac{y_i^T g_{i+1}}{y_i^T s_i}. \tag{9}$$

It can be shown (see e.g. [107]) that this value already assures mutual orthogonality of direction vectors and gradients and finding a minimum of a strictly convex quadratic function after a finite number of steps if a step-length is exact. We call this property a quadratic termination.

If the step-length is exact, we can write by (8)

$$y_i^T s_i = g_{i+1}^T s_i - g_i^T s_i = -g_i^T s_i = g_i^T g_i - \beta_{i-1} g_i^T s_{i-1} = g_i^T g_i.$$

Moreover, if the minimized function is quadratic, then its gradients are mutually orthogonal, and so

$$y_i^T g_{i+1} = g_{i+1}^T g_{i+1} - g_i^T g_{i+1} = g_{i+1}^T g_{i+1}.$$

This fact implies that we can use three different denominators and two different numerators in expression (9) without violation a quadratic termination property. Thus we obtain six basic conjugate gradient methods:

$$\beta_i^{HS} = \frac{y_i^T g_{i+1}}{y_i^T s_i}, \quad \beta_i^{PR} = \frac{y_i^T g_{i+1}}{g_i^T g_i}, \quad \beta_i^{LS} = \frac{y_i^T g_{i+1}}{|g_i^T s_i|} \tag{10}$$

(HS – Hestenes and Stiefel [62], PR – Polak and Ribiére [94], LS – Liu and Storey [74]),

$$\beta_i^{DY} = \frac{g_{i+1}^T g_{i+1}}{y_i^T s_i}, \quad \beta_i^{FR} = \frac{g_{i+1}^T g_{i+1}}{g_i^T g_i}, \quad \beta_i^{CD} = \frac{g_{i+1}^T g_{i+1}}{|g_i^T s_i|} \tag{11}$$

(DY – Dai and Yuan [32], FR – Fletcher and Reeves [50], CD – conjugate descent [49]).

These methods can be divided into two groups by the numerator used. Methods of the first group (HS, PR, LS) are more suitable for practical computations but they are globally convergent only with necessary modifications. Methods of the second group (DY, FR, CD) are globally convergent under certain assumptions (put on a choice of a step-length) but the direction vectors stay worse conjugate if a step-length is inexact and the minimized function is not quadratic.

The properties of methods (10) can be improved by eliminating negative values, so

$$\beta_i^{HS+} = \max(0, \beta_i^{HS}), \quad \beta_i^{PR+} = \max(0, \beta_i^{PR}), \quad \beta_i^{LS+} = \max(0, \beta_i^{LS}). \tag{12}$$

We use nonnegative values in order to prevent possible cycling [96]. Methods (10) can also be combined with methods (11). Such combined methods use relations

$$\begin{aligned}\beta_i^{HSC} &= \max(0, \min(\beta_i^{HS}, \beta_i^{DY})),\\ \beta_i^{PRC} &= \max(0, \min(\beta_i^{PR}, \beta_i^{FR})),\\ \beta_i^{LSC} &= \max(0, \min(\beta_i^{LS}, \beta_i^{CD})).\end{aligned} \tag{13}$$

Combined methods HSC, PRC, LSC are globally convergent under the same conditions as methods DY, FR, CD. Furthermore, they are efficient for practical computations because values β_i^{HS}, β_i^{PR}, β_i^{LS} are used sufficiently often.

2 Modification of conjugate gradient methods

Relation (8) can be variously modified in order to improve effectiveness of conjugate gradient methods. It is usually performed so that we add terms proportional to $s_i^T g_{i+1}$, which vanish in case of the exact choice of a step-length and the quadratic termination property stays unchanged. One possibility, used in [130], is to replace (8) with

$$s_1 = -g_1 \quad and \quad s_{i+1} = -\left(1 + \beta_i \frac{g_{i+1}^T s_i}{g_{i+1}^T g_{i+1}}\right) g_{i+1} + \beta_i s_i \quad for \quad i \in N, \tag{14}$$

where β_i is one of the values in (10) or (11).

Theorem 1 *Modified conjugate gradient method (14) has the quadratic termination property. Moreover, for $i \in N$ we have*

$$-g_{i+1}^T s_{i+1} = g_{i+1}^T g_{i+1}. \tag{15}$$

Proof If a step-length is exact, one has $g_{i+1}^T s_i = 0$, so (14) will change into (8) and the quadratic termination property will stay unchanged. Multiplying (14) by a vector g_{i+1}, we obtain equality (15). □

If we substitute the value β_i^{CD} into (14), we will get $s_{i+1} = -\vartheta_i^{CD} g_{i+1} + \beta_i^{CD} s_i$, where $\vartheta_i^{CD} = -y_i^T s_i / g_i^T s_i$. Method FR can be modified in a similar way, see [131]. These modifications allow to weaken substantially conditions for global convergence.

Theorem 2 *Consider modified methods DY, FR, CD given by the rule*

$$s_1 = -g_1 \quad and \quad s_{i+1} = -\vartheta_i g_{i+1} + \beta_i s_i \quad for \quad i \in N, \tag{16}$$

where the values β_i^{DY}, β_i^{FR}, β_i^{CD} are determined by (11) and

$$\vartheta_i^{DY} = \frac{y_i^T s_i}{y_i^T s_i} = 1, \quad \vartheta_i^{FR} = \frac{y_i^T s_i}{g_i^T g_i}, \quad \vartheta_i^{CD} = \frac{y_i^T s_i}{|g_i^T s_i|}. \tag{17}$$

These methods have the quadratic termination property. If a function $F : R^n \to R$ satisfies conditions (4)–(5) and if we use generalized Wolfe conditions (2)–(3) with $0 < \varepsilon_1 < \varepsilon_2 < 1$ and $0 \le \varepsilon_3 < \infty$ during a choice of a step-length, then these methods are globally convergent.

Proof If a step-length is exact, then $y_i^T s_i = -g_i^T s_i = g_i^T g_i$, or $\vartheta_i^{DY} = \vartheta_i^{FR} = \vartheta_i^{CD} = 1$, so (16) changes into (8) and the quadratic termination property stays unchanged. Now we will prove global convergence.

(a) Since $\vartheta_i^{DY} = 1$, method DY is unchanged using (16), so global convergence follows from the theorem proved in [32].

(b) For modified method FR we have

$$g_{i+1}^T s_{i+1} = -y_i^T s_i \frac{g_{i+1}^T g_{i+1}}{g_i^T g_i} + \frac{g_{i+1}^T g_{i+1}}{g_i^T g_i} g_{i+1}^T s_i = \frac{g_{i+1}^T g_{i+1}}{g_i^T g_i} g_i^T s_i < 0.$$

Since $g_1^T s_1 = -g_1^T g_1$, we obtain equality (15) with sequential substituting into the previous relation (by induction). Thus modified method FR is identical to modified method CD and equality (15) is fulfilled for both methods.

(c) For modified method CD, equality (15) is fulfilled. Therefore, direction vectors s_i, $i \in N$, are descent and (6) holds with $\varepsilon_0 = 1$, which implies inequality (7). Because generalized Wolfe conditions (2)–(3) are used during a choice of a step-length, we have

$$y_i^T s_i = g_{i+1}^T s_i - g_i^T s_i \le \varepsilon_3 |g_i^T s_i| - g_i^T s_i = (1+\varepsilon_3)|g_i^T s_i|,$$

or $\vartheta_i \le 1+\varepsilon_3$. If we use this estimate together with relations (15)–(17), we can write

$$\begin{aligned}
\|s_{i+1}\|^2 &= \left(-\vartheta_i g_{i+1} + \frac{\|g_{i+1}\|^2}{|g_i^T s_i|} s_i\right)^T \left(-\vartheta_i g_{i+1} + \frac{\|g_{i+1}\|^2}{|g_i^T s_i|} s_i\right) \\
&= \vartheta_i^2 \|g_{i+1}\|^2 - 2\vartheta_i \frac{\|g_{i+1}\|^2}{|g_i^T s_i|} g_{i+1}^T s_i + \frac{\|g_{i+1}\|^4}{|g_i^T s_i|^2} \|s_i\|^2 \\
&\le (1+\varepsilon_3)^2 \|g_{i+1}\|^2 + 2\varepsilon_2(1+\varepsilon_3)\|g_{i+1}\|^2 + \frac{\|g_{i+1}\|^4}{|g_i^T s_i|^2}\|s_i\|^2,
\end{aligned}$$

or

$$\frac{\|s_{i+1}\|^2}{\|g_{i+1}\|^4} \le \frac{(1+\varepsilon_3)(1+2\varepsilon_2+\varepsilon_3)}{\|g_{i+1}\|^2} + \frac{\|s_i\|^2}{\|g_i\|^4}.$$

Now suppose that

$$\liminf_{i\to\infty} \|g_i\| = 0$$

does not hold. Then there exists a constant $\underline{\varepsilon} > 0$ such that $\|g_i\| \ge \underline{\varepsilon}\ \forall i \in N$, so from the previous inequality it follows that

$$\frac{\|s_{i+1}\|^2}{\|g_{i+1}\|^4} \le \frac{(1+\varepsilon_3)(1+2\varepsilon_2+\varepsilon_3)}{\underline{\varepsilon}^2} + \frac{\|s_i\|^2}{\|g_i\|^4} \le \frac{(1+\varepsilon_3)(1+2\varepsilon_2+\varepsilon_3)}{\underline{\varepsilon}^2}(i+1)$$

(we assume without loss of generality that $\underline{\varepsilon}^2\|s_1\|^2/\|g_1\|^4 \le (1+\varepsilon_3)(1+2\varepsilon_2+\varepsilon_3)$). Thus

$$\sum_{i=1}^{\infty} \frac{\|g_i\|^4}{\|s_i\|^2} \ge \frac{\underline{\varepsilon}^2}{(1+\varepsilon_3)(1+2\varepsilon_2+\varepsilon_3)} \sum_{i=1}^{\infty} \frac{1}{i} = \infty,$$

which is in contradiction with inequality (7). $\square$

It follows from Theorem 2 that modification (16) allows to weaken conditions for global convergence of methods FR [1] and CD [30]. It suffices to choose a step-length by generalized Wolfe conditions (2) and (3), where $\varepsilon_3 \geq 0$ is arbitrarily large but finite number. This condition does not differ much from the weak Wolfe conditions, where $\varepsilon_3 = \infty$.

Relation (16) can also be used to improve conjugation of direction vectors in methods PR and LS.

Theorem 3 *Consider modifications of methods HS, PR, LS given by the rule*

$$s_1 = -g_1 \quad and \quad s_{i+1} = -\vartheta_i g_{i+1} + \beta_i s_i \quad for \quad i \in N,$$

where the values β_i^{HS}, β_i^{PR}, β_i^{LS} are determined by (10) and

$$\vartheta_i^{HS} = \frac{y_i^T s_i}{y_i^T s_i} = 1, \quad \vartheta_i^{PR} = \frac{y_i^T s_i}{g_i^T g_i}, \quad \vartheta_i^{LS} = \frac{y_i^T s_i}{|g_i^T s_i|}. \tag{18}$$

Then the quadratic termination property stays unchanged and moreover,

$$y_i^T s_{i+1} = 0 \quad for \quad i \in N. \tag{19}$$

Proof As in the proof of Theorem 2 we have $\vartheta_i^{HS} = \vartheta_i^{PR} = \vartheta_i^{LS} = 1$, if a step-length is exact. So (16) changes into (8) and the quadratic termination property stays unchanged. Method HS, for which (19) holds, is unchanged. In case of methods PR and LS we obtain

$$y_i^T s_{i+1} = -\frac{y_i^T s_i}{g_i^T g_i} y_i^T g_{i+1} + \frac{y_i^T g_{i+1}}{g_i^T g_i} y_i^T s_i = 0$$

and

$$y_i^T s_{i+1} = -\frac{y_i^T s_i}{|g_i^T s_i|} y_i^T g_{i+1} + \frac{y_i^T g_{i+1}}{|g_i^T s_i|} y_i^T s_i = 0.$$

$\square$

Formula (16) does not assure a descent of direction vectors of methods HS, PR, LS. This requirement is guaranteed by relation (14) or by formula

$$s_1 = -g_1 \quad \text{and} \quad s_{i+1} = -g_{i+1} + \beta_i s_i - \gamma_i y_i \quad \text{for} \quad i \in N, \tag{20}$$

where the values β_i^{HS}, β_i^{PR}, β_i^{LS} are determined by (10) and

$$\gamma_i^{HS} = \frac{g_{i+1}^T s_i}{y_i^T s_i}, \quad \gamma_i^{PR} = \frac{g_{i+1}^T s_i}{g_i^T g_i}, \quad \gamma_i^{LS} = \frac{g_{i+1}^T s_i}{|g_i^T s_i|}, \tag{21}$$

see [132]. Multiplying (20) by a vector g_{i+1} we can easily check a validity of (15). From the practical point of view, formula (20) is less efficient than (14).

Basic conjugate gradient methods can also be combined so that we choose

$$\beta_i = \frac{\lambda_i^1 g_{i+1}^T y_i + \lambda_i^2 g_{i+1}^T g_{i+1}}{\mu_i^1 y_i^T s_i + \mu_i^2 g_i^T g_i - \mu_i^3 g_i^T s_i} = \frac{g_{i+1}^T (g_{i+1} - \lambda_i^1 g_i)}{\mu_i^1 y_i^T s_i + \mu_i^2 g_i^T g_i - \mu_i^3 g_i^T s_i}, \tag{22}$$

where λ_i^1, λ_i^2, μ_i^1, μ_i^2, μ_i^3 are nonnegative numbers such that $\lambda_i^1 + \lambda_i^2 = 1$ and $\mu_i^1 + \mu_i^2 + \mu_i^3 = 1$. One possibility is the choice $\lambda_i^1 = \min(1, \|g_{i+1}\|/\|g_i\|)$, see [117], [122], which leads to modifications

$$\beta_i^{HSM} = \frac{g_{i+1}^T \tilde{y}_i}{y_i^T s_i}, \quad \beta_i^{PRM} = \frac{g_{i+1}^T \tilde{y}_i}{g_i^T g_i}, \quad \beta_i^{LSM} = \frac{g_{i+1}^T \tilde{y}_i}{|g_i^T s_i|}, \tag{23}$$

where

$$\tilde{y}_i = g_{i+1} - \min\left(1, \frac{\|g_{i+1}\|}{\|g_i\|}\right) g_i. \tag{24}$$

Effectiveness of conjugate gradient methods can be improved by suitable restarts. This is performed so that we test fulfilling a prescribed condition after computation of a direction vector. If this condition is not satisfied, then the computed direction vector is replaced with a negative gradient (which corresponds to a choice $\beta_i = 0$). It is very convenient to test a uniform descent condition $-g_{i+1}^T s_{i+1} \geq \varepsilon_0 \|g_{i+1}\| \|s_{i+1}\|$, where $\varepsilon_0 > 0$ is a small number (e.g. $\varepsilon_0 = 10^{-8}$). Such a modified conjugate gradient method is globally convergent without occurring restarts too often. If methods (11) are used, then it is suitable to test a conjugation of direction vectors. In this case, we interrupt the iteration process if the condition

$$y_i^T s_{i+1} \leq \eta_1 \|s_{i+1}\| \|y_i\| \tag{25}$$

does not hold, where the value η_1 depends on the Wolfe conditions chosen. It is also possible to test orthogonality of gradients. The iteration process is restarted if

$$g_i^T g_{i+1} \leq \eta_2 \|g_{i+1}\| \|g_i\| \tag{26}$$

does not hold, where the value η_2 again depends on the Wolfe conditions chosen. If the number of variables is sufficiently large, then it is worth interrupting the iteration process after every n steps counted from the last restart.

3 Numerical experiments

We present a numerical comparison of conjugate gradient methods for minimization of 60 test functions taken from [76] with 1000 variables (NIT is a total number of iterations, NFV is a total number of function evaluations, F is a total number of failures, and T is a total computational time in seconds). The first table contains the results for methods using strong Wolfe conditions (2)–(3) with $\varepsilon_1 = 10^{-4}$, $\varepsilon_2 = 10^{-1}$ and $\varepsilon_3 = 10^{-1}$; the value $\eta_1 = 0.05$ is used in condition (25). The second table contains the results for methods using weak Wolfe conditions (2)–(3) with $\varepsilon_1 = 10^{-4}$, $\varepsilon_2 = 0.9$ and $\varepsilon_3 = \infty$; the value $\eta_1 = 0.2$ is used in condition (25). The third table contains the results for methods using a special line search described in [57].

From the data stated in Tables 1-3 we can deduce several conclusions:

Realization	Method HS NIT	NFV	F	T	Method PR NIT	NFV	F	T	Method LS NIT	NFV	F	T
(10)	139774	212632	-	49.26	174261	283226	2	58.61	168932	265329	2	53.94
(13)	119210	180182	-	40.63	161040	249186	2	55.91	147376	226064	2	46.45
(14)	124991	192584	-	45.04	135756	205897	-	45.66	135960	206329	-	46.86
(16)	139774	212632	-	49.17	141557	225649	-	50.20	142055	221960	-	47.20
(20)	138044	225193	-	50.52	146081	234783	-	52.39	145463	234396	-	52.22
(23)	135025	201969	-	44.91	170704	276052	2	55.48	172540	272644	2	58.91
	Method DY				Method FR				Method CD			
Realization	NIT	NFV	F	T	NIT	NFV	F	T	NIT	NFV	F	T
(11)	205447	267994	4	52.13	227498	321939	5	64.92	257187	346818	5	73.49
(14)	269027	367475	6	68.78	211859	277978	4	53.12	215700	282033	5	56.02
(16)	205447	267994	4	52.17	211783	278629	4	55.93	210948	272413	5	54.56
(11) + (25)	136535	218387	1	46.86	150147	234202	2	50.67	140509	222035	1	44.91
(14) + (25)	142833	228778	1	47.40	147208	221234	2	45.91	141765	221279	2	51.25
(16) + (25)	136535	218387	1	46.75	139604	223049	-	48.05	141318	217956	1	48.50

Table 1: The strong Wolfe conditions.

Realization	Method HS NIT	NFV	F	T	Method PR NIT	NFV	F	T	Method LS NIT	NFV	F	T
(10)	278645	350503	2	65.84	239625	315005	1	49.40	254874	338751	1	56.36
(13)	309881	386999	3	77.77	275419	354945	2	60.77	318272	404547	4	73.03
(14)	298873	371150	3	74.45	249197	309460	1	59.55	267303	332440	1	60.00
(16)	278645	350503	2	66.50	99198	229502	-	47.72	300630	374527	3	71.94
(20)	419046	619145	6	116.39	303858	406525	3	75.53	303835	405806	3	71.49
(23)	313271	362428	4	77.94	264157	348745	1	63.61	285764	375206	3	66.36
	Method DY				Method FR				Method CD			
Realization	NIT	NFV	F	T	NIT	NFV	F	T	NIT	NFV	F	T
(11)	267710	272646	4	56.38	371666	451181	5	89.36	450286	509099	10	106.05
(14)	513154	588460	9	123.89	275391	280049	4	58.13	286283	291727	5	63.76
(16)	267710	272646	4	56.31	270054	274863	4	57.53	276589	281664	4	60.75
(11) + (25)	192988	206408	1	51.00	249538	302095	2	61.09	255854	301273	1	59.34
(14) + (25)	368022	439971	6	75.36	231142	245748	2	55.53	195636	209153	1	50.39
(16) + (25)	192988	206408	1	50.84	186556	199588	1	47.20	196423	210158	1	51.24

Table 2: The weak Wolfe conditions.

Realization	Method HS NIT	NFV	F	T	Method PR NIT	NFV	F	T	Method LS NIT	NFV	F	T
(10)	100585	300068	-	72.78	104260	308173	-	74.28	102661	307106	-	69.94
(13)	89728	268513	-	62.41	89847	268416	-	58.94	96601	289303	-	69.89
(14)	93631	282202	-	60.84	98031	293426	-	71.72	105638	315271	1	83.44
(16)	100614	300229	-	72.70	103395	308503	-	61.99	92335	276284	-	62.92
(20)	101023	300800	1	73.56	94165	280980	-	69.16	101486	301235	-	79.20
(23)	93373	277795	-	68.81	96783	288230	-	63.99	101383	303307	-	71.19
	Method DY				**Method FR**				**Method CD**			
Realization	NIT	NFV	F	T	NIT	NFV	F	T	NIT	NFV	F	T
(11)	163046	489494	5	96.78	165868	496121	5	98.31	177997	530560	4	107.88
(14)	169538	502837	2	122.53	162684	486854	4	94.77	161931	484030	4	92.84
(16)	163046	489494	5	96.88	162315	485926	5	92.76	165392	494846	4	99.43
(11) + (25)	108705	328674	1	81.20	96632	292572	1	58.92	135110	404208	1	83.34
(14) + (25)	119501	356686	2	88.36	105748	320766	1	74.63	108234	328114	1	83.95
(16) + (25)	109326	328889	1	81.29	109797	332741	1	81.97	108563	329042	1	78.84

Table 3: The special Hager–Zhang line search.

- It is advantageous to use the strong Wolfe conditions with $\varepsilon_2 = 10^{-1}$ (this value was obtained experimentally) at a realization of conjugate gradient methods, particularly methods HS, PR, LS, and their modifications.
- In case that we use the strong Wolfe conditions, method HS gives the best results. Combination (13) or modifications (14) and (16) (particularly (14)) improve effectiveness of methods HS, PR, LS. Modification (20) improves effectiveness of methods PR and LS. Modification (23) slightly improves effectiveness of method HS.
- In case that we use the strong Wolfe conditions, methods DY, FR, CD give worse results than methods HS, PR, LS. The properties of methods DY, FR, CD are considerably improved if they are restarted each time condition (25) is not fulfilled. The choice of a value η_1 in (25) depends on the Wolfe conditions used (a suitable value must be determined experimentally).
- In general, modifications (14) and (16) considerably improve effectiveness of methods FR and CD. This observation is independent of a choice of the Wolfe conditions which confirms a significance of Theorem 2. If we supplement the stated modifications with conjugation test (25), then the resulting methods are competitive with the best modifications of methods HS and PR. Modification (14) is unsuitable for method DY.
- In case that we use the weak Wolfe conditions, method PR gives better results than methods HS (particularly if we use modification (16)). Modifications (20) and (23) are unsuitable.
- A special choice of a step-length described in [57] allows to find a more accurate solution. Properties of individual methods and their modifications are in this case in accord with the previous conclusions.

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