DIDO'S PROBLEM AND PARETO OPTIMALITY

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The birth of the theory of extremal problems is usually tied with the mythical Phoenician Princess Dido. Virgil told about the escape of Dido from her treacherous brother in the first chapter of *The Aeneid*. Dido had to decide about the choice of a tract of land near the future city of Carthage, while satisfying the famous constraint of selecting "a space of ground, which (Byrsa call'd, from the bull's hide) they first inclos'd." By the legend, Phoenicians cut the oxhide into thin strips and enclosed a large expanse. Now it is customary to think that the decision by Dido was reduced to the isoperimetric problem of finding a figure of greatest area among those surrounded by a curve whose length is given. It is not excluded that Dido and her subjects solved the practical versions of the problem when the tower was to be located at the sea coast and part of the boundary coastline of the tract was somehow prescribed in advance.

The foundation of Carthage is usually dated to the ninth century B.C.E. when there was no hint of the Euclidean geometry, the cadastral surveying was the job of harpedonaptae, and measuring the tracts of land was used in decision making. Rope-stretching around stakes leads to convex figures. The Dido problem has a unique solution in the class of convex figures provided that the fixed nonempty part of the boundary is a convex polygonal line.

Decision making has become a science in the twentieth century. The presence of many contradictory conditions and conflicting interests is the main particularity of the social situations under control of today. Management by objectives is an exceptional instance of the stock of rather complicated humanitarian problems of goal agreement which has no candidates for a unique solution.

The extremal problems of optimizing several parameters simultaneously are collected nowadays under the auspices of *vector* or *multiobjective optimization*. Search for control in these circumstances is *multiple criteria decision making*. The mathematical apparatus of these areas of research is not rather sophisticated at present (see [1, 2] and the references therein).

The today's research deals mostly with the concept of *Pareto optimality* (e.g., [3–6]). Let us explain this approach by the example of a bunch of economic agents each of which intends to maximize his own income. The Pareto principle asserts that as an effective agreement of the conflicting goals it is reasonable to take any state in which nobody can increase his income in any way other than diminishing the income of at least one of the other fellow members. Formally speaking, this implies the search of the maximal elements of the set comprising the tuples of incomes of the agents at every state; i.e., some vectors of a finite-dimensional arithmetic space endowed with the coordinatewise order. Clearly, the concept of Pareto optimality was already abstracted to arbitrary ordered vector spaces (for more details see [7–10]).

The variational principles of mechanics, precursors of variational calculus, served at least partly to justifying the Christian belief in the unicity and beauty of the act of creation. The extremal problems, generously populating all branches of mathematics, use only scalar targets. Problems with many objectives have become the topic of research rather recently and noticeably beyond mathematics, which explains the substantial gap between the levels of complexity and power of the mathematical tools available for single objective and multiple objective problems. This challenges the task of enriching the stock of vector optimization problems within the theoretical core of mathematics.

For the sake of simplicity, it stands to reason to start with the problems using the concept of Pareto optimality. The point is that such a problem is in fact equivalent to a parametric family of single objective problems which can be inspected by the classical methods. For instance, there is a curve joining the Legendre and Chebyshev polynomials which consists of the polynomials "Pareto-optimal" with respect to the uniform and mean square metrics. Clearly, some physical processes admit description in terms of vector optimization. For instance, we may treat the Leidenfrost effect of evaporation of a liquid drop in the spheroidal state (see [11, pp. 300–303]) as the problem of simultaneous minimization of the surface area and the width of a drop of a given volume.

Under study in this article is the class of geometrically meaningful vector optimization problems whose solutions can be found explicitly to some extend in terms of conditions on surface area measures. As model examples we give explicit solutions of the Urysohn-type problems aggravated by the flattening condition or the requirement to optimize the convex hull of a few figures. Technically speaking, everything reduces to the parametric programming of isoperimetric type problems with many subsidiary constraints along the lines of the approach developed in [12, 13]. At the same time, the functional-analytical technique of settling the extremal problem of convex geometry is still insufficiently popular, and so its somewhat uncommon applications could be of use in bridging the gaps between the research within mathematics and the application of mathematics in the art and science of multiple criteria decision making.

1. The Space of Convex Figures

1.1. The classical *Minkowski duality* is well known to consist in identifying a *convex figure*, i.e. a compact convex subset $\mathfrak{x}$ of the space $\mathbb{R}^N$, and the *support function* of $\mathfrak{x}$ that is defined as $\mathfrak{x}(z) := \sup\{(x, z) \mid x \in \mathfrak{x}\}$ for all $z \in \mathbb{R}^N$. Considering the members of $\mathbb{R}^N$ as singletons, we assume that $\mathbb{R}^N$ lies in the set of all convex figures $\mathcal{V}_N$ of $\mathbb{R}^N$. The Minkowski duality makes $\mathcal{V}_N$ into a cone in the space $C(S_{N-1})$ of continuous functions on the unit Euclidean sphere S_{N-1}, the boundary of the ball $\mathfrak{z}_N$. This parametrization is called the *Minkowski structure* on $\mathcal{V}_N$. The addition of the support functions of $\mathfrak{x}$ and $\mathfrak{y}$ corresponds to the algebraic sum $\mathfrak{x} + \mathfrak{y}$ called the *Minkowski sum* of $\mathfrak{x}$ and $\mathfrak{y}$. Note that the *linear span* $[\mathcal{V}_N]$ of $\mathcal{V}_N$ is dense in $C(S_{N-1})$. All these circumstances were mentioned in the classical papers on the theory of mixed volumes by Alexandrov who constantly used the ideas and tools of functional analysis in his geometrical writings (see [14]). Later the embedding of the set of convex figures into function spaces became the topic of research of many authors.

1.2. A *convex body* is a solid convex figure, i.e. a compact convex set with an interior point. The boundary of a convex body is a *(complete) convex surface*. The coset $\{z + \mathfrak{x} \mid z \in \mathbb{R}^N\}$ of all translates of a convex body $\mathfrak{x}$ is identified with the corresponding measure $\mu(\mathfrak{x})$ on the sphere S_{N-1} which is called the *surface area measure* of each member of the class. To the complete polyhedral convex surface $\mathfrak{x}$ given by the unit outer normals $z_1, \dots, z_m$ of the facets (i.e., $(N-1)$-dimensional faces) of area $s_1, \dots, s_m$ there corresponds the weighed sum of the Dirac deltas at $z_1, \dots, z_m$. In other words, $\mu(\mathfrak{x}) = \sum_{k=1}^{M} s_k \varepsilon_{z_k}$. The surface area measure of an arbitrary convex body $\mathfrak{x}$ may be defined as the weak limit of the inclusion-ordered net of inscribed convex polyhedra in $\mathfrak{x}$. Each surface area measure is an *Alexandrov measure*. So we call a positive measure on S_{N-1} which is not supported by any great hypersphere (the intersection of S_{N-1} with a hypersubspace) and which annihilates points (i.e. vanishes at the restrictions to S_{N-1} of linear functionals on $\mathbb{R}^N$). The soundness of this identification bases on the celebrated Alexandrov theorem on the reconstruction of a convex surface from its surface area measure (see [14, p. 108]). This theorem was published in 1938.

Each Alexandrov measure annihilates the restrictions of the support functions of singletons to the unit Euclidean sphere. In convex geometry this property of a linear functional is called *translation invariance*. The cone of translation-invariant positive linear functionals in the *dual space* $C'(S_{N-1})$ is denoted by $\mathcal{A}_N$. We now specify some of the concepts to be of use in the sequel.

1.3. Let $\mathcal{V}_N$ be the set of convex figures in $\mathbb{R}^N$. Given $\mathfrak{x}, \mathfrak{y} \in \mathcal{V}_N$, we write $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ in case $\mathfrak{x}$ and $\mathfrak{y}$ are translates of one another. We may say that $=_{\mathbb{R}^N}$ is the equivalence induced by the preorder $\geq_{\mathbb{R}^N}$ on $\mathcal{V}_N$ that expresses the possibility of inserting one convex figure into the other by parallel translation. Consider the factor-set $\mathcal{V}_N/\mathbb{R}^N$ comprising the cosets of translates of the members of $\mathcal{V}_N$. Clearly, $\mathcal{V}_N/\mathbb{R}^N$ is a cone in the factor-space $[\mathcal{V}_N]/\mathbb{R}^N$ of $[\mathcal{V}_N]$ by $\mathbb{R}^N$.

1.4. There is a natural bijection between $\mathcal{V}_N/\mathbb{R}^N$ and $\mathcal{A}_N$. The coset of singletons is identified with the zero measure. To the coset of the straight line segment with endpoints x and y there corresponds the measure $|x-y|(\varepsilon_{(x-y)/|x-y|} + \varepsilon_{(y-x)/|x-y|})$, where $|\cdot|$ is the Euclidean length of a vector in $\mathbb{R}^N$. If the dimension of the *affine hull* $\operatorname{aff}(\mathfrak{x})$ of a member $\mathfrak{x}$ of some coset in $\mathcal{V}_N/\mathbb{R}^N$ is greater than 1 then we view $\operatorname{aff}(\mathfrak{x})$ as a subspace of $\mathbb{R}^N$ and identify the coset of $\mathfrak{x}$ with the surface area measure $\mathfrak{x}$ in $\operatorname{aff}(\mathfrak{x})$ which is now some measure on $S_{N-1} \cap \operatorname{aff}(\mathfrak{x})$. Extending this measure trivially to the measure on S_{N-1}, we come to the member of $\mathcal{A}_N$ corresponding to the coset of $\mathfrak{x}$. The bijective property of this correspondence is an easy consequence of the Alexandrov theorem.

The structure of a vector space in the set of regular Borel measures on S_{N-1} induces in $\mathcal{A}_N$ (hence, in $\mathcal{V}_N/\mathbb{R}^N$) the structure of a cone or, in more detail, the structure of an $\mathbb{R}_+$-operator commutative semigroup with cancellation. This is the *Blaschke structure* on $\mathcal{V}_N/\mathbb{R}^N$. Observe that the sum of the surface area measures of convex bodies $\mathfrak{x}$ and $\mathfrak{y}$ generates the unique class $\mathfrak{x}\#\mathfrak{y}$ which is called the *Blaschke* *sum* of $\mathfrak{x}$ and $\mathfrak{y}$. In details this construction was described by W. Firey in [15]. About the procedures of constructing Blaschke sums see [16].

1.5. Let $C(S_{N-1})/\mathbb{R}^N$ be the factor space of $C(S_{N-1})$ by the subspace of the restrictions of linear functionals on $\mathbb{R}^N$ to S_{N-1}. Denote by $[\mathcal{A}_N]$ the space $\mathcal{A}_N - \mathcal{A}_N$ of translation-invariant signed measures. Clearly, $[\mathcal{A}_N]$ is the linear span of the set of all Alexandrov measures. The spaces $C(S_{N-1})/\mathbb{R}^N$ and $[\mathcal{A}_N]$ are paired by the canonical bilinear form

$$\langle f, \mu \rangle = \frac{1}{N} \int\limits_{S_{N-1}} f\, d\mu \quad (f \in C(S_{N-1})/\mathbb{R}^N,\ \mu \in [\mathcal{A}_N]).$$

Given $\mathfrak{x} \in \mathcal{V}_N/\mathbb{R}^N$ and $\mathfrak{y} \in \mathcal{A}_N$, we see that $\langle \mathfrak{x}, \mathfrak{y} \rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y}, \mathfrak{x})$. In particular, if $\mathfrak{z}_N$ is the unit Euclidean ball in $\mathbb{R}^N$ then $V_1(\mathfrak{x}, \mathfrak{z}_N)$ is proportional to the *surface area* of $\mathfrak{x}$ and $V_1(\mathfrak{z}_N, \mathfrak{x})$, to the *integral breadth* of $\mathfrak{x}$. Recall that the *breadth* $b_z(\mathfrak{x})$ of a convex figure $\mathfrak{x}$ in the direction $z \in \mathbb{R}_N$ is defined as $b_z(\mathfrak{x}) := \frac{1}{2}(\mathfrak{x}(z) + \mathfrak{x}(-z))$. Note also that $V_1(\mathfrak{x}, \mathfrak{x})$ is the *volume* of $\mathfrak{x}$. The space $[\mathcal{A}_N]$ is usually endowed with the weak topology with respect to the above-defined duality between $[\mathcal{A}_N]$ and $C(S_{N-1})/\mathbb{R}^N$.

The importance of the above constructions spreads far beyond the definition of a new sum of convex surfaces. The presence of the dual pair of nonreflexive Banach spaces is curiously combined with the Alexandrov theorem establishing an uncommon and powerful isomorphism between the ordering cones in these spaces. These phenomena are exceptional in functional analysis, opening up extra opportunities to apply abstract methods. Considering the convex surfaces $\mathfrak{x}$ with the same *support* $\operatorname{supp}(\mathfrak{x})$ of the surface area measure $\mu(\mathfrak{x})$, we see that they comprise a (punctured) cone in the Blaschke structure. If the support consists of finitely many points then we deal with the collection of all polyhedra with the prescribed outer normals of facets. Geometry knows the *Lindelöf problem* which leads to the extremal property of the polyhedron circumscribed about the ball.

1.6. We do not distinguish between a convex figure, the corresponding coset of translates in $\mathcal{V}_N/\mathbb{R}^N$ and the respective measure in $\mathcal{A}_N$ whenever this leads to no confusion. In this event it is customary to use the same symbol for each of the hypostases of a geometrical object.

Note that the *volume* $V(\mathfrak{x}) := \langle \mathfrak{x}, \mathfrak{x} \rangle$ of $\mathfrak{x}$ is an N-degree homogeneous polynomial with respect to the Minkowski addition. Therefore, it is an easy matter to calculate the subdifferential of its relevant power. However, the volume loses this property with respect to the Blaschke addition in the space of dimension at least 3. In what follows we will use the mappings $p : \mathfrak{x} \mapsto V^{1/N}(\mathfrak{x})$ for $\mathfrak{x} \in \mathcal{V}_N/\mathbb{R}^N$ and $\hat{p} : \mathfrak{x} \mapsto V^{(N-1)/N}(\mathfrak{x})$ for $\mathfrak{x} \in \mathcal{A}_N$. Hence, the *Minkowski inequality* may be rephrased as $\langle \mathfrak{x}, \mathfrak{y} \rangle \geq p(\mathfrak{x})\hat{p}(\mathfrak{y})$. By the Brunn–Minkowski theorem p is a superlinear functional on $\mathcal{V}_N$ with respect to the Minkowski addition. This implies that $\hat{p}$ is superlinear on $\mathcal{A}_N$ with respect to the Blaschke addition. Since the *surface area* of $\mathfrak{x}$ is written as $S(\mathfrak{x}) = \langle \mathfrak{z}_N, \mathfrak{x} \rangle$, the isoperimetric problem[1] turns into a convex program in the Blaschke structure.

[1]It seems somewhat pedantic to use the term "isovolume problem."

1.7. Isoperimetric Problem:

$$\mathfrak{x} \in \mathcal{A}_N;\ \langle \mathfrak{z}_N, \mathfrak{x}\rangle = \langle \mathfrak{z}_N, \bar{\mathfrak{x}}\rangle;\ \hat{p}(\mathfrak{x}) \to \sup.$$

The simplest convex program in the Minkowski structure is the *Urysohn problem* of maximizing the volume of $\mathfrak{x}$ given the integral breadth of $\mathfrak{x}$ (see [17]).

1.8. Urysohn Problem:

$$\mathfrak{x} \in \mathcal{V}_N;\ \langle \mathfrak{x}, \mathfrak{z}_N\rangle = \langle \bar{\mathfrak{x}}, \mathfrak{z}_N\rangle;\ p(\mathfrak{x}) \to \sup.$$

Recall that to a convex subset U of a real vector space X and a point $\bar{x}$ of U there corresponds the set

$$K_{\bar{x}} := \operatorname{Fd}(U, \bar{x}) := \{h \in X \mid (\exists \alpha \geq 0)\ x + \alpha h \in U\},$$

called the *cone of feasible directions* of U at $\bar{x}$. Clearly, the search for optimal solutions of the above problems requires the calculations of the derivatives of the volume of a convex figure in the feasible directions of $\mathcal{V}_N$ and $\mathcal{A}_N$. Let $p_{\bar{\mathfrak{x}}}$ stand for the derivative of $\mathfrak{y} \in \mathcal{V}_N \mapsto \hat{p}(\bar{\mathfrak{x}})p(\mathfrak{y})$ at $\bar{\mathfrak{x}}$. Similarly, we let $\hat{p}_{\bar{\mathfrak{x}}}$ stand for the derivative of $\mathfrak{y} \in \mathcal{A}_N \mapsto \hat{p}(\mathfrak{y})p(\mathfrak{x})$ at $\bar{\mathfrak{x}}$.

1.9. *We have*
(1) $\hat{p}_{\bar{\mathfrak{x}}}(\mathfrak{g}) = \langle \bar{\mathfrak{x}}, \mathfrak{g}\rangle$ *for all* $\mathfrak{g} \in \mathcal{A}_{N,\bar{\mathfrak{x}}}$*;*
(2) $p_{\bar{\mathfrak{x}}}(g) = \langle g, \bar{\mathfrak{x}}\rangle$ *for all* $g \in \mathcal{V}_{N,\bar{\mathfrak{x}}}$ *(see* [18]*).*

1.10. The details and references on convex geometry are collected in [19]. The Brunn–Minkowski theory is thoroughly presented in [20–22].

2. Linear Majorization and Dual Cones

2.1. By the *dual cone* or *polar* K^* of a given cone K in some vector space X that is in duality with a vector space Y we understand the set of all positive linear functionals on K; i.e., $K^* := \{y \in Y \mid \langle x, y\rangle\}$.

In the situation under study the description of the dual cones we need is readily available.

2.2. *The dual cone* $\mathcal{A}_N^*$ *is the positive cone of* $C(S_{N-1})/\mathbb{R}^N$*.*

2.3. *If* $\bar{\mathfrak{x}} \in \mathcal{A}_N$ *then the dual cone* $\mathcal{A}^*_{n,\bar{\mathfrak{x}}}$ *of the cone of feasible directions of* $\mathcal{A}_N$ *at* $\mathfrak{x}$ *may be written as* $\mathcal{A}^*_{n,\bar{\mathfrak{x}}} = \{f \in \mathcal{A}_N^* \mid \langle \bar{\mathfrak{x}}, f\rangle = 0\}$*.*

2.4. In the long passed year of 1954 Reshetnyak suggested in his unpublished thesis [23] to compare positive measures on the Euclidean sphere S_{N-1} as follows:

A measure μ *linearly majorizes* a measure ν provided that to each partition of S_{N-1} into finitely many disjoint Borel subsets $U_1, \dots, U_m$ there are measures $\mu_1, \dots, \mu_m$ with sum μ such that every difference $\mu_k - \nu|_{U_k}$ annihilates the restrictions to S_{N-1} of all linear functionals on $\mathbb{R}^N$. In this event we write $\mu \gg_{\mathbb{R}^N} \nu$.

2.5. Reshetnyak proved that

$$\int_{S_{N-1}} p d\mu \geq \int_{S_{N-1}} p d\nu$$

for every sublinear functional p on $\mathbb{R}^N$ in case $\mu \gg_{\mathbb{R}^N} \nu$. Thus, he discovered an important tool for generating positive Minkowski-linear functionals over various classes of convex surfaces and functions.

2.6. Loomis suggested an analogous construction in Choquet theory in 1962 (see [24]).

A measure μ *affinely majorizes* a measure ν (both given on a compact convex subset Q of an arbitrary locally convex space X) provided that to each decomposition of ν into finitely many addends $\nu_1, \dots, \nu_m$ there are measures $\mu_1, \dots, \mu_m$ with sum μ such that every difference $\mu_k - \nu_k$ annihilates all restrictions to Q of the affine functions on X. In this event we write $\mu \gg_{\operatorname{Aff}(Q)} \nu$. Many applications of affine majorization are collected in [25].

Cartier, Fell, and Meyer demonstrated in [26] that

$$\int_{Q} f d\mu \geq \int_{Q} f d\nu$$

for every continuous convex function f on Q if and only if $\mu \gg_{\operatorname{Aff}(Q)} \nu$. An analogous necessity claim for linear majorization was established in [27].

By linear majorization it is easy to give some descriptions for the dual cones needed in the analysis of the extremal problems of convex geometry.

2.7. *Let $\mathfrak{x}$ and $\mathfrak{y}$ be convex bodies.*
(1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \mathcal{V}_N^* \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$.
(2) *If* $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ *then* $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$.
(3) $\mathfrak{x} \geq_{\mathbb{R}^2} \mathfrak{y} \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^2} \mu(\mathfrak{y})$.
(4) *If* $\mathfrak{y} - \bar{\mathfrak{x}} \in \mathcal{A}^*_{N,\bar{\mathfrak{x}}}$ *then* $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$.
(5) *If* $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \mathcal{V}^*_{N,\bar{\mathfrak{x}}}$ *then* $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$.
(6) *If* $\bar{\mathfrak{x}}$ *is a regular convex surface then* $\mathcal{V}^*_{N,\bar{\mathfrak{x}}} = 0$.

2.8. The geometrical meaning of the relation $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$ remains vague in case $N \geq 3$. It is worth observing that the converse of 2.7 (2) fails in general. Indeed, no translates of $\mathfrak{x} := \mathfrak{z}_N$ and $\mathfrak{y} := \mathfrak{z}_N + \alpha \mathfrak{z}_{N-2}$ are comparable by inclusion, provided that $\alpha = \beta^{1/(N-2)}$ and β satisfies $2^{N-1}/(2^{N-1}-1) > \beta > 1$. However, $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$.

In fact, applications require the following more detailed version of majorization (see [28]):

2.9. Decomposition Theorem. *Let $H_1, \dots, H_m$ be cones in a vector lattice X. Assume that f and g are positive functionals on X. The inequality*

$$f(h_1 \vee \dots \vee h_m) \geq g(h_1 \vee \dots \vee h_m)$$

holds for all $h_k \in H_k$ *and* $k := 1, \dots, m$ *if and only if to each decomposition of* g *into the sum of* m *positive addends* $g = g_1 + \dots + g_N$ *there corresponds a decomposition of* f *into the sum of* m *positive addends* $f = f_1 + \dots + f_m$ *such that*

$$f_k(h_k) \geq g_k(h_k) \quad (h_k \in H_k;\ k := 1, \dots, m).$$

We may use this theorem for calculating the subdifferential and directional derivative of the integral breadth of the convex hull of a few convex figures and similar aggregates with other mixed volumes.

2.10. *Let* $\nu_1, \dots, \nu_m$ *be positive Borel measures on* S_{N-1}. *The inequality*

$$\sum_{k=1}^{m} \int\limits_{S_{N-1}} \mathfrak{x}_k d\nu_k \leq \int\limits_{S_{N-1}} \operatorname{co}\{\mathfrak{x}_1, \dots, \mathfrak{x}_m\} d\mu(\mathfrak{z}_N)$$

holds for all $\mathfrak{x}_k \in \mathcal{V}_N$ $(k := 1, \dots, m)$ *if and only if there are positive Borel measures* $\mu_1, \dots, \mu_m$ *on* S_{N-1} *such that*

$$\mu_1 + \dots + \mu_m = \mu(\mathfrak{z}_N); \quad \mu_k \gg_{\mathbb{R}^N} \nu_k \ (k := 1, \dots, m).$$

Proof. The right-hand side of the inequality in question is a sublinear functional with respect to the variables $\mathfrak{x}_1, \dots, \mathfrak{x}_m$ which is similar to that in 2.9, while the left-hand side is a member of the subdifferential of the sublinear functional over the mth power of $\mathcal{V}_N$. It suffices to refer to 2.7 (1) and 2.9.

3. Pareto Optimality

Here we will formally discuss some modern concepts of optimality in the problems of multiple criteria decision making.

3.1. Assume that X is a vector space, E is an ordered vector space, $f : X \to E$ is a convex operator, and $C \subset X$ is a convex subset of X. A *vector convex program* we call a pair (C, f), writing it symbolically as

$$x \in C; \quad f(x) \to \inf.$$

A vector program is also referred to as a *multiobjective extremal problem*. The operator f is the *target*, *goal*, or *objective* of (C, f); and C is the *constraint* of (C, f). Each $x \in C$ is called a *feasible solution*. The above record of a vector program reflects the fact that under study is the following extremal problem: Find the greatest lower bound of the values of f on C. In case $C = X$ it is customary to speak about an unconditional or unconstrained problem.

The constraints of an extremal program are usually given as some equalities and inequalities. Assume that $g : X \to F$ is a convex operator; and Λ is a linear operator, a member of the space $L(X, Y)$; while $y \in Y$. Here Y is a vector space and F is an ordered vector space. If some constraints C_1 and C_2 are of the form

$$C_1 := \{x \in C \mid g(x) \leq 0\};$$
$$C_2 := \{x \in X \mid g(x) \leq 0,\ \Lambda x = y\};$$

then we replace (C_1, f) and (C_2, f) with (C, g, f) and (Λ, g, f) or use the more expressive record

$$x \in C; \quad g(x) \le 0; \quad f(x) \to \inf;$$
$$\Lambda x = y; \quad g(x) \le 0; \quad f(x) \to \inf.$$

3.2. An element $e := \inf_{x \in C} f(x)$ (if existent) is the *value* of (C, f). A feasible element x_0 is an *ideal optimum* or *ideal solution* of (C, f) provided that $e = f(x_0)$. Therefore, x_0 is an ideal optimum if and only if x_0 is feasible and $f(x_0)$ is the bottom of the image $f(C)$; i.e., $f(C) \subset f(x_0) + E^+$. As usual, E^+ is the positive cone of E.

It might seem that we observe ideal optima only in the case of scalar problems. Indeed, it is rather improbable that a few real functions attain a minimum at the same point. Notwithstanding this commonsense argument, it is easy to suggest an abstract formalism that treats the various minimum points of different functions as a unique element. Such an abstraction must be viewed as *generalization by dilution.*[2] From a rational point of view, the ideal is practically unattainable and one of the minimal feasible elements should be considered as a reasonable approximation to the ideal.

3.3. Let state the corresponding conception of optimality preciser. It is convenient to assume that E is a preordered vector space; i.e., the positive cone E^+ may fail to be *salient.* In other words, the apex subspace $E_0 := E^+ \cap (-E^+)$ may differ from zero. Given $u \in E$, put $[u] := \{v \in E : u \le v, \ \ v \le u\}$. We write $u \sim v$ whenever $[u] = [v]$. A feasible point x_0 is *ε-Pareto-optimal* for (C, f), where ε is a positive element in the target space E, provided that $f(x_0)$ is a minimal element of the set $f(C) + \varepsilon$; i.e., $(f(x_0) - E^+) \cap (f(C) + \varepsilon) = [f(x_0)]$. In more detail, x_0 is ε-Pareto-optimal provided that $x_0 \in C$ and for all $x \in C$ the inequality $f(x_0) \ge f(x) + \varepsilon$ implies that $f(x_0) \sim f(x) + \varepsilon$. If $\varepsilon = 0$ then we speak about *Pareto-optimal* points. Study of Pareto optimality often bases on *scalarization*, reduction of the vector problem under consideration to a scalar extremal problem with a single goal. Scalarization can be implemented in various fashions. Let us consider one of the available approaches.

3.4. Assume that the preorder $\le$ on E is given by the formula

$$u \le v \ \leftrightarrow \ (\forall l \in \partial q) \ lu \le lv,$$

where $q : E \to \mathbb{R}$ is a sublinear functional, and ∂q is the subdifferential of q. In other words, the cone E^+ has the form $E^+ := \{u \in E \mid (\forall l \in \partial q) \ lu \ge 0\}$.

A feasible solution x_0 is ε-Pareto-optimal for (C, f) if and only if for each $x \in C$ either $f(x_0) \sim f(x)+\varepsilon$ or there is $l \in \partial q$ satisfying $lf(x_0) < l(f(x)+\varepsilon)$. In particular, for a ε-Pareto-optimal point $x_0 \in C$ we have $\inf_{x \in C} q(f(x) - f(x_0) + \varepsilon) \ge 0$. The converse is false, since the last inequality amounts to the following weaker concept of optimality.

3.5. Call $x_0 \in C$ *weakly ε-Pareto-optimal* provided that to each $x \in C$ there is $l \in \partial q$ satisfying $l(f(x) - f(x_0) + \varepsilon) \ge 0$; i.e., the simultaneous strict inequalities $lf(x_0) < l(f(x) + \varepsilon)$, with $l \in \partial q$, are incompatible for any $x \in C$. Clearly, the weak ε-Pareto optimality of x_0 is equivalent to the fact that $q(f(x) - f(x_0) + \varepsilon) \ge 0$ for

[2] This term belongs to Polya as mentioned by Weyl.

all $x \in C$, and the concept is nontrivial only if $0 \notin \partial q$. It is possible to proceed along these lines in the wake of Robinsonian nonstandard analysis on using infinitesimal parameters ε (particularities and details are collected in [10, Ch. 5]).

3.6. Subdifferential calculus shows that the Pareto-optimal points are solutions of some problem of parametric programming in the simplest case of an optimization problem with finitely many scalar goals. For instance, in the case of two goals each Pareto-optimal point minimizes a weighted sum of the goals.

4. Multiobjective Model Problems

The above facts enable us to address the multiple criteria extremal problems of geometry which involve the goals and constraints with the available directional derivatives and the duals of the cones of feasible directions. Transition to Pareto-optimality actually involves the scalar problems with bulkier objectives. The manner of combining the geometrical and functional-analytical tools remains practically the same as in the case of a single goal typical of an isoperimetric-type problem. Therefore, we will proceed by way of example and present here a few model multiobjective problems that are connected with the Blaschke and Minkowski structures.

4.1. Vector Isoperimetric Problem: Given are convex bodies $\mathfrak{y}_1, \dots, \mathfrak{y}_m$. Find a convex body $\mathfrak{x}$ encompassing a given volume and minimizing each of the mixed volumes $V_1(\mathfrak{x}, \mathfrak{y}_1), \dots, V_1(\mathfrak{x}, \mathfrak{y}_m)$. In symbols,

$$\mathfrak{x} \in \mathcal{A}_N;\ \hat{p}(\mathfrak{x}) \geq \hat{p}(\bar{\mathfrak{x}});\ (\langle \mathfrak{y}_1, \mathfrak{x} \rangle, \dots, \langle \mathfrak{y}_m, \mathfrak{x} \rangle) \to \inf.$$

Clearly, this is a Slater regular convex program in the Blaschke structure. Hence, the following holds.

4.2. Theorem. *Each Pareto-optimal solution $\bar{\mathfrak{x}}$ of the vector isoperimetric problem has the form*

$$\bar{\mathfrak{x}} = \alpha_1 \mathfrak{y}_1 + \dots + \alpha_m \mathfrak{y}_m,$$

where $\alpha_1, \dots, \alpha_m$ are positive reals.

Let us illustrate 4.2 for the *Leidenfrost effect*, the spheroidal state of a liquid drop on a heated horizontal plate.

4.3. Leidenfrost Problem. Given the volume of a three-dimensional convex figure, minimize its surface area and vertical breadth.

By symmetry everything reduces to an analogous plane two-objective problem, whose every Pareto-optimal solution is by 4.2 a *stadium*, a weighted Minkowski sum of a disk and a horizontal straight line segment.

4.4. Theorem. *A plane spheroid, a Pareto-optimal solution of the Leidenfrost problem, is the result of rotation of a stadium around the vertical axis through the center of the stadium.*

4.5. Internal Urysohn Problem with Flattening. Given are some convex body $\mathfrak{x}_0 \in \mathcal{V}_N$ and some flattening direction $\bar{z} \in S_{N-1}$. Among the convex bodies lying in $\mathfrak{x}_0$ and having fixed integral breadth, find $\mathfrak{x}$ maximizing the volume of $\mathfrak{x}$ and minimizing the breadth in the flattening direction:

$$\mathfrak{x} \in \mathcal{V}_N;\ \mathfrak{x} \subset \mathfrak{x}_0;\ \langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle;\ (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf.$$

4.6. Theorem. *For a feasible convex body $\bar{\mathfrak{x}}$ to be Pareto-optimal in the internal Urysohn problem with the flattening direction $\bar{z}$ it is necessary and sufficient that there be positive reals α, β and a convex figure $\mathfrak{x}$ satisfying*

$$\mu(\bar{\mathfrak{x}}) = \mu(\mathfrak{x}) + \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}});$$
$$\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \operatorname{supp}(\mu(\mathfrak{x}))).$$

Proof. By way of illustration we will derive the optimality criterion in somewhat superfluous detail. In actually, it would suffice to appeal for instance to [10, § 5.2] or the other numerous sources treating Pareto optimality in slightly less generality.

Note firstly that the internal Urysohn problem with flattening may be rephrased in $C(S_{N-1})$ as the following two-objective program

$$\mathfrak{x} \in \mathcal{V}_N;$$
$$\max\{\mathfrak{x}(z) - \mathfrak{x}_0(z) \mid z \in S_{N-1}\} \le 0;$$
$$\langle \mathfrak{x}, \mathfrak{z}_N \rangle \ge \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle;$$
$$(-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf.$$

The problem of Pareto optimization reduces to the scalar program

$$\mathfrak{x} \in \mathcal{V}_N;$$
$$\max\{\max\{\mathfrak{x}(z) - \mathfrak{x}_0(z) \mid z \in S_{N-1}\}, \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle - \langle \mathfrak{x}, \mathfrak{z}_N \rangle\} \le 0;$$
$$\max\{-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})\} \to \inf.$$

The last program is Slater-regular and so we may apply the *Lagrange principle*. In other words, the value of the program under consideration coincides with the value of the unconstrained minimization problem for an appropriate Lagrangian:

$$\mathfrak{x} \in \mathcal{V}_N;$$
$$\max\{-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})\} + \gamma \max\{\max\{\mathfrak{x}(z) - \mathfrak{x}_0(z) \mid z \in S_{N-1}\}, \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle - \langle \mathfrak{x}, \mathfrak{z}_N \rangle\} \to \inf.$$

Here γ is a positive Lagrange multiplier.

We are left with differentiating the Lagrangian along the feasible directions and appealing to 1.9 (2) and 2.7 (5). Note in particular that the relation

$$\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \operatorname{supp}(\mu(\mathfrak{x})))$$

is the *complementary slackness condition* standard in mathematical programming. The proof of the optimality criterion for the Urysohn problem with flattening is complete.

Assume that a plane convex figure $\mathfrak{x}_0 \in \mathcal{V}_2$ has the symmetry axis $A_{\bar{z}}$ with generator $\bar{z}$. Assume further that $\mathfrak{x}_{00}$ is the result of rotating $\mathfrak{x}_0$ around the symmetry axis $A_{\bar{z}}$ in $\mathbb{R}^3$. In this event we come to the following problem.

4.7. The Case of Rotational Symmetry:

$$\begin{gathered}\mathfrak{x} \in \mathcal{V}_3;\ \mathfrak{x} \text{ is a convex body of rotation around } A_{\bar{z}};\\ \mathfrak{x} \supset \mathfrak{x}_{00};\ \langle \mathfrak{z}_N, \mathfrak{x}\rangle \ge \langle \mathfrak{z}_N, \bar{\mathfrak{x}}\rangle;\\ (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf.\end{gathered}$$

By rotational symmetry, the three-dimensional problem reduces to an analogous two-dimensional problem. The integral breadth and perimeter are proportional on the plane, and we come to the already settled problem 4.5. Thus, we have the following.

4.8. Theorem. *Each Pareto-optimal solution of 4.7 is the result of rotating around the symmetry axis a Pareto-optimal solution of the plane internal Urysohn problem with flattening in the direction of the axis.*

Little is known about the analogous problems in arbitrary dimensions (for instance, see [29–31] and the references therein). An especial place is occupied by the Porogelov article [32]) where it is demonstrated that the "soap bubble" in a tetrahedron has the form of the result of the rolling of a ball over a solution of the internal Urysohn problem, i.e. the weighted Blaschke sum of a tetrahedron and a ball. As regards the forms of "real" soap bubbles and films see [33, 34].

4.9. External Urysohn Problem with Flattening. Given are some convex body $\mathfrak{x}_0 \in \mathcal{V}_N$ and some flattening direction $\bar{z} \in S_{N-1}$. Among the convex bodies encompassing $\mathfrak{x}_0$ and having fixed integral breadth, find a convex body $\mathfrak{x}$ maximizing volume and minimizing breadth in the flattening direction:

$$\mathfrak{x} \in \mathcal{V}_N;\ \mathfrak{x} \supset \mathfrak{x}_0;\ \langle \mathfrak{x}, \mathfrak{z}_N\rangle \ge \langle \bar{\mathfrak{x}}, \mathfrak{z}_N\rangle;\ (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \to \inf.$$

4.10. Theorem. *For a feasible convex body $\bar{\mathfrak{x}}$ to be a Pareto-optimal solution of the external Urysohn problem with flattening it is necessary and sufficient that there be positive reals α, β and a convex figure $\mathfrak{x}$ satisfying*

$$\begin{gathered}\mu(\bar{\mathfrak{x}}) + \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}});\\ V(\bar{\mathfrak{x}}) + V_1(\mathfrak{x}, \bar{\mathfrak{x}}) = \alpha V_1(\mathfrak{z}_N, \bar{\mathfrak{x}}) + 2N\beta b_{\bar{z}}(\bar{\mathfrak{x}});\\ \bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \operatorname{supp}(\mu(\mathfrak{x}))).\end{gathered}$$

Proof. Demonstration proceeds by analogy to the internal Urysohn problem with flattening. The extra equality for mixed volumes appears as deciphering of the complementary slackness condition.

4.11. The above list may be continued with the multiobjective generalization of many scalar problems such as problems with zone constraints and current hyperplanes, problems over centrally symmetric convex figures, Lindelöf type problems, etc. (see [35–37]). These problems are usually convex with respect to Blaschke or Minkowski structures. Of greater complexity are the nonconvex parametric problems stemming from the extremal properties of the Reuleaux triangle. These problems require extra tools and undergone only a fragmentary study (in particular, see [38–40] and the references therein). In closing we dwell upon the problems of another type where we seek for the form of several convex figures simultaneously.

4.12. Optimal Convex Hulls. Given are convex bodies $\mathfrak{y}_1, \dots, \mathfrak{y}_m$ in $\mathbb{R}^N$. Place a convex figure $\mathfrak{x}_k$ within $\mathfrak{y}_k$, for $k := 1, \dots, m$, so as to simultaneously maximize the volume of each of the figures $\mathfrak{x}_1, \dots, \mathfrak{x}_m$ and minimize the integral breadth of the convex hull of the union of these figures:

$$\mathfrak{x}_k \subset \mathfrak{y}_k \ (k := 1, \dots, m);$$
$$(-p(\mathfrak{x}_1), \dots, -p(\mathfrak{x}_m), \langle \mathrm{co}\{\mathfrak{x}_1, \dots, \mathfrak{x}_m\}, \mathfrak{z}_N \rangle) \to \inf .$$

4.13. Theorem. *For some feasible convex bodies $\bar{\mathfrak{x}}_1, \dots, \bar{\mathfrak{x}}_m$ to have a Pareto-optimal convex hull it is necessary and sufficient that there be positive reals $\alpha_1, \dots, \alpha_m$ not vanishing simultaneously and two collections of positive Borel measures $\mu_1, \dots, \mu_m$ and $\nu_1, \dots, \nu_m$ on S_{N-1} such that*

$$\nu_1 + \dots + \nu_m = \mu(\mathfrak{z}_N);$$
$$\bar{\mathfrak{x}}_k(z) = \mathfrak{y}_k(z) \quad (z \in \mathrm{supp}(\mu_k)); \quad \alpha_k \mu(\bar{\mathfrak{x}}_k) = \mu_k + \nu_k \ (k := 1, \dots, m).$$

Proof. The criterion appears along the lines of 4.6 on considering 2.10.

Is Dido's Problem solved?

From a utilitarian standpoint, the answer is definitely in the affirmative. There is no evidence that Dido experienced any difficulties, showed indecisiveness, and procrastinated the choice of the tract of land. Dido had met a particular problem of decision making and settled it successfully as witnessed by Virgil. The decision was taken and Carthage was founded, which is beyond any doubt.

The study of the isoperimetric problems of geometry is commonly tracked back to Dido, which we have mentioned at the very beginning. Isoperimetry led to variational calculus and the modern conceptions of control and optimization. Men indulges in exaggerating his capabilities and achievements. The firm belief is universal that the Dido problem is a historical anecdote rather than a topic of the modern science. In fact the matter is quite different. The hypothesis that mathematics has a method of solving Dido's problem is beneath all criticism.

Mathematics deals with abstract objects. Application of mathematics to practice bases on choosing some adequate models of real situations. Solution of a particular problem differs from the presence of a general method of solution. Finding the tangent of a parabola at the origin is rather far off the rules of differential calculus. Comprehension of the essence of some phenomenon requires free mind that is not confined with particularities, ignores random features, and seeks for general laws and interrelationships whereas excluding neither stochasticity nor multiplicity of possibilities. Each solution method basing on comprehension must apply to a rather broad class of similar problems, giving some algorithms or at least reasonable recommendations for solution.

Returning to Dido, let us assume that she had known the isoperimetric property of the circle and had been aware of the symmetrization principles that were elaborated in the nineteenth century. Would this knowledge be sufficient for Dido to choose the tract of land? Definitely, it would not. The real coastline may be rather ragged and craggy. The photo snaps of coastlines are exhibited as the most visual examples of fractality. From a theoretical standpoint, the free boundary in the plane Dido problem may be nonrectifiable, and so the concept of area as the quantity to be optimized is itself rather ambiguous. Practically speaking, the situation in which Dido made her decision was not as primitive as it seems at the first glance. Choosing the tract of land, Dido had no right to trespass the territory under the control of the local sovereign. She had to choose the tract so as to encompass the camps of her subjects and satisfy some fortification requirements. Clearly, this generality is unavailable in the mathematical models known as the classical isoperimetric problem.

Dido's problem inspiring our ancestors remains the same intellectual challenge as Kant's starry heavens and moral law.

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