ON MIXING SETS ARISING IN CHANCE-CONSTRAINED PROGRAMMING

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Abstract. The mixing set with a knapsack constraint arises in deterministic equivalent of chance-constrained programming problems with finite discrete distributions. We first consider the case that the chance-constrained program has equal probabilities for each scenario. We study the resulting mixing set with a cardinality constraint and propose facet-defining inequalities that subsume known explicit inequalities for this set. We extend these inequalities to obtain valid inequalities for the mixing set with a knapsack constraint. In addition, we propose a compact extended reformulation (with polynomial number of variables and constraints) that characterizes a linear programming equivalent of a single chance constraint with equal scenario probabilities. We introduce a *blending* procedure to find valid inequalities for intersection of multiple mixing sets. We propose a polynomial-size extended formulation for the intersection of multiple mixing sets with a knapsack constraint that is stronger than the original mixing formulation. We also give a compact extended linear program for the intersection of multiple mixing sets and a cardinality constraint for a special case. We illustrate the effectiveness of the proposed inequalities in our computational experiments with probabilistic lot-sizing problems.

Key words: Mixed-integer programming, facets, compact extended formulations, chance constraints, lot-sizing, computation.

1. Introduction

Many optimization problems in practice contain quality of service (QoS) or reliability constraints that result in probabilistic (chance) constraints. In this paper, we consider mixed-integer programming (MIP) reformulations of chance-constrained programs with joint probabilistic constraints in which the right-hand-side vector is random with a finite discrete distribution (Ruszczyński, 2002, Luedtke et al., 2010). The reformulation contains the mixing set (Günlük and Pochet, 2001) with an additional cardinality/knapsack constraint as a substructure. We first study the mixing set with a cardinality constraint and propose facet-defining inequalities that subsume the explicit inequalities given by Luedtke et al. (2010). In addition, we propose a compact

Date: March 26, 2010.

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extended reformulation (with polynomial number of variables and constraints) that characterizes a linear programming equivalent of a single inequality in the probabilistic constraint for a special case. This is in contrast to an exponential extended formulation proposed in Luedtke et al. (2010). We extend the results derived for the mixing set with a cardinality constraint to obtain valid inequalities for the mixing set with a knapsack constraint. In addition, we introduce a *blending* procedure to find valid inequalities for intersection of multiple mixing sets.

Charnes et al. (1958) were first to define a chance-constrained program with disjoint probabilistic constraints. Miller and Wagner (1965) study chance-constrained programming with joint probabilistic constraints for independent random variables. Joint probabilistic constraints with dependent random variables were introduced in Prékopa (1973). Sen (1992) studies chance-constrained programs with discrete distributions and gives a disjunctive programming reformulation by using so-called $(1-\tau)$-efficient points (Prékopa, 1990). Valid inequalities are proposed based on the extreme points of the reverse polar of the disjunctive program. The computational challenges of this approach are the enumeration of the $(1-\tau)$-efficient points and the solution of a linear program for each cut generation. Dentcheva et al. (2000) use $(1-\tau)$-efficient points to obtain various reformulations for chance-constrained programming with discrete random variables and to derive valid bounds on the optimal objective function value. Ruszczyński (2002) uses the concept of $(1-\tau)$-efficient points to derive consistent orders on different scenarios representing the discrete distribution. The consistent ordering is represented with precedence constraints and valid inequalities for the resulting precedence-constrained knapsack set are proposed. Beraldi and Ruszczyński (2002a) propose a branch-and-bound method for chance-constrained integer programs using a partial enumeration of the $(1-\tau)$-efficient points.

Some recent applications of chance-constrained programs with discrete distributions are probabilistic set covering (Beraldi and Ruszczyński, 2002b, Saxena et al., 2010), probabilistic lot/batch sizing (Beraldi and Ruszczyński, 2002a, Lulli and Sen, 2004), and probabilistic production and distribution planning (Lejeune and Ruszczyński, 2007).

The particular MIP reformulation of the chance-constrained programs of interest in this paper is proposed in Luedtke et al. (2010). This reformulation contains the mixing set as a substructure. Günlük and Pochet (2001) first introduced the mixing set and gave valid inequalities that define the convex hull of feasible solutions. Because this is a fundamental substructure arising in different contexts, various extensions of the mixing set has been studied, such as the continuous mixing set (Miller and Wolsey, 2003, van Vyve, 2005), mixing set with flows (Conforti et al., 2007) and mixing set with divisible capacities (Zhao and de Farias Jr, 2008).

Let ξ denote a d-variate random variable with a known finite discrete cumulative distribution function, $F(z) = P(\xi \le z)$. Given A, a $d \times n$ matrix, c, an n-dimensional cost vector, τ, a threshold probability with $0 \le \tau \le 1$, and $X \subseteq \mathbb{R}^{n_1} \times \mathbb{Z}^{n_2}$, where $n_1 + n_2 = n$, the chance-constrained programming problem is

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & P(Ax \ge \xi) \ge 1 - \tau \\ & x \in X, \end{aligned}$$

or equivalently

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & y = Ax \\ & P(y \ge \xi) \ge 1 - \tau \\ & x \in X. \end{aligned}$$

Suppose that the random vector ξ has finitely many realizations (scenarios) given by $\mathbf{h^1}, \mathbf{h^2}, \ldots, \mathbf{h^n}$, where $\mathbf{h^i} = (h_{1i}, h_{2i}, \ldots, h_{di})$, with probabilities $\pi_1, \pi_2, \ldots, \pi_n$, respectively. By definition, $0 < \pi_1, \pi_2, \ldots, \pi_n < 1$ and $\sum_{i=1}^n \pi_i = 1$. Throughout, we assume, without loss of generality, that $h_{ti} \ge 0$ for all $t = 1, \ldots, d$ and $i = 1, \ldots, n$. (For each $t = 1, \ldots, d$, if there exists $i = \arg\min\{h_{ti} : i = 1, \ldots, n\}$ with $h_{ti} < 0$, then we can replace h_{tj} by $h_{tj} - h_{ti}$ for all $j = 1, \ldots, n$ and let $y = Ax - h_{ti} e_t$, where e_t is the unit vector of size d, with tth entry equal to 1 and the other entries equal to 0.) Throughout, we let $[i, j] := \{t \in \mathbb{Z} : i \le t \le j\}$. A deterministic equivalent of the chance-constrained program is

$$\min \quad c^T x$$

$$\text{s.t.} \quad y = Ax \tag{1}$$

$$y_t \ge h_{ti}(1 - z_i) \qquad t \in [1, d], i \in [1, n] \tag{2}$$

$$\textstyle\sum_{i=1}^n \pi_i z_i \le \tau \tag{3}$$

$$x \in X, \mathbf{0} \le z \le \mathbf{1} \tag{4}$$

$$z \in \mathbb{Z}^n, \tag{5}$$

where $z_i = 0$ implies that under scenario i we have no violated inequality in the probabilistic constraint (i.e., $y = Ax \ge \mathbf{h^i}$) at the solution (y, x). If at least one inequality in the probabilistic constraint is violated (i.e., $y = Ax \not\ge \mathbf{h^i}$) in a feasible solution, then $z_i = 1$. When $z_i = 1$, we have $y_t \ge 0$, which trivially follows from the assumption that $h_{ti} \ge 0$ for all $t = 1, \ldots, d$, $i = 1, \ldots, n$. The total probability of violating the joint chance constraint is then given by $P(Ax \not\ge \xi) \le \sum_{i=1}^n \pi_i z_i$, which must not exceed the threshold τ. Note that the inequalities (2)–(3) contain the intersection of d mixing sets with a knapsack constraint as a substructure. We study this set in more detail in Sections 3 and 5.

Outline. In Section 2, we review earlier results from the study of related mixing sets. In Section 3, we give facet-defining inequalities for the mixing set with a cardinality constraint that subsume the known inequalities for this set. In Section 4, we give a compact extended formulation that characterizes a linear programming equivalent of a single probabilistic constraint with equal scenario probabilities. In Section 5, we extend our results to give valid inequalities for the mixing set with a knapsack constraint. In Section 6, we introduce a blending approach and reformulations for intersection of multiple mixing sets with a cardinality/knapsack constraint. In Section 7 we illustrate the effectiveness of the proposed inequalities in our computational experiments with probabilistic lot-sizing problems. We conclude with Section 8.

2. Mixing sets arising in chance-constrained programming

For $t = 1, \dots, d$, let

$$\mathcal{K}_t = \{(y_t, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{n} \pi_i z_i \leq \tau, y_t + h_{ti} z_i \geq h_{ti}, i \in [1,p]\}.$$

The set $\mathcal{K}_t$ is a mixing set with a knapsack constraint. We are interested in studying the polyhedral structure of the intersection of mixing sets with a (single) knapsack constraint given by $\cap_{t=1}^{d} \mathcal{K}_t$, which arises in deterministic equivalent of chance-constrained programs (see (2)–(3)).

First, we consider a single mixing set with a knapsack constraint, i.e., $d = 1$. Dropping the subscript t we get

$$\mathcal{K} = \{(y, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{n} \pi_i z_i \leq \tau, y + h_i z_i \geq h_i, i \in [1,n]\}.$$

We assume that h_i are in non-increasing order, $h_1 \geq h_2 \geq \cdots \geq h_n$. As observed by Luedtke et al. (2010), for ν such that $\sum_{i=1}^{\nu} \pi_i \leq \tau$ and $\sum_{i=1}^{\nu+1} \pi_i > \tau$, we must have $y \geq h_{\nu+1}$. Then constraints $y + h_i z_i \geq h_i$ for $i = \nu+1, \dots, n$ are redundant. Furthermore, given that $y \geq h_{\nu+1}$ in any solution, $y + (h_i - h_{\nu+1}) z_i \geq h_i$ is valid and at least as strong as $y + h_i z_i \geq h_i$. To see this, note that for $z_i = 0$ the two inequalities are equivalent, and for $z_i = 1$ the former reduces to $y \geq h_{\nu+1}$, whereas the latter reduces to $y \geq 0$. Therefore, we can rewrite $\mathcal{K}$ as $\mathcal{K} = \{(y, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{n} \pi_i z_i \leq \tau, y + (h_i - h_{\nu+1}) z_i \geq h_i, i \in [1, \nu]\}$. Note that we do not drop the variables z_i for $i = \nu+1, \dots, n$ because they are necessary when we consider the intersection of multiple mixing sets, $\mathcal{K}_t$, $t = 1, \dots, d$.

2.1. **Basic mixing set.** The basic mixing set is first defined in Günlük and Pochet (2001). The mixing set arising in chance-constrained programming is given by

$$\mathcal{S} = \{(y, z) \in \mathbb{R}_+ \times \{0, 1\}^n : y + (h_i - h_{\nu+1})z_i \geq h_i, i \in [1, \nu]\}.$$

Theorem 1 (Günlük and Pochet (2001), Atamtürk et al. (2000))**.** *For* $T = \{t_1, t_2, \ldots, t_a\} \subseteq \{1, \ldots, \nu\}$*, the inequalities*

$$y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}})z_{t_j} \geq h_{t_1}, \tag{6}$$

where $t_1 < t_2 < \cdots < t_a$ *and* $h_{t_{a+1}} = h_{\nu+1}$*, are valid for* $\mathcal{S}$ *and facet-defining for* conv$(\mathcal{S})$ *when* $t_1 = 1$*.*

We illustrate inequalities (6) in an example.

Example 1. Let $h = (40, 38, 34, 31, 26, 16, 8, 4, 2, 1)$ for $n = 10$, and $\nu = 6$.

$$\begin{aligned} y + 32z_1 &\geq 40 \\ y + 30z_2 &\geq 38 \\ y + 26z_3 &\geq 34 \\ y + 23z_4 &\geq 31 \\ y + 18z_5 &\geq 26 \\ y + 8z_6 &\geq 16. \end{aligned}$$

For $T = \{1, 2, 4\}$, the mixing inequality is

$$y + (40 - 38)z_1 + (38 - 31)z_2 + (31 - 8)z_4 \geq 40.$$

□

2.2. **Mixing set with a cardinality constraint.** Consider the chance-constrained program for which the scenarios are empirically approximated through i.i.d. sampling. In this case, $\mathbf{h^i}$ are independent observations of ξ with $\pi_i = 1/n$ for all $i = 1, \ldots, n$. For example, Luedtke and Ahmed (2008) give a sample approximation approach to get bounds for chance-constrained programs in which the original distribution is replaced by an empirical distribution obtained by independent Monte-Carlo sampling.

When $\pi_i = 1/n$ for all i, the knapsack constraint (3) can be written as a cardinality constraint:

$$\sum_{i=1}^{n} z_i \leq \lfloor n\tau \rfloor = p, \tag{7}$$

and $\nu = p$, where ν is such that $\sum_{i=1}^{\nu} \pi_i \leq \tau$ and $\sum_{i=1}^{\nu+1} \pi_i > \tau$. Let

$$\mathcal{Q} = \{(y, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{n} z_i \leq p, y + h_i z_i \geq h_i, i \in [1, n]\}.$$

Also, for $t = 1, \ldots, d$ let

$$\mathcal{Q}_t = \{(y_t, z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{n} z_i \leq p, y_t + h_{ti} z_i \geq h_{ti}, i \in [1, n]\}.$$

Theorem 2 (Luedtke et al. (2010)). *For $m \in \mathbb{Z}_+$ with $m \leq p$, let $T = \{t_1, t_2, \ldots, t_a\} \subseteq \{1, \ldots, m\}$ where $t_1 < t_2 < \cdots < t_a$ and $Q = \{q_1, q_2, \ldots, q_{p-m}\} \subseteq \{p+1, \ldots, n\}$, the inequalities*

$$y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} + \sum_{i=1}^{p-m} \Delta_i^m (1 - z_{q_i}) \geq h_{t_1}, \tag{8}$$

where for $m < p$

$$\Delta_i^m = \begin{cases} h_{m+1} - h_{m+2} & i = 1 \\ \max\{\Delta_{i-1}^m, h_{m+1} - h_{m+i+1} - \sum_{j=1}^{i-1} \Delta_j^m\} & i \in [2, p-m], \end{cases} \tag{9}$$

and $h_{t_{a+1}} := h_{m+1}$, are valid for $\mathcal{Q}$ and facet-defining for conv($\mathcal{Q}$) when $t_1 = 1$.

Example 1 (cont.) For $T = \{1, 2\}$ and $Q = \{7, 8, 9\}$, $m = 3$, inequality (8) is

$$y + (40 - 38) z_1 + (38 - 31) z_2 + (31 - 26)(1 - z_7) + (31 - 16 - 5)(1 - z_8) + 10(1 - z_9) \geq 40.$$

$\square$

3. Proposed Valid Inequalities for the Mixing set with a cardinality constraint

In this section, we give a class of inequalities that contains inequalities (8) as a special case.

Theorem 3. *For $m \in \mathbb{Z}_+$ such that $m \leq p$, let $T = \{t_1, t_2, \ldots, t_a\} \subseteq \{1, \ldots, m\}$ with $t_1 < t_2 < \cdots < t_a$, $L \subseteq \{m+2, \ldots, n\}$ and a permutation of the elements in L, $\Pi_L = \{\ell_1, \ell_2, \ldots, \ell_{p-m}\}$ such that $\ell_j \geq m + 1 + j$. The (T, Π_L) inequalities*

$$y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} + \sum_{j=1}^{p-m} \alpha_j (1 - z_{\ell_j}) \geq h_{t_1}, \tag{10}$$

are valid for $\mathcal{Q}$, where $t_{a+1} = m + 1$ and for $m < p$

$$\alpha_j = \begin{cases} h_{m+1} - h_{m+1+j} & j = 1 \\ \max\{\alpha_{j-1}, h_{m+1} - h_{m+1+j} - \sum_{i: i<j \text{ and } \ell_i \geq m+1+j} \alpha_i\} & j \in [2, p-m]. \end{cases} \tag{11}$$

Proof. First note that $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_{p-m}$. If $y \ge h_{t_1}$ then inequality (10) is trivially satisfied. If $y \ge h_{t_i}$ for some $i = 2, \ldots, a$ and $y < h_{t_j}$ for all $j \in [1, i-1]$, then we must have $z_{t_j} = 1$ for all $j \in [1, i-1]$. Thus,

$$y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} \ge h_{t_i} + \sum_{j=1}^{i-1} (h_{t_j} - h_{t_{j+1}}) = h_{t_1} \ge h_{t_1} - \sum_{j=1}^{p-m} \alpha_j (1 - z_{\ell_j}),$$

and inequality (10) is satisfied. Therefore, we assume that $y < h_{t_a}$ and $z_{t_j} = 1$ for all $j = 1, \ldots, a$ in the rest of the proof. Hence,

$$\sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} = h_{t_1} - h_{m+1}. \tag{12}$$

Now suppose that $y \ge h_{m+1}$. Then

$$y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} \ge h_{m+1} + h_{t_1} - h_{m+1} \ge h_{t_1} - \sum_{j=1}^{p-m} \alpha_j (1 - z_{\ell_j}),$$

and inequality (10) is valid. Otherwise, we must have $h_{m+i'} > y \ge h_{m+i'+1}$ for some $i' = 1, \ldots, p-m$. Thus, $z_j = 1$ for all $j = 1, \ldots, m+i'$. Because $\sum_{j=1}^{n} z_j \le p$, we have

$$\sum_{j=m+i'+1}^{n} z_j \le p - m - i'. \tag{13}$$

Let $i'' = |\{j : j \in [1, p-m] \text{ and } \ell_j \le m + i'\}|$. Note that, due to the choice of the ordering in L, Π_L, if $\ell_j \le m + i'$, then we must have $j < i'$. As a result, $i'' = |\{j : j \in [1, i'-1] \text{ and } \ell_j \le m + i'\}| < i'$. So in the set $L \setminus [1, m+i']$ there are $p - m - i''$ elements. For $j \in L \setminus [1, m+i']$ we have $|\{j \in L \setminus [1, m+i'] : z_j = 1\}| \le p - m - i'$ (from (13)), and so $|\{j \in L \setminus [1, m+i'] : z_j = 0\}| \ge i' - i''$. Thus,

$$\begin{aligned} \sum_{j=1}^{p-m} \alpha_j (1 - z_{\ell_j}) = \sum_{j: \ell_j \ge m+1+i'} \alpha_j (1 - z_{\ell_j}) &\ge \alpha_{i'} + \sum_{j: j < i', \ell_j \ge m+1+i'} \alpha_j \\ &\ge h_{m+1} - h_{m+1+i'}. \end{aligned} \tag{14}$$

To see the first inequality in (14), note that the coefficients, α, are in increasing order, so the $i' - i''$ elements of the set $\{j \in [1, i'] : \ell_j \ge m + i' + 1\}$ have the smallest α_j among all $\ell_j \in L \setminus [1, m+i']$. From (12), (14) and the assumption that $y \ge h_{m+i'+1}$, we have

$$y+\sum_{j=1}^{a}(h_{t_j}-h_{t_{j+1}})z_{t_j}+\sum_{j=1}^{p-m}\alpha_j(1-z_{\ell_j})\geq h_{m+1+i'}+h_{t_1}-h_{m+1}+h_{m+1}-h_{m+1+i'}$$
$$=h_{t_1}.$$

□

Theorem 4. *Inequality* (10) *is facet-defining for conv*($\mathcal{Q}$) *if and only if* $t_1 = 1$*. Furthermore, for a given* $i = 1, \ldots, d$*, assume without loss of generality, that* $h_{i1} \geq h_{i2} \geq \cdots \geq h_{in}$*. Then the* (T, Π_L) *inequality:*

$$y_i+\sum_{j=1}^{a}(h_{it_j}-h_{it_{j+1}})z_{t_j}+\sum_{j=1}^{p-m}\alpha_j(1-z_{\ell_j})\geq h_{it_1}, \tag{15}$$

valid for $\mathcal{Q}_i$ *is facet-defining for* $\text{conv}(\cap_{i=1}^{d}\mathcal{Q}_i)$ *if and only if* $t_1 = 1$*, where* T, L *and* $\Pi_L = \{\ell_1, \ldots, \ell_{p-m}\}$ *are as previously defined, and* α *is given by* (11) *with* $h_j = h_{ij}$ *for* $j \in T \cup L$*.*

Proof. Note that $\mathcal{Q}$ is full-dimensional. First, we show that $t_1 = 1$ is a necessary facet condition. Given a (T, Π_L) inequality (10) where $t_1 > 1$, consider the $(T', \Pi_{L'})$ inequality with $T' = T \cup \{1\}$ and $L' = L \setminus \{\ell_{p-m}\}$:

$$y+(h_1-h_{t_1})z_1+\sum_{j=1}^{a}(h_{t_j}-h_{t_{j+1}})z_{t_j}+\sum_{j=1}^{p-m-1}\alpha_j(1-z_{\ell_j})\geq h_1,$$

or equivalently,

$$(h_1-h_{t_1})(z_1-1)-\alpha_{p-m}(1-z_{\ell_{p-m}})+y+\sum_{j=1}^{a}(h_{t_j}-h_{t_{j+1}})z_{t_j}+\sum_{j=1}^{p-m}\alpha_j(1-z_{\ell_j})\geq h_{t_1}.$$

As $(h_1 - h_{t_1})(z_1 - 1) - \alpha_{p-m}(1 - z_{\ell_{p-m}}) \leq 0$, $(T', \Pi_{L'})$ inequality is at least as strong as the (T, Π_L) inequality.

To show that inequalities (10) are facet-defining for conv($\mathcal{Q}$) when $t_1 = 1$ we give $n + 1$ affinely independent points on the face defined by the inequality (10). First, let $y^0 = h_{t_1} = h_1$, $z_j^0 = 1$ if $j \in L$ and $z_j^0 = 0$ otherwise. Next, for each $j \notin (T \cup L)$, consider the point $(y^j, \mathbf{z}^j) = (y^0, \mathbf{z}^0 + e_j)$, where e_j is the unit vector of size n, with jth entry equal to 1 and the other entries equal to 0. This point is feasible, because $t_1 = 1$ implies that $a \geq 1$, so $\sum_{i=1}^{n} z_i^j = p - a + 1 \leq p$. For each $j \in [1, a]$, let $y^{t_j} = h_{t_{j+1}}$, $z_i^{t_j} = 1$ if $i = 1, \ldots, t_{j+1} - 1$ or $i \in L$, and $z_i^{t_j} = 0$ otherwise. Let $y^{\ell_1} = h_{m+2}$, $z_i^{\ell_1} = 1$ if $i = 1, \ldots, m + 1$ and $z_{\ell_i}^{\ell_1} = 1$ for $i > 1$; $z_i^{\ell_1} = 0$ for all other values of i. For each $j = 2, \ldots, p - m$ such that $\alpha_{\ell_j} = h_{m+1} - h_{m+1+j} - \sum_{i:i<j \text{ and } \ell_i \geq m+1+j} \alpha_i$, let $y^{\ell_j} = h_{m+1+j}$, $z_i^{\ell_j} = 1$ if $i = 1, \ldots, m + j$ and $z_{\ell_i}^{\ell_j} = 1$ for $i > j$; $z_i^{\ell_j} = 0$ for all other values of i. Finally, for each $j = 2, \ldots, p-m$ such that $\alpha_j = \alpha_{j-1}$, let $(y^{\ell_j}, \mathbf{z}^{\ell_j}) = (y^{\ell_{j-1}}, \mathbf{z}^{\ell_{j-1}} + e_{\ell_{j-1}} - e_{\ell_j})$. As $z^{\ell_{j-1}}_{\ell_{j-1}} = 0$ and $z^{\ell_{j-1}}_{\ell_j} = 1$, we have $z^{\ell_j}_{\ell_{j-1}} = 1$ and $z^{\ell_j}_{\ell_j} = 0$. These $n+1$ points on the face defined by inequality (10) are affinely independent.

To prove the second part of the theorem for inequality (15), valid for $\mathcal{Q}_i$ for some $i = 1, \ldots, d$, we first construct $n+1$ affinely independent points $(\mathbf{y^j}, \mathbf{z^j})$, $j = 0, \ldots, n$, from the $n+1$ affinely independent points $(y_i^j, \mathbf{z^j})$, $j = 0, \ldots, n$ listed above by letting $y_t^j = h_{t[1]_t}$ for $t = 1, \ldots, d$ and $t \neq i$, where $[1]_t = \arg\max\{h_{ti} : i = 1, \ldots, n\}$. The corresponding $(\mathbf{y^j}, \mathbf{z^j})$, $j = 0, \ldots, n$, are feasible in $\cap_{t=1}^d \mathcal{Q}_t$. Let $(\hat{\mathbf{y}}, \hat{\mathbf{z}})$ be one of these points. Now consider the $d-1$ additional points, $(\hat{\mathbf{y}}, \hat{\mathbf{z}}) + \epsilon e_j$ for $\epsilon > 0$, for each $j = 1, \ldots, d$ and $j \neq i$, where e_j is the jth unit vector of size $n+d$. These points are affinely independent and hence inequality (10) is facet-defining for $\mathrm{conv}(\cap_{t=1}^d \mathcal{Q}_t)$. The necessity of the facet condition $t_1 = 1$ in this case follows similarly to the case of a single mixing set.

□

Note that if $L = \emptyset$ then inequalities (10) are equivalent to inequalities (6). In addition, inequality (8) is a special case of inequality (10) with $L \subseteq [p+1, n]$ and $\ell_1 \leq \ell_2 \leq \cdots \leq \ell_{p-m}$.

Example 1 (cont.) For $T = \{1\}$, $m = 1$ and $L = \{4, 6, 7, 8, 9\}$ the (T, Π_L) inequalities corresponding to different permutations Π_L are

(16)
$$\begin{aligned}
&y + (h_1 - h_2)z_1 + (h_2 - h_3)(1 - z_4) + (h_2 - h_3)(1 - z_6) + (h_2 - h_5 - \alpha_6)(1 - z_7)\\
&\quad + (h_2 - h_6 - \alpha_6 - \alpha_7)(1 - z_8) + (h_2 - h_7 - \alpha_7 - \alpha_8)(1 - z_9) \geq h_1,\\
&y + (h_1 - h_2)z_1 + (h_2 - h_3)(1 - z_4) + (h_2 - h_3)(1 - z_6) + (h_2 - h_5 - \alpha_6)(1 - z_7)\\
&\quad + (h_2 - h_7 - \alpha_7 - \alpha_9)(1 - z_8) + (h_2 - h_6 - \alpha_6 - \alpha_7)(1 - z_9) \geq h_1,\\
&y + (h_1 - h_2)z_1 + (h_2 - h_3)(1 - z_4) + (h_2 - h_3)(1 - z_6) + (h_2 - h_5 - \alpha_6)(1 - z_9)\\
&\quad + (h_2 - h_6 - \alpha_6 - \alpha_9)(1 - z_8) + (h_2 - h_7 - \alpha_8 - \alpha_9)(1 - z_7) \geq h_1,\\
&y + (h_1 - h_2)z_1 + (h_2 - h_3)(1 - z_4) + (h_2 - h_5 - \alpha_7)(1 - z_6) + (h_2 - h_3)(1 - z_7)\\
&\quad + (h_2 - h_6 - \alpha_6 - \alpha_7)(1 - z_8) + (h_2 - h_7 - \alpha_7 - \alpha_8)(1 - z_9) \geq h_1,\\
&y + (h_1 - h_2)z_1 + (h_2 - h_3)(1 - z_4) + (h_2 - h_3)(1 - z_8) + (h_2 - h_5 - \alpha_8)(1 - z_6)\\
&\quad + (h_2 - h_6 - \alpha_6 - \alpha_8)(1 - z_9) + (h_2 - h_7 - \alpha_8 - \alpha_9)(1 - z_7) \geq h_1.
\end{aligned}$$

For example, in the first inequality $\Pi_L = \{4, 6, 7, 8, 9\}$, whereas in the last inequality $\Pi_L = \{4, 8, 6, 9, 7\}$.

Even though we propose a large class of facet-defining inequalities for $\text{conv}(\mathcal{Q})$, we show that the proposed inequalities are not enough to give the convex hull of solutions in the original space of variables. The convex hull representation in the original space of variables proves to be much richer. In particular, the following inequalities are valid and facet-defining for this example:

$$\begin{aligned}
&y + (h_1 - h_2)z_1 + (h_2 - h_3)z_2 + (h_3 - h_6 - \alpha_7)z_3 \\
&\quad + (h_6 - h_7)(1 - z_7) + (h_6 - h_7)(1 - z_9) \geq h_1, \\
&y + (h_1 - h_3)z_1 + (h_3 - h_6 - \alpha_7)z_3 + (h_6 - h_7)(1 - z_7) + (h_6 - h_7)(1 - z_9) \geq h_1, \\
&y + (h_1 - h_2)z_1 + (h_2 - h_6 - \alpha_7)z_2 + (h_6 - h_7)(1 - z_7) + (h_6 - h_7)(1 - z_9) \geq h_1, \\
&y + (h_1 - h_3)z_1 + (h_3 - h_4)(1 - z_4) + (h_1 - h_5 - \alpha_1)((1 - z_6) + (1 - z_7)) \\
&\quad + (h_1 - h_7 - \alpha_1 - \alpha_5 - \alpha_7)(1 - z_9) + (h_1 - h_6 - \alpha_1 - \alpha_6 - \alpha_7)z_5 \geq h_1, \\
&y + (h_1 - h_2)z_1 + (h_1 - h_3 - \alpha_1)(1 - z_4) + \frac{h_1 - h_7 - \alpha_1}{2}((1 - z_7) + (1 - z_9)) \\
&\quad + (h_1 - h_5 - \alpha_1 - \alpha_6)(1 - z_5) + (h_1 - h_6 - \alpha_1 - \alpha_7)(1 - z_6) \geq h_1.
\end{aligned}$$

These inequalities are different than the (T, Π_L) inequalities (10). In the first four inequalities, the coefficient of the last element in T depends on the coefficient of elements in L, whereas in inequality (10), the coefficient of the last element in T depends only on the cardinality of T. Finally, the last inequality is different because of the coefficient $\frac{h_1 - h_7 - \alpha_1}{2}$. Although we are able to prove the validity of these inequalities for this example, we were not able to obtain a general form of these inequalities. (See Appendix A for a proof of validity of the last inequality listed.) □

3.1. **Separation of (T, Π_L) Inequalities.** In this section, we give a polynomial time exact separation algorithm for a special case of the (T, Π_L) inequalities. This algorithm is used in our computational experiments in Section 7. The special case we consider has $S = \{m+2, \ldots, m+r+1\}$ for $m, r \in \mathbb{Z}_+$ with $m + r \leq p$, and $Q \subseteq [p+1, n]$ such that $L = S \cup Q$. Note that with this choice of S, we must have $\ell_j = m+1+j$ for $j = 1, \ldots, r$ as the first r elements in the permutation Π_L. As a result, α_j in equation (11) simplifies as $\alpha_j = \max\{\alpha_{j-1}, h_{m+1} - h_{m+1+j}\}$ for $j = 2, \ldots, r$. (If S is not contiguous, then this simplification does not hold.) Therefore, it is easy to calculate, in advance, all of the coefficients α_j for all $j = 1, \ldots, r$, which do not depend on the choice of Q. Next, observe that for $\ell_i \in Q \subseteq [p+1, n]$, $\ell_i \geq p+1 \geq m+1+j$ for all $j = r+1, \ldots, p-m$. As a result, α_j in equation (11) simplifies as $\alpha_j = \max\{\alpha_{j-1}, h_{m+1} - h_{m+1+j} - \sum_{i=1}^{j-1} \alpha_i\}$ for $j = r+1, \ldots, p-m$. Note that, assuming $S = \{m+2, \ldots, m+r+1\}$, the coefficients α_j, $j = r+1, \ldots, p-m$, do not depend on a particular choice of Q, but depend only on α_r.

Let (y^*, z^*) be a fractional solution. For given $m, r \in \mathbb{Z}_+$ with $m+r \le p$, we give an algorithm to identify the most violated inequality (10) with $S = \{m+2, \dots, m+r+1\}$. Note that the problem of finding the best set T in inequalities (10) can be solved as a shortest path problem on a directed acyclic graph, $G = (V, A)$, where $V = \{1, \dots, m+1\}$. There exists an arc $(i, j) \in A$ for all $1 \le i < j \le m+1$ with a cost of $(h_i - h_j)z_i^*$. There are $O(p^2)$ arcs in G. The vertices visited in the shortest path on this graph, starting from node 1 before reaching the sink $m+1$, give the set T in the most violated (T, Π_L) inequalities. Note that we always include $1 \in T$ to obtain violated facets, as this is a necessary and sufficient facet condition (Theorem 4).

For a given $m, r \in \mathbb{Z}_+$ with $m + r \le p$, S is fixed. To find the set Q that gives the most violated inequality (10) in the desired form, we keep an ordered list of the elements in $\{p+1, \dots, n\}$, denoted by $Z = \{q_1, q_2, \dots, q_{n-p}\}$, in increasing order of $(1 - z_j^*)$ for $j = p+1, \dots, n$ and we choose the first $p - m - r$ elements in the list Z to be in the set Q. This order also determines the order of the last $p - m - r$ elements in the permutation Π_L. In other words, $\ell_{r+i} = q_i$ for $i = 1, \dots, p - m - r$

As a result, for a given $m, r \in \mathbb{Z}_+$ with $m + r \le p$, the above algorithm runs in $O(p^3)$. Therefore, for a given $m \le p$ we can find the most violated inequality (10) with $L = S \cup Q$, $S = \{m+2, \dots, m+r+1\}$ and $Q \subseteq [p+1, n]$ in $O(p^4)$ by searching over r, $0 \le r < p - m$. Note that for $m = p$, the algorithm gives the most violated basic mixing inequality (6), and for $r = 0$ and Q such that $q_1 < q_2 < \dots < q_k$, it gives the most violated inequality (8).

4. A compact extended formulation for the mixing set with a cardinality constraint

In this section, we give a compact (polynomial-size) formulation for the mixing set with a cardinality constraint based on disjunctive programming. Note that the extended formulation given by Luedtke et al. (2010) for the mixing set with a cardinality constraint has exponentially many inequalities, which can be separated in polynomial time.

Theorem 5. *The set* $\mathcal{D} = \{(y, z, \lambda, \omega) \in \mathbb{R}^{2n+p+np+2} : (17) - (23)\}$, *where*

$$\sum_{j=1}^{p+1} \lambda_j = 1 \tag{17}$$
$$0 \leq \omega_i^j \leq \lambda_j \qquad j \in [1, p+1], i \in [1, n] \tag{18}$$
$$y \geq \sum_{j=1}^{p+1} h_j \lambda_j \tag{19}$$
$$z_i = \sum_{j=1}^{p+1} \omega_i^j \qquad i \in [1, n] \tag{20}$$
$$\sum_{i=j}^{n} \omega_i^j \leq (p - j + 1)\lambda_j \qquad j \in [1, p+1] \tag{21}$$
$$\omega_i^j \geq \lambda_j \qquad j \in [1, p+1], i \in [1, j-1] \tag{22}$$
$$\lambda_j \geq 0 \qquad j \in [1, p+1] \tag{23}$$

is a compact extended formulation of the set $\text{conv}(\mathcal{Q})$ *and* $\text{conv}(\mathcal{Q}) = \text{proj}_{y,z}(\mathcal{D})$.

Proof. Observe that y takes at most $p+1$ distinct values, $h_1, \ldots, h_{p+1}$, in extreme points of $\text{conv}(\mathcal{Q})$. For $j \in [1, p+1]$ such that $h_i > h_j$ for all $i < j$, let $\mathcal{Q}(h_j) = \{(y, z) \in \mathcal{Q} : y = h_j\}$. Note that, all feasible points in $\mathcal{Q}(h_j)$ have $z_i = 1$ for all $i = 1, \ldots, j-1$. Therefore,

$$\mathcal{Q}(h_j) = \{(y, z) \in \{h_j\} \times \{0, 1\}^n : \sum_{i=j}^{n} z_i \leq p - j + 1, z_i \geq 1, i \in [1, j-1]\}. \tag{24}$$

Observe that

$$\text{conv}(\mathcal{Q}(h_j)) = \{(y, z) \in \{h_j\} \times \mathbb{R}_+^n : \sum_{i=j}^{n} z_i \leq p - j + 1, z_i \geq 1, i \in [1, j-1], z \leq \mathbf{1}\},$$

because the constraint matrix defining $\mathcal{Q}(h_j)$ is totally unimodular.

As $y \in \{h_1, \ldots, h_{p+1}\}$ in extreme points of $\text{conv}(\mathcal{Q})$, we have

$$\text{conv}(\mathcal{Q}) = \text{conv}(\cup_{j=1}^{p+1} \text{conv}(\mathcal{Q}(h_j))) + \mathcal{C},$$

where

$$\mathcal{C} = \{(y, z) \in \mathbb{R}^{n+1} : z = \mathbf{0}, y \geq 0\}$$

is the recession cone of the linear programming relaxation of $\mathcal{Q}$. The theorem now follows from Theorem 2.1 of Balas (1998) on union of polyhedra (see also Theorem 4 in Cornuéjols (2008)).

□

Theorem 5 is a case when a compact formulation can be obtained as a union of polyhedra as observed for related polyhedra without cardinality constraints (Miller and Wolsey, 2003, Atamtürk, 2006, Conforti and Wolsey, 2008).

5. Valid inequalities for the mixing set with a knapsack constraint

Until now we studied the mixing set with a cardinality constraint (7), $\mathcal{Q}$, corresponding to the chance-constrained program with equal scenario probabilities $\pi_1 = \cdots = \pi_n$. For the more general case that scenarios have unequal probabilities, if we can find p such that the cardinality constraint (7) is valid for the set $\mathcal{K}$, then we can derive (T, Π_L) inequalities (10) valid for $\mathcal{K}$. Let $\langle 1\rangle, \langle 2\rangle, \ldots, \langle n\rangle$ be a nondecreasing order of scenario probabilities, i.e, $\pi_{\langle 1\rangle} \leq \pi_{\langle 2\rangle} \leq \cdots \leq \pi_{\langle n\rangle}$. Also let p be such that $\sum_{i=1}^{p} \pi_{\langle i\rangle} \leq \tau$ and $\sum_{i=1}^{p+1} \pi_{\langle i\rangle} > \tau$. Then the extended (knapsack) cover inequality

$$\sum_{i=1}^{n} z_i \leq p$$

is valid (cf. Wolsey (1998)) and can be used as a cardinality constraint to derive inequalities (10) valid for $\mathcal{K}$. Recall that $h_1 \geq h_2 \geq \cdots \geq h_n$, by assumption, and ν is such that $\sum_{i=1}^{\nu} \pi_i \leq \tau$ and $\sum_{i=1}^{\nu+1} \pi_i > \tau$. Note that unlike the equal probability case, ν is not necessarily equal to p and we have $y \geq h_{\nu+1}$ in every feasible solution. Therefore, we can further strengthen inequalities (10) for the set $\mathcal{K}$ when $\nu < p$.

Theorem 6. *For $m \in \mathbb{Z}_+$ such that $m \leq \nu$, let $T = \{t_1, t_2, \ldots, t_a\} \subseteq \{1, \ldots, m\}$ with $t_1 < t_2 < \cdots < t_a$, $L \subseteq \{m+2, \ldots, n\}$ and a permutation of the elements in L, $\Pi_L = \{\ell_1, \ell_2, \ldots, \ell_{p-m}\}$ such that $\ell_j \geq m+1+j$. For $\nu < p$, the strengthened (T, Π_L) inequalities*

$$y + \sum_{j=1}^{a} (h_{t_j} - h_{t_{j+1}}) z_{t_j} + \sum_{j=1}^{p-m} \alpha'_j (1 - z_{\ell_j}) \geq h_{t_1}, \tag{25}$$

are valid for $\mathcal{K}$, where $t_{a+1} = m+1$, $\alpha'_1 = h_{m+1} - h_{\min\{\nu+1, m+2\}}$, and for $j = 2, \ldots, p-m$

$$\alpha'_j = \max\{\alpha'_{j-1}, h_{m+1} - h_{\min\{\nu+1, m+1+j\}} - \sum_{i: i<j \text{ and } \ell_i \geq m+1+j} \alpha'_i\}.$$

Proof. Note that for $\nu = p$, inequality (25) is equivalent to inequality (10). Therefore, we consider the case $\nu < p$. The proof for the cases in which $y \geq h_{t_i}$ for $i = 1, \ldots, a$, or $h_{m+i'} > y \geq h_{m+i'+1} \geq h_{\nu+1}$ for $i' = 1, \ldots, p-m$, is the same as that of Theorem 3. Therefore, we assume that $z_{t_i} = 1$ for all $i = 1, \ldots, a$ and so (12) holds. For the cases in which $h_{m+i'} > y \geq h_{\nu+1} > h_{m+i'+1}$ for some $i' = 1, \ldots, p-m$, inequality (13) holds. Hence,

$$\sum_{j=1}^{p-m} \alpha'_j (1 - z_{\ell_j}) \geq \alpha'_{i'} + \sum_{j: j<i', \ell_j \geq m+1+i'} \alpha'_j \geq h_{m+1} - h_{\nu+1},$$

following a similar argument to the proof of Theorem 3. Consequently,

$$y + \sum_{j=1}^{a}(h_{t_j} - h_{t_{j+1}})z_{t_j} + \sum_{j=1}^{p-m} \alpha'_j(1 - z_{\ell_j}) \geq h_{\nu+1} + h_{t_1} - h_{m+1} + h_{m+1} - h_{\nu+1}$$
$$= h_{t_1}.$$

□

Note that as $h_{\min\{\nu+1,m+1+j\}} \geq h_{m+1+j}$, $\alpha'_j \leq \alpha_j$ and inequality (25) is at least as strong as inequality (10) when $\nu < p$.

Example 1 (cont.) Suppose that we have $\tau = 0.5$ and $\pi_1 = \pi_2 = \cdots = \pi_4 = \tau/4$ and $\pi_5 = \pi_6 = \cdots = \pi_{10} = \tau/6$. Thus, $\nu = 4$ and $p = 6$. The strengthened (T, Π_L) inequality with $T = \{1\}, L = \Pi_L = \{4, 6, 7, 8, 9\}$ is

$$y + (h_1 - h_2)z_1 + (h_2 - h_3)(1 - z_4) + (h_2 - h_3)(1 - z_6) + (h_2 - h_5 - \alpha'_6)(1 - z_7)$$
$$+ (h_2 - h_5 - \alpha'_6)(1 - z_8) + (h_2 - h_5 - \alpha'_6)(1 - z_9) \geq h_1.$$

This inequality is stronger than inequality (16) for the same choice of (T, Π_L), because $\alpha'_j < \alpha_j$ for $j = 8, 9$. In fact, we can show that this inequality is facet-defining for the convex hull of feasible solutions to the set $\mathcal{Q}$ with the additional constraint $y \geq h_{\nu+1}$. □

6. Intersection of multiple mixing sets

Until now, we considered a single mixing set with a cardinality or a knapsack constraint. The single mixing set with a knapsack constraint, given by $\mathcal{K}_t$, corresponds to the deterministic equivalent of a single inequality in the probabilistic constraint. In this section, we consider the case of a joint probabilistic constraint that contains $d > 1$ inequalities, defined by an intersection of d mixing sets and a knapsack constraint, $\cap_{t=1}^{d}\mathcal{K}_t$. Inequalities (25) are valid for $\cap_{t=1}^{d}\mathcal{K}_t$. We also showed in Theorem 4 that inequalities (15), with $\arg\max\{h_{ij}, j = 1, \ldots, n\} \in T$, are facet-defining for $\text{conv}(\cap_{t=1}^{d}\mathcal{Q}_t)$ when $\pi_t = 1/n$ for all $t = 1, \ldots, n$ (i.e., when the knapsack constraint (3) reduces to the cardinality constraint (7)). Furthermore, considering the intersection of multiple mixing sets, we can derive new mixing sets and valid inequalities for them. In particular, for $\beta \in \mathbb{Z}^d_+$, consider the single mixing set with a knapsack constraint given by

$$\mathcal{K}^\beta = \{(y', z) \in \mathbb{R}_+ \times \{0,1\}^n : \sum_{i=1}^{d} \pi_i z_i \leq \tau, y' + h'_i z_i \geq h'_i, i \in [1, n]\}, \quad (26)$$

where $y' = \sum_{t=1}^{d} \beta_t y_t$ and $h'_i = \sum_{t=1}^{d} \beta_t h_{ti}$. We call this the *blending set* with proportions β. Note that using scaling arguments we can assume $\beta \in \mathbb{Z}_+^d$ without loss of generality. Inequalities (25) valid for the mixing set $\mathcal{K}^\beta$ are valid for $\cap_{t=1}^{d} \mathcal{K}_t$. In Example 2 in Section 6.1, we illustrate that they may define facets that are not given by inequalities (25) valid for each individual mixing set $\mathcal{K}_t$, $t = 1, \ldots, d$.

Next we give a formal definition of $(1-\tau)$-efficient points. Using $(1-\tau)$-efficient points, we give conditions to find blending proportions for the intersection of two mixing sets that may provide a violated inequality for a given fractional point. Throughout, let $h_{t[1]_t} \geq h_{t[2]_t} \geq \cdots \geq h_{t[n]_t}$ for each $t = 1, \ldots, d$. Reordering h', let $h'_{1'} \geq h'_{2'} \geq \cdots \geq h'_{n'}$. Finally, let ν_β be such that $\sum_{i=1}^{\nu_\beta} \pi_{i'} \leq \tau$ and $\sum_{i=1}^{\nu_\beta+1} \pi_{i'} > \tau$. Recall that the finite discrete cumulative distribution function of the random right-hand-side vector ξ is given by $F(z) = P(\xi \leq z)$.

Definition 1. (Prékopa, 1990) Let $\theta^i \in \mathbb{R}_+^d$, $i = 1, \ldots, S$ be such that $F(\theta^i) \geq 1-\tau$ and $F(\theta^i - \epsilon) < 1-\tau$ for any infinitesimally small $\epsilon \geq \mathbf{0}$, $\epsilon \neq \mathbf{0}$. The points θ^i, $i = 1, \ldots, S$ are called $(1-\tau)$-efficient.

Note that all $(1-\tau)$-efficient points can be obtained by total enumeration of all possible outcomes for each right-hand-side. Therefore, the total number of $(1-\tau)$-efficient points, S, is $O(n^d)$. However, the number of distinct values of any two components in all $(1-\tau)$-efficient points is at most $O(n^2)$. Without loss of generality, we consider the first two components of θ^i, $i = 1, \ldots, S$. We reorder the $(1-\tau)$-efficient points θ^i, $i = 1, \ldots, S$ such that the vectors (θ_1^i, θ_2^i) are distinct for $i = 1, \ldots, S'$.

Proposition 7. *Let $\bar{y} \in \mathbb{R}^2$ be a given a point with $\bar{y}_j \in \text{proj}_{y_j}(\text{conv}(\mathcal{K}_j))$, $j = 1, 2$ and $\bar{y} \notin \text{proj}_y(\text{conv}(\mathcal{K}_1 \cap \mathcal{K}_2))$, and let $\theta^i \in \mathbb{R}^2$, $i = 1, \ldots, S'$ be the distinct values of (θ_1^i, θ_2^i) in all $(1-\tau)$-efficient points. If $\beta^\top \theta^j = h'_{(\nu_\beta+1)'}$ for some $j = 1, \ldots, S'$ and*

$$\max_{i=1,\ldots,S':\theta_1^i - \bar{y}_1 > 0, \theta_2^i - \bar{y}_2 < 0} \left\{ \frac{\theta_2^i - \bar{y}_2}{\bar{y}_1 - \theta_1^i} \right\} < \frac{\beta_1}{\beta_2} < \min_{i=1,\ldots,S':\theta_1^i - \bar{y}_1 < 0, \theta_2^i - \bar{y}_2 > 0} \left\{ \frac{\theta_2^i - \bar{y}_2}{\bar{y}_1 - \theta_1^i} \right\}, \tag{27}$$

then $\beta^\top \bar{y} \notin \text{proj}_{y'}(\text{conv}(\mathcal{K}^\beta))$ for $\beta \in \mathbb{Z}_+^2$ with $\beta > \mathbf{0}$.

Proof. Observe that $\mathcal{K}^\beta$ is a relaxation of $\mathcal{K}_1 \cap \mathcal{K}_2$ for $\beta > \mathbf{0}$. As $\theta^i \in \text{proj}_y(\mathcal{K}_1 \cap \mathcal{K}_2)$, for all $i = 1, \ldots, S'$, we have $\beta^\top \theta^i \in \text{proj}_{y'}(\mathcal{K}^\beta)$. Therefore, we have $\beta^\top \theta^i \geq h'_{(\nu_\beta+1)'}$ for all $i = 1, \ldots, S'$. Note that $\bar{y} \notin \text{proj}_y(\text{conv}(\mathcal{K}_1 \cap \mathcal{K}_2))$ implies that we do not have $\bar{y}_j \geq \theta_j^i$, $j = 1, 2$, for any $i = 1, \ldots, S'$. For all $i = 1, \ldots, S'$ such that $\bar{y}_j < \theta_j^i$ for $j = 1, 2$, we have $\beta^\top \bar{y} < \beta \theta^i$ for *any* $\beta > \mathbf{0}$. Similarly, for all i such that $\bar{y}_1 = \theta_1^i$ and $\bar{y}_2 < \theta_2^i$ or $\bar{y}_1 < \theta_1^i$ and $\bar{y}_2 = \theta_2^i$, we have $\beta^\top \bar{y} < \beta^\top \theta^i$ for *any* $\beta > \mathbf{0}$. For all $i = 1, \ldots, S'$ such that $\bar{y}_1 > \theta_1^i$ and $\bar{y}_2 < \theta_2^i$, the condition $\frac{\beta_1}{\beta_2} < \frac{\theta_2^i - \bar{y}_2}{\bar{y}_1 - \theta_1^i}$ in (27) implies that $\beta^\top \bar{y} < \beta^\top \theta^i$ for such i. Similarly, for all $i = 1, \ldots, S'$ such that $\bar{y}_1 < \theta_1^i$ and $\bar{y}_2 > \theta_2^i$, the condition $\frac{\beta_1}{\beta_2} > \frac{\theta_2^i - \bar{y}_2}{\bar{y}_1 - \theta_1^i}$ in (27) implies that $\beta^\top \bar{y} < \beta^\top \theta^i$ for such i. As a result, $\beta^\top \bar{y} < \beta^\top \theta^i$

for *all* $i = 1, \dots, S'$ when β satisfies (27). In addition, $\beta^\top \bar{y} < \beta^\top \theta^j = h'_{(\nu_\beta+1)'}$ for some $j = 1, \dots, S'$. Therefore, $\beta^\top \bar{y} \notin \text{proj}_{y'}(\text{conv}(\mathcal{K}_\beta))$, as all feasible points (y', z) of $\text{conv}(\mathcal{K}_\beta)$ have $y' \geq h'_{(\nu_\beta+1)'}$.

□

As a result, for a point $\bar{y} \notin \text{proj}_y(\text{conv}(\cap_{t=1}^d \mathcal{K}_t))$, if the conditions in Proposition 7 hold for β_1, β_2, then $\beta^\top \bar{y} \notin \text{proj}_{y'}(\text{conv}(K^\beta))$ for $\beta \in \mathbb{Z}_+^d$ with $\beta = (\beta_1, \beta_2, 0, \dots, 0)$. We illustrate this on Example 2. In what follows, we give a strong reformulation for $\cap_{t=1}^d \mathcal{K}_t$. The reformulation can be further strengthened using blending set reformulations.

Theorem 8. *The formulation*

$$\sum_{j=1}^{\nu_t+1} \lambda_{tj} = 1 \qquad t \in [1, d] \tag{28}$$

$$0 \leq \omega_{ti}^j \leq \lambda_{tj} \qquad t \in [1, d], j \in [1, \nu_t + 1], i \in [1, n] \tag{29}$$

$$y_t \geq \sum_{j=1}^{\nu_t+1} h_{t[j]_t} \lambda_{tj} \qquad t \in [1, d] \tag{30}$$

$$z_{[i]_t} = \sum_{j=1}^{\nu_t+1} \omega_{t[i]_t}^j \qquad t \in [1, d], i \in [1, n] \tag{31}$$

$$\sum_{i=j}^{n} \omega_{t[i]_t}^j \leq (p - j + 1)\lambda_{tj} \qquad t \in [1, d], j \in [1, \nu_t + 1] \tag{32}$$

$$\omega_{t[i]_t}^j \geq \lambda_{tj} \qquad t \in [1, d], j \in [1, \nu_t + 1], i \in [1, j - 1] \tag{33}$$

$$\sum_{i=1}^{n} \pi_i z_i \leq \tau \tag{34}$$

$$Ax = y \tag{35}$$

$$x \in X, \mathbf{0} \leq \lambda \leq \mathbf{1} \tag{36}$$

$$\lambda \in \mathbb{Z}^{\sum_{t=1}^d \nu_t + d}, \tag{37}$$

is an extended formulation for the set given by (1)–(5)*. The continuous relaxation of the extended formulation defined by* (28)–(36) *is at least as strong as the continuous relaxation of the mixing set formulation defined by* (1)–(4)*.*

Proof. The validity of this formulation follows from the validity of the reformulation given in Theorem 5 for a single mixing set. To show that formulation (28)–(36) is at least as strong as the formulation given by (1)–(4), we show that for any $(y, x, z, \lambda, \omega)$ satisfying (28)–(36), the vector (y, x, z) satisfies (1)–(4). Clearly, (y, x, z) satisfies (1), (3)–(4). We show that inequalities (2) are also satisfied by this choice of (y, x, z). For each $t = 1, \dots, d$ and $i = 1, \dots, \nu_t + 1$ from inequality (30) we have

$$
\begin{aligned}
y_t &\geq \sum_{j=1}^{\nu_t+1} h_{t[j]_t}\lambda_{tj} \\
&\geq h_{t[i]_t}\sum_{j=1}^{i}\lambda_{tj} \\
&\geq h_{t[i]_t}\sum_{j=1}^{i}(\lambda_{tj} - \omega^j_{t[i]_t}) \\
&= h_{t[i]_t}(\sum_{j=1}^{i}(\lambda_{tj} - \omega^j_{t[i]_t}) - z_{[i]_t} + \sum_{j=1}^{\nu_t+1}\omega^j_{t[i]_t}) && \text{(from (31))} \\
&= h_{t[i]_t}(\sum_{j=1}^{i}\lambda_{tj} + \sum_{j=i+1}^{\nu_t+1}\omega^j_{t[i]_t} - z_{[i]_t}) \\
&= h_{t[i]_t}(1 - \sum_{j=i+1}^{\nu_t+1}\lambda_{tj} + \sum_{j=i+1}^{\nu_t+1}\omega^j_{t[i]_t} - z_{[i]_t}) && \text{(from (28))} \\
&= h_{t[i]_t}(1 - \sum_{j=i+1}^{\nu_t+1}\omega^j_{t[i]_t} + \sum_{j=i+1}^{\nu_t+1}\omega^j_{t[i]_t} - z_{[i]_t}) && \text{(from (29) and (33))} \\
&= h_{t[i]_t}(1 - z_{[i]_t}).
\end{aligned}
$$

From (30), y_t is a convex combination of $h_{t[1]_t}, h_{t[2]_t}, \ldots, h_{t[\nu_t+1]_t}$. Therefore, $y_t \geq h_{t[\nu_t+2]_t}$ in any feasible solution and inequalities $y_t \geq h_{t[i]_t}(1 - z_{[i]_t})$ are trivially satisfied for $i = \nu_t + 2, \ldots, n$.

□

As a result, the set of feasible solutions given by (28)–(36) is a subset of the set of feasible solutions given by (1)–(4). We show that the former set could be a strict subset in Example 2 in Section 6.1. Observe that, we can strengthen the formulation (28)–(36) further by appending it with the extended formulation of the set $\mathcal{K}^\beta$ for $\beta \in \mathbb{R}^d_+$. We illustrate this strengthening in Example 2.

Note that unlike in the single mixing set with a cardinality constraint, we must have integer λ in formulation (28)–(37), as relaxing integrality does not necessarily result in integral λ for the intersection of multiple mixing sets, even when the knapsack constraint is a cardinality constraint. However, for the special case when $h_{t1} \geq h_{t2} \geq \cdots \geq h_{tn}$ for all $t = 1, \ldots, d$, we give a more compact extended formulation that describes the intersection of mixing sets with a cardinality constraint as a linear program.

Theorem 9. *Suppose that* $h_{t1} \geq h_{t2} \geq \cdots \geq h_{tp+1}$ *for all* $t = 1, \ldots, d$ *and* $\pi_i = 1/n$ *for* $i = 1, \ldots, n$. *A compact extended formulation of the polyhedron given by* $\mathrm{conv}(\cap_{t=1}^d \mathcal{Q}_t)$ *is*

$$\sum_{j=1}^{p+1} \lambda_j = 1 \tag{38}$$

$$0 \leq \omega_i^j \leq \lambda_j \qquad j \in [1, p+1], i \in [1, n] \tag{39}$$

$$\mathbf{y} \geq \sum_{i=1}^{p+1} \mathbf{h_i} \lambda_i \tag{40}$$

$$z_i = \sum_{j=1}^{p+1} \omega_i^j \qquad i \in [1, n] \tag{41}$$

$$\sum_{i=j}^{n} \omega_i^j \leq (p - j + 1)\lambda_j \qquad j \in [1, p+1] \tag{42}$$

$$\omega_i^j \geq \lambda_j \qquad j \in [1, p+1], i \in [1, j-1] \tag{43}$$

$$\lambda_j \geq 0 \qquad j \in [1, p+1] \tag{44}$$

$$\mathbf{y} \in \mathbb{R}_+^d, \tag{45}$$

where $\mathbf{h_i} \in \mathbb{R}_+^d$ *for* $i = 1, \ldots, p+1$.

Proof. Note that if $h_{t1} \geq h_{t2} \geq \cdots \geq h_{tp+1}$ for all $t = 1, \ldots, d$, we have $\nu_t = p$ for $t = 1, \ldots, d$. In an extreme point of the convex hull of the intersection of mixing sets with a cardinality constraint, the vector $\mathbf{y} \in \mathbb{R}^d$ is one of at most $p+1$ vectors $\mathbf{h_j} = (h_{1j}, h_{2j}, \ldots, h_{dj})$ for $j = 1, \ldots, p+1$. Therefore,

$$\mathcal{Q}(\mathbf{h_j}) = \{(\mathbf{y}, z) \in \{\mathbf{h_j}\} \times \{0,1\}^n : \sum_{i=j}^{n} z_i \leq p - j + 1, z_i \geq 1, i \in [1, j-1]\}.$$

Observe that

$$\mathrm{conv}(\mathcal{Q}(\mathbf{h_j})) = \{(\mathbf{y}, z) \in \{\mathbf{h_j}\} \times \mathbb{R}_+^n : \sum_{i=j}^{n} z_i \leq p - j + 1, z_i \geq 1, i \in [1, j-1], z \leq \mathbf{1}\},$$

because the constraint matrix defining $\mathcal{Q}(h_j)$ is totally unimodular.

As $\mathbf{y} \in \{\mathbf{h_1}, \ldots, \mathbf{h_{p+1}}\}$ in extreme points of $\mathrm{conv}(\cap_{t=1}^d \mathcal{Q}_t)$, we have

$$\mathrm{conv}(\cap_{t=1}^d \mathcal{Q}_t) = \mathrm{conv}(\cup_{j=1}^{p+1} \mathrm{conv}(\mathcal{Q}(\mathbf{h_j}))) + \mathcal{C},$$

where

$$\mathcal{C} = \{(\mathbf{y}, z) \in \mathbb{R}^{d+n} : z = \mathbf{0}, \mathbf{y} \geq 0\}. \tag{46}$$

Then the theorem follows from the result of Balas (1998) on union of polyhedra. □

Note that the linear programming reformulation for the special case described in Theorem 9 has $p+1$ many λ variables as compared to $\sum_{t=1}^d (\nu_t + 1)$ many λ variables in the MIP reformulation given in Theorem 8 for the general case. Finally, note that $(h_{tj}, \hat{\mathbf{z}}^{\mathbf{j}})$, $j = 1, \ldots, p+1$, with $\hat{z}_i^j = 1$ for $i < j$ and $\hat{z}_i^j = 0$ for $i \geq j$ are all extreme point solutions of $\text{conv}(\mathcal{Q}_t)$ for all $t = 1, \ldots, d$ if $h_{t1} \geq h_{t2} \geq \cdots \geq h_{tp+1}$ for all $t = 1, \ldots, d$. Also, $(y_t, \mathbf{z}) = (1, \mathbf{0})$ is the extreme ray of $\text{conv}(\mathcal{Q}_t)$ for each $t = 1, \ldots, d$ and the conical combination of these extreme rays give $\mathcal{C}$ in (46). Therefore, we have the following result.

Corollary 10. *$\text{conv}(\cap_{t=1}^{d} \mathcal{Q}_t) = \cap_{t=1}^{d} \text{conv}(\mathcal{Q}_t)$ if $h_{t1} \geq h_{t2} \geq \cdots \geq h_{tp+1}$ for all $t = 1, \ldots, d$.*

6.1. **An example on the strength of alternative reformulations.** In this section, we give a slight modification of the example in Sen (1992) to illustrate the strength of the alternative reformulations. While the reformulation proposed in Sen (1992) is stronger in most cases, there are several computational challenges in obtaining this reformulation. In this approach, first all $(1-\tau)$-efficient points need to be enumerated to obtain an equivalent disjunctive programming reformulation. In general, it is computationally intensive to enumerate all $(1-\tau)$-efficient points (Beraldi and Ruszczyński, 2002a). The $(1-\tau)$-efficient points are also used to define the reverse polar of this disjunctive program whose extreme points give valid inequalities that define a linear inequality reformulation of this disjunctive set (Sen, 1992). It is also not practical to list all extreme points of the reverse polar, in general.

Example 2. Consider the chance-constrained program

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & P\left\{ \begin{array}{rcl} 2x_1 - x_2 & \geq & \xi_1 \\ x_1 + 2x_2 & \geq & \xi_2 \end{array} \right\} \geq 0.6 = 1 - \tau \\ & x \geq 0, \end{aligned}$$

where ξ_1 and ξ_2 are dependent random variables with joint probability density function given in Table 1.

TABLE 1. Joint probability density function of ξ

Scenario	1	2	3	4	5	6	7	8	9
ξ_1	0.75	0.5	0.5	0.25	0.25	0.25	0	0	0
ξ_2	1.25	1.5	1.25	1.75	1.5	1.25	2	1.5	1.25
Probability	0.2	0.14	0.06	0.06	0.06	0.3	0.04	0.04	0.1

Observe that the set of all $(1-\tau)$-efficient points, obtained by enumerating all possible combinations of ξ_1 and ξ_2 and checking the condition in Definition 1, is $\{(0.25, 2), (0.5, 1.5), (0.75, 1.25)\}$. For example, $\theta^1 = (0.25, 2)$ is $(1-\tau)$-efficient, because the cumulative distribution function evaluated at this point, $F(\theta^1) = P(\xi_j \leq \theta^i_j, i = 1, 2) = 0.6 = 1 - \tau$ and $F(\theta^i - \epsilon) < 0.6$ for any infinitesimally small $\epsilon \geq \mathbf{0}$, $\epsilon \neq \mathbf{0}$.

Note that the $(1-\tau)$-efficient point $\theta^1 = (0.25, 2)$ is not given by any realization $\mathbf{h^i}$, $i = 1, \ldots, n$. Using the list of all $(1-\tau)$-efficient points, an alternative reformulation of this chance-constrained program is given by the disjunctive program (Sen, 1992):

$$
\begin{aligned}
\min \quad & x_1 + x_2 \\
\text{s.t.} \quad & \left\{ \begin{array}{lll} y_1 & \geq & 0.25 \\ y_2 & \geq & 2 \end{array} \right\} \text{ or} \\
& \left\{ \begin{array}{lll} y_1 & \geq & 0.5 \\ y_2 & \geq & 1.5 \end{array} \right\} \text{ or} \\
& \left\{ \begin{array}{lll} y_1 & \geq & 0.75 \\ y_2 & \geq & 1.25 \end{array} \right\} \\
& y_1 = 2x_1 - x_2 \\
& y_2 = x_1 + 2x_2 \\
& x \geq 0.
\end{aligned}
$$

The optimal solution is $(x, y) = (0.55, 0.35, 0.75, 1.25)$ with objective value 0.9. Next, we illustrate the reformulations proposed in this paper on this example.

For this example, $\tau = 0.4$, $p = 6$, $\nu_1 = 3$, $\nu_2 = 5, y_1 = 2x_1 - x_2$ and $y_2 = x_1 + 2x_2$, and the mixing set reformulation is

$$
\begin{array}{llcll}
y_1 + 0.75z_1 & \geq 0.75 & \qquad & y_2 + 2.00z_7 & \geq 2 \\
y_1 + 0.50z_2 & \geq 0.5 & & y_2 + 1.75z_4 & \geq 1.75 \\
y_1 + 0.50z_3 & \geq 0.5 & & y_2 + 1.50z_2 & \geq 1.5 \\
y_1 + 0.25z_4 & \geq 0.25 & & y_2 + 1.50z_5 & \geq 1.5 \\
y_1 + 0.25z_5 & \geq 0.25 & & y_2 + 1.50z_8 & \geq 1.5 \\
y_1 + 0.25z_6 & \geq 0.25 & & y_2 + 1.25z_1 & \geq 1.25 \\
\vdots & & & \vdots &
\end{array}
$$

$$\sum_{i=1}^{n} \pi_i z_i \leq 0.4 = \tau.$$

The initial linear programming (LP) relaxation solution of the mixing set reformulation is $(x, y) = (0.49, 0.38, 0.6, 1.25)$ with an objective value 0.87. After adding the following violated cuts (25)

$$y_1 + 0.25z_1 + 0.25z_3 \geq 0.75$$
$$y_1 + 0.25z_1 + 0.25(1 - z_4 + 1 - z_5 + 1 - z_7 + 1 - z_8) \geq 0.75$$
$$y_2 + 0.25z_7 + 0.25z_4 + 0.25z_5 \geq 2$$

in that order, we get a solution $(x, y) = (0.52, 0.365, 0.675, 1.25)$ with an objective value 0.885. There is no violated inequality (25) at this point valid for either of the two individual mixing sets. Note that for $\beta_1 = \beta_2 = 1$ we have $1 = \frac{\beta_1}{\beta_2} < \min\{\frac{2-1.25}{0.675-0.25}, \frac{1.5-1.25}{0.675-0.5}\}$ and $\beta^\top \theta^2 = h'_{(\nu_\beta+1)'} = 2$, where $\nu_\beta = 3$ for $\mathcal{K}^\beta$. So, to obtain violated inequalities using Proposition 7, we consider the blending set formed by $y' = y_1 + y_2$ with $\beta_1 = \beta_2 = 1$ in (26):

$$\begin{aligned} y' + 2z_1 &\geq 2 \\ y' + 2z_2 &\geq 2 \\ y' + 2z_4 &\geq 2 \\ y' + 2z_7 &\geq 2 \\ y' + 1.75z_3 &\geq 1.75 \\ &\vdots \end{aligned}$$

The violated inequality (25) is

$$y_1 + y_2 \geq 2,$$

and it is facet-defining for $\text{conv}(\cap_{t=1}^{d} \mathcal{K}_t)$. After adding this inequality, we get the solution $(x, y) = (0.55, 0.35, 0.75, 1.25)$, which is optimal. However, $z_1 = 0.3$ and $z_2 = z_4 = z_5 = z_7 = z_8 = 1$ and there are no violated inequalities (25) at this point.

In contrast, solving the LP relaxation of the extended reformulation of the chance-constrained program given by (28)–(37), we get $(x, y) = (0.52, 0.365, 0.675, 1.25)$ with an objective function value 0.885. This example illustrates that the extended formulation is a stronger formulation than the original mixing set formulation. Furthermore, adding the extended formulation for the mixing set given by $y_1 + y_2$ to this formulation, we get the optimal solution with integral λ, z.

Finally, consider the linear programming relaxation of the extended formulation proposed in Luedtke et al. (2010) given by the additional constraints:

$$y_t + \sum_{i=1}^{\nu_t} (h_{t[i]_t} - h_{t[i+1]_t}) w_{t[i]_t} \geq h_{t[1]_t} \quad t \in [1, d]$$
$$w_{t[i]_t} \geq w_{t[i+1]_t} \quad t \in [1, d], i \in [1, \nu_t]$$
$$z_i \geq w_{ti} \quad t \in [1, d], i \in \{[1]_t, \ldots, [\nu_t]_t\}$$

where $w_{t[i]_t} = 1$ if scenario $[i]_t$ is violated for the single constraint t and $w_{t[\nu_t+1]_t} = 0$. The LP relaxation solution to this extended formulation has an objective function value 0.8769, with $(x, y) = (0.504, 0.373, 0.635, 1.25)$, which shows that this is a weaker formulation. Luedtke et al. (2010) propose a class of valid inequalities for this formulation, which results in an exponential-size LP extended formulation for the case that $d = 1$.

□

In the next section, we summarize our computational experience in solving larger probabilistic lot-sizing problems effectively with a branch-and-cut algorithm incorporating inequalities (25).

7. Computations

To test the effectiveness of the proposed inequalities in solving chance-constrained programs with finite discrete distributions, we implement a branch-and-cut algorithm that incorporates inequalities (25). All computations are done on a 3.2 GHz Sun workstation with 4 GB RAM, under 3600 CPU seconds time limit.

We test our methods on the probabilistic lot-sizing problem described in Beraldi and Ruszczyński (2002a), where the right-hand-sides, h_{ti}, represent cumulative demands in time period $t = 1, \ldots, d$ under scenario $i = 1, \ldots, n$, and the probabilistic constraint represents a service level requirement on the joint probability of a stock-out in any time period. We assume that the demand in a time period is Uniform(1,50). Therefore, the right hand sides, h_{1i}, for row 1 of the probabilistic constraint is generated from discrete uniform distribution between 1 and 50 for each scenario $i = 1, \ldots, n$. To obtain the right-hand-side h_{ti}, we add a Uniform(1,50) random variable to the value of $h_{(t-1)i}$ for each $t = 2, \ldots, d$ and $i = 1, \ldots, n$. As a result, we have dependency between the rows of the probabilistic constraints.

The variable production costs are generated from a discrete uniform distribution between 0 and 10. We let $\tau \in \{5, 10, 15, 20\}$ be the threshold percentage on the probabilistic constraint. In addition, production setup costs follow a discrete uniform distribution between 0 and $1000f$, for $f \in \{0, 1\}$. In other words, when $f = 0$, there are no setup costs and we get a chance-constrained linear program, whereas when $f = 1$, we get a chance-constrained mixed-integer program. To test the performance of our branch-and-cut algorithm for varying cost parameters and probability thresholds, we generate five random instances for each combination of f and τ and report the averages.

A summary of these experiments with $d = 50, n = 500$ is reported in Table 2. In column `gap`, we report the average integrality gap, which is $100 \times (\texttt{zub} - \texttt{zinit})/\texttt{zub}$, where `zinit` is the objective value of the initial LP relaxation and `zub` is the objective value of the best integer solution. In column `% gapimp`, we compare the average percentage improvement of the integrality gap at the root node, which is $100 \times (\texttt{zroot} - \texttt{zinit})/(\texttt{zub} - \texttt{zinit})$, where `zroot` is the objective value of the LP at the root node after the cuts are added. Columns `cuts` and `nodes` compare the average number of cuts added, and the average number of branch-and-cut tree nodes explored, respectively. In the last column, we report the average CPU time elapsed (in seconds). We indicate the case that none of the five problem instances could be solved within an hour with `T`. If the problem is not solved within the time limit, then we also report, in parenthesis, the average percentage gap between the best lower bound and the best integer solution found in the search tree (`endgap`). Except for percentage gaps, all table entries are rounded to the nearest integer.

The set of experiments summarized in Table 2, is on solving probabilistic lot-sizing problems with scenario probabilities generated from Uniform(0,1) distribution. We implement a branch-and-cut algorithm using a separation algorithm for a subset of inequalities (25) as described in Section 3.1 with the restriction that $p-m \in \{0, 1, 2, 3\}$. The problem instances are solved with the MIP solver of CPLEX[1] Version 10.1. The experiments with the branch-and-cut algorithm using inequalities (25) are summarized under the columns `TL`. We solve the same instances with the default settings of CPLEX (`CPX`) without adding any user cuts. We also report our experiments using mixing inequalities (6) instead of inequalities (25) under the columns `Mix`.

In our experiments with no setup costs ($f = 0$), we observe that even though the initial gaps are small, the default CPLEX reaches the time limit in instances with large τ, whereas the branch-and-cut algorithm using a subset of inequalities (25) takes less than a few minutes on average for all problem instances. This can be attributed to the close to 100 per cent gap improvement at the root node as compared to the less than 26 percent improvement made by default CPLEX. While adding inequalities (6) also improve the percentage gap close to 100 per cent, `Mix` takes longer to solve. As the objective function does not include the z variables, we see that even though the gap improvement is almost always the same for `Mix` and `TL`, we get more fractional z's using `Mix` than using `TL`. CPLEX default adds about half the number of inequalities in all problem instances, however, these inequalities are not very effective in closing the integrality gap and CPLEX resorts to enumerating thousands of nodes in the branch-and-cut tree. The problems with setup costs ($f = 1$) are harder to solve for all methods as we have additional binary variables in the formulation. In addition, we observe that for both $f = 0$ and $f = 1$, the problems are harder to solve for larger τ.

We have also tested the extended formulation proposed in Section 6. We found that while the bounds given by this formulation are much stronger, the formulation is very large to make it practical for large instances. This addresses a question posed in Conforti and Wolsey (2008) regarding the practicality of similar extended formulations.

[1]CPLEX is a trademark of ILOG, Inc.

Table 2. Probabilistic lot-sizing experiments

f	τ	gap	% gapimp			cuts			nodes			time (endgap)		
			CPX	Mix	TL	CPX	Mix	TL	CPX	Mix	TL	CPX	Mix	TL
0	5	1.3	23	90	90	85	559	199	1139	117	81	130	255	59
	10	1.7	26	97	97	131	833	333	10493	171	68	821	529	104
	15	2.0	24	98	98	217	1360	559	36765	265	209	T(0.4)	1387	341
	20	2.4	18	97	97	248	1527	779	25479.0	291	418	T(1.0)	1764	967
1	5	2.4	47	74	74	235	574	359	72889	3258	15663	T(0.3)	T(0.4)	3476(0.2)
	10	2.7	43	78	77	288	846	499	39773	1323	4169	T(0.7)	T(0.5)	T(0.4)
	15	3.1	39	77	76	373	1334	707	22837	492	1831	T(1.4)	T(0.7)	T(0.6)
	20	3.5	36	75	76	452	1849	1089	17031	245	939	T(1.7)	T(0.9)	T(0.7)

8. Conclusion

In this paper, we study the mixing set with a cardinality constraint arising in chance-constrained programs and propose facet-defining inequalities that subsume the explicit inequalities given by Luedtke et al. (2010). We extend the results derived for the mixing set with a cardinality constraint to obtain valid inequalities for the mixing set with a knapsack constraint. Our computational tests illustrate the efficacy of a branch-and-cut algorithm using these inequalities. In addition, we propose a compact extended reformulation (with polynomial number of variables and constraints) that characterizes a linear programming equivalent of a single inequality in the probabilistic constraint. We propose an extended formulation for the intersection of multiple mixing sets with a knapsack constraint that is stronger than the original mixing formulation and is polynomial in size. We also give a compact extended linear program for the intersection of multiple mixing sets and a cardinality constraint for a special case.

The complete linear description of the single mixing set with a cardinality constraint, in its original space, remains an open question. In addition, an efficient method for finding blending proportions β for the intersection of multiple mixing sets merits further research. In this paper, we give a simple condition on β for blending two mixing sets.

Acknowledgements

We thank Kai Huang for discussions on probabilistic lot-sizing, and Suvrajeet Sen for bringing Sen (1992) to our attention. We are grateful to two anonymous referees for their careful reading of the paper, and for their suggestions which improved the content and the presentation of the results. We are also grateful to Juan Pablo Vielma for his corrections of Proposition 7 in an earlier version of the paper.

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Appendix A. Example 1 (cont.)

In this section, we prove the validity of one of the inequalities that cannot be expressed as a (T, Π_L) inequality:

$$y + (h_1 - h_2)z_1 + (h_1 - h_3 - \alpha_1)(1 - z_4) + \frac{h_1 - h_7 - \alpha_1}{2}((1 - z_7) + (1 - z_9))$$
$$+ (h_1 - h_5 - \alpha_1 - \alpha_6)(1 - z_5) + (h_1 - h_6 - \alpha_1 - \alpha_7)(1 - z_6) \geq h_1. \tag{47}$$

Consider each feasible value for $y = h_i$, $i = 1, \dots, 7$ and a feasible assignment of z values that minimizes the left-hand-side (LHS) of an inequality:

$$y + \alpha_1 z_1 + \alpha_4(1 - z_4) + \alpha_5(1 - z_5) + \alpha_6(1 - z_6) + \alpha_7(1 - z_7) + \alpha_9(1 - z_9) \geq h_1. \tag{48}$$

Case 1. For $y = h_1$, a valid assignment that minimizes the LHS of (48) is $z_1 = 0, z_4 = z_5 = z_6 = z_7 = z_9 = 1$. In this case, inequality (48) is tight.

Case 2. For $y = h_2$, a valid assignment that minimizes the LHS of (48) is $z_1 = z_4 = z_5 = z_6 = z_7 = z_9 = 1$. In this case, we must have $h_2 + \alpha_1 \geq h_1$, or

$$\alpha_1 \geq h_1 - h_2. \tag{49}$$

Case 3. For $y = h_3$, a valid assignment that minimizes the LHS of (48) is $z_4 = 0$ $z_1 = z_2 = z_5 = z_6 = z_7 = z_9 = 1$. In this case, we must have $h_3 + \alpha_1 + \alpha_4 \geq h_1$, or

$$\alpha_1 + \alpha_4 \geq h_1 - h_3. \tag{50}$$

Case 4. For $y = h_4$, a valid assignment that minimizes the LHS of (48) is $z_4 = z_5 = 0$ $z_1 = z_2 = z_3 = z_6 = z_7 = z_9 = 1$. In this case, we must have $h_4 + \alpha_1 + \alpha_4 + \alpha_5 \geq h_1$, or

$$\alpha_1 + \alpha_4 + \alpha_5 \geq h_1 - h_4. \tag{51}$$

Case 5. For $y = h_5$, a valid assignment that minimizes the LHS of (48) is $z_5 = z_6 = 0$ $z_1 = z_2 = z_3 = z_4 = z_7 = z_9 = 1$. In this case, we must have $h_5 + \alpha_1 + \alpha_5 + \alpha_6 \geq h_1$, or

$$\alpha_1 + \alpha_5 + \alpha_6 \geq h_1 - h_5. \tag{52}$$

Case 6. For $y = h_6$, a valid assignment that minimizes the LHS of (48) is $z_6 = z_7 = 0$ $z_1 = z_2 = z_3 = z_4 = z_5 = z_9 = 1$. In this case, we must have $h_6 + \alpha_1 + \alpha_6 + \alpha_9 \geq h_1$, or

$$\alpha_1 + \alpha_6 + \alpha_9 \geq h_1 - h_6. \tag{53}$$

Alternatively, another valid assignment that minimizes the LHS of (48) is $z_6 = z_9 = 0$ $z_1 = z_2 = z_3 = z_4 = z_5 = z_7 = 1$. In this case, we must have $h_6 + \alpha_1 + \alpha_6 + \alpha_7 \geq h_1$, or

$$\alpha_1 + \alpha_6 + \alpha_7 \geq h_1 - h_6. \tag{54}$$

Case 7. For $y = h_7$, a valid assignment that minimizes the LHS of (48) is $z_7 = z_9 = 0$ $z_1 = z_2 = z_3 = z_4 = z_5 = z_6 = 1$. In this case, we must have $h_7 + \alpha_1 + \alpha_7 + \alpha_9 \geq h_1$, or

$$\alpha_1 + \alpha_7 + \alpha_9 \geq h_1 - h_7. \tag{55}$$

To show validity of inequality (47), we select the six coefficients α in (48) such that six of the seven inequalities (49)–(55) hold at equality and the remaining inequality is satisfied. Assuming that inequalities (49)–(50) and (52)–(55) hold at equality and solving for α, we get a unique solution for α that gives the inequality (47). With this choice of α, $\alpha_1 + \alpha_4 + \alpha_5 > h_1 - h_4$ and (51) is satisfied. Therefore, inequality (47) is a valid inequality for this example. We can also show that it is facet-defining.