

A continuation method for nonlinear complementarity problems over symmetric cone

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ABSTRACT. In this paper, we introduce a new P -type condition for nonlinear functions defined over Euclidean Jordan algebras, and study a continuation method for nonlinear complementarity problems over symmetric cones. This new P -type condition represents a new class of nonmonotone nonlinear complementarity problems that can be solved numerically.

1. INTRODUCTION

The nonlinear complementarity problem (NCP) is the problem of finding, for a given map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, a nonnegative vector $x \in \mathbb{R}^n$ such that

$$f(x) \geq 0 \quad \text{and} \quad x^T f(x) = 0.$$

Both NCP and its special case when f is affine—known as the linear complementarity problem (LCP)—are well documented in the literature (e.g., [12, 20]). When solving the NCP, one usually reformulates it as a system of nonlinear equations via either the *min-map* or *normal map* [20]. In the first approach, the conditions

$$x_i \geq 0, \quad f_i(x) \geq 0 \quad \text{and} \quad x_i f_i(x) = 0$$

are equivalently written as the non-differentiable equations $\min\{x_i, f_i(x)\} = 0$. In the latter, every solution to the NCP corresponds exactly to a solution to the normal map equations (NME)

$$f(z^+) + z^- = 0$$

via $x = z^+$ and $z = x - f(x)$. Here, and henceforth, z^+ denotes the component-wise maximum of the zero vector and z , and z^- denotes the component-wise minimum (or, equivalently, $z - z^+$).

These reformulations can be extended to nonlinear complementarity problems over general convex cones, which are problems of finding, for a given map $f : \mathbb{E} \rightarrow \mathbb{E}$ and some given closed convex cone $K \subseteq \mathbb{E}$, $x \in K$ such that

$$f(x) \in K^\# \quad \text{and} \quad \langle x, f(x) \rangle = 0.$$

Here, and henceforth, \mathbb{E} denotes a Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and $K^\#$ denotes the closed dual cone

$$\{s \in \mathbb{E} : \langle s, x \rangle \geq 0 \quad \forall x \in K\}.$$

Using the Löwner partial order where $x \geq_K y$ means $x - y \in K$, this nonlinear complementarity problem is equivalently described as

$$x \geq_K 0, \quad f(x) \geq_{K^\#} 0 \quad \text{and} \quad \langle x, f(x) \rangle = 0.$$

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(We will also use $x >_K y$ to mean $x - y \in \text{int}(K)$, the interior of K .) In this general setting, the min-map formulation becomes the fixed-point equation [14]

$$x = \text{Proj}_K(x - f(x)),$$

and the normal map equation is

$$(1) \quad f(\text{Proj}_K(z)) - \text{Proj}_{K^\sharp}(-z) = 0.$$

Here, and henceforth, Proj_K denotes the Euclidean projection onto K .

In this paper, we focus on solving the normal map formulation in the setting where K is the closure of a *symmetric cone*. A symmetric cone is a self-dual (i.e., $K = K^\sharp$) open convex cone whose linear automorphism group acts transitively on it. Symmetric cones have been completely classified as the direct sum of cones from five irreducible groups [15]:

1. the quadratic cones $\{x \in \mathbb{R}^{n+1} : x_{n+1} > \sqrt{x_1^2 + \cdots + x_n^2}\}$ for $n \geq 2$;
2. the cones of real symmetric positive definite $n \times n$ matrices for $n \geq 1$;
3. the cones of complex Hermitian positive definite $n \times n$ matrices for $n \geq 2$;
4. the cones of Hermitian positive definite $n \times n$ matrices of quaternions for $n \geq 2$;
5. the cone of Hermitian positive definite 3×3 matrices of octonions.

We shall rely heavily on Euclidean Jordan algebraic characterization of symmetric cones. Thus we identify the Euclidean space \mathbb{E} with a Euclidean Jordan algebra \mathfrak{J} associated with the symmetric cone $\text{int}(K)$. We refer the readers to Section 2 for more details on Euclidean Jordan algebras.

The nonlinear complementarity problem over the cone K shall be denoted by $\text{NCP}_K(f)$. When f is affine, say $f(x) = l(x) + q$ for some linear transformation $l : \mathbb{E} \rightarrow \mathbb{E}$ and some vector $q \in \mathbb{E}$, we may also write $\text{LCP}_K(l, q)$ or $\text{LCP}_K(M, q)$, where M is a matrix representation of l , instead of $\text{NCP}_K(f)$. In this case, the problem is called a linear complementarity problem over the cone K . We may also drop the subscript K when the cone is \mathbb{R}_+^n .

The main difficulty in solving the normal map equation (1) is the nonsmoothness of the Euclidean projector Proj_K [32]. Among various methods proposed to overcome this difficulty is the use of smoothing approximations of the Euclidean projector. Proposed by Chen and Mangasarian [5], a class of parametric smooth function approximating Euclidean projector for nonnegative orthants has had a great success in smoothing methods for the NCP. See, e.g., [1, 3, 4, 7, 28] and the references therein. Chen et al. [4] proposed a continuation method for the NCP via normal maps and reported some encouraging numerical results. In this paper, by employing a subclass of Chen and Mangasarian's smoothing functions to approximate the NME, we study a continuation method for solving $\text{NCP}_K(f)$. We show that this method is globally convergent under some suitable P -type property on f . In general, we find that this P -type property lies between the concept of P -property and uniform P -property when K is polyhedral.

It is noted that there are many existing algorithms for solving $\text{NCP}_K(f)$. These include algorithms using merit functions extended from the context of NCP [33, 34], smoothing Newton methods [6, 9, 11, 16, 21, 24, 29], interior-point method [30], and non-interior continuation methods [10, 22]. All these algorithms require either the monotonicity of f or the nonsingularity of the Jacobians of the systems involved. Thus, an interesting question is to identify a class of nonmonotone NCP_K which can be solved without any nonsingularity assumption. Related work has been done by Chen and Qi [8] by introducing the concept of Cartesian P -property for $\text{LCP}_K(l, q)$, where K is a direct sum of cones of symmetric positive semidefinite matrices. The natural extension of the Cartesian

P -property to the general case where K is the closure of a symmetric cone is

$$(2) \quad \max_{1 \leq \nu \leq \kappa} \langle x_\nu, l(x)_\nu \rangle > 0 \quad \forall x \neq 0,$$

where x_ν denotes the ν -th component of x in the direct sum $K = K_1 \oplus \cdots \oplus K_\kappa$ of irreducible symmetric cones. In one extreme case where the K_ν 's are isomorphic to \mathbb{R} , the Cartesian P -property reduces to the P -property of the matrix representation of l . However, in general, we show that the Cartesian P -property implies our P -type property. Thus, our P -type property gives a wider class of nonmonotone nonlinear complementarity problems over symmetric positive definite cones (and, in general, symmetric cones) that can be solved numerically, without requiring any nonsingularity assumptions.

The paper is organized as follows. In the next section, we briefly review relevant concepts in the theory of Euclidean Jordan algebras. In Section 3, we formulate the nonsmooth NME as a system of smooth equations. In Section 4, we introduce a new equivalent definition of P -matrix which results in a new property that lies between P - and uniform P -properties when extended to nonlinear function. We then extend this new property to functions defined on Euclidean Jordan algebras. In Section 5, we discuss the boundedness and uniqueness of solution trajectory for the continuation method. The continuation algorithm and its convergence analysis will be studied in Section 6.

2. EUCLIDEAN JORDAN ALGEBRAS

In this section, we review concepts in the theory of Euclidean Jordan algebras that are necessary for the purpose of this paper. Interested readers are referred to Chapters II–IV of [15] for a more comprehensive discussion on the theory of Euclidean Jordan algebras.

Definition 2.1 (Jordan algebra). An algebra (\mathfrak{J}, \circ) over the field \mathbb{R} or \mathbb{C} is said to be a *Jordan algebra* if it is commutative and the endomorphisms $y \mapsto x \circ y$ and $y \mapsto (x \circ x) \circ y$ commute for each $x \in \mathfrak{J}$.

Definition 2.2 (Euclidean Jordan algebra). A finite dimensional Jordan algebra (\mathfrak{J}, \circ) with unit e is said to be *Euclidean* if there exists a positive definite symmetric bilinear form on \mathfrak{J} that is associative; i.e., \mathfrak{J} has an inner product $\langle \cdot, \cdot \rangle$ such that

$$\langle x \circ y, z \rangle = \langle y, x \circ z \rangle \quad \forall x, y, z \in \mathfrak{J}.$$

Henceforth, (\mathfrak{J}, \circ) shall denote a Euclidean Jordan algebra, and e shall denote its unit. We shall identify \mathfrak{J} with a Euclidean space equipped with the inner product $\langle \cdot, \cdot \rangle$ in the above definition. For each $x \in \mathfrak{J}$, we shall use L_x to denote the linear endomorphism $y \mapsto x \circ y$, and use P_x to denote $2L_x^2 - L_{x \circ x}$. By the definition of Euclidean Jordan algebra, L_x , whence P_x , is symmetric under $\langle \cdot, \cdot \rangle$. The linear endomorphism P_x is called the *quadratic representation* of x .

Definition 2.3 (Jordan frame). An *idempotent* of \mathfrak{J} is a nonzero element $c \in \mathfrak{J}$ satisfying $c \circ c = c$. An idempotent is said to be *primitive* if it cannot be written as the sum of two idempotents. Two idempotents c and d are said to be *orthogonal* if $c \circ d = 0$. A *complete system of orthogonal idempotents* is a set of idempotents that are pair-wise orthogonal and sum to the unit e . A *Jordan frame* is a complete system of primitive idempotents. The number of elements in any Jordan frame is an invariant called the *rank* of \mathfrak{J} (see paragraph immediately after Theorem III.1.2 of [15]).

Remark 2.1. Orthogonal idempotents are indeed orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$ since

$$\langle c, d \rangle = \langle c \circ e, d \rangle = \langle e, c \circ d \rangle.$$

Primitive idempotents have unit norm.

Henceforth, r shall denote the rank of \mathfrak{J} .

Theorem 2.1 (Spectral decomposition of type I). *Each element x of the Euclidean Jordan algebra (\mathfrak{J}, \circ) has a spectral decomposition of type I*

$$x = \sum_{i=1}^k \lambda_i c_i,$$

where $\lambda_1 > \dots > \lambda_k$, and $\{c_1, \dots, c_k\} \subset \mathfrak{J}$ forms a complete system of idempotents. Moreover, the λ_i 's and c_i 's are uniquely determined by x .

Proof. See Theorem III.1.1 of [15]. □

Theorem 2.2 (Spectral decomposition of type II). *Each element x of the Euclidean Jordan algebra (\mathfrak{J}, \circ) has a spectral decomposition of type II*

$$x = \sum_{i=1}^r \lambda_i c_i,$$

where $\lambda_1 \geq \dots \geq \lambda_r$ (with their multiplicities) are uniquely determined by x , and $\{c_1, \dots, c_r\} \subset \mathfrak{J}$ forms a Jordan frame.

The coefficients $\lambda_1, \dots, \lambda_r$ are called the eigenvalues of x , and they are denoted by $\lambda_1(x), \dots, \lambda_r(x)$.

Proof. See Theorem III.1.2 of [15]. □

Remark 2.2. The two spectral decompositions are related as follows: If $x = \sum_{i=1}^k \mu_i d_i$ and $x = \sum_{i=1}^r \lambda_i(x) c_i$ are spectral decompositions of type I and II, respectively, then for each $i \in \{1, \dots, k\}$, we have $d_i = \sum_{j: \lambda_j(x) = \mu_i} c_j$.

Remark 2.3. Two elements share the same Jordan frames in their type II spectral decompositions precisely when they *operator commute* [15, Lemma X.2.2]; i.e., when the linear endomorphisms L_x and L_y commute.

Henceforth, all spectral decompositions are of type II, unless stated otherwise.

Theorem 2.3 (Characterization of symmetric cones). *A cone is symmetric if and only if it is linearly isomorphic to the interior of the cone of squares*

$$K(\mathfrak{J}) := \{x \circ x : x \in \mathfrak{J}\}$$

of a Euclidean Jordan algebra (\mathfrak{J}, \circ) . Moreover, the interior $\text{int}(K(\mathfrak{J}))$ of the cone of squares coincides with the following equivalent sets:

- (i) the set $\{x \in \mathfrak{J} : L_x \text{ is positive definite under } \langle \cdot, \cdot \rangle\}$;
- (ii) the set $\{x \in \mathfrak{J} : \lambda_i(x) > 0 \forall i\}$.

Proof. See Theorems III.2.1 and III.3.1 of [15]. □

The cone of squares $K(\mathfrak{J})$ can alternatively be described as the set of elements with nonnegative eigenvalues (see proof of Theorem III.2.1 of [15]), whence every symmetric cone can be identified with the set of elements with positive eigenvalues in certain Euclidean Jordan algebra.

For each idempotent $c \in \mathfrak{J}$, the only possible eigenvalues of L_c are 0, $\frac{1}{2}$ and 1; see Theorem III.1.3 of [15]. We shall use $V(c, 0)$, $V(c, \frac{1}{2})$ and $V(c, 1)$ to denote the eigenspaces of L_c corresponding to the eigenvalues 0, $\frac{1}{2}$ and 1, respectively. If μ is not an eigenvalue of L_c , then we use the convention $V(c, \mu) = \{0\}$.

Theorem 2.4 (Peirce decomposition). *Given a Jordan frame $\{c_1, \dots, c_r\}$, the space \mathfrak{J} decomposes into the orthogonal direct sum*

$$\mathfrak{J} = \bigoplus_{i=1}^r \mathfrak{J}_i \oplus \bigoplus_{1 \leq i < j \leq r} \mathfrak{J}_{ij},$$

where $\mathfrak{J}_i := \mathfrak{J}(c_i, 1) = \mathbb{R}c_i$ and $\mathfrak{J}_{ij} := \mathfrak{J}(c_i, \frac{1}{2}) \cap \mathfrak{J}(c_j, \frac{1}{2})$ for $i < j$, such that the orthogonal projector onto \mathfrak{J}_i is P_{c_i} , and that onto \mathfrak{J}_{ij} is $4L_{c_i}L_{c_j}$.

The decomposition of $x \in \mathfrak{J}$ into

$$x = \sum_{i=1}^r x_i c_i + \sum_{(i,j): 1 \leq i < j \leq r} x_{ij}$$

with $x_i c_i = P_{c_i}(x)$ and $x_{ij} = 4L_{c_i}(L_{c_j}(x))$ is called its Peirce decomposition with respect to the Jordan frame $\{c_1, \dots, c_r\}$.

Proof. See Theorem IV.2.1 of [15]. □

Remark 2.4. It is straightforward to check that if $\{c_1, \dots, c_r\}$ is the Jordan frame in a spectral decomposition $x = \sum \lambda_i(x)c_i$ of x , then the Peirce decomposition of x with respect to $\{c_1, \dots, c_r\}$ coincide with this spectral decomposition.

Remark 2.5. Since both L_{c_i} and P_{c_i} are both continuous, it follows that the maps $x \mapsto x_i$ and $x \mapsto x_{ij}$ are continuous for each i, j .

We conclude this section with the following result.

Corollary 2.1. *For each $x \in \mathfrak{J}$, $x \in K(\mathfrak{J})$ if and only if x_i is nonnegative in every Peirce decomposition $x = \sum x_i c_i + \sum x_{ij}$.*

Proof. The ‘‘if’’ part follows from using a spectral decomposition of x . For the ‘‘only if’’ part, the orthogonality of the direct sum in Peirce decompositions implies that $x_i \langle c_i, c_i \rangle = \langle x, c_i \rangle = \langle x, c_i \circ c_i \rangle = \langle L_x(c_i), c_i \rangle$, and $x \in K(\mathfrak{J}) = \{y \in \mathfrak{J} : L_y \text{ is positive semidefinite}\}$ further implies $x_i \geq 0$. □

3. SMOOTHING APPROXIMATION

In [5], Chen and Mangasarian proposed to approximate the plus function $z^+ := \max\{0, z\}$ by a parametric smoothing function $p : \mathbb{R} \times \mathbb{R}_{++} \rightarrow \mathbb{R}_+$ such that $p(z, u) \rightarrow z^+$ as $u \downarrow 0$. More specifically, the function p is defined by double integrating a probability density function d with parameter u . In this paper, we are interested in a subclass of the Chen and Mangasarian smoothing function, whose probability density function d satisfies the following assumptions.

(A1) $d(t)$ is symmetric and piecewise continuous with finite number of pieces.

(A2) $E[|t|]_{d(t)} = \int_{-\infty}^{+\infty} |t|d(t)dt < +\infty$.

It is not difficult to verify that, under assumptions (A1) and (A2), the function p is the same as defining

$$(3) \quad p(z, u) = \int_{-\infty}^z (z - t)d(t, u)dt,$$

where $d(t, u) := \frac{1}{u}d(\frac{t}{u})$ (see proof of Proposition 2.1 of [5]). If, in addition, $d(t)$ has an infinite support, then $p(z, u)$ has the following nice properties.

Proposition 3.1. *Let $d(t)$ satisfy (A1), (A2) and has an infinite support. The following properties hold for the function $p(z, u)$ defined in (3).*

- (1) $p(z, u)$ is convex and continuously differentiable.
- (2) For each $z \in \mathbb{R}$, the function $u \in \mathbb{R}_{++} \mapsto p(z, u)$ is Lipschitz continuous; moreover, the Lipschitz constant is uniformly bounded above over all $z \in \mathbb{R}$.
- (3) $\lim_{z \rightarrow -\infty} p(z, u) = 0$, $\lim_{z \rightarrow \infty} p(z, u)/z = 1$, $0 < p'(z, u) < 1$, and $p'(-z, u) = 1 - p'(z, u)$, for all $u > 0$.
- (4) For each $u \geq 0$ and each $b > 0$, $p(z, u) = b$ has a unique solution.

Proof. See Proposition 1 of [4]. □

The following are two well-known smoothing functions derived from probability density functions with infinite support and satisfy assumptions (A1) and (A2).

Example 3.1. *Neural network smoothing function* [5].

$$p(z, u) = z + u \log(1 + e^{-\frac{z}{u}}),$$

where $d(t) = e^{-t}/(1 + e^{-t})^2$.

Example 3.2. *Chen-Harker-Kanzow-Smale (CHKS) function* [2, 23, 31].

$$p(z, u) = (z + \sqrt{z^2 + 4u})/2,$$

where $d(t) = 2/(t^2 + 4)^{\frac{3}{2}}$.

Throughout this paper, we shall assume that $p(z, u)$ has all the properties in Proposition 3.1.

From the smoothing approximation function p , we define the smooth approximation of the Euclidean projector $\text{Proj}_K(z)$ as the Löwner operator

$$p^\diamond(\cdot, u) : z \in \mathfrak{J} \mapsto \sum_{i=1}^r p(\lambda_i(z), u)c_i,$$

where $z = \sum_{i=1}^r \lambda_i(z)c_i$ is a spectral decomposition of z . For instance, the Löwner operator obtained from the CHKS smoothing function is

$$p(z, u) = (z + \sqrt{z^2 + 4ue})/2,$$

where \sqrt{x} denotes the unique $y \in \text{int}(K)$ with $y^2 = x$. We list below some properties of $p^\diamond(z, u)$ that are useful in this paper.

Proposition 3.2. *The following statements are true:*

- (a) $\lim_{u \downarrow 0} p^\diamond(z, u) = \text{Proj}_K(z)$.
- (b) $p^\diamond(z, u)$ is continuously differentiable.
- (c) For each $z \in \mathfrak{J}$, $u \in \mathbb{R}_{++} \mapsto p^\diamond(z, u)$ is Lipschitz continuous; moreover, the Lipschitz constant is uniformly bounded above over all $z \in \mathfrak{J}$.
- (d) For each $u > 0$, the Jacobian of the map $z \mapsto p^\diamond(z, u)$ is

$$w \mapsto \sum d_i w_i c_i + \sum d_{ij} w_{ij}$$

where $z = \sum \lambda_i(z)c_i$ is a spectral decomposition, $w = \sum w_i c_i + \sum w_{ij}$ is the Peirce decomposition, $d_i = p'(\lambda_i(z), u)$ and

$$d_{ij} = \begin{cases} \frac{p(\lambda_i(z), u) - p(\lambda_j(z), u)}{\lambda_i - \lambda_j} & \text{if } \lambda_i \neq \lambda_j, \\ p'(\lambda_i(z), u) & \text{if } \lambda_i = \lambda_j. \end{cases}$$

Moreover, $d_i, d_{ij} \in (0, 1)$.

- (e) For each $u \geq 0$ and each $b >_K 0$, $p^\diamond(z, u) = b$ has a unique solution.

Proof. (a) Follows from $\text{Proj}_K(z) = \sum_{i=1}^r \lambda_i(z)^+ c_i$ and the special case $K = \mathbb{R}_+$.

(b) Follows from Theorem 3.2 of [32] and corresponding property of p .

(c) Straightforward from the definition of p^\diamond and corresponding property of p .

(d) See [25, p. 74].

(e) Let $b = \sum_{i=1}^k \lambda_i c_i$ be a spectral decomposition of type I. Note that z solves $p^\diamond(z, u) = b$ if and only if the type I spectral decomposition of z is $\sum_{i=1}^k \mu_i c_i$ with $p(\mu_i, u) = \lambda_i$. \square

Since $\text{Proj}_K(z)$ and $\text{Proj}_{K^\#}(-z)$ can be approximated by $p^\diamond(z, u)$ and $p^\diamond(-z, u)$, respectively, the NME (1) can be approximated by the following parametric equation, called the Smooth Normal Map Equation (SNME):

$$(4) \quad (1 - u)f(p^\diamond(z, u)) - p^\diamond(-z, u) + ub = 0,$$

where $b >_K 0$, $u \in (0, 1]$. When $u = 1$, the SNME becomes

$$-p^\diamond(-z, u) + b = 0,$$

which has a unique solution by the above proposition. On the other hand, when $u = 0$, the SNME reduces to the NME (1). Therefore, if there exists a monotone trajectory from the unique solution at $u = 1$ to a solution at $u = 0$, we can apply standard homotopy techniques to find the solution of the NME, and hence a solution of $\text{NCP}_K(f)$. In [4], the uniform P -property is a sufficient condition to ensure the existence of such trajectory when K is polyhedral. In the next section, we study similar P -type properties.

4. P -TYPE PROPERTIES

4.1. Functions on \mathbb{R}^n . In the theory of LCPs [12], the P -property of a matrix plays a very important role and it can be defined in a number of ways. Here, we summarize below some known equivalent conditions for a matrix $M \in \mathbb{R}^{n \times n}$ to be a P -matrix:

(1) For every nonzero $x \in \mathbb{R}^n$, there is an index $i \in \{1, 2, \dots, n\}$ such that

$$x_i(Mx)_i > 0.$$

(2) Every principal minor of M is positive.

(3) $\text{LCP}(M, q)$ has a unique solution for every $q \in \mathbb{R}^n$ [27].

(4) The solution map of $\text{LCP}(M, q)$ is locally Lipschitzian with respect to data (M, q) [17].

The following lemma illustrates another equivalent characterization of P -property of a matrix.

Lemma 4.1. *A matrix $M \in \mathbb{R}^{n \times n}$ is a P -matrix if and only if*

$$(5) \quad \exists \alpha > 0, \forall d_1, \dots, d_n \geq 0, \forall x \in \mathbb{R}^n \quad \left\| Mx + \sum_{i=1}^n d_i x_i e_i \right\| \geq \alpha \|x\|,$$

where e_i denotes the i -th standard unit vector of \mathbb{R}^n .

Proof. “Only if”: Suppose M is a P -matrix. Then the continuity of $x \mapsto \max_i x_i(Mx)_i$ and the compactness of $\{x \in \mathbb{R}^n : \|x\| = 1\}$ implies $\alpha := \inf\{\max_i x_i(Mx)_i : x \in \mathbb{R}^n, \|x\| = 1\} > 0$. Thus for any $0 \neq x \in \mathbb{R}^n$, there is an index i with $x_i(Mx)_i \geq \alpha \|x\|^2 > 0$, whence for any $d_1, \dots, d_n \geq 0$,

$$\left\| Mx + \sum d_j x_j e_j \right\| \geq |(Mx)_i + d_i x_i| \geq |(Mx)_i| \geq \alpha \frac{\|x\|^2}{|x_i|} \geq \alpha \|x\|.$$

“If:” Suppose M satisfies (5). For any $0 \neq x \in \mathbb{R}^n$, the norm $\|Mx + \sum d_i x_i e_i\|$ is minimized over $(d_1, \dots, d_n) \in \mathbb{R}_+^n$ at

$$d_i = \begin{cases} 0 & \text{if } x_i(Mx)_i \geq 0, \\ -\frac{(Mx)_i}{x_i} & \text{if } x_i(Mx)_i < 0, \end{cases}$$

with minimum value $\sqrt{\sum_{i: x_i(Mx)_i \geq 0} (Mx)_i^2}$. With the $\alpha > 0$ in (5), it then follows that $\sum_{i: x_i(Mx)_i \geq 0} (Mx)_i^2 \geq \alpha^2 \|x\|^2$ for all $x \in \mathbb{R}^n$. For each $x \in \mathbb{R}^n$, by considering the perturbation $x - \varepsilon \Delta x$ with $(\Delta x)_i = \text{sgn}((Mx)_i)$ and taking the limit $\varepsilon \downarrow 0$, we deduce $\sum_{i: x_i(Mx)_i > 0} (Mx)_i^2 \geq \alpha^2 \|x\|^2$. Since the right hand side is positive when $x \neq 0$, we conclude that M is a P -matrix. \square

The P -property has also been extensively studied for the nonlinear functions in \mathbb{R}^n .

Definition 4.1. Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \rightarrow \mathbb{R}^n$. The function is said to be

- a P -function over Ω if there is an index $i \in \{1, \dots, n\}$ such that

$$(x_i - y_i)(f_i(x) - f_i(y)) > 0$$

for all $x, y \in \Omega$, $x \neq y$;

- a uniform P -function with modulus $\alpha > 0$ over Ω if there is an index $i \in \{1, \dots, n\}$ such that

$$(x_i - y_i)(f_i(x) - f_i(y)) \geq \alpha \|x - y\|^2.$$

for all $x, y \in \Omega$.

It is easy to see that, when f is affine, say $f(x) = Mx + q$ for some matrix $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the concept of P -function and uniform P -function coincide with the P -property of M . However, the uniform P -property is strictly stronger than the P -property in general. For more related discussion on P -functions and uniform P -functions, we refer the reader to [14].

Motivated by the new characterization for the P -property of a matrix, we introduce the following new P -type property for functions $f : \Omega \rightarrow \mathbb{R}^n$ defined on an open domain $\Omega \subset \mathbb{R}^n$.

Property 4.1. There is a constant $\alpha > 0$ such that for any nonnegative d_1, \dots, d_n and any $x, y \in \Omega$,

$$\left\| f(x) - f(y) + \sum_{i=1}^n d_i (x_i - y_i) e_i \right\| \geq \alpha \|x - y\|,$$

where e_i denotes the i -th standard unit vector of \mathbb{R}^n .

The nonlinear counter-part to Lemma 4.1 is given in the following proposition.

Proposition 4.1. Let $f : \Omega \rightarrow \mathbb{R}^n$ be a continuous function defined on the open domain $\Omega \subset \mathbb{R}^n$.

- If f is a uniform P -function, then it satisfies Property 4.1.
- If f satisfies Property 4.1, then it is a P -function.

Proof. The proofs of both statements are similar to the argument used in the proof of Lemma 4.1.

(a) Suppose f is a uniform P -function with modulus α . Then for any $x, y \in \Omega$ with $x \neq y$, there is an index i with $(f_i(x) - f_i(y))(x_i - y_i) \geq \alpha \|x - y\|^2 > 0$, whence for any

$d_1, \dots, d_n \geq 0$,

$$\begin{aligned} \left\| f(x) - f(y) + \sum d_j(x_j - y_j)e_j \right\| &\geq |f_i(x) - f_i(y) + d_i(x_i - y_i)| \\ &\geq |f_i(x) - f_i(y)| \geq \alpha \frac{\|x - y\|^2}{|x_i - y_i|} \geq \alpha \|x - y\|. \end{aligned}$$

(b) Suppose f satisfies Property 4.1. For any $x, y \in \Omega$ with $x \neq y$, the norm $\|f(x) - f(y) + \sum d_i(x_i - y_i)e_i\|$ is minimized over $(d_1, \dots, d_n) \in \mathbb{R}_+^n$ at

$$d_i = \begin{cases} 0 & \text{if } (x_i - y_i)(f_i(x) - f_i(y)) \geq 0, \\ -\frac{f_i(x) - f_i(y)}{x_i - y_i} & \text{if } (x_i - y_i)(f_i(x) - f_i(y)) < 0, \end{cases}$$

with minimum value $\sqrt{\sum_{i:(x_i - y_i)(f_i(x) - f_i(y)) \geq 0} (f_i(x) - f_i(y))^2}$. With the $\alpha > 0$ in (5), it then follows that $\sum_{i:(x_i - y_i)(f_i(x) - f_i(y)) \geq 0} (f_i(x) - f_i(y))^2 \geq \alpha^2 \|x - y\|^2$ for all $x, y \in \Omega$. For each $x, y \in \Omega$, by considering the perturbation $x \mapsto x - \varepsilon \Delta x$ with $(\Delta x)_i = \text{sgn}((f_i(x) - f_i(y)))$ and taking the limit $\varepsilon \downarrow 0$, we deduce, via the continuity of f , $\sum_{i:(x_i - y_i)(f_i(x) - f_i(y)) > 0} (f_i(x) - f_i(y))^2 \geq \alpha^2 \|x - y\|^2$. Since the right hand side is positive when $x \neq y$, we conclude that f is a P -function. \square

It is well known that the $\text{NCP}(f)$ has a unique solution if f is a continuous uniform P -function over the open domain $\Omega \supset \mathbb{R}_+$; see, e.g., [20, Theorem 3.9]. Our next result shows the same conclusion when f satisfies Property 4.1.

Proposition 4.2. *If f is a continuous function over the open domain $\Omega \supset \mathbb{R}_+$ satisfying Property 4.1, then $\text{NCP}(f)$ has unique solution.*

Proof. “Existence:” In view of Theorem 3.4 in [26], it suffices to show that for any $M > 0$, there is a constant $r > 0$ such that

$$\max_{1 \leq i \leq n} \frac{x_i f_i(x)}{\|x\|} \geq M,$$

for all $\|x\| \geq r$ and $x \in \mathbb{R}_+^n$. Suppose $M > 0$ is given, and let $r_0 = \max\{2M\sqrt{n}/\alpha, 3/4\}$. For each $x \in \mathbb{R}_+^n$, we define an index set $I(x) := \{i : x_i \geq r_0\}$ and define a corresponding vector \tilde{x} as

$$\tilde{x}_i = \begin{cases} 0 & \text{if } i \in I(x), \\ x_i & \text{if } i \notin I(x). \end{cases}$$

Suppose that $\|x\| > \sqrt{n}r_0$. Then the index set $I(x)$ is nonempty. For some $\varepsilon \in (0, 1)$ to be determined later, define another vector y as

$$y_i = \begin{cases} \tilde{x}_i (= 0) & \text{if } i \in I(x), \\ \tilde{x}_i + \varepsilon \text{sgn}(f_i(x) - f_i(\tilde{x})) & \text{if } i \notin I(x). \end{cases}$$

We only consider ε sufficiently small so that $y \in \Omega$. It is straightforward to deduce from this definition that $\|y - \tilde{x}\| \leq \sqrt{n}\varepsilon$. Moreover, from the assumption $\|x\| > \sqrt{n}r_0$, we deduce that $\|x - \tilde{x}\| \geq r_0$, whence $\|x - y\| \geq r_0 - \sqrt{n}\varepsilon$. Fix ε sufficiently small so that $(f_i(x) - f_i(\tilde{x}))(f_i(x) - f_i(y)) > 0$ whenever $f_i(x) - f_i(\tilde{x}) \neq 0$ and $|f_i(x) - f_i(y)| < \frac{\alpha}{\sqrt{n}}(r_0 - \sqrt{n}\varepsilon) \leq \frac{\alpha}{\sqrt{n}}\|x - y\|$ whenever $f_i(x) - f_i(\tilde{x}) = 0$; such ε exists by the continuity

of f . Now, we let

$$d_i := \begin{cases} 0 & \text{if } i \in I(x) \text{ and } x_i(f_i(x) - f_i(\tilde{x})) \geq 0, \\ -\frac{f_i(x) - f_i(y)}{x_i} & \text{if } i \in I(x) \text{ and } x_i(f_i(x) - f_i(\tilde{x})) < 0, \\ 0 & \text{if } i \notin I(x) \text{ and } (f_i(x) - f_i(\tilde{x})) = 0, \\ -\frac{f_i(x) - f_i(y)}{\varepsilon \operatorname{sgn}(f_i(x) - f_i(\tilde{x}))} & \text{if } i \notin I(x) \text{ and } (f_i(x) - f_i(\tilde{x})) \neq 0. \end{cases}$$

Then $d_i \geq 0$. By Property 4.1, there is an index i such that $|f_i(x) - f_i(y) + d_i(x_i - y_i)| \geq \frac{\alpha}{\sqrt{n}}\|x - y\|$. Moreover, from the construction of d_i and the choice of ε , it must happen that $i \in I(x)$, $d_i = 0$ and $f_i(x) - f_i(y) > 0$, whence

$$(6) \quad f_i(x) - f_i(y) \geq \frac{\alpha}{\sqrt{n}}\|x - y\|.$$

It is straightforward to deduce from its definition that $\|y\| \leq \sqrt{n}(r_0 + \varepsilon)$. Now, suppose further that

$$(7) \quad \|x\| \geq \max \left\{ 3\sqrt{n}(r_0 + 1), \frac{\alpha^2}{144n}M^2, \frac{16n}{\alpha^2}M_1 \right\},$$

where $M_1 := \max\{\|f(z)\| : z \in [-1, r_0 + 1]^n\}$. Then $\|x\| \leq \|y\| + \|x - y\| \leq \sqrt{n}(r_0 + \varepsilon) + \|x - y\|$ implies that $\|x - y\| \geq 2\sqrt{n}(r_0 + 1) \geq 2\|y\|$, and subsequently,

$$(8) \quad \|x - y\| \geq \frac{2}{3}\|x\|.$$

In the case $x_i \geq \frac{1}{2}\|x\|^{\frac{1}{2}}$, we obtain from (6) that $f_i(x)x_i \geq \frac{\alpha x_i}{\sqrt{n}}\|x - y\| - f_i(y)x_i \geq \frac{\alpha}{2\sqrt{n}}\|x - y\|\|x\|^{\frac{1}{2}} - \|f(y)\|\|x\|$. Hence, together with (7) and (8), we conclude that $|f_i(x)x_i| \geq \frac{\alpha}{12\sqrt{n}}\|x\|^{\frac{3}{2}} \geq M\|x\|$.

In the case $r_0 \leq x_i < \frac{1}{2}\|x\|^{\frac{1}{2}}$, we again obtain from (6) that $f_i(x)x_i \geq \frac{\alpha r_0}{\sqrt{n}}\|x - y\| - \frac{1}{2}\|f(y)\|\|x\|^{\frac{1}{2}}$. Hence, together with (7), (8) and the definition of r_0 , we conclude that $f_i(x)x_i \geq \frac{2\alpha r_0}{3\sqrt{n}}\|x\| - \frac{\alpha}{8\sqrt{n}}\|x\| \geq \frac{\alpha r_0}{2\sqrt{n}}\|x\| \geq M\|x\|$.

“Uniqueness.” By Proposition 4.1, f is a P -function. Uniqueness is a straightforward consequence of the P -property; see, e.g., [20, Theorem 3.9]. \square

From above results, it seems that functions satisfying Property 4.1 behave like uniform P -functions. Therefore, a natural question is: Are uniform P -property functions characterized by Property 4.1? The following example shows that this is not always true in general. Therefore, our P -type property lies between the concept of P -property and uniform P -property.

Example 4.1. Let $\Omega = (1, \infty)^2$ and $f : \Omega \rightarrow \mathbb{R}^2$ be defined as:

$$f(x_1, x_2) := (x_1^2 + x_2, -x_1^3 + x_2^2).$$

Given any $\alpha \in (0, 1)$, we take $x = (\frac{4}{\alpha}, \frac{4}{\alpha})$ and $y = (\frac{4}{\alpha} - \frac{\alpha}{4}, \frac{4}{\alpha} - 1)$. It is easy to see that $x, y \in \Omega$ and

$$\max_{i=1,2} (x_i - y_i)(f_i(x) - f_i(y)) < \alpha\|x - y\|^2$$

Therefore, f is not a uniform P -function.

Fix any $x, y \in \Omega$ and any $d_1, d_2 \geq 0$. If $(x_1 - y_1)(x_2 - y_2) \geq 0$, then

$$\begin{aligned}
& \left\| f(x) - f(y) + \sum d_i(x_i - y_i)e_i \right\|^2 \\
& \geq |(f_1(x) - f_1(y)) + d_1(x_1 - y_1)|^2 \\
& = \left| \frac{\partial f_1(x_\theta, y_\theta)}{\partial x_1}(x_1 - y_1) + \frac{\partial f_1(x_\theta, y_\theta)}{\partial x_2}(x_2 - y_2) + d_1(x_1 - y_1) \right|^2 \\
& \geq (2 + d_1)^2(x_1 - y_1)^2 + (x_2 - y_2)^2 \\
& \geq \|x - y\|^2,
\end{aligned}$$

where $x_\theta = x_2 + \theta(x_1 - x_2)$ and $y_\theta = y_2 + \theta(y_1 - y_2)$ for some $\theta \in (0, 1)$. Similarly, if $(x_1 - y_1)(x_2 - y_2) < 0$, we have

$$\begin{aligned}
& \left\| f(x) - f(y) + \sum d_i(x_i - y_i)e_i \right\|^2 \\
& \geq |(f_2(x) - f_2(y)) + d_2(x_1 - y_1)|^2 \\
& = \left| \frac{\partial f_2(x_\theta, y_\theta)}{\partial x_1}(x_1 - y_1) + \frac{\partial f_2(x_\theta, y_\theta)}{\partial x_2}(x_2 - y_2) + d_2(x_2 - y_2) \right|^2 \\
& \geq 9(x_1 - y_1)^2 + (2 + d_2)^2(x_2 - y_2)^2 \\
& \geq \|x - y\|^2.
\end{aligned}$$

Combining the above two cases, we have shown that f satisfies Property 4.1.

So far, we have only discussed the P -property defined on the space \mathbb{R}^n . In the remaining of this section, we investigate the P -property for transformations in the setting of LCP_K and NCP_K where K is the closure of a symmetric cone.

4.2. Functions on Euclidean Jordan algebras. Motivated by the significance of P -matrices in the theory of LCP, Gowda et al. [18] introduced the following P -property for linear transformations on Euclidean Jordan algebras \mathfrak{J} to study LCP_K where K is the closure of a symmetric cone.

Definition 4.2. A linear transformation $l : \mathfrak{J} \rightarrow \mathfrak{J}$ is said to possess the P -property if x and $l(x)$ operator commute¹ and $x \circ l(x) \leq_K 0 \implies x = 0$.

For the convenience of our discussion, we introduce several more definitions.

Definition 4.3. Let K be the cone of squares of \mathfrak{J} . A linear transformation $l : \mathfrak{J} \rightarrow \mathfrak{J}$ is said to possess

- (1) the R_0 -property if the zero vector is the only solution of $\text{LCP}_K(l, 0)$;
- (2) the Q -property if for every $q \in \mathfrak{J}$, $\text{LCP}_K(l, q)$ has a solution;
- (3) the globally uniquely solvable property (**GUS**-property, for short) if for all $q \in \mathfrak{J}$, $\text{LCP}_K(l, q)$ has a unique solution.

It is not too difficult to see that if a linear transformation l has the P -property, then it has the R_0 -property. Moreover, the P -property implies the Q -property [18, Theorem 12]. Recall that $\text{LCP}(M, q)$ has a unique solution for any $q \in \mathbb{R}^n$ if M is a P -matrix. Unfortunately, this result can not be carried over to $\text{LCP}_K(l, q)$; i.e., the **GUS**-property may not hold even if l has the P -property. However, the converse is always true.

¹See Remark 2.3.

Proposition 4.3. *If a linear transformation $l : \mathfrak{J} \rightarrow \mathfrak{J}$ has the **GUS**-property, then it has the P -property.*

Proof. See proof of Theorem 14 of [18]. □

A natural extension of (5) to linear transformations in \mathfrak{J} is the following property:

Property 4.2. There exists $\alpha > 0$ such that for any nonnegative d_1, \dots, d_r and any $x \in \mathfrak{J}$,

$$\left\| l(x) + \sum_{i=1}^r d_i \lambda_i(x) c_i \right\| \geq \alpha \|x\|,$$

for any spectral decomposition $x = \sum_{i=1}^r \lambda_i(x) c_i$.

The following proposition shows that the above property is equivalent to the P -property.

Proposition 4.4. *A linear transformation $l : \mathfrak{J} \rightarrow \mathfrak{J}$ has the P -property if and only if it satisfies Property 4.2.*

Proof. “Only if”: We shall prove the contra-positive.

Suppose l does not satisfy Property 4.2. Let $\{\alpha_k\}$ be a positive sequence converging to 0. Then for each α_k , there exist an $x^k \in \mathfrak{J}$ and a sequence d_1^k, \dots, d_r^k of nonnegative real numbers such that

$$\left\| l(x^k) + \sum d_i^k \lambda_i(x^k) c_i^k \right\| < \alpha_k \|x^k\|,$$

where $x^k = \sum_{i=1}^r \lambda_i(x^k) c_i^k$ is a spectral decomposition of x^k . Without any loss of generality, we may assume, by scaling if necessary, that $\|x^k\| = 1$. Thus x^k has a convergent subsequence, and we may assume, by taking a subsequence if necessary, that $x^k \rightarrow x$. The continuity of Peirce decomposition² implies $\lambda_i(x^k) \rightarrow \lambda_i(x)$. Moreover, since $\{c_i^k\}$ is bounded for each i , we can further assume that $c_i^k \rightarrow c_i$ for each i such that $x = \sum_{i=1}^r \lambda_i(x) c_i$. Subsequently, we deduce that $\{d_i^k\}$ is bounded for each i , whence can be assumed to be convergent, say to $d_i \geq 0$. Passing to the limit $k \rightarrow \infty$, we see that $l(x) = -\sum d_i \lambda_i(x) c_i$ operator commutes with x . Moreover, $x \circ l(x) = -\sum d_i \lambda_i(x)^2 c_i \leq_K 0$. Thus l does not have the P -property.

“If.” The proof for this part is similar to the argument in the proof of Lemma 4.1.

Suppose l satisfies Property 4.2. For any $0 \neq x \in \mathfrak{J}$ having x and $l(x)$ operator commute, and any spectral decomposition $x = \sum_{i=1}^r \lambda_i(x) c_i$ with $l(x) = \sum_{i=1}^r l_i c_i$ being the Peirce decomposition, the norm $\|l(x) + \sum d_i \lambda_i(x) c_i\|$ is minimized over $(d_1, \dots, d_n) \in \mathbb{R}_+^n$ at

$$d_i = \begin{cases} 0 & \text{if } \lambda_i(x) l_i \geq 0, \\ -\frac{l_i}{\lambda_i(x)} & \text{if } \lambda_i(x) l_i < 0, \end{cases}$$

with minimum value $\sqrt{\sum_{i: \lambda_i(x) l_i \geq 0} l_i^2}$. With the $\alpha > 0$ in Property 4.2, it then follows that $\sum_{i: \lambda_i(x) l_i \geq 0} l_i^2 \geq \alpha^2 \|x\|^2$. By considering the perturbation $x - \varepsilon \Delta x$ with $\Delta x = \sum_{i=1}^r \text{sgn}(l_i) c_i$ and taking the limit $\varepsilon \downarrow 0$, we deduce $\sum_{i: \lambda_i(x) l_i > 0} l_i^2 \geq \alpha^2 \|x\|^2$. If $x \circ l(x) = \sum_{i=1}^r \lambda_i(x) l_i c_i$ has nonpositive eigenvalues, then the index set $\{i : \lambda_i(x) l_i > 0\}$ must be empty, whence $x = 0$. Therefore l has the P -property. □

Corollary 4.1. *Given a linear transformation $l : \mathfrak{J} \rightarrow \mathfrak{J}$, the following statements hold:*

- (a) *if l satisfies Property 4.2, then it has the R_0 - and Q -properties.*

²See Remark 2.5.

(b) if l has the **GUS**-property, then it satisfies Property 4.2.

Next, we extend Property 4.2 to nonlinear transformations $f : \Omega \rightarrow \mathfrak{J}$ defined on an open subset $\Omega \subseteq \mathfrak{J}$.

Property 4.3. There exists $\alpha > 0$ such that for any $d_1, \dots, d_r \geq 0$, any $d_{ij} \geq 0$, any Jordan frame $\{c_1, \dots, c_r\}$, and every $x, y \in \Omega$

$$\left\| f(x) - f(y) + \sum_{i=1}^r d_i(x_i - y_i)c_i + \sum_{(i,j):1 \leq i < j \leq r} d_{ij}(x_{ij} - y_{ij}) \right\| \geq \alpha \|x - y\|,$$

where $x = \sum_{i=1}^r x_i c_i + \sum_{(i,j):1 \leq i < j \leq r} x_{ij}$ and $y = \sum_{i=1}^r y_i c_i + \sum_{(i,j):1 \leq i < j \leq r} y_{ij}$ are Peirce decompositions.

In the subsequent two sections, we will use Property 4.3 to give a global analysis of an extension of the continuation method in [4] to solve NCP_K .

Note that when f is linear, Property 4.3 is a strengthening of Property 4.2. It is not clear if this strengthening is related to the **GUS**-property. On the other hand, it can be shown that Property 4.3 is implied by the Cartesian P -property when f is linear.

Proposition 4.5. If a linear transformation $l : \mathfrak{J} \rightarrow \mathfrak{J}$ satisfies the Cartesian P -property (2), then it satisfies Property 4.3.

Proof. Suppose that l does not satisfy Property 4.3. Then, following an argument similar to the proof of Proposition 4.4, we deduce sequences $\alpha_k \downarrow 0$, $x^k \rightarrow x$ with $\|x^k\| = \|x\| = 1$, a sequence of nonnegative tuples $\{(d_1^k, \dots, d_r^k, d_{12}^k, \dots, d_{r-1,r}^k)\}$, and a sequence of Jordan frames $\{c_1^k, \dots, c_r^k\}$ such that

$$\left\| l(x^k) + \sum_{i=1}^r d_i^k x_i^k c_i + \sum_{(i,j):1 \leq i < j \leq r} d_{ij}^k x_{ij}^k \right\| < \alpha_k \|x^k\|,$$

where $x^k = \sum_{i=1}^r x_i^k c_i^k + \sum_{(i,j):1 \leq i < j \leq r} x_{ij}^k$ is the Peirce decomposition. Subsequently, with w^k denoting the sum $l(x^k) + \sum_{i=1}^r d_i^k x_i^k c_i + \sum_{(i,j):1 \leq i < j \leq r} d_{ij}^k x_{ij}^k$, we deduce

$$\langle x_\nu^k, l(x^k)_\nu \rangle = \langle x_\nu^k, w_\nu^k \rangle - \sum_{i=1}^r d_i^k (x_i^k)_\nu^2 - \sum_{(i,j):1 \leq i < j \leq r} d_{ij}^k \|(x_{ij}^k)_\nu\|^2 \leq \langle x_\nu^k, w_\nu^k \rangle$$

for each $\nu \in \{1, \dots, \kappa\}$. Passing to the limit $k \rightarrow \infty$, we see that $\langle x_\nu, l(x)_\nu \rangle \leq 0$ for all ν , whence l does not have the Cartesian P -property. \square

5. BOUNDEDNESS AND UNIQUENESS OF TRAJECTORY

We now return to the problem $\text{NCP}_K(f)$, where K is the closure of a symmetric cone with associated Jordan algebra \mathfrak{J} , and $f : \Omega \rightarrow \mathfrak{J}$ is a continuously differentiable function defined over an open domain Ω containing K . Let $h : \mathfrak{J} \times \mathbb{R} \rightarrow \mathfrak{J}$ be defined by

$$(9) \quad (z, u) \mapsto (1 - u)f(p^\diamond(z, u)) - p^\diamond(-z, u) + ub,$$

where $p^\diamond(z, u)$ is the smoothing approximation of $\text{Proj}_K(z)$. Recall that the SNME is given by $h(z, u) = 0$, where $b >_K 0$, $u \in (0, 1]$.

Let S denote the set $\{(z, u) \in \mathfrak{J} \times (0, 1] : h(z, u) = 0\}$. Define the solution path T as the connected component of S emanating from the unique solution of $h(z, 1) = 0$. In this section, we show that T forms a smooth and bounded trajectory that is monotone with respect to u . Thus, there exists at least a limit point as u is reduced to zero along

the trajectory. We further show that every limit point is a solution of the NME. The boundedness of the level set is studied first.

Proposition 5.1. *Let $S_{c,\delta}$ denote the level set*

$$\{(z, u) \in \mathfrak{J} \times (0, 1 - \delta] : \|h(z, u)\| \leq c\}$$

If f satisfies Property 4.3, then for all c the level set $S_{c,\delta}$ is bounded for each $\delta > 0$. If, in addition, $c < \lambda_r(b)$, then the level set $S_{c,0}$ is also bounded.

Proof. Suppose on the contrary that $S_{c,\delta}$ is unbounded for some c and some $\delta > 0$. Then there exists an unbounded sequence $\{z^k\}$ such that $\|h(z^k, u_k)\| \leq c$ for some $u_k \in (0, 1 - \delta]$.

For each k , let $z^k = \sum_{i=1}^r z_i^k c_i^k$ be a spectral decomposition of z^k . Denote by x^k and y^k , respectively, $p^\diamond(z^k, u_k)$ and $p^\diamond(-z^k, u_k)$. Then $x^k = \sum_{i=1}^r x_i^k c_i^k$ and $y^k = \sum_{i=1}^r y_i^k c_i^k$, with $x_i^k = p(z_i^k, u_k) > 0$ and $y_i^k = p(-z_i^k, u_k) > 0$, are Peirce decompositions. Let $b = \sum_{i=1}^r b_i^k c_i^k + \sum_{(i,j):1 \leq i < j \leq r} b_{ij}^k$, $h(z^k, u_k) = \sum_{i=1}^r h_i^k c_i^k + \sum_{(i,j):1 \leq i < j \leq r} h_{ij}^k$ and $f(x^k) = \sum_{i=1}^r f_i^k c_i^k + \sum_{(i,j):1 \leq i < j \leq r} f_{ij}^k$ be Peirce decompositions.

If the sequence $\{x^k\}$ is bounded, then we deduce from $\|h(z^k, u_k)\| \leq c$ that the sequence $\{y^k\}$ is bounded, so that $\|z^k\|^2 \leq \|x^k\|^2 + \|y^k\|^2$ for all k results in a contradiction with the unboundedness of $\{z^k\}$. Thus the index set $I := \{i : \{x_i^k\} \text{ is unbounded}\}$ is nonempty.

Consider the bounded sequence $\{\tilde{x}^k := \sum_{i \notin I} x_i^k c_i^k\}$. Let $f(x^k) - f(\tilde{x}^k) = \sum_{i=1}^r \tilde{g}_i^k c_i^k + \sum_{(i,j):1 \leq i < j \leq r} \tilde{g}_{ij}^k$ be a Peirce decomposition. Consider another bounded sequence

$$\left\{ \hat{x}^k := \tilde{x}^k + \sum_{i \notin I} \frac{\varepsilon}{|\tilde{g}_i^k| + 1} \tilde{g}_i^k c_i^k + \sum_{(i,j):1 \leq i < j \leq r} \frac{\varepsilon}{\|\tilde{g}_{ij}^k\| + 1} \tilde{g}_{ij}^k \right\},$$

where $\varepsilon = \frac{1}{2}$. Let $f(x^k) - f(\hat{x}^k) = \sum_{i=1}^r \hat{g}_i^k c_i^k + \sum_{(i,j):1 \leq i < j \leq r} \hat{g}_{ij}^k$ be a Peirce decomposition. For $i \in I$, let $d_i^k = \max\left\{0, -\frac{\tilde{g}_i^k}{x_i^k}\right\}$; for $i \notin I$, let $d_i^k = \frac{1}{\varepsilon}(|\tilde{g}_i^k| + 1)$. Let $d_{ij}^k = \frac{1}{\varepsilon}(\|\tilde{g}_{ij}^k\| + 1)$ for $1 \leq i < j \leq r$. By construction, $d_i^k \geq 0$ and $d_{ij}^k > 0$. Recall Property 4.3:

$$\left\| \begin{aligned} & \sum_{i=1}^r \hat{g}_i^k c_i^k + \sum_{(i,j):1 \leq i < j \leq r} \hat{g}_{ij}^k + \sum_{i \in I} d_i^k x_i^k c_i^k \\ & - \sum_{i \notin I} d_i^k \frac{\varepsilon}{|\tilde{g}_i^k| + 1} \tilde{g}_i^k c_i^k - \sum_{(i,j):1 \leq i < j \leq r} d_{ij}^k \frac{\varepsilon}{\|\tilde{g}_{ij}^k\| + 1} \tilde{g}_{ij}^k \end{aligned} \right\| \geq \alpha \|x^k - \hat{x}^k\|,$$

which simplifies to

$$\left\| f(\tilde{x}^k) - f(\hat{x}^k) + \sum_{i \in I, \tilde{g}_i^k > 0} \tilde{g}_i^k c_i^k \right\| \geq \alpha \|x^k - \hat{x}^k\|.$$

As $k \rightarrow +\infty$, the right hand side tends to $+\infty$ while $\{(f(\tilde{x}^k), f(\hat{x}^k))\}$ remains bounded. Thus, by taking a subsequence if necessary, we conclude that there is a index $i \in I$ such that $|\tilde{g}_i^k| \rightarrow +\infty$ and $\tilde{g}_i^k > 0$ for all k ; i.e., there is some index i with $x_i^k \rightarrow +\infty$ and $\tilde{g}_i^k \rightarrow +\infty$. For this i , we then have $z_i^k \rightarrow +\infty$, whence $y_i^k \rightarrow 0$. Moreover, since $f(\tilde{x}^k)$ is bounded, $\tilde{g}_i^k \rightarrow +\infty$ implies $f_i^k \rightarrow +\infty$. Thus $\lim(1 - u_k)(f_i^k - b_i^k) = +\infty$ since $|b_i^k| \leq \|b\|$ and $1 - u_k \geq \delta > 0$. Together with (9) and $\|h(z^k, u_k)\| \leq c$, we get the contradiction

$$c + \|b\| \geq \liminf_k (\|h(z^k, u_k)\| - b_i^k) \geq \liminf_k (h_i^k - b_i^k) = \liminf_k (1 - u_k)(f_i^k - b_i^k) = +\infty.$$

“Moreover”: If, instead, we suppose that $S_{c,0}$ is unbounded for some $c < \lambda_r(b)$, then we can only deduce $\liminf_k (1 - u_k)(f_i^k - b_i^k) \geq 0$. Nonetheless, together with (9) and

$\|h(z^k, u_k)\| \leq c < \lambda_r(b)$, we still get a contradiction:

$$\begin{aligned} 0 > c - \lambda_r(b) &\geq \liminf_k (\|h(z^k, u_k)\| - b_i^k) \\ &\geq \liminf_k (h_i^k - b_i^k) = \liminf_k (1 - u_k)(f_i^k - b_i^k) \geq 0, \end{aligned}$$

where the second inequality follows from $b \geq_K \lambda_r(b)e$ and Corollary 2.1. \square

Corollary 5.1. *Suppose that f satisfies Property 4.3, then the solution set S is bounded.*

The following proposition proves the nonsingularity of the Jacobian $J_z h(z, u)$ under the assumption that f satisfies Property 4.3.

Proposition 5.2. *Suppose f satisfies Property 4.3, and $h(z, u)$ is as defined in (9). Then the Jacobian $J_z h(z, u)$ is nonsingular for any $(z, u) \in \mathfrak{J} \times (0, 1]$.*

Proof. Fix any $z \in \mathfrak{J}$ and let $z = \sum_{i=1}^r \lambda_i c_i$ be a spectral decomposition of z . We shall show that $J_z h(z, u)w = 0$ only has the trivial solution. Let $w \in \mathfrak{J}$ be a solution; i.e., w satisfies

$$(10) \quad (1-u)Jf(p^\diamond(z, u)) \left[\sum d_i w_i c_i + \sum d_{ij} w_{ij} \right] + \sum (1-d_i) w_i c_i + \sum (1-d_{ij}) w_{ij} = 0$$

where $w = \sum w_i c_i + \sum w_{ij}$ is a Peirce decomposition, $d_i = p'(\lambda_i, u)$ and

$$d_{ij} = \begin{cases} \frac{p(\lambda_i, u) - p(\lambda_j, u)}{\lambda_i - \lambda_j} & \text{if } \lambda_i \neq \lambda_j, \\ p'(\lambda_i, u) & \text{if } \lambda_i = \lambda_j. \end{cases}$$

Note that $d_i \in (0, 1)$ and $d_{ij} \in (0, 1)$; see Propositions 3.1 and 3.2. If $u = 1$, then (10) implies $w = 0$.

For the remainder of this proof, we assume $u \in (0, 1)$. Let $y = \sum d_i w_i c_i + \sum d_{ij} w_{ij}$. Then, in the Peirce decomposition $y = \sum y_i c_i + \sum y_{ij}$ we have $y_i = d_i w_i$ and $y_{ij} = d_{ij} w_{ij}$. Rewriting (10) in terms of y gives

$$(11) \quad (1-u)Jf(p^\diamond(z, u))[y] + \sum \left(\frac{1}{d_i} - 1 \right) y_i c_i + \sum \left(\frac{1}{d_{ij}} - 1 \right) y_{ij} = 0.$$

Note that $\frac{1}{d_i} - 1 > 0$ and $\frac{1}{d_{ij}} - 1 > 0$.

Since f satisfies Property 4.3, we have

$$\begin{aligned} &\left\| \frac{f(p^\diamond(z, u) + ty) - f(p^\diamond(z, u))}{t} + \sum \frac{1}{1-u} \left(\frac{1}{d_i} - 1 \right) y_i c_i + \sum \frac{1}{1-u} \left(\frac{1}{d_{ij}} - 1 \right) y_{ij} \right\| \\ &\geq \alpha \|y\|, \end{aligned}$$

which, in the limit as $t \rightarrow 0$, becomes

$$\left\| Jf(p^\diamond(z, u))[y] + \sum \frac{1}{1-u} \left(\frac{1}{d_i} - 1 \right) y_i c_i + \sum \frac{1}{1-u} \left(\frac{1}{d_{ij}} - 1 \right) y_{ij} \right\| \geq \alpha \|y\|.$$

From (11), we then have $y = 0$, and hence $w = 0$. \square

Now, we are ready to establish the uniqueness of the trajectory.

Proposition 5.3. *Let $h(z, u)$ be as defined in (9) with $b >_K 0$. If f satisfies Property 4.3, then the following statements are true.*

- For each $u \in (0, 1]$, the SNME (4) has a unique solution $z(u)$; hence, the trajectory can be rewritten as $T = \{(z(u), u) : u \in (0, 1]\}$, which is monotone with respect to u .*
- The trajectory T is bounded; hence, the trajectory T has at least one accumulation point with (z_*, u_*) with $u_* = 0$.*

(c) Every accumulation point $(z_*, 0)$ of T gives a solution z_* to the NME (1).

Proof. (a). Under the hypotheses of the proposition, S is bounded Corollary 5.1.

Let U denote the set of $u \in (0, 1]$ for which $h(z, \bar{u}) = 0$ has a unique solution for each $\bar{u} \in [u, 1]$. The set U is nonempty as it contains 1. Thus the infimum \tilde{u} of U exists. If $\tilde{u} = 0$, the desired result follows. Consider the case $\tilde{u} > 0$. Pick a sequence $\{u_k\} \subset U$ such that $u_k \downarrow \tilde{u}$ as $k \rightarrow \infty$. Corresponding to each u_k , let z^k denote the unique solution to $h(z^k, u_k) = 0$. Since S is bounded, the sequence $\{z^k\}$ has a limit point \tilde{z} . It follows from the continuity of H that $(\tilde{z}, \tilde{u}) \in S$. As we assumed that $\tilde{u} > 0$, the Jacobian $J_z h(z, \tilde{u})$ is nonsingular for all $z \in \mathfrak{J}$. Thus we may apply the Implicit Function Theorem to the equation $h(z, u) = 0$ at (\tilde{z}, \tilde{u}) . This gives a $\delta \in (0, \tilde{u})$ and a continuous function $u \mapsto \tilde{z}(u)$ with $\tilde{z}(\tilde{u}) = \tilde{z}$ such that $h(\tilde{z}(u), u) = 0$ for all $u \in (\tilde{u} - \delta, \tilde{u} + \delta)$. Moreover, there is an $\varepsilon > 0$ such that for each $u \in (\tilde{u} - \delta, \tilde{u} + \delta)$, $z(u)$ is the only solution of $h(z, u) = 0$ satisfying $\|z - \tilde{z}\| < \varepsilon$.

For each k , $\tilde{u} - \frac{1}{k}\delta \notin U$. By definition of U , there exists $v_k \in (\tilde{u} - \frac{1}{k}\delta, \tilde{u})$ such that $h(z, v_k) = 0$ does not have a unique solution. Since $v_k \in (\tilde{u} - \delta, \tilde{u} + \delta)$, the equation $h(z, v_k) = 0$ does have a solution $z(v_k)$, whence there is another solution, say, \bar{z}^k . Since $z(v_k)$ is the only solution to $h(z, v_k) = 0$ satisfying $\|z - \tilde{z}\| < \varepsilon$, we must have $\|\bar{z}^k - \tilde{z}\| \geq \varepsilon$. As S is bounded, so is the sequence $\{\bar{z}^k\}$. Hence it has a limit point, say \bar{z}^∞ , different from \tilde{z} . By continuity of H , \bar{z}^∞ solves $h(z, \tilde{u}) = 0$. Now apply the Implicit Function Theorem to $h(z, u) = 0$ at $(\bar{z}^\infty, \tilde{u})$. This gives a $\bar{\delta} \in (0, \delta)$ and a continuous function $u \mapsto \bar{z}(u)$ with $\bar{z}(\tilde{u}) = \bar{z}^\infty$ such that $h(\bar{z}(u), 0) = 0$ for all $u \in (\tilde{u} - \bar{\delta}, \tilde{u} + \bar{\delta})$. Since $h(z, u_k) = 0$ has a unique solution for every k and $u_k \downarrow \tilde{u}$, it follows that $\tilde{z}(u_k) = \bar{z}(u_k)$ for all k sufficiently large (so that $u_k \in (\tilde{u} - \bar{\delta}, \tilde{u} + \bar{\delta})$). Thus by continuity we arrive at the contradiction

$$\tilde{z} = \lim_{k \rightarrow \infty} \tilde{z}(u_k) = \lim_{k \rightarrow \infty} \bar{z}(u_k) = \bar{z}^\infty.$$

(b). This follows from (a) and Corollary 5.1.

(c). Let z_* be any limit point of $z(u)$ as $u \downarrow 0$. By continuity of H , it follows that $h(z_*, 0) = 0$, i.e., $f(z_*^+) + z_*^- = 0$. This shows that z_* is a solution of the NME. \square

6. CONTINUATION METHOD

In this section, we describe the continuation method for solving the SNME (4).

Define the merit function $\theta : (z, u) \in \mathfrak{J} \times \mathbb{R} \mapsto \|h(z, u)\|^2$.

Algorithm 6.1.

Step 0. Given $z^0 \in \mathfrak{J}$. Choose $u_0 \in (0, 1]$, $\beta, \delta \in (0, 1)$, nonnegative integer M and $b >_K 0$. Set $k = 0$.

Step 1. Solve for v^k in

$$h(z^k, u_k) + J_z h(z^k, u_k) v^k = 0.$$

Step 2. Let i be the smallest nonnegative integer such that

$$(12) \quad \theta(z^k + \beta^i v^k, u_k) \leq \mathcal{W} + \delta \beta^i J_z \theta(z^k, u_k) v^k.$$

where \mathcal{W} is any value satisfying

$$(13) \quad \theta(z^k, u_k) \leq \mathcal{W} \leq \max_{j=0,1,\dots,m_k} \theta(z^{k-j}, u_{k-j}),$$

and m_k is a nonnegative integer no more than $\min\{m_{k-1} + 1, M\}$. Set $\alpha_k = \beta^i$.

Step 3. Set

$$z^{k+1} = z^k + \alpha_k v^k, \quad u_{k+1} = \delta u_k, \quad k \leftarrow k + 1.$$

Go to Step 1.

Remark 6.1. Since

$$\begin{aligned} \theta(z^k + \alpha v^k, u_k) - \delta J_z \theta(z^k, u_k) v^k &= \theta(z^k, u_k) + (1 - \delta) \alpha J_z \theta(z^k, u_k) v^k + o(\alpha) \\ &\leq \mathcal{W} + (1 - \delta) \alpha J_z \theta(z^k, u_k) v^k + o(\alpha) \end{aligned}$$

with $(1 - \delta) J_z \theta(z^k, u_k) v^k = -(1 - \delta) \|h(z^k, u_k)\|^2 < 0$ whenever $v^k \neq 0$, the nonnegative integer i in Step 2 of Algorithm 6.1 always exist.

Remark 6.2. When m_k is always zero, the algorithm performs monotone line search. Otherwise, a nonmonotone line search is used instead. For more description of nonmonotone line search, we refer the reader to [13, 19].

Next, we show the the global convergence of Algorithm 6.1.

Proposition 6.1. *Suppose that f satisfies Property 4.3, and $\{(z^k, u_k)\}$ is a sequence generated by Algorithm 6.1. If the sequence $\{\theta(z^k, u_k)\}$ is bounded, then $\{z^k\}$ has a limit point and every limit point of $\{z^k\}$ solves the NME (1).*

Proof. In view of Proposition 5.2, the sequence $\{(z^k, u_k)\}$ is well-defined. Assume that the sequence $\{\theta(z^k, u_k)\}$ is bounded from above, say by Θ . Together with $u_k \downarrow 0$, we conclude from Proposition 5.1 that the sequence $\{z^k\}$ is bounded. Thus it has a limit point.

From Proposition 3.2, we know that the functions $u \in \mathbb{R}_{++} \mapsto p^\diamond(z, u)$ and $u \in \mathbb{R}_{++} \mapsto p^\diamond(-z, u)$ are Lipschitz continuous with Lipschitz constants bounded from above by, say, l_p . Therefore the set $\bigcup_k \{p^\diamond(z^k, u) : u \in (0, 1]\}$ is bounded, and hence is contained in some compact $C \subset \mathfrak{J}$. The function f is continuously differentiable, whence is Lipschitz continuous on C with Lipschitz constant, say, l . Subsequently for any $u_1, u_2 \in (0, 1]$,

$$\begin{aligned} &\|h(z^k, u_1) - h(z^k, u_2)\| \\ &= \left\| \begin{aligned} &(1 - u_1)[f(p^\diamond(z^k, u_1)) - f(p^\diamond(z^k, u_2))] \\ &+ (u_1 - u_2)[f(p^\diamond(z^k, u_2)) - f((z^k)^+)] + (u_1 - u_2)(f((z^k)^+) + b) \\ &+ (p^\diamond(-z^k, u_2) - p^\diamond(-z^k, u_1)) \end{aligned} \right\| \\ &\leq (1 - u_1) \|f(p^\diamond(z^k, u_1)) - f(p^\diamond(z^k, u_2))\| \\ &\quad + |u_1 - u_2| \|f(p^\diamond(z^k, u_2)) - f((z^k)^+)\| + |u_1 - u_2| \|f((z^k)^+) + b\| \\ &\quad + \|p^\diamond(-z^k, u_2) - p^\diamond(-z^k, u_1)\| \\ &\leq (1 - u_1) l \cdot l_p |u_1 - u_2| + |u_1 - u_2| l \cdot l_p u_2 \\ &\quad + |u_1 - u_2| \|f((z^k)^+) + b\| + l_p |u_1 - u_2| \end{aligned}$$

shows that $u \in (0, 1] \mapsto h(z, u)$ is Lipschitz continuous with Lipschitz constant no more than $(2l + 1)l_p + \|f((z^k)^+) + b\|$. The function f is continuous, whence the sequence $\{(2l + 1)l_p + \|f((z^k)^+) + b\|\}$ is bounded from above, say by \bar{l} . Thus

$$(14) \quad \begin{aligned} \theta(z^{k+1}, u_{k+1}) &= (\|h(z^{k+1}, u_k)\| + \|h(z^{k+1}, u_{k+1})\| - \|h(z^{k+1}, u_k)\|)^2 \\ &\leq (\|h(z^{k+1}, u_k)\| + \gamma_k)^2, \end{aligned}$$

where $\gamma_k := \bar{l}|u_k - u_{k+1}| = \bar{l}(1 - \delta)\delta^k u_0$. Let $l(k) \in \{k - m_k, \dots, k\}$ be an integer such that

$$\theta(z^{l(k)}, u_{l(k)}) = \max_{j=0,1,\dots,m_k} \theta(z^{k-j}, u_{k-j}).$$

For convenience of notation, we shall denote $\theta(z^k, u_k)$ by θ_k . In these notations, the inequalities (12) and (13) imply that

$$\|h(z^k, u_{k-1})\| \leq \sqrt{\theta_{l(k-1)} + \alpha_{k-1} J_z \theta(z^{k-1}, u_{k-1}) v^{k-1}},$$

and hence, it further follows from (14) that

$$(15) \quad \sqrt{\theta_k} \leq \sqrt{\theta_{l(k-1)} + \alpha_{k-1} J_z \theta(z^{k-1}, u_{k-1}) v^{k-1}} + \gamma_{k-1},$$

Thus $\sqrt{\theta_k} \leq \sqrt{\theta_{l(k-1)}} + \gamma_{k-1}$ since $\alpha_k > 0$ and $J_z \theta(z^k, u_k) v^k \leq 0$ for all k (see Remark 6.1). Subsequently, using $m_k \leq m_{k-1} + 1$, we deduce

$$\begin{aligned} \sqrt{\theta_{l(k)}} &= \sqrt{\max_{j=0,1,\dots,m_k} \theta_{k-j}} \leq \sqrt{\max_{j=0,1,\dots,m_{k-1}+1} \theta_{k-1-(j-1)}} \\ &= \sqrt{\max\{\theta_{l(k-1)}, \theta_k\}} \leq \sqrt{\theta_{l(k-1)}} + \gamma_{k-1}. \end{aligned}$$

Being a bounded sequence, $\{\theta_{l(k)}\}$ has a convergent subsequence $\{\theta_{l(k_i)}\}$. By induction on i , it follows from the above inequality that

$$\sqrt{\theta_{l(k_{i+1})}} \leq \sqrt{\theta_{l(k_{i+1}-j)}} + \sum_{k=k_{i+1}-j}^{k_{i+1}-1} \gamma_k \leq \sqrt{\theta_{l(k_i)}} + \sum_{k=k_i}^{k_{i+1}-1} \gamma_k$$

for any $j \in \{1, \dots, k_{i+1} - k_i\}$. Together with the fact that both the subsequence $\{\theta_{l(k_i)}\}$ and the series $\sum_{k=1}^{\infty} \gamma_k$ are convergent, we then conclude that the sequence $\{\theta_{l(k)}\}$ is convergent. Taking $k = l(k')$ followed by the limit $k' \rightarrow \infty$ in (15), we have

$$(16) \quad \lim_{k \rightarrow \infty} \alpha_{l(k)-1} J_z \theta(z^{l(k)-1}, u_{l(k)-1}) v^{l(k)-1} = 0.$$

Since $\{z^k\}$ is bounded, so are the eigenvalues of z^k . Thus, by the continuous differentiability of the smoothing function p , the d_i 's and d_{ij} 's in the proof of Proposition 5.2 are bounded below away from zero; i.e., $d_i, d_{ij} \geq d$ for some $d > 0$. Thus, from the proof of Proposition 5.2, we have

$$\|J_z h(z^{l(k)-1}, u_{l(k)-1}) v^{l(k)-1}\| \geq \alpha d (1 - u_{l(k)-1}) \|v^{l(k)-1}\|,$$

which implies

$$\begin{aligned} \alpha_{l(k)-1} J_z \theta(z^{l(k)-1}, u_{l(k)-1}) v^{l(k)-1} &= -\alpha_{l(k)-1} \|h(z^{l(k)-1}, u_{l(k)-1})\|^2 \\ &= -\alpha_{l(k)-1} \|J_z h(z^{l(k)-1}, u_{l(k)-1}) v^{l(k)-1}\|^2 \\ &\leq -\alpha_{l(k)-1} \alpha^2 d^2 (1 - u_{l(k)-1})^2 \|v^{l(k)-1}\|^2 \\ &\leq -\alpha_{l(k)-1}^2 \alpha^2 d^2 (1 - u_{l(k)-1})^2 \|v^{l(k)-1}\|^2, \end{aligned}$$

where the last inequality follows from $\alpha_k \leq 1$ for all k . Since $u_k \rightarrow 0$, we obtain from (16) that

$$(17) \quad \lim_{k \rightarrow \infty} \alpha_{l(k)-1} \|v^{l(k)-1}\| = 0.$$

Next, adapting the arguments employed for the proof of the theorem in [19, pp 709–711], we prove that

$$(18) \quad \lim_{k \rightarrow \infty} \alpha_k J_z \theta(z^k, u_k) v^k = -\lim_{k \rightarrow \infty} \alpha_k \|h(z^k, u_k)\|^2 = 0.$$

Let $\hat{l}(k) = l(k + M + 2)$, so that $\hat{l}(k) \geq (k + M + 2) - M = k + 2$. We first show, by induction, that for any integer $j \geq 1$

$$(19) \quad \lim_{k \rightarrow \infty} \alpha_{\hat{l}(k)-j} \|v^{\hat{l}(k)-j}\| = 0$$

and

$$(20) \quad \lim_{k \rightarrow \infty} \theta_{\hat{l}(k)-j} = \lim_{k \rightarrow \infty} \theta_{l(k)}.$$

If $j = 1$, since $\{\hat{l}(k)\} \subset \{l(k)\}$, (19) follows from (17). This in turn implies $\|(z^{\hat{l}(k)}, u_{\hat{l}(k)}) - (z^{\hat{l}(k)-1}, u_{\hat{l}(k)-1})\| := \|z^{\hat{l}(k)} - z^{\hat{l}(k)-1}\| + |u_{\hat{l}(k)} - u_{\hat{l}(k)-1}| \rightarrow 0$, so that (20) holds for $j = 1$ by the uniform continuity of θ on the compact set $C \times [0, 1]$. Assume now that (19) and (20) hold for some given j . By (15) one can write

$$\sqrt{\theta_{\hat{l}(k)-j}} \leq \sqrt{\theta_{l(\hat{l}(k)-j-1)} + \alpha_{\hat{l}(k)-j-1} J_z \theta(z^{\hat{l}(k)-j-1}, u_{\hat{l}(k)-j-1}) v^{\hat{l}(k)-j-1} + \gamma_{\hat{l}(k)-j-1}}.$$

Taking limit for $k \rightarrow \infty$, we obtain from (20) that

$$\lim_{k \rightarrow \infty} \alpha_{\hat{l}(k)-(j+1)} J_z \theta(z^{\hat{l}(k)-(j+1)}, u_{\hat{l}(k)-(j+1)}) v^{\hat{l}(k)-(j+1)} = 0.$$

Using the same argument for deriving (17) from (16), we get

$$\lim_{k \rightarrow \infty} \alpha_{\hat{l}(k)-(j+1)} \|v^{\hat{l}(k)-(j+1)}\| = 0.$$

Moreover, this implies $\|(z^{\hat{l}(k)-j}, u_{\hat{l}(k)-j}) - (z^{\hat{l}(k)-(j+1)}, u_{\hat{l}(k)-(j+1)})\| \rightarrow 0$, so that by (20) and the uniform continuity of θ on $C \times [0, 1]$:

$$\lim_{k \rightarrow \infty} \theta_{\hat{l}(k)-(j+1)} = \lim_{k \rightarrow \infty} \theta_{\hat{l}(k)-j} = \lim_{k \rightarrow \infty} \theta_{l(k)}.$$

Therefore, (19) and (20) hold for any $j \geq 1$.

For any k , we have

$$z^{\hat{l}(k)} - z^{k+1} = \sum_{j=1}^{\hat{l}(k)-k-1} \alpha_{\hat{l}(k)-j} v^{\hat{l}(k)-j},$$

and $\hat{l}(k) - k - 1 = l(k + M + 2) - k - 1 \leq M + 1$, so it follows from (19) that

$$\lim_{k \rightarrow \infty} \|(z^{k+1}, u_{k+1}) - (z^{\hat{l}(k)}, u_{\hat{l}(k)})\| = 0.$$

Since $\{\theta_{l(k)}\}$ is convergent, then the uniform continuity of θ on $C \times [0, 1]$ yields

$$\lim_{k \rightarrow \infty} \theta_k = \lim_{k \rightarrow \infty} \theta_{\hat{l}(k)} = \lim_{k \rightarrow \infty} \theta_{l(k)}.$$

Taking limit as $k \rightarrow \infty$ in (15), we obtain (18) as desired.

Now, suppose that z_* is a limit point of $\{z^k\}$, say it is the limit of the subsequence $\{z^k : k \in K\}$. By taking a subsequence if necessary, we may assume, without any loss of generality, that the subsequence $\{\alpha_k : k \in K\}$ converges. If $\lim_{k \rightarrow \infty, k \in K} \alpha_k > 0$, it follows from (18) that $h(z_*, 0) = 0$. In the other case where $\lim_{k \rightarrow \infty, k \in K} \alpha_k = 0$, we have that, from the definition of α_k , for each $k \in K$,

$$\theta(z^k + \alpha_k \beta^{-1} v^k, u_k) > \theta(z^k, u_k) + \delta \alpha_k \beta^{-1} J_z \theta(z^k, u_k) v^k,$$

implying that

$$(21) \quad (\delta - 1) \alpha_k \beta^{-1} \|h(z^k, u_k)\|^2 + o(\alpha_k \beta^{-1} \|v^k\|) > 0.$$

Notice that

$$\|h(z^k, u_k)\| = \|J_z h(z^k, u_k) v^k\| \geq \alpha d (1 - u_k) \|v^k\|,$$

and $\|h(z^k, u^k)\|$ is bounded, thus $\{v^k\}$ is also bounded. Dividing both sides of (21) by $\alpha_k\beta^{-1}$ and taking limits as $k \rightarrow \infty, k \in K$, we obtain

$$(\delta - 1)\|h(z_*, 0)\|^2 \geq 0.$$

Since $\delta - 1 < 0$ and $\|h(z^k, u_k)\|^2 \geq 0$, it must happen that $h(z_*, 0) = 0$, and this completes the proof. \square

7. CONCLUSION

Based on a different characterization of P -matrices, we proposed a new P -type property for functions defined over Euclidean Jordan algebras, and established global and linear convergence of a continuation method for solving nonlinear complementarity problems over symmetric cones. Our P -type property represents a new class of nonmonotone nonlinear complementarity problems that can be solved numerically. It might be interesting to investigate if our P -type property can be used in other numerical methods such as smoothing Newton methods, non-interior continuation methods and merit function methods.

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