

CONVERGENCE RATE OF STOCHASTIC GRADIENT SEARCH IN THE CASE OF MULTIPLE AND NON-ISOLATED MINIMA

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Abstract. The convergence rate of stochastic gradient search is analyzed in this paper. Using arguments based on differential geometry and Lojasiewicz inequalities, tight bounds on the convergence rate of general stochastic gradient algorithms are derived. As opposed to the existing results, the results presented in this paper allow the objective function to have multiple, non-isolated minima, impose no restriction on the values of the Hessian (of the objective function) and do not require the algorithm estimates to have a single limit point. Applying these new results, the convergence rate of recursive prediction error identification algorithms is studied. The convergence rate of supervised and temporal-difference learning algorithms is also analyzed using the results derived in the paper.

Key words. Stochastic gradient algorithms, rate of convergence, Lojasiewicz inequalities, system identification, recursive prediction error, ARMA models, machine learning, supervised learning, temporal-difference learning.

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1. Introduction. Stochastic gradient algorithms are a recursive optimization method of the stochastic approximation type. This method is commonly used to compute minima (or maxima) of a function whose values are available only through noise-corrupted observations. It has found a wide range of applications in the areas such as automatic control, system identification, signal processing, machine learning, operations research, statistical inference, economics and management (to name a few). For further details, see [8], [18], [19], [24], [26], [27], [28] and the references cited therein.

Due to their practical importance, the asymptotic behavior of stochastic gradient algorithms has been thoroughly studied in a large number of papers and books. A significant attention has been given to the rate of convergence, as this property directly characterizes the efficiency and enables a construction of reliable stopping rules (see [2], [16], [18], [26], [28] and the references given therein). Although the existing results on the convergence rate provide a good insight into the efficiency and asymptotic behavior of stochastic gradient algorithms, they hold under very restrictive conditions. More specifically, the existing results require the algorithm estimates to converge to an isolated minimum of the objective function at which the Hessian (of the objective function) is strictly positive definite. Unfortunately, such conditions are practically impossible to verify for complex, high-dimensional and high-nonlinear stochastic gradient algorithms.

In this paper, the rate of convergence of stochastic gradient algorithms is analyzed for the case when the objective function has multiple, non-isolated minima (note that the Hessian can be only semi-definite at a non-isolated minimum) and when the algorithm estimates do not necessarily converge to a single limit point. Using arguments based on differential geometry and Lojasiewicz inequalities, relatively tight upper bounds on the convergence rate are derived. The obtained results cover a broad class of complex stochastic gradient algorithms. We show how they can be used to evaluate the convergence rate of recursive prediction error algorithms for identification of linear stochastic dynamical systems. We also show how the convergence rate

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of supervised and temporal-difference learning algorithms can be assessed using the results derived in the paper.

The paper is organized as follows. The main results are presented in Section 2, where stochastic gradient algorithms with additive noise are considered. In Section 3, the convergence rate of stochastic gradient algorithms with Markovian dynamics is analyzed. Sections 4 and 6 are devoted to examples of the results presented in Sections 2 and 3. In Section 4, supervised learning algorithms for feedforward neural networks and their convergence rate are studied, while the rate of convergence of temporal-difference learning algorithms is considered in Section 5. The convergence rate of recursive prediction error algorithms for the identification of linear stochastic systems is analyzed in Section 6. Sections 7 – 11 contain the proofs of the results presented in Sections 2 – 6.

2. Main Results. In this section, the rate of convergence of the following algorithm is analyzed:

$$\theta_{n+1} = \theta_n - \alpha_n(\nabla f(\theta_n) + w_n), \quad n \geq 0. \quad (2.1)$$

In this recursion, $f : \mathbb{R}^{d_\theta} \rightarrow \mathbb{R}$ is a differentiable function, while $\{\alpha_n\}_{n \geq 0}$ is a sequence of positive real numbers. While θ_0 is an \mathbb{R}^{d_θ} -valued random variable defined on a probability space (Ω, \mathcal{F}, P) , while $\{w_n\}_{n \geq 0}$ is an \mathbb{R}^{d_θ} -valued stochastic process defined on the same probability space. To allow more generality, we assume that for each $n \geq 0$, w_n is a random function of $\theta_0, \dots, \theta_n$. In the area of stochastic optimization, recursion (2.1) is known as a stochastic gradient algorithm (or stochastic gradient search), while function $f(\cdot)$ is referred to as an objective function. For further details see [24], [28] and references given therein.

Throughout the paper, unless otherwise stated, the following notation is used. The Euclidean norm is denoted by $\|\cdot\|$, while $d(\cdot, \cdot)$ stands for the distance induced by the Euclidean norm. S and C are the sets of stationary and critical points of $f(\cdot)$, i.e.,

$$S = \{\theta \in \mathbb{R}^{d_\theta} : \nabla f(\theta) = 0\}, \quad C = \{f(\theta) : \theta \in S\}.$$

Sequence $\{\gamma_n\}_{n \geq 0}$ is defined by $\gamma_0 = 0$ and

$$\gamma_n = \sum_{i=0}^{n-1} \alpha_i$$

for $n \geq 1$. For $t \in (0, \infty)$ and $n \geq 0$, $a(n, t)$ is an integer defined as

$$a(n, t) = \max \{k \geq n : \gamma_k - \gamma_n \leq t\}.$$

Algorithm (2.1) is analyzed under the following assumptions:

ASSUMPTION 2.1. $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

ASSUMPTION 2.2. *There exists a real number $r \in (0, \infty)$ such that*

$$w = \limsup_{n \rightarrow \infty} \max_{n \leq k < a(n, 1)} \left\| \sum_{i=n}^k \alpha_i \gamma_i^T w_i \right\| < \infty$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$.

ASSUMPTION 2.3. For any compact set $Q \subset \mathbb{R}^{d_\theta}$ and any $a \in f(Q)$, there exist real numbers $\delta_{Q,a} \in (0, 1)$, $\mu_{Q,a} \in (1, 2]$, $M_{Q,a} \in [1, \infty)$ such that

$$|f(\theta) - a| \leq M_{Q,a} \|\nabla f(\theta)\|^{\mu_{Q,a}} \quad (2.2)$$

for all $\theta \in Q$ satisfying $|f(\theta) - a| \leq \delta_{Q,a}$.

ASSUMPTION 2.4. For any compact set $Q \subset \mathbb{R}^{d_\theta}$, there exist real numbers $\nu_Q \in (0, 1]$, $N_Q \in [1, \infty)$ such that

$$d(\theta, S) \leq N_Q \|\nabla f(\theta)\|^{\nu_Q} \quad (2.3)$$

for all $\theta \in Q$.

REMARK. In order to show that Assumption 2.3 holds, it is sufficient to demonstrate its ‘local version,’ i.e., that there exists an open vicinity U of S with the following property: For any compact set $Q \subset U$ and any $a \in f(Q)$, there exist real numbers $\delta_{Q,a} \in (0, 1]$, $\mu_{Q,a} \in (1, 2]$, $M_{Q,a} \in [1, \infty)$ such that (2.2) holds for all $\theta \in Q$ satisfying $|f(\theta) - a| \leq \delta_{Q,a}$ (for details see the appendix at the end of the paper). Similar conclusions apply to Assumption 2.4.

Assumption 2.1 correspond to the sequence $\{\alpha_n\}_{n \geq 0}$ and is widely used in the asymptotic analysis of stochastic gradient and stochastic approximation algorithms. Assumption 2.2 is a noise condition. In this or a similar form, it is involved in most of the results on the convergence rate of stochastic gradient search and stochastic approximation. It holds for algorithms with Markovian dynamics (see the next section). It is also satisfied when $\{w_n\}_{n \geq 0}$ is a martingale-difference sequence. Assumptions 2.3 and 2.4 are related to the stability of the gradient flow $d\theta/dt = -\nabla f(\theta)$, or more specifically, to the geometry of the set of stationary points S . In the area of differential geometry, relations (2.2) and (2.3) are known as the Lojasiewicz inequalities (see [20] and [21] for details). They hold if $f(\cdot)$ is analytic or subanalytic in an open vicinity of S (see [6], [21] for the proof; for the form of Lojasiewicz inequality appeared in Assumption 2.3 see [15, Theorem LI, p. 775]; for the definition and properties of analytic and subanalytic functions, consult [6], [14]). Although analyticity and subanalyticity are fairly strong conditions, they hold for the objective functions of many stochastic gradient algorithms commonly used in the areas of system identification, signal processing, machine learning, operations research and statistical inference. E.g., in this paper, we show that the objective functions associated with supervised and temporal-difference learning are analytical (Sections 4 and 5). We also demonstrate the same property for recursive prediction error identification (Section 6). Furthermore, in [31], we show analyticity for the objective functions associated with recursive identification methods for hidden Markov models. It is also worth mentioning that the objective functions associated with recursive algorithms for principal and independent component analysis (as well as with many other adaptive signal processing algorithms) are usually polynomial or rational, and hence, analytic, too (see e.g., [10] and references cited therein).

In order to state the main results of this section, we need further notation. For a compact set $Q \subset \mathbb{R}^{d_\theta}$, $C_Q \in [1, \infty)$ stands for an upper bound of $\|\nabla f(\cdot)\|$ on Q and for a Lipschitz constant of $\nabla f(\cdot)$ on the same set. \hat{A} denotes the set of accumulation points of $\{\theta_n\}_{n \geq 0}$ (notice that \hat{A} is a random set), while

$$\hat{f} = \liminf_{n \rightarrow \infty} f(\theta_n).$$

\hat{Q} is a random set defined as

$$\hat{Q} = \begin{cases} \{\theta : d(\theta, \hat{A}) \leq \rho\}, & \text{if } \sup_{n \geq 0} \|\theta_n\| < \infty \\ \hat{A}, & \text{otherwise} \end{cases}$$

where ρ is an arbitrary positive (deterministic or random) quantity. $\hat{\delta}$, $\hat{\mu}$, $\hat{\nu}$, \hat{C} , \hat{M} and \hat{N} are random quantities defined by

$$\hat{\delta} = \delta_{\hat{Q}, \hat{f}}, \quad \hat{\mu} = \mu_{\hat{Q}, \hat{f}}, \quad \hat{\nu} = \mu_{\hat{Q}, \hat{f}} \nu_{\hat{Q}} / 2, \quad \hat{C} = C_{\hat{Q}}, \quad \hat{M} = M_{\hat{Q}, \hat{f}}, \quad \hat{N} = N_{\hat{Q}} \quad (2.4)$$

when $\sup_{n \geq 0} \|\theta_n\| < \infty$ and by

$$\hat{\delta} = 1, \quad \hat{\mu} = 2, \quad \hat{\nu} = 1, \quad \hat{C} = 1, \quad \hat{M} = 1, \quad \hat{N} = 1 \quad (2.5)$$

otherwise (symbol $\hat{\cdot}$ is used to emphasize the dependence on \hat{f} and \hat{Q}). Moreover, let

$$\hat{r} = \begin{cases} 1/(2 - \hat{\mu}), & \text{if } \hat{\mu} < 2 \\ \infty, & \text{if } \hat{\mu} = 2 \end{cases}, \quad \hat{p} = \hat{\mu} \min\{r, \hat{r}\}, \quad \hat{q} = \hat{\nu} \min\{r, \hat{r}\}. \quad (2.6)$$

Furthermore, let

$$\phi(w) = \begin{cases} w, & \text{if } r < \hat{r} \\ 1 + w, & \text{if } r = \hat{r} \\ 1, & \text{if } r > \hat{r} \end{cases}$$

REMARK. Since $\hat{f} \in f(\hat{Q})$ when $\sup_{n \geq 0} \|\theta_n\| < \infty$, it is obvious that random quantities $\hat{\delta}$, $\hat{\mu}$, $\hat{\nu}$, \hat{p} , \hat{q} , \hat{r} , \hat{C} , \hat{M} , \hat{N} are well-defined. Moreover, it is easy to conclude that inequalities $0 < \hat{\delta} \leq 1$, $1 < \hat{\mu} \leq 2$, $\hat{p} > \min\{1, r\}$, $\hat{q} > 1$, $\hat{r} > 1$, $1 \leq \hat{C}, \hat{M}, \hat{N} < \infty$ hold everywhere (i.e., on entire Ω). It can also be demonstrated that (Lojasiewicz coefficients) $\delta_{Q,a}$, $\mu_{Q,a}$, ν_Q , $M_{Q,a}$, N_Q have ‘measurable versions’ such that $\hat{\delta}$, $\hat{\mu}$, $\hat{\nu}$, \hat{p} , \hat{q} , \hat{r} , \hat{M} , \hat{N} are random variables in probability space (Ω, \mathcal{F}, P) (i.e., measurable with respect to \mathcal{F} ; details are provided in the appendix at the end of the paper). Furthermore, as a consequence of Assumption 2.3, we have

$$|f(\theta) - \hat{f}| \leq \hat{M} \|\nabla f(\theta)\|^{\hat{\mu}} \quad (2.7)$$

on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$ for all $\theta \in \hat{Q}$ satisfying $|f(\theta) - \hat{f}| \leq \hat{\delta}$.

Our main results on the convergence and convergence rate of the recursion (2.1) are contained in the next two theorems.

THEOREM 2.1. *Let Assumptions 2.1 – 2.3 hold. Then, $\lim_{n \rightarrow \infty} \nabla f(\theta_n) = 0$ and $\lim_{n \rightarrow \infty} f(\theta_n) = \hat{f}$ w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$.*

THEOREM 2.2. *Let Assumptions 2.1 – 2.3 hold. Then, there exists a random quantity \hat{K} (which is a deterministic function of \hat{C}, \hat{M}) such that $1 \leq \hat{K} < \infty$ everywhere and such that*

$$\limsup_{n \rightarrow \infty} \gamma_n^{\hat{p}} \|\nabla f(\theta_n)\|^2 \leq \hat{K} (\phi(w))^{\hat{\mu}}, \quad (2.8)$$

$$\limsup_{n \rightarrow \infty} \gamma_n^{\hat{p}} |f(\theta_n) - \hat{f}| \leq \hat{K} (\phi(w))^{\hat{\mu}} \quad (2.9)$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$. If additionally, Assumption 2.4 is satisfied, then, there exists another random quantity \hat{L} (which is a deterministic function of $\hat{C}, \hat{M}, \hat{N}$) such that $1 \leq \hat{L} < \infty$ everywhere and such that

$$\limsup_{n \rightarrow \infty} \gamma_n^{\hat{q}} d(\theta_n, S) \leq \hat{L}(\phi(w))^{\hat{v}} \quad (2.10)$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$.

The proofs are provided in Section 7. As an immediate consequence of the previous theorems, we get the following corollaries:

COROLLARY 2.3. *Let Assumptions 2.1 – 2.4 hold. Then, the following is true:*

$$\|\nabla f(\theta_n)\|^2 = o(\gamma_n^{-\hat{p}}), \quad d(f(\theta_n), C) = o(\gamma_n^{-\hat{p}}), \quad d(\theta_n, S) = o(\gamma_n^{-\hat{q}})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \cap \{w = 0, \hat{r} > r\}$, and

$$\|\nabla f(\theta_n)\|^2 = O(\gamma_n^{-\hat{p}}), \quad d(f(\theta_n), C) = O(\gamma_n^{-\hat{p}}), \quad d(\theta_n, S) = O(\gamma_n^{-\hat{q}})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \cap \{w = 0, \hat{r} > r\}^c$.

COROLLARY 2.4. *Let Assumptions 2.1 – 2.3 hold. Then,*

$$\|\nabla f(\theta_n)\|^2 = o(\gamma_n^{-p}), \quad d(f(\theta_n), C) = o(\gamma_n^{-p})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$, where $p = \min\{1, r\}$.

In the literature on stochastic and deterministic optimization, the asymptotic behavior of gradient search is usually characterized by the gradient, objective and estimate convergence, i.e., by the convergence of sequences $\{\nabla f(\theta_n)\}_{n \geq 0}$, $\{f(\theta_n)\}_{n \geq 0}$ and $\{\theta_n\}_{n \geq 0}$ (see e.g., [4], [5], [25], [26] are references quoted therein). Similarly, the convergence rate can be described by the rates at which $\{\nabla f(\theta_n)\}_{n \geq 0}$, $\{f(\theta_n)\}_{n \geq 0}$ and $\{\theta_n\}_{n \geq 0}$ tend to the sets of their limit points. Theorem 2.2 and Corollary 2.3 provide relatively tight upper bounds on these rates in the terms of the asymptotic properties of noise $\{w_n\}_{n \geq 0}$ and the gradient flow $d\theta/dt = -\nabla f(\theta)$. Basically, the theorem and its corollary claim that the convergence rate of $\{\|\nabla f(\theta_n)\|^2\}_{n \geq 0}$ and $\{f(\theta_n)\}_{n \geq 0}$ is the slower of the rates $O(\gamma_n^{-\hat{r}\hat{\mu}})$ (the rate of the gradient flow $d\theta/dt = -\nabla f(\theta)$ sampled at instants $\{\gamma_n\}_{n \geq 0}$) and $O(\gamma_n^{-r\hat{\mu}})$ (the rate of the noise averages $\max_{k \geq n} \|\sum_{i=n}^k \alpha_i w_i\|^{\hat{\mu}}$). Apparently, the rates provided in Theorem 2.1 and Corollary 2.3 are of a local nature: They hold only on the event where algorithm (2.1) is stable (i.e., where sequence $\{\theta_n\}_{n \geq 0}$ is bounded). Stating results on the convergence rate in such a local form is quite reasonable due to the following reasons. The stability of stochastic gradient search is based on well-understood arguments which are rather different from the arguments used in the analysis of the convergence rate. Moreover and more importantly, it is straightforward to get a global version of the rates provided in Theorem 2.1 and Corollary 2.3 by combining the theorem with the methods used to verify or ensure the stability (e.g., with the results of [7] and [9]).

Due to its practical and theoretical importance, the rate of convergence of stochastic gradient search (and stochastic approximation) has been the subject of a large number of papers and books (see see [2], [16], [18], [26], [28] and references cited therein). Although the existing results provide a good insight into the asymptotic behavior and efficiency of stochastic gradient algorithms, they are based on fairly restrictive assumptions: Literally, they all require the objective function $f(\cdot)$ to have an isolated minimum $\hat{\theta}$ (sometimes even to be strongly unimodal) such that Hessian

$\nabla^2 f(\hat{\theta})$ is strictly positive definite and $\lim_{n \rightarrow \infty} \theta_n = \hat{\theta}$ w.p.1. Unfortunately, in the case of high-dimensional and high-nonlinear stochastic gradient algorithms (such as online machine learning and recursive identification), it is hard (if not impossible at all) to show even the existence of an isolated minimum, let alone the definiteness of $\nabla^2 f(\cdot)$ and the point-convergence of $\{\theta_n\}_{n \geq 0}$. Relying on the Lojasiewicz inequalities, Theorem 2.1 and Corollary 2.3 overcome these difficulties: Both the theorem and its corollary allow the objective function $f(\cdot)$ to have multiple, non-isolated minima, impose no restriction on the values of $\nabla^2 f(\cdot)$ (notice that $\nabla^2 f(\cdot)$ cannot be strictly definite at a non-isolated minimum or maximum) and permit $\{\theta_n\}_{n \geq 0}$ to have multiple limit points. Moreover, they cover a broad class of complex stochastic gradient algorithms (see Sections 4 and 6; see also [31]). To the best of our knowledge, these are the only results on the convergence rate with such features.

Regarding the results of Theorem 2.1 and Corollary 2.3, it is worth mentioning that they are not just a combination of the Lojasiewicz inequalities and the existing techniques for the asymptotic analysis of stochastic gradient search and stochastic approximation. On the contrary, the existing techniques seem to be inapplicable to the case of multiple non-isolated minima. The reason comes out of the fact that these techniques crucially rely on the Lyapunov function $u(\theta) = (\theta - \hat{\theta})^T \nabla^2 f(\hat{\theta})(\theta - \hat{\theta})$, where $\hat{\theta}$ is an isolated minimum such that $\lim_{n \rightarrow \infty} \theta_n = \hat{\theta}$ w.p.1 and $\nabla^2 f(\cdot)$ is strictly positive definite. Unfortunately, in the case of multiple, non-isolated minima, neither does $\{\theta_n\}_{n \geq 0}$ necessarily have a single limit point (limit cycles can occur), nor $\nabla^2 f(\cdot)$ can be a strictly positive definite matrix. In order to overcome this problem, we use a ‘singular’ Lyapunov function $v(\theta) = 1/(f(\theta) - \hat{f})^{1/p}$, where $p \in (0, \hat{\mu}/(2 - \hat{\mu})]$ and $\theta \in \{\vartheta \in \mathbb{R}^{d_\theta} : f(\vartheta) > \hat{f}\}$. Although subtle techniques are needed to handle such a Lyapunov function (see Section 7), $v(\cdot)$ provides intuitively clear explanation of the results of Theorem 2.2 and Corollary 2.3. The explanation is based on the heuristic analysis of the following two cases.

Case 1: $\sup_{n \geq 0} \|\theta_n\| < \infty$ and $\liminf_{n \rightarrow \infty} \gamma_n^{\hat{\mu}}(f(\theta_n) - \hat{f}) = -\infty$.

In this case, there exists an increasing integer sequence $\{n_k\}_{k \geq 0}$ such that $f(\theta_{n_k}) < \hat{f}$ for each $k \geq 0$ and $\lim_{n \rightarrow \infty} \gamma_{n_k}^{\hat{\mu}}(f(\theta_{n_k}) - \hat{f}) = -\infty$. Therefore, Assumption 2.3 implies $\lim_{n \rightarrow \infty} \gamma_{n_k}^r \|\nabla f(\theta_{n_k})\| = \infty$. Since $\max_{k \geq n} \left\| \sum_{i=n}^k \alpha_i w_i \right\| = O(\gamma_n^{-r})$ (see Lemma 7.1), there exists a large integer $m \gg 1$ such that $f(\theta_m) < \hat{f}$ and $\max_{n \geq m} \left\| \sum_{i=m}^n \alpha_i w_i \right\| \leq \|\nabla f(\theta_m)\|/2$. Then, for $n \geq a(m, 1)$, Taylor formula yields

$$\begin{aligned} f(\theta_n) &\approx f(\theta_m) - (\nabla f(\theta_m))^T \sum_{i=m}^{n-1} \alpha_i (\nabla f(\theta_i) + w_i) \\ &\approx f(\theta_m) - \|\nabla f(\theta_m)\|^2 (\gamma_n - \gamma_m) - (\nabla f(\theta_m))^T \sum_{i=m}^{n-1} \alpha_i w_i \\ &\leq f(\theta_m) - \frac{\|\nabla f(\theta_m)\|^2}{2} - \|\nabla f(\theta_m)\| \left(\frac{\|\nabla f(\theta_m)\|}{2} - \left\| \sum_{i=m}^{n-1} \alpha_i w_i \right\| \right) \\ &\leq f(\theta_m) \end{aligned}$$

(notice that $\gamma_n - \gamma_m \geq 1$). Hence, $f(\theta_n) \leq f(\theta_m) < \hat{f}$ for $n \geq a(m, 1)$, which is impossible as $\lim_{n \rightarrow \infty} f(\theta_n) = \hat{f}$.

Case 2: $\sup_{n \geq 0} \|\theta_n\| < \infty$ and $\limsup_{n \rightarrow \infty} \gamma_n^{\hat{\mu}}(f(\theta_n) - \hat{f}) = \infty$.

Similarly as in the previous case, there exists an increasing integer sequence $\{n_k\}_{k \geq 0}$

such that $f(\theta_{n_k}) > \hat{f}$ for each $k \geq 0$ and $\lim_{n \rightarrow \infty} \gamma_{n_k}^{r\hat{\mu}}(f(\theta_{n_k}) - \hat{f}) = \infty$. Consequently, Assumption 2.3 yields $\lim_{k \rightarrow \infty} \gamma_{n_k}^r \|\nabla f(\theta_{n_k})\| = \infty$ and

$$\frac{\|\nabla f(\theta_{n_k})\|^2}{(f(\theta_{n_k}) - \hat{f})^{1+1/p}} \geq \frac{1}{\hat{M}^{2/\hat{\mu}}(f(\theta_{n_k}) - \hat{f})^{1+1/p-2/\hat{\mu}}}$$

for $k \geq 0$. Since $1 + 1/p \geq 2/\hat{\mu}$, $\lim_{n \rightarrow \infty} f(\theta_n) = \hat{f}$ and $\max_{k \geq n} \left\| \sum_{i=n}^k \alpha_i w_i \right\| = O(\gamma_n^{-r})$, there exists a large integer $m \gg 1$ such that $\max_{n \geq m} \left\| \sum_{i=m}^n \alpha_i w_i \right\| \leq \|\nabla f(\theta_m)\|/2$, $f(\theta_m) \geq \hat{f}$ and

$$\frac{\|\nabla f(\theta_m)\|^2}{(f(\theta_m) - \hat{f})^{1+2/p}} \geq \frac{1}{\hat{M}^{2/\hat{\mu}}}.$$

Then, for any $n \geq a(m, 1)$ satisfying $f(\theta_n) > \hat{f}$, Taylor formula implies

$$\begin{aligned} v(\theta_n) &\approx v(\theta_m) - (\nabla v(\theta_m))^T \sum_{i=m}^{n-1} \alpha_i (\nabla f(\theta_i) + w_i) \\ &\approx v(\theta_m) + \frac{\|\nabla f(\theta_m)\|^2}{p(f(\theta_m) - \hat{f})^{1+1/p}} (\gamma_n - \gamma_m) + \frac{(\nabla f(\theta_m))^T}{p(f(\theta_m) - \hat{f})^{1+1/p}} \sum_{i=m}^{n-1} \alpha_i w_i \\ &\geq v(\theta_m) + \frac{1}{2p\hat{M}^{2/\hat{\mu}}} (\gamma_n - \gamma_m) + \frac{\|\nabla f(\theta_m)\|}{p(f(\theta_m) - \hat{f})^{1+1/p}} \left(\frac{1}{2} - \left\| \sum_{i=m}^{n-1} \alpha_i w_i \right\| \right) \\ &\geq \frac{1}{2p\hat{M}^{2/\hat{\mu}}} (\gamma_n - \gamma_m). \end{aligned}$$

Thus, $f(\theta_n) - \hat{f} \leq (2p\hat{M})^{2p} (\gamma_n - \gamma_m)^{-p}$ for $n \geq a(m, 1)$ (notice that $\hat{\mu} > 1$).

Following the reasoning outlined in the above cases, it can easily be concluded that the slower of $O(\gamma_n^{-p})$ and $O(\gamma_n^{-r\hat{\mu}})$ is the rate at which $f(\theta_n)$ tends to \hat{f} . Since p can be any number from $(0, \hat{r}\hat{\mu}]$ (in the proof of Theorem 2.1, Section 7, value $p = \hat{p} = \hat{\mu} \min\{r, \hat{r}\}$ is used), it is also straightforward to deduce that $O(\gamma_n^{-\hat{p}})$ is the convergence rate of $\{f(\theta_n)\}_{n \geq 0}$. In addition to this, the previously described heuristics indicate that in the terms of r and $\hat{\mu}$, $O(\gamma_n^{-\hat{p}})$ is probably the tightest estimate of the convergence rate of $\{f(\theta_n)\}_{n \geq 0}$. The same conclusion is suggested by the following two special cases:

Case (a): $w_n = 0$ for each $n \geq 0$.

Due to Assumption 2.3, we have

$$\frac{d(f(\theta(t)) - \hat{f})}{dt} = -\|\nabla f(\theta(t))\|^2 \leq -\left(1/\hat{M}\right)^{2/\hat{\mu}} (f(\theta(t)) - \hat{f})^{2/\hat{\mu}}$$

for a solution $\theta(\cdot)$ of $d\theta/dt = -\nabla f(\theta)$ satisfying $\theta(t) \in \hat{Q}$ for all $t \in [0, \infty)$ and $\lim_{t \rightarrow \infty} f(\theta(t)) = \hat{f}$. Consequently, $f(\theta(t)) - \hat{f} = O(t^{-\hat{\mu}/(2-\hat{\mu})}) = O(t^{-\hat{r}\hat{\mu}})$. As $\{\theta_n\}_{n \geq 0}$ is asymptotically equivalent to $\theta(\cdot)$ sampled at time instances $\{\gamma_n\}_{n \geq 0}$, we get $f(\theta_n) - \hat{f} = O(\gamma_n^{-\hat{r}\hat{\mu}})$. The same result is implied by Theorem 2.1 and Corollary 2.3.

Case (b): $f(\theta) = \theta^T A \theta$ and A is a strictly positive definite matrix.

Recursion (2.1) reduces to a linear stochastic approximation algorithm in this case. For such an algorithm, it is known that the tightest estimate of the convergence rate is $f(\theta_n) = O(\gamma_n^{-2r})$ if $w > 0$, and $f(\theta_n) = o(\gamma_n^{-2r})$ for $w = 0$ (see [30]). The same rate is provided by Theorem 2.2 and Corollary 2.3.

3. Stochastic Gradient Algorithms with Markovian Dynamics. In order to illustrate the results of Section 2 and to set up a framework for the analysis carried out in Sections 4 and 6, we apply Theorems 2.1, 2.2 and Corollaries 2.3, 2.4 to stochastic gradient algorithms with Markovian dynamics. These algorithms are defined by the following difference equation:

$$\theta_{n+1} = \theta_n - \alpha_n F(\theta_n, \xi_{n+1}), \quad n \geq 0. \quad (3.1)$$

In this recursion, $F : \mathbb{R}^{d_\theta} \times \mathbb{R}^{d_\xi} \rightarrow \mathbb{R}^{d_\theta}$ is a Borel-measurable function, while $\{\alpha_n\}_{n \geq 0}$ is a sequence of positive real numbers. θ_0 is an \mathbb{R}^{d_θ} -valued random variable defined on a probability space (Ω, \mathcal{F}, P) , while $\{\xi_n\}_{n \geq 0}$ is an \mathbb{R}^{d_ξ} -valued stochastic process defined on the same probability space. $\{\xi_n\}_{n \geq 0}$ is a Markov process controlled by $\{\theta_n\}_{n \geq 0}$, i.e., there exists a family of transition probability kernels $\{\Pi_\theta(\cdot, \cdot)\}_{\theta \in \mathbb{R}^{d_\theta}}$ defined on \mathbb{R}^{d_ξ} such that

$$P(\xi_{n+1} \in B | \theta_0, \xi_0, \dots, \theta_n, \xi_n) = \Pi_{\theta_n}(\xi_n, B)$$

w.p.1 for any Borel-measurable set $B \subseteq \mathbb{R}^{d_\xi}$ and $n \geq 0$. In the context of stochastic gradient search, $F(\theta_n, \xi_{n+1})$ is regarded to as an estimator of $\nabla f(\theta_n)$.

The algorithm (3.1) is analyzed under the following assumptions.

ASSUMPTION 3.1. $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\limsup_{n \rightarrow \infty} |\alpha_{n+1}^{-1} - \alpha_n^{-1}| < \infty$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$. There exists a real number $r \in (0, \infty)$ such that $\sum_{n=0}^{\infty} \alpha_n^2 \gamma_n^{2r} < \infty$.

ASSUMPTION 3.2. There exist a differentiable function $f : \mathbb{R}^{d_\theta} \rightarrow \mathbb{R}$ and a Borel-measurable function $\tilde{F} : \mathbb{R}^{d_\theta} \times \mathbb{R}^{d_\xi} \rightarrow \mathbb{R}^{d_\theta}$ such that $\nabla f(\cdot)$ is locally Lipschitz continuous and such that

$$F(\theta, \xi) - \nabla f(\theta) = \tilde{F}(\theta, \xi) - (\Pi \tilde{F})(\theta, \xi)$$

for each $\theta \in \mathbb{R}^{d_\theta}$, $\xi \in \mathbb{R}^{d_\xi}$, where $(\Pi \tilde{F})(\theta, \xi) = \int \tilde{F}(\theta, \xi') \Pi_\theta(\xi, d\xi')$.

ASSUMPTION 3.3. For any compact set $Q \subset \mathbb{R}^{d_\theta}$ and $s \in (0, 1)$, there exists a Borel-measurable function $\varphi_{Q,s} : \mathbb{R}^{d_\xi} \rightarrow [1, \infty)$ such that

$$\begin{aligned} \max\{\|F(\theta, \xi)\|, \|\tilde{F}(\theta, \xi)\|, \|(\Pi \tilde{F})(\theta, \xi)\|\} &\leq \varphi_{Q,s}(\xi), \\ \|(\Pi \tilde{F})(\theta', \xi) - (\Pi \tilde{F})(\theta'', \xi)\| &\leq \varphi_{Q,s}(\xi) \|\theta' - \theta''\|^s \end{aligned}$$

for all $\theta, \theta', \theta'' \in Q$, $\xi \in \mathbb{R}^{d_\xi}$.

ASSUMPTION 3.4. Given a compact set $Q \subset \mathbb{R}^{d_\theta}$ and $s \in (0, 1)$,

$$\sup_{n \geq 0} E(\varphi_{Q,s}^2(\xi_n) I_{\{\tau_Q \geq n\}} | \theta_0 = \theta, \xi_0 = \xi) < \infty$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $\xi \in \mathbb{R}^{d_\xi}$, where $\tau_Q = \inf\{n \geq 0 : \theta_n \notin Q\}$.

The main results on the convergence rate of recursion (3.1) are in the next theorem.

THEOREM 3.1. Let Assumptions 3.1 – 3.4 hold, and suppose that $f(\cdot)$ (introduced in Assumption 3.2) satisfies Assumptions 2.3 and 2.4. Then,

$$\|\nabla f(\theta_n)\|^2 = o(\gamma_n^{-p}), \quad d(f(\theta_n), C) = o(\gamma_n^{-p})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$. Moreover, the following is true:

$$\|\nabla f(\theta_n)\|^2 = o(\gamma_n^{-\hat{p}}), \quad d(f(\theta_n), C) = o(\gamma_n^{-\hat{p}}), \quad d(\theta_n, S) = o(\gamma_n^{-\hat{q}})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \cap \{\hat{r} > r\}$, and

$$\|\nabla f(\theta_n)\|^2 = O(\gamma_n^{-\hat{p}}), \quad d(f(\theta_n), C) = O(\gamma_n^{-\hat{p}}), \quad d(\theta_n, S) = O(\gamma_n^{-\hat{q}})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \cap \{\hat{r} \leq r\}$.

The proof is provided in Section 8. $C, S, p, \hat{p}, \hat{q}$ and \hat{r} are defined in Section 2.

Assumption 3.1 is related to the sequence $\{\alpha_n\}_{n \geq 0}$. It holds if $\alpha_n = 1/n^a$ for $n \geq 1$, where $a \in (1/2, 1]$ is a constant. On the other side, Assumptions 3.2 – 3.4 correspond to the stochastic process $\{\xi_n\}_{n \geq 0}$ and are quite standard for the asymptotic analysis of stochastic approximation algorithms with Markovian dynamics. Assumptions 3.2 – 3.4 have been introduced by Metivier and Priouret in [22] (see also [2, Part II]), and later generalized by Kushner and his co-workers (see [16] and references cited therein). However, neither the results of Metivier and Priouret, nor the results of Kushner and his co-workers provide any information on the convergence rate of stochastic gradient search in the case of multiple, non-isolated minima.

Regarding Theorem 3.1, the following note is also in order. As already mentioned in the beginning of the section, the purpose of the theorem is illustrating the results of Theorem 2.1 and providing a framework for studying the examples presented in the next sections. Since these examples perfectly fit into the framework developed by Metivier and Priouret, more general assumptions and settings of [16] are not considered here in order just to keep the exposition as concise as possible.

4. Example 1: Supervised Learning. In this section, online algorithms for supervised learning in feedforward neural networks are analyzed using the results of Theorems 2.2 and 3.1.

To state the problem of supervised learning and to define the corresponding algorithms, we need the following notation. N_1 and N_2 are positive integers, while $d_\theta = N_1(N_2 + 1)$. $\phi_1, \phi_2 : \mathbb{R} \rightarrow \mathbb{R}$ are differentiable functions, while $\psi_1, \dots, \psi_{N_2} : \mathbb{R}^{d_x} \rightarrow \mathbb{R}$ are Borel-measurable functions. For $a'_1, \dots, a'_{N_1} \in \mathbb{R}$, $a''_{1,1}, \dots, a''_{N_1, N_2} \in \mathbb{R}$, $x \in \mathbb{R}^{d_x}$, let

$$G_\theta(x) = \phi_1 \left(\sum_{i_1=1}^{N_1} a'_{i_1} \phi_2 \left(\sum_{i_2=1}^{N_2} a''_{i_1, i_2} \psi_{i_2}(x) \right) \right),$$

where $\theta = [a'_1 \cdots a'_{N_1} \ a''_{1,1} \cdots a''_{N_1, N_2}]^T$. Moreover, $\pi(\cdot, \cdot)$ denotes a probability measure on $\mathbb{R}^{d_x} \times \mathbb{R}$, while

$$f(\theta) = \frac{1}{2} \int (y - G_\theta(x))^2 \pi(dx, dy)$$

for $\theta \in \mathbb{R}^{d_\theta}$. Then, the mean-square error based supervised learning in feedforward neural networks can be described as the minimization of $f(\cdot)$ in a situation when only samples from $\pi(\cdot, \cdot)$ are available. In this context, $G_\theta(\cdot)$ represents the input-output function (i.e., the architecture) of the feedforward neural network to be trained. $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are the network activation functions, while θ is the vector of the network parameters to be tuned through the process of supervised learning. For more details on neural networks and supervised learning, see e.g., [11], [12] and references cited therein.

Function $f(\cdot)$ is usually minimized by the following stochastic gradient algorithm:

$$\theta_{n+1} = \theta_n + \alpha_n (y_n - G_{\theta_n}(x_n)) H_{\theta_n}(x_n), \quad n \geq 0. \quad (4.1)$$

In this recursion, $\{\alpha_n\}_{n \geq 0}$ is a sequence of positive real numbers, while $H_\theta(\cdot) = \nabla_\theta G_\theta(\cdot)$. θ_0 is an \mathbb{R}^{d_θ} -valued random variable defined on a probability space (Ω, \mathcal{F}, P) , while $\{(x_n, y_n)\}_{n \geq 0}$ is an $\mathbb{R}^{d_\theta} \times \mathbb{R}$ -valued stochastic process defined on the same probability space. In the context of supervised learning, $\{x_n, y_n\}_{n \geq 0}$ is regarded to as a training sequence.

The asymptotic behavior of algorithm (4.1) is analyzed under the following assumptions:

ASSUMPTION 4.1. $\phi_1(\cdot)$ and $\phi_2(\cdot)$ are real-analytic. Moreover, $\phi_1(\cdot)$ and $\phi_2(\cdot)$ have (complex-valued) continuations $\hat{\phi}_1(\cdot)$ and $\hat{\phi}_2(\cdot)$ (respectively) with the following properties:

- (i) $\phi_1(z)$ and $\phi_2(z)$ map $z \in \mathbb{C}$ into \mathbb{C} (\mathbb{C} denotes the set of complex numbers).
- (ii) $\hat{\phi}_1(x) = \phi_1(x)$ and $\hat{\phi}_2(x) = \phi_2(x)$ for all $x \in \mathbb{R}$.
- (iii) There exist real numbers $\varepsilon \in (0, 1)$, $K \in [1, \infty)$ such that $\hat{\phi}_1(\cdot)$ and $\hat{\phi}_2(\cdot)$ are analytic on $\hat{V}_\varepsilon = \{z \in \mathbb{C} : d(z, \mathbb{R}) \leq \varepsilon\}$, and such that

$$\begin{aligned} |\hat{\phi}_1(z)| &\leq K(1 + |z|), \\ \max\{|\hat{\phi}_1'(z)|, |\hat{\phi}_2(z)|, |\hat{\phi}_2'(z)|\} &\leq K \end{aligned}$$

for all $z \in \hat{V}_\varepsilon$ ($\hat{\phi}_1(\cdot)$, $\hat{\phi}_2(\cdot)$ are the derivatives of $\hat{\phi}_1(\cdot)$, $\hat{\phi}_2(\cdot)$).

ASSUMPTION 4.2. $\{(x_n, y_n)\}_{n \geq 0}$ are i.i.d. random variables distributed according the probability measure $\pi(\cdot, \cdot)$. There exists a real number $L \in [1, \infty)$ such that $\max_{1 \leq k \leq N_2} |\psi_k(x_0)| \leq L$ and $|y_0| \leq L$ w.p.1.

Our main results on the properties of objective function $f(\cdot)$ and algorithm (4.1) are contained in the next two theorems.

THEOREM 4.1. Let Assumptions 4.1 and 4.2 hold. Then, $f(\cdot)$ is analytic on entire \mathbb{R}^{d_θ} , i.e., it satisfies Assumptions 2.3 and 2.4.

THEOREM 4.2. Let Assumptions 3.1, 4.1 and 4.2 hold. Then,

$$\|\nabla f(\theta_n)\|^2 = o(\gamma_n^{-p}), \quad d(f(\theta_n), C) = o(\gamma_n^{-p})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$. Moreover, the following is true:

$$\|\nabla f(\theta_n)\|^2 = o(\gamma_n^{-\hat{p}}), \quad d(f(\theta_n), C) = o(\gamma_n^{-\hat{p}}), \quad d(\theta_n, S) = o(\gamma_n^{-\hat{q}})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \cap \{\hat{r} > r\}$, and

$$\|\nabla f(\theta_n)\|^2 = O(\gamma_n^{-\hat{p}}), \quad d(f(\theta_n), C) = O(\gamma_n^{-\hat{p}}), \quad d(\theta_n, S) = O(\gamma_n^{-\hat{q}})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \cap \{\hat{r} \leq r\}$.

The proofs are provided in Section 9. $C, S, p, \hat{p}, \hat{q}$ and \hat{r} are defined in Section 2.

Assumption 4.1 is related to the neural network being trained. It covers some of the most popular feedforward architectures such as backpropagation networks with logistic activations¹ and radial basis function networks with Gaussian activations².

¹Since

$$|1 + \exp(-z)|^2 = 1 + \exp(-2\operatorname{Re}(z)) + 2\exp(-\operatorname{Re}(z)) \cos(\operatorname{Im}(z)) \geq 1 + \exp(-2\operatorname{Re}(z))$$

when $|\operatorname{Im}(z)| \leq \pi/2$, complex-valued logistic function $h(z) = (1 + \exp(-z))^{-1}$ is analytical on $\{z \in \mathbb{C} : d(z, \mathbb{R}) \leq \pi/2\}$. Due to the same reason, $\max\{|h(z)|, |h'(z)|\} \leq 1$ on $\{z \in \mathbb{C} : d(z, \mathbb{R}) \leq \pi/2\}$.

²Complex-valued Gaussian activation $h(z) = (2\pi)^{-1/2} \exp(-z^2/2)$ is analytical on entire \mathbb{C} . As

$$(1 + |z|) \exp(-z^2/2) \leq (1 + |\operatorname{Re}(z)| + |\operatorname{Im}(z)|) \exp(-\operatorname{Re}^2(z)/2 + \operatorname{Im}^2(z)/2) \leq 3e$$

when $|\operatorname{Im}(z)| \leq 1$, we have $\max\{|h(z)|, |h'(z)|\} \leq 3e$ on $\{z \in \mathbb{C} : d(z, \mathbb{R}) \leq 1\}$.

On the other side, Assumption 4.2 corresponds to the training sequence $\{x_n, y_n\}_{n \geq 0}$, and is quite common for the analysis of supervised learning.

The asymptotic properties of supervised learning algorithms have been studied in a large number of papers (see [11], [12] and references cited therein). Unfortunately, the available literature does not provide any information on the rate of convergence which can be verified for the feedforward networks with nonlinear activation functions. The main difficulty comes out of the fact that the existing results on the convergence rate of stochastic gradient search require the objective function to have an isolated minimum at which the Hessian is strictly positive definite. Since the objective function is highly nonlinear in the case of supervised learning algorithms, it is hard (if not impossible) to show even the existence of isolated minima, let alone the definiteness of the Hessian. As opposed to the existing results, Theorem 4.2 does not invoke any of these requirements and covers some of the most widely used feedforward neural networks.

5. Example 2: Temporal Difference Learning. In this section, the results of Theorems 2.2 and 3.1 are illustrated by applying them to the analysis of temporal-difference learning algorithms.

In order to explain temporal-difference learning and to define the corresponding algorithms, we use the following notation. $N > 1$ is an integer, while $\mathcal{X} = \{1, \dots, N\}$. $\{x_n\}_{n \geq 0}$ is an \mathcal{X} -valued Markov chain defined on a probability space (Ω, \mathcal{F}, P) , while $\{c(i)\}_{i \in \mathcal{X}}$ are real numbers. $\beta \in (0, 1)$ is a constant, while

$$g(i) = E \left(\sum_{n=0}^{\infty} \beta^n c(x_n) \mid x_0 = i \right)$$

for $i \in \mathcal{X}$. For each $i \in \mathcal{X}$, $G_\theta(i)$ is a real-valued differentiable function of $\theta \in \mathbb{R}^{d_\theta}$, while

$$f(\theta) = \frac{1}{2} \lim_{n \rightarrow \infty} E(g(x_n) - G_\theta(x_n))^2$$

for $\theta \in \mathbb{R}^{d_\theta}$. With this notation, the problem of temporal-difference learning can be posed as the minimization of $f(\cdot)$ in a situation when only a realization of $\{x_n\}_{n \geq 0}$ is available. In this context, $c(i)$ is considered as a cost of visiting state i , while $g(i)$ is regarded to as a total discounted cost incurred by $\{x_n\}_{n \geq 0}$ when $\{x_n\}_{n \geq 0}$ starts from state i . $G_\theta(\cdot)$ is a parameterized approximation of $g(\cdot)$, while θ is the parameter to be tuned through the process of temporal-difference learning. For more details on temporal-difference learning, see e.g., [3], [27], [29] and references cited therein.

Function $f(\cdot)$ can be minimized by the following algorithm:

$$\theta_{n+1} = \theta_n + \alpha_n (c(x_n) + \beta G_{\theta_n}(x_{n+1}) - G_{\theta_n}(x_n)) y_n, \quad (5.1)$$

$$y_{n+1} = \beta y_n + H_{\theta_n}(x_{n+1}), \quad n \geq 0. \quad (5.2)$$

In this recursion, $\{\alpha_n\}_{n \geq 0}$ is a sequence of positive reals, while $H_\theta(\cdot) = \nabla_\theta G_\theta(\cdot)$. θ_0 is an \mathbb{R}^{d_θ} -valued random variable, which is defined on probability space (Ω, \mathcal{F}, P) and independent of $\{x_n\}_{n \geq 0}$. In the literature on reinforcement learning, recursion (5.1), (5.2) is known as *TD(1)* temporal-difference learning algorithm with a nonlinear function approximation, while $G_\theta(\cdot)$ is referred to as a function approximation, or just as an ‘approximator.’

We analyze algorithm (5.1), (5.2) under the following assumptions:

ASSUMPTION 5.1. $\{x_n\}_{n \geq 0}$ is geometrically ergodic.

ASSUMPTION 5.2. For each i , $G_\theta(i)$ is analytic in θ on entire \mathbb{R}^{d_θ} .

Our main results on the properties of $f(\cdot)$ and asymptotic behavior of the algorithm (5.1), (5.2) are presented in the next two theorems.

THEOREM 5.1. Let Assumptions 5.1 and 5.2 hold. Then, $f(\cdot)$ is analytic on entire \mathbb{R}^{d_θ} , i.e., it satisfies Assumptions 2.3 and 2.4.

THEOREM 5.2. Let Assumptions 3.1, 5.1 and 5.2 hold. Then,

$$\|\nabla f(\theta_n)\|^2 = o(\gamma_n^{-p}), \quad d(f(\theta_n), C) = o(\gamma_n^{-p})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\}$. Moreover, the following is true:

$$\|\nabla f(\theta_n)\|^2 = o(\gamma_n^{-\hat{p}}), \quad d(f(\theta_n), C) = o(\gamma_n^{-\hat{p}}), \quad d(\theta_n, S) = o(\gamma_n^{-\hat{q}})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \cap \{\hat{r} > r\}$, and

$$\|\nabla f(\theta_n)\|^2 = O(\gamma_n^{-\hat{p}}), \quad d(f(\theta_n), C) = O(\gamma_n^{-\hat{p}}), \quad d(\theta_n, S) = O(\gamma_n^{-\hat{q}})$$

w.p.1 on $\{\sup_{n \geq 0} \|\theta_n\| < \infty\} \cap \{\hat{r} \leq r\}$.

The proofs are provided in Section 10. $C, S, p, \hat{p}, \hat{q}$ and \hat{r} are defined in Section 2.

Assumption 5.1 corresponds to the stability of Markov chain $\{x_n\}_{n \geq 0}$. In this or similar form, it is involved in any result on the asymptotic behavior of temporal-difference learning. On the other side, Assumption 5.2 is related to the properties of $G_\theta(\cdot)$. It covers some of the most popular function approximations used in the area of reinforcement learning (e.g., polynomial approximations and feedforward neural networks with analytic activation functions; for details see [3], [27], [29]).

Asymptotic properties of temporal-difference learning have been the subject of a number of papers (see [3], [27] and references cited therein). However, the available literature on reinforcement learning does not offer any information on the rate of convergence of the algorithm (5.1), (5.2) in the case when $G_\theta(\cdot)$ is nonlinear in θ . Similarly as in the case of supervised learning, the main difficulty is caused by the fact that the existing results on the convergence rate of stochastic gradient search require $f(\cdot)$ to have an isolated minimum at which $\nabla^2 f(\cdot)$ is strictly positive definite. Unless $G_\theta(\cdot)$ is linear in θ , $f(\cdot)$ is so complex that these requirements are practically impossible to show. On the other side, Theorem 5.2 does not impose any restriction on the topological properties of the minima of $f(\cdot)$, or on the values of $\nabla^2 f(\cdot)$. Moreover, it can be applied to many temporal-difference learning algorithms met in practice.

Regarding the results of this section, the following note is also in order. Using the arguments Theorems 4.1 and 5.2 are based on, it is possible (at the cost of increasing significantly the amount of technical details) to generalize Theorems 5.1 and 5.2 to the case when $\{x_n\}_{n \geq 0}$ is a continuous state Markov chain, as well as to actor-critic learning algorithms proposed in [13].

6. Example 2: Identification of Linear Stochastic Dynamical Systems.

In this section, the general results presented in Sections 2 and 3 are applied to the asymptotic analysis of recursive prediction error algorithms for identification of linear stochastic dynamical systems. To avoid unnecessary technical details and complicated notation, only the identification of one dimensional ARMA models is considered here. However, it is straightforward to generalize the obtained results to any linear stochastic dynamical system.

In order to state the problem of recursive prediction error identification in ARMA models, we use the following notation. M and N are positive integers, while $d_\theta = M + N$. For $a_1, \dots, a_M \in \mathbb{R}$ and $b_1, \dots, b_N \in \mathbb{R}$, let

$$A_\theta(z) = 1 - \sum_{k=1}^M a_k z^{-k}, \quad B_\theta(z) = 1 + \sum_{k=1}^N b_k z^{-k},$$

where $\theta = [a_1 \cdots a_M \ b_1 \cdots b_N]^T$ and $z \in \mathbb{C}$ (\mathbb{C} denotes the set of complex numbers). Moreover, let

$$\Theta = \{\theta \in \mathbb{R}^{d_\theta} : B_\theta(z) = 0 \Rightarrow |z| > 1\}.$$

On the other side, $\{y_n\}_{n \geq 0}$ is a real-valued signal generated by the actual system (i.e., by the system being identified). For $\theta \in \Theta$, $\{y_n^\theta\}_{n \geq 0}$ is the output of the ARMA model

$$A_\theta(q)y_n^\theta = B_\theta(q)e_n, \quad n \geq 0, \quad (6.1)$$

where $\{e_n\}_{n \geq 0}$ is a real-valued white noise and q^{-1} is the backward time-shift operator. $\{\varepsilon_n^\theta\}_{n \geq 0}$ is the process generated by the recursion

$$B_\theta(q)\varepsilon_n^\theta = A_\theta(q)y_n, \quad n \geq 0, \quad (6.2)$$

while $\hat{y}_n^\theta = y_n - \varepsilon_n^\theta$ and

$$f(\theta) = \frac{1}{2} \lim_{n \rightarrow \infty} E((\varepsilon_n^\theta)^2).$$

Then, \hat{y}_n^θ is a mean-square optimal estimate of y_n given y_0, \dots, y_{n-1} (which the model (6.1) can provide; see e.g., [18], [19]). Consequently, ε_n^θ can be interpreted as the estimation error.

The parametric identification in ARMA models can be defined as the following estimation problem: Given a realization of $\{y_n\}_{n \geq 0}$, estimate the values of θ for which the model (6.1) provides the best approximation to the signal $\{y_n\}_{n \geq 0}$. If the identification is based on the prediction error principle, the estimation problem reduces to the minimization of $f(\cdot)$ over Θ . As the asymptotic value of the second moment of ε_n^θ is rarely available analytically, $f(\cdot)$ is minimized by a stochastic gradient (or stochastic Newton) algorithm. Such an algorithm is defined by the following difference equations:

$$\phi_n = [y_n \cdots y_{n-M+1} \ \varepsilon_n \cdots \varepsilon_{n-N+1}]^T, \quad (6.3)$$

$$\varepsilon_{n+1} = y_{n+1} - \phi_n^T \theta_n, \quad (6.4)$$

$$\psi_{n+1} = \phi_n - [\psi_n \cdots \psi_{n-N+1}]^T A_0 \theta_n, \quad (6.5)$$

$$\theta_{n+1} = \theta_n + \alpha_n \psi_{n+1} \varepsilon_{n+1}, \quad n \geq 0. \quad (6.6)$$

In this recursion, $\{\alpha_n\}_{n \geq 0}$ denotes a sequence of positive reals, while A_0 is a composite matrix defined as $A_0 = [0_{N \times M} \ I_{N \times N}]$. $\{y_n\}_{n \geq -M}$ is a real-valued stochastic process defined on a probability space (Ω, \mathcal{F}, P) , while $\theta_0 \in \Theta$, $\varepsilon_0, \dots, \varepsilon_{1-N} \in \mathbb{R}$ and $\psi_0, \dots, \psi_{1-N} \in \mathbb{R}^{d_\theta}$ are random variables defined on the same probability space. $\theta_0, \varepsilon_0, \dots, \varepsilon_{1-N}, \psi_0, \dots, \psi_{1-N} \in \mathbb{R}^{d_\theta}$ represent the initial conditions of the algorithm (6.3) – (6.6).

In the literature on system identification, recursion (6.3) – (6.6) is known as the recursive prediction error algorithm for ARMA models (for more details [18], [19] and references cited therein). It usually involves a projection (or truncation) device which ensures that estimates $\{\theta_n\}_{n \geq 0}$ remain in Θ . However, in order to avoid unnecessary technical details and to keep the exposition as concise as possible, this aspect of algorithm (6.3) – (6.6) is not discussed here. Instead, similarly as in [17] – [19], we state our asymptotic results (Theorem 6.2) in a local form.

Algorithm (6.3) – (6.6) is analyzed under the following assumptions:

ASSUMPTION 6.1. *There exist a positive integer L , a matrix $A \in \mathbb{R}^{L \times L}$, a vector $b \in \mathbb{R}^L$ and \mathbb{R}^L -valued stochastic processes $\{x_n\}_{n > -M}$, $\{w_n\}_{n > -M}$ (defined on (Ω, \mathcal{F}, P)) such that the following holds:*

- (i) $x_{n+1} = Ax_n + w_n$ and $y_n = b^T x_n$ for $n > -M$.
- (ii) The eigenvalues of A lie in $\{z \in \mathbb{C} : |z| < 1\}$.
- (iii) $\{w_n\}_{n \geq -M}$ are i.i.d. and independent of $\theta_0, x_{1-M}, \varepsilon_0, \dots, \varepsilon_{1-N}, \psi_0, \dots, \psi_{1-N}$.
- (iv) $E\|w_0\|^4 < \infty$.

ASSUMPTION 6.2. *For any compact set $Q \subset \Theta$,*

$$\sup_{n \geq 0} E \left((\varepsilon_n^4 + \|\psi_n\|^4) I_{\{\tau_Q \geq n\}} \right) < \infty, \quad (6.7)$$

where $\tau_Q = \inf\{n \geq 0 : \theta_n \notin Q\}$.

Our main result on the analyticity of $f(\cdot)$ is contained in the next theorem.

THEOREM 6.1. *Suppose that $\{y_n\}_{n \geq 0}$ is a weakly stationary process such that*

$$\sum_{n=0}^{\infty} |\text{Cov}(y_0, y_n)| < \infty.$$

Then, $f(\cdot)$ is analytic on entire Θ , i.e., the following is true: For any compact set $Q \subset \Theta$ and any $a \in f(Q)$, there exist real numbers $\delta_{Q,a}, \mu_{Q,a} \in (1, 2], \nu_Q \in (0, 1], M_{Q,a} \in [1, \infty), N_Q$ such that (2.3) holds for all $\theta \in Q$ and such that (2.2) is satisfied for each $\theta \in Q$ fulfilling $|f(\theta) - a| \leq \delta_{Q,a}$.

In order to state our main result of the convergence rate of algorithm (6.3) – (6.6), we use the following notation. Λ is the event defined by

$$\Lambda = \left\{ \sup_{n \geq 0} \|\theta_n\| < \infty, \inf_{n \geq 0} d(\theta_n, \partial\Theta) > 0 \right\}.$$

\hat{A} is the set of accumulation points of $\{\theta_n\}_{n \geq 0}$, while

$$\hat{\rho} = 2^{-1} d(\hat{A}, \partial\Theta) I_{\Lambda}, \quad \hat{f} = \liminf_{n \rightarrow \infty} f(\theta_n).$$

\hat{Q} is the random set defined as

$$\hat{Q} = \begin{cases} \left\{ \theta \in \mathbb{R}^{d_\theta} : d(\theta, \hat{A}) \hat{\rho} \right\}, & \text{on } \Lambda \\ \hat{A}, & \text{otherwise} \end{cases}.$$

$\hat{\delta}, \hat{\mu}, \hat{\nu}$ are random quantities defined by (2.4) on Λ and by (2.5) otherwise. Random quantities $\hat{p}, \hat{q}, \hat{r}$ are defined by (2.6). With this notation, our main result on the convergence rate of algorithm (6.3) – (6.6) reads as follows.

THEOREM 6.2. *Let Assumptions 3.1, 6.1 and 6.2 hold. Then,*

$$\|\nabla f(\theta_n)\|^2 = o(\gamma_n^{-p}), \quad d(f(\theta_n), C) = o(\gamma_n^{-p})$$

w.p.1 on Λ . Moreover, the following is true:

$$\|\nabla f(\theta_n)\|^2 = o(\gamma_n^{-\hat{p}}), \quad d(f(\theta_n), C) = o(\gamma_n^{-\hat{p}}), \quad d(\theta_n, S) = o(\gamma_n^{-\hat{q}})$$

w.p.1 on $\Lambda \cap \{\hat{r} > r\}$, and

$$\|\nabla f(\theta_n)\|^2 = O(\gamma_n^{-\hat{p}}), \quad d(f(\theta_n), C) = O(\gamma_n^{-\hat{p}}), \quad d(\theta_n, S) = O(\gamma_n^{-\hat{q}})$$

w.p.1 on $\Lambda \cap \{\hat{r} \leq r\}$.

The proofs are provided in Section 11. C and S are defined in Section 2.

Assumption 6.1 corresponds to the signal $\{y_n\}_{n \geq 0}$. It is quite common for the asymptotic analysis of recursive identification algorithm (see e.g., [2, Part I]) and cover all stable linear Markov models. Assumption 6.2 is related to the stability of subrecursion (6.3) – (6.5) and its output $\{\varepsilon_n\}_{n \geq 0}$, $\{\psi_n\}_{n \geq 0}$. In this or a similar form, Assumption 6.2 is involved in most of the asymptotic results on the recursive prediction error identification algorithms. E.g., [18, Theorems 4.1 – 4.3] (which are probably the most general and famous results of this kind) require sequence $\{(\varepsilon_n, \psi_n)\}_{n \geq 0}$ to visit a fixed compact set infinitely often w.p.1 on event Λ . When $\{y_n\}_{n \geq 0}$ is generated by a stable linear Markov system, such a requirement is practically equivalent to (6.7).

Various aspects of recursive prediction error identification in linear stochastic dynamical systems have been the subject of numerous papers and books (see [18], [19] and references cited therein). Despite providing a deep insight into the asymptotic behavior of recursive prediction error identification algorithms, the available results do not offer information about the convergence rate which can be verified for models of a moderate or high order (e.g., M and N are three or above). The main difficulty is the same as in the case of supervised learning. The existing results on convergence rate of stochastic gradient search require $f(\cdot)$ to have an isolated minimum which is the limit of $\{\theta_n\}_{n \geq 0}$ and at which $\nabla^2 f(\cdot)$ is strictly positive definite. Unfortunately, $f(\cdot)$ is so complex (even for relatively small M and N) that these requirements are practically impossible to verify. Apparently, Theorem 6.2 relies on none of them.

Regarding Theorems 6.1 and 6.2, it should be mentioned that these results can be generalized in several ways. E.g., it is straightforward to extend them to practically any stable multiple-input, multiple-output linear system. Moreover, it is possible to show that the results also hold for signals $\{y_n\}_{n \geq 0}$ satisfying mixing conditions of the type [18, Condition S1, p. 169].

7. Proof of Theorems 2.1 and 2.2. In this section, the following notation is used. Let Λ be the event

$$\Lambda = \left\{ \sup_{n \geq 0} \|\theta_n\| < \infty \right\}.$$

For $\varepsilon \in (0, \infty)$, let

$$\phi_\varepsilon(w) = \phi(w) + \varepsilon.$$

For $\theta \in \mathbb{R}^{d_\theta}$, let

$$u(\theta) = f(\theta) - \hat{f}, \quad v(\theta) = \begin{cases} (f(\theta) - \hat{f})^{-1/\hat{p}}, & \text{if } f(\theta) > \hat{f} \\ 0, & \text{otherwise} \end{cases}$$

(\hat{p} is introduced in Section 2). On the other side, for $0 \leq n < k$, let $u_{n,n} = 0$, $v_{n,n} = v'_{n,n} = v''_{n,n} = 0$ and

$$\begin{aligned} u_{n,k} &= \sum_{i=n}^{k-1} \alpha_i w_i, \\ v'_{n,k} &= -(\nabla f(\theta_n))^T \sum_{i=1}^{k-1} \alpha_i (\nabla f(\theta_i) - \nabla f(\theta_n)), \\ v''_{n,k} &= \int_0^1 (\nabla f(\theta_n + s(\theta_k - \theta_n)) - \nabla f(\theta_n))^T (\theta_k - \theta_n) ds, \\ v_{n,k} &= v'_{n,k} + v''_{n,k}. \end{aligned}$$

Then, it is straightforward to show

$$f(\theta_k) - f(\theta_n) = -(\gamma_k - \gamma_n) \|\nabla f(\theta_n)\|^2 - (\nabla f(\theta_n))^T u_{n,k} + v_{n,k} \quad (7.1)$$

for $0 \leq n \leq k$.

Regarding the notation, the following note is also in order: \sim symbol is used for locally defined quantities, i.e., for a quantity whose definition holds only in the proof where such a quantity appears.

LEMMA 7.1. *Let Assumptions 2.1 and 2.2 hold. Then, there exists an event $N_0 \in \mathcal{F}$ such that $P(N_0) = 0$ and*

$$\limsup_{n \rightarrow \infty} \gamma_n^r \max_{n \leq k \leq a(n,1)} \|u_{n,k}\| \leq w < \infty \quad (7.2)$$

on $\Lambda \setminus N_0$.

Proof. It is straightforward to verify

$$u_{n,k} = \sum_{i=n}^{k-1} (\gamma_i^{-r} - \gamma_{i+1}^{-r}) \left(\sum_{j=n}^i \alpha_j \gamma_j^r w_j \right) + \gamma_k^{-r} \sum_{i=n}^{k-1} \alpha_i \gamma_i^r w_i$$

for $0 \leq n < k$. Consequently,

$$\|u_{n,k}\| \leq \left(\gamma_k^{-r} + \sum_{i=n}^{k-1} (\gamma_i^{-r} - \gamma_{i+1}^{-r}) \right) \max_{n \leq j < k} \left\| \sum_{i=n}^j \alpha_i \gamma_i^r w_i \right\| = \gamma_n^{-r} \max_{n \leq j < k} \left\| \sum_{i=n}^j \alpha_i \gamma_i^r w_i \right\|$$

for $0 \leq n < k$. Thus,

$$\gamma_n^r \|u_{n,k}\| \leq \max_{n \leq j < a(n,1)} \left\| \sum_{i=n}^j \alpha_i \gamma_i^r w_i \right\|$$

for $0 \leq n \leq k \leq a(n,1)$. Then, the lemma's assertion directly follows from Assumption 2.2. \square

LEMMA 7.2. *Suppose that Assumptions 2.1 – 2.3 hold. Moreover, let $\varepsilon \in (0, \infty)$ be an arbitrary positive real number. Then, there exist random quantities \hat{C}_1 , \hat{t} (which are deterministic functions of \hat{C} ; \hat{C} is defined in Section 2) and a non-negative integer-valued random variable σ_ε such that $1 \leq \hat{C} < \infty$, $0 < \hat{t} \leq 1$, $0 \leq \sigma_\varepsilon < \infty$ everywhere*

and such that

$$\max_{n \leq k \leq a(n, \hat{t})} (f(\theta_k) - f(\theta_n)) \leq \gamma_n^{-\hat{p}/\hat{\mu}} \|\nabla f(\theta_n)\| \phi_\varepsilon(w) + \hat{C}_1 \gamma_n^{-2\hat{p}/\hat{\mu}} (\phi_\varepsilon(w))^2, \quad (7.3)$$

$$f(\theta_{a(n, \hat{t})}) - f(\theta_n) \leq -\hat{t} \|\nabla f(\theta_n)\|^2 / 2 + \gamma_n^{-\hat{p}/\hat{\mu}} \|\nabla f(\theta_n)\| \phi_\varepsilon(w) + \hat{C}_1 \gamma_n^{-2\hat{p}/\hat{\mu}} (\phi_\varepsilon(w))^2 \quad (7.4)$$

on $\Lambda \setminus N_0$ for $n > \sigma_\varepsilon$ ($\hat{\mu}$ is introduced in Section 3).

Proof. Let $\hat{C}_1 = 12\hat{C}^3 \exp(2\hat{C})$, $\hat{t} = 1/(4\hat{C}_1)$, while

$$\begin{aligned} \tilde{\sigma}_1 &= \max \left(\{n \geq 0 : \theta_n \notin \hat{Q}\} \cup \{0\} \right), \\ \tilde{\sigma}_2 &= \max \left(\{n \geq 0 : \alpha_n > \hat{t}/3\} \cup \{0\} \right), \\ \tilde{\sigma}_{3, \varepsilon} &= \max \left(\left\{ n \geq 0 : \max_{n \leq k \leq a(n, 1)} \|u_{n, k}\| > \gamma_n^{-\hat{p}/\hat{\mu}} \phi_\varepsilon(w) \right\} \cup \{0\} \right) \end{aligned}$$

and $\sigma_\varepsilon = \max\{\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_{3, \varepsilon}\} I_{\Lambda \setminus N_0}$. Then, it is obvious that σ_ε is well-defined. On the other side, Lemma 7.1 yields

$$\limsup_{n \rightarrow \infty} \gamma_n^{\hat{p}/\hat{\mu}} \max_{n \leq k \leq a(n, 1)} \|u_{n, k}\| = \limsup_{n \rightarrow \infty} \gamma_n^r \max_{n \leq k \leq a(n, 1)} \|u_{n, k}\| = w < \phi_\varepsilon(w)$$

on $(\Lambda \setminus N_0) \cap \{\hat{r} \geq r\}$ (notice that if $r \leq \hat{r}$, then $\hat{p}/\hat{\mu} = r$ and $\phi_\varepsilon(w) \geq w + \varepsilon > w$) and

$$\limsup_{n \rightarrow \infty} \gamma_n^{\hat{p}/\hat{\mu}} \max_{n \leq k \leq a(n, 1)} \|u_{n, k}\| = \limsup_{n \rightarrow \infty} \gamma_n^{\hat{p}/\hat{\mu} - r} w = 0 < \phi_\varepsilon(w)$$

on $(\Lambda \setminus N_0) \cap \{\hat{r} < r\}$ (notice that if $r > \hat{r}$, then $\hat{p}/\hat{\mu} = \hat{r} < r$ and $\phi_\varepsilon(w) \geq \varepsilon > 0$). Thus, $0 \leq \sigma_\varepsilon < \infty$ everywhere. Moreover, we have

$$\max_{n \leq k \leq a(n, 1)} \|u_{n, k}\| \leq \gamma_n^{-\hat{p}/\hat{\mu}} \phi_\varepsilon(w), \quad (7.5)$$

$$\hat{t} \geq \gamma_{a(n, \hat{t})} - \gamma_n = \gamma_{a(n, \hat{t})+1} - \gamma_n - \alpha_{a(n, \hat{t})} \geq 2\hat{t}/3 \quad (7.6)$$

on $\Lambda \setminus N_0$ for $n > \sigma_\varepsilon$. On the other side, (7.5) yields

$$\begin{aligned} \|\nabla f(\theta_k)\| &\leq \|\nabla f(\theta_n)\| + \|\nabla f(\theta_k) - \nabla f(\theta_n)\| \\ &\leq \|\nabla f(\theta_n)\| + \hat{C} \|\theta_k - \theta_n\| \\ &\leq \|\nabla f(\theta_n)\| + \hat{C} \sum_{i=n}^{k-1} \alpha_i \|\nabla f(\theta_i)\| + \hat{C} \|u_{n, k}\| \\ &\leq \|\nabla f(\theta_n)\| + \hat{C} \gamma_n^{-\hat{p}/\hat{\mu}} \phi_\varepsilon(w) + \hat{C} \sum_{i=n}^{k-1} \alpha_i \|\nabla f(\theta_i)\| \end{aligned}$$

on Λ for $\sigma_\varepsilon < n \leq k$. Then, Bellman-Gronwall inequality implies

$$\begin{aligned} \|\nabla f(\theta_k)\| &\leq \left(\|\nabla f(\theta_n)\| + \hat{C} \gamma_n^{-\hat{p}/\hat{\mu}} \phi_\varepsilon(w) \right) \exp \left(\hat{C} (\gamma_{a(n, 1)} - \gamma_n) \right) \\ &\leq \hat{C} \exp(\hat{C}) \left(\|\nabla f(\theta_n)\| + \gamma_n^{-\hat{p}/\hat{\mu}} \phi_\varepsilon(w) \right) \end{aligned}$$

on $\Lambda \setminus N_0$ for $\sigma_\varepsilon < n \leq k \leq a(n, 1)$ (notice that $\gamma_{a(n,1)} - \gamma_n \leq 1$). Consequently, (7.5) gives

$$\begin{aligned} \|\theta_k - \theta_n\| &\leq \sum_{i=n}^{k-1} \alpha_i \|\nabla f(\theta_i)\| + \|u_{n,k}\| \\ &\leq \hat{C} \exp(\hat{C}) \left(\|\nabla f(\theta_n)\| + \gamma_n^{-\hat{p}/\hat{\mu}} \phi_\varepsilon(w) \right) (\gamma_k - \gamma_n) + \gamma_n^{-\hat{p}/\hat{\mu}} \phi_\varepsilon(w) \\ &\leq 2\hat{C} \exp(\hat{C}) \left((\gamma_k - \gamma_n) \|\nabla f(\theta_n)\| + \gamma_n^{-\hat{p}/\hat{\mu}} \phi_\varepsilon(w) \right) \end{aligned}$$

on $\Lambda \setminus N_0$ for $\sigma_\varepsilon < n \leq k \leq a(n, 1)$. Therefore,

$$\begin{aligned} |v'_{n,k}| &\leq \hat{C} \|\nabla f(\theta_n)\| \sum_{i=n}^{k-1} \alpha_i \|\theta_i - \theta_n\| \\ &\leq 2\hat{C}^2 \exp(\hat{C}) \left((\gamma_k - \gamma_n)^2 \|\nabla f(\theta_n)\|^2 + \gamma_n^{-\hat{p}/\hat{\mu}} (\gamma_k - \gamma_n) \|\nabla f(\theta_n)\| \phi_\varepsilon(w) \right) \\ &\leq 4\hat{C}^2 \exp(\hat{C}) \left((\gamma_k - \gamma_n)^2 \|\nabla f(\theta_n)\|^2 + \gamma_n^{-2\hat{p}/\hat{\mu}} (\phi_\varepsilon(w))^2 \right), \\ |v''_{n,k}| &\leq \hat{C} \|\theta_k - \theta_n\|^2 \\ &\leq 4\hat{C}^3 \exp(2\hat{C}) \left((\gamma_k - \gamma_n) \|\nabla f(\theta_n)\| + \gamma_n^{-\hat{p}/\hat{\mu}} \phi_\varepsilon(w) \right)^2 \\ &\leq 8\hat{C}^3 \exp(2\hat{C}) \left((\gamma_k - \gamma_n)^2 \|\nabla f(\theta_n)\|^2 + \gamma_n^{-2\hat{p}/\hat{\mu}} (\phi_\varepsilon(w))^2 \right) \end{aligned}$$

on $\Lambda \setminus N_0$ for $\sigma_\varepsilon < n \leq k \leq a(n, 1)$. Thus,

$$|v_{n,k}| \leq \hat{C}_1 \left((\gamma_k - \gamma_n)^2 \|\nabla f(\theta_n)\|^2 + \gamma_n^{-2\hat{p}/\hat{\mu}} (\phi_\varepsilon(w))^2 \right) \quad (7.7)$$

on $\Lambda \setminus N_0$ for $\sigma_\varepsilon < n \leq k \leq a(n, 1)$. Since

$$\hat{C}_1(\gamma_k - \gamma_n) \leq \hat{C}_1(\gamma_{a(n,\hat{t})} - \gamma_n) \leq \hat{C}_1 \hat{t} \leq 1/4$$

for $0 \leq n \leq k \leq a(n, \hat{t})$ (due to (7.6)), (7.1), (7.5) and (7.7) yield

$$\begin{aligned} f(\theta_k) - f(\theta_n) &\leq -(\gamma_k - \gamma_n) \left(1 - \hat{C}_1(\gamma_k - \gamma_n) \right) \|\nabla f(\theta_n)\|^2 \\ &\quad + \gamma_n^{-\hat{p}/\hat{\mu}} \|\nabla f(\theta_n)\| \phi_\varepsilon(w) + \hat{C}_1 \gamma_n^{-2\hat{p}/\hat{\mu}} (\phi_\varepsilon(w))^2 \\ &\leq -3(\gamma_k - \gamma_n) \|\nabla f(\theta_n)\|^2 / 4 \\ &\quad + \gamma_n^{-\hat{p}/\hat{\mu}} \|\nabla f(\theta_n)\| \phi_\varepsilon(w) + \hat{C}_1 \gamma_n^{-2\hat{p}/\hat{\mu}} (\phi_\varepsilon(w))^2 \end{aligned} \quad (7.8)$$

on $\Lambda \setminus N_0$ for $\sigma_\varepsilon < n \leq k \leq a(n, \hat{t})$. As an immediate consequence of (7.6), (7.8), we get that (7.3), (7.4) hold on $\Lambda \setminus N_0$ for $n > \sigma_\varepsilon$. \square

LEMMA 7.3. *Suppose that Assumptions 2.1 – 2.3 hold. Then, $\lim_{n \rightarrow \infty} \nabla f(\theta_n) = 0$ on $\Lambda \setminus N_0$.*

Proof. The lemma's assertion is proved by contradiction. We assume that $\limsup_{n \rightarrow \infty} \|\nabla f(\theta_n)\| > 0$ for some sample $\omega \in \Lambda \setminus N_0$ (notice that all formulas which follow in the proof correspond to this ω). Then, there exists $a \in (0, \infty)$ and an increasing sequence $\{l_k\}_{k \geq 0}$ such that $\liminf_{k \rightarrow \infty} \|\nabla f(\theta_{l_k})\| > a$. Since

$\liminf_{k \rightarrow \infty} f(\theta_{a(l_k, \hat{t})}) \geq \hat{f}$, Lemma 7.2 (inequality (7.4)) gives

$$\begin{aligned} \hat{f} - \liminf_{k \rightarrow \infty} f(\theta_{l_k}) &\leq \limsup_{k \rightarrow \infty} (f(\theta_{a(l_k, \hat{t})}) - f(\theta_{l_k})) \\ &\leq -(\hat{t}/2) \liminf_{k \rightarrow \infty} \|\nabla f(\theta_{l_k})\|^2 \\ &\leq -a^2 \hat{t}/2. \end{aligned}$$

Therefore, $\liminf_{k \rightarrow \infty} f(\theta_{l_k}) \geq \hat{f} + a\hat{t}^2/2$. Consequently, there exist $b, c \in \mathbb{R}$ such that $\hat{f} < b < c < \hat{f} + a\hat{t}^2/2$, $b < \hat{f} + \hat{\delta}$ and $\limsup_{n \rightarrow \infty} f(\theta_n) > c$. Thus, there exist sequences $\{m_k\}_{k \geq 0}$, $\{n_k\}_{k \geq 0}$ with the following properties: $m_k < n_k < m_{k+1}$, $f(\theta_{m_k}) < b$, $f(\theta_{n_k}) > c$ and

$$\max_{m_k < n \leq n_k} f(\theta_n) \geq b \quad (7.9)$$

for $k \geq 0$. Then, Lemma 7.2 (inequality (7.3)) implies

$$\limsup_{k \rightarrow \infty} (f(\theta_{m_{k+1}}) - f(\theta_{m_k})) \leq 0, \quad (7.10)$$

$$\limsup_{k \rightarrow \infty} \max_{m_k \leq n \leq a(m_k, \hat{t})} (f(\theta_n) - f(\theta_{m_k})) \leq 0. \quad (7.11)$$

Since

$$b > f(\theta_{m_k}) = f(\theta_{m_{k+1}}) - (f(\theta_{m_{k+1}}) - f(\theta_{m_k})) \geq b - (f(\theta_{m_{k+1}}) - f(\theta_{m_k}))$$

for $k \geq 0$, (7.10) yields $\lim_{k \rightarrow \infty} f(\theta_{m_k}) = b$. As $f(\theta_{n_k}) - f(\theta_{m_k}) > c - b$ for $k \geq 0$, (7.11) implies $a(m_k, \hat{t}) < n_k$ for all, but infinitely many k (otherwise, $\liminf_{k \rightarrow \infty} (f(\theta_{n_k}) - f(\theta_{m_k})) \leq 0$ would follow from (7.11)). Consequently, $\liminf_{k \rightarrow \infty} f(\theta_{a(m_k, \hat{t})}) \geq b$ (due to (7.9)), while Lemma 7.2 (inequality (7.4)) gives

$$\begin{aligned} 0 &\leq \limsup_{k \rightarrow \infty} f(\theta_{a(m_k, \hat{t})}) - b = \limsup_{k \rightarrow \infty} (f(\theta_{a(m_k, \hat{t})}) - f(\theta_{m_k})) \\ &\leq -(\hat{t}/2) \liminf_{k \rightarrow \infty} \|\nabla f(\theta_{m_k})\|^2. \end{aligned}$$

Therefore, $\lim_{k \rightarrow \infty} \|\nabla f(\theta_{m_k})\| = 0$. Thus, there exists $k_0 \geq 0$ such that $\theta_{m'_k} \in \hat{Q}$ and $f(\theta_{m_k}) \geq (\hat{f} + b)/2$ for $k \geq k_0$ (notice that $\lim_{k \rightarrow \infty} f(\theta_{m_k}) = b > (\hat{f} + b)/2$). Consequently, $\theta_{m_k} \in \hat{Q}$ and $0 < (b - \hat{f})/2 \leq f(\theta_{m_k}) - \hat{f} \leq \hat{\delta}$ for $k \geq k_0$ (notice that $f(\theta_{m_k}) < b < \hat{f} + \hat{\delta}$ for $k \geq 0$). Then, owing to (2.7) (i.e., to Assumption 3.3), we have

$$0 < (b - \hat{f})/2 \leq f(\theta_{m_k}) - \hat{f} \leq \hat{M} \|\nabla f(\theta_{m_k})\|^{\hat{\mu}}$$

for $k \geq k_0$. However, this directly contradicts the fact $\lim_{k \rightarrow \infty} \|\nabla f(\theta_{m_k})\| = 0$. Hence, $\lim_{n \rightarrow \infty} \nabla f(\theta_n) = 0$ on $\Lambda \setminus N_0$. \square

LEMMA 7.4. *Suppose that Assumptions 2.1 – 2.3 hold. Then, $\lim_{n \rightarrow \infty} f(\theta_n) = \hat{f}$ on $\Lambda \setminus N_0$.*

Proof. We use contradiction to prove the lemma's assertion: Suppose that $\hat{f} < \limsup_{n \rightarrow \infty} f(\theta_n)$ for some sample $\omega \in \Lambda \setminus N_0$ (notice that all formulas which follow in the proof correspond to this ω). Then, there exists $a \in \mathbb{R}$ such that $\hat{f} < a < \hat{f} + \hat{\delta}$ and $\limsup_{n \rightarrow \infty} f(\theta_n) > a$. Thus, there exists an increasing sequence $\{n_k\}_{k \geq 0}$ such that

$f(\theta_{n_k}) < a$ and $f(\theta_{n_{k+1}}) \geq a$ for $k \geq 0$. On the other side, Lemma 7.2 (inequality (7.3)) implies

$$\limsup_{k \rightarrow \infty} (f(\theta_{n_{k+1}}) - f(\theta_{n_k})) \leq 0. \quad (7.12)$$

Since

$$a > f(\theta_{n_k}) = f(\theta_{n_{k+1}}) - (f(\theta_{n_{k+1}}) - f(\theta_{n_k})) \geq a - (f(\theta_{n_{k+1}}) - f(\theta_{n_k}))$$

for $k \geq 0$, (7.12) yields $\lim_{k \rightarrow \infty} f(\theta_{n_k}) = a$. Consequently, there exists $k_0 \geq 0$ such that $\theta_{n_k} \in \hat{Q}$ and $f(\theta_{n_k}) \geq (\hat{f} + a)/2$ for $k \geq k_0$ (notice that $\lim_{k \rightarrow \infty} f(\theta_{n_k}) = a > (\hat{f} + a)/2$). Thus, $\theta_{n_k} \in \hat{Q}$ and $0 < (a - \hat{f})/2 \leq f(\theta_{n_k}) - \hat{f} \leq \hat{\delta}$ for $k \geq k_0$ (notice that $f(\theta_{n_k}) < a < \hat{f} + \hat{\delta}$ for $k \geq 0$). Then, due to (2.7) (i.e., to Assumption 2.3), we have

$$0 < (a - \hat{f})/2 \leq f(\theta_{n_k}) - \hat{f} \leq \hat{M} \|\nabla f(\theta_{n_k})\|^{\hat{\mu}}$$

for $k \geq k_0$. However, this directly contradicts the fact $\lim_{n \rightarrow \infty} \nabla f(\theta_n) = 0$. Hence, $\lim_{n \rightarrow \infty} f(\theta_n) = \hat{f}$ on $\Lambda \setminus N_0$. \square

LEMMA 7.5. *Suppose that Assumptions 2.1 – 2.3 hold. Moreover, let $\varepsilon \in (0, \infty)$ be an arbitrary positive real number. Then, there exist random quantities \hat{C}_2, \hat{C}_3 (which are deterministic functions of r, \hat{C}, \hat{M}) and a non-negative integer-valued random variable τ_ε such that $1 \leq \hat{C}_2, \hat{C}_3 < \infty, 0 \leq \tau_\varepsilon < \infty$ everywhere and such that the following is true:*

$$\left(u(\theta_{a(n, \hat{t})}) - u(\theta_n) + \hat{t} \|\nabla f(\theta_n)\|^2 / 4 \right) I_{A_{n, \varepsilon}} \leq 0, \quad (7.13)$$

$$\left(u(\theta_{a(n, \hat{t})}) - u(\theta_n) + (\hat{t} / \hat{C}_3) u(\theta_n) \right) I_{B_{n, \varepsilon}} \leq 0, \quad (7.14)$$

$$\left(v(\theta_{a(n, \hat{t})}) - v(\theta_n) - (\hat{t} / \hat{C}_3) (\phi_\varepsilon(w))^{-\hat{\mu} / \hat{p}} \right) I_{C_{n, \varepsilon}} \geq 0 \quad (7.15)$$

on $\Lambda \setminus N_0$ for $n \geq \tau_\varepsilon$, where

$$A_{n, \varepsilon} = \left\{ \gamma_n^{\hat{p}} |u(\theta_n)| \geq \hat{C}_2 (\phi_\varepsilon(w))^{\hat{\mu}} \right\} \cup \left\{ \gamma_n^{\hat{p}} \|\nabla f(\theta_n)\|^2 \geq \hat{C}_2 (\phi_\varepsilon(w))^{\hat{\mu}} \right\},$$

$$B_{n, \varepsilon} = \left\{ \gamma_n^{\hat{p}} u(\theta_n) \geq \hat{C}_2 (\phi_\varepsilon(w))^{\hat{\mu}} \right\} \cap \{ \hat{\mu} = 2 \},$$

$$C_{n, \varepsilon} = \left\{ \gamma_n^{\hat{p}} u(\theta_n) \geq \hat{C}_2 (\phi_\varepsilon(w))^{\hat{\mu}} \right\} \cap \left\{ u(\theta_{a(n, \hat{t})}) > 0 \right\} \cap \{ \hat{\mu} < 2 \}.$$

REMARK. *Inequalities (7.13) – (7.15) can be represented in the following equivalent form: Relations*

$$\begin{aligned} & \left(\gamma_n^{\hat{p}} |u(\theta_n)| \geq \hat{C}_2 (\phi_\varepsilon(w))^{\hat{\mu}} \vee \gamma_n^{\hat{p}} \|\nabla f(\theta_n)\|^2 \geq \hat{C}_2 (\phi_\varepsilon(w))^{\hat{\mu}} \right) \wedge n > \tau_\varepsilon \\ & \implies u(\theta_{a(n, \hat{t})}) \leq u(\theta_n) - \hat{t} \|\nabla f(\theta_n)\|^2 / 4, \end{aligned} \quad (7.16)$$

$$\begin{aligned} & \gamma_n^{\hat{p}} u(\theta_n) \geq \hat{C}_2 (\phi_\varepsilon(w))^{\hat{\mu}} \wedge \hat{\mu} = 2 \wedge n > \tau_\varepsilon \\ & \implies u(\theta_{a(n, \hat{t})}) \leq \left(1 - \hat{t} / \hat{C}_3 \right) u(\theta_n), \end{aligned} \quad (7.17)$$

$$\begin{aligned} & \gamma_n^{\hat{p}} u(\theta_n) \geq \hat{C}_2 (\phi_\varepsilon(w))^{\hat{\mu}} \wedge u(\theta_{a(n, \hat{t})}) > 0 \wedge \hat{\mu} < 2 \wedge n > \tau_\varepsilon \\ & \implies v(\theta_{a(n, \hat{t})}) \geq v(\theta_n) + (\hat{t} / \hat{C}_3) (\phi_\varepsilon(w))^{-\hat{\mu} / \hat{p}} \end{aligned} \quad (7.18)$$

are true on $\Lambda \setminus N_0$.

Proof. Let $\tilde{C} = 8\hat{C}_1^{1/2}/\hat{t}$, $\hat{C}_2 = \tilde{C}^2\hat{M}$ and $\hat{C}_3 = 8\hat{M}^2 \max\{1, r\}$, while

$$\begin{aligned}\tilde{\tau}_1 &= \max\left(\left\{n \geq 0 : \theta_n \notin \hat{Q}\right\} \cup \{0\}\right), \\ \tilde{\tau}_2 &= \max\left(\left\{n \geq 0 : |u(\theta_n)| > \hat{\delta}\right\} \cup \{0\}\right), \\ \tilde{\tau}_{3,\varepsilon} &= \max\left(\left\{n \geq 0 : \gamma_n^{-\hat{p}/2}(\phi_\varepsilon(w))^{\hat{\mu}/2} < \gamma_n^{-\hat{p}/\hat{\mu}}\phi_\varepsilon(w)\right\} \cup \{0\}\right)\end{aligned}\quad (7.19)$$

and $\tau_\varepsilon = \max\{\sigma_\varepsilon, \tilde{\tau}_1, \tilde{\tau}_2, \tilde{\tau}_{3,\varepsilon}\}I_{\Lambda \setminus N_0}$. Obviously, τ_ε is well-defined. On the other side, Lemmas 7.3, 7.5 imply $0 \leq \tau_\varepsilon < \infty$ everywhere (in order to conclude that $\tilde{\tau}_2$ is finite, notice that $\lim_{n \rightarrow \infty} u(\theta_n) = 0$ on $\Lambda \setminus N_0$; in order to deduce that $\tilde{\tau}_{3,\varepsilon}$ is finite, notice that $\hat{p}/2 < \hat{p}/\hat{\mu}$ when $\hat{\mu} < 2$, and that the left and right hand sides of the inequality in (7.19) are equal when $\hat{\mu} = 2$). Moreover, we have

$$\gamma_n^{-\hat{p}/2}(\phi_\varepsilon(w))^{\hat{\mu}/2} \geq \gamma_n^{-\hat{p}/\hat{\mu}}\phi_\varepsilon(w) \quad (7.20)$$

on $\Lambda \setminus N_0$ for $n > \tau_\varepsilon$. Since $\tau_\varepsilon \geq \sigma_\varepsilon$ on $\Lambda \setminus N_0$, Lemma 7.2 (inequality (7.4)) yields

$$u(\theta_{a(n,\hat{t})}) - u(\theta_n) \leq -\hat{t}\|\nabla f(\theta_n)\|^2/2 + \gamma_n^{-\hat{p}/\hat{\mu}}\|\nabla f(\theta_n)\|\phi_\varepsilon(w) + \hat{C}_1\gamma_n^{-2\hat{p}/\hat{\mu}}(\phi_\varepsilon(w))^2 \quad (7.21)$$

on $\Lambda \setminus N_0$ for $n > \tau_\varepsilon$. As $\theta_n \in \hat{Q}$ and $|u(\theta_n)| \leq \hat{\delta}$ on $\Lambda \setminus N_0$ for $n > \tau_\varepsilon$, (2.7) (i.e., Assumption 2.3) implies

$$|u(\theta_n)| \leq \hat{M}\|\nabla f(\theta_n)\|^{\hat{\mu}} \quad (7.22)$$

on $\Lambda \setminus N_0$ for $n > \tau_\varepsilon$.

Let ω be an arbitrary sample from $\Lambda \setminus N_0$ (notice that all formulas which follow in the proof correspond to this ω). First, we show (7.13). We proceed by contradiction: Suppose that (7.13) is violated for some $n > \tau_\varepsilon$. Therefore,

$$u(\theta_{a(n,\hat{t})}) - u(\theta_n) > -\hat{t}\|\nabla f(\theta_n)\|^2/4 \quad (7.23)$$

and at least one of the following two inequalities is true:

$$|u(\theta_n)| \geq \hat{C}_2\hat{M}\gamma_n^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}}, \quad (7.24)$$

$$\|\nabla f(\theta_n)\|^2 \geq \hat{C}_2\gamma_n^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}}. \quad (7.25)$$

If (7.24) holds, then (7.22) implies

$$\|\nabla f(\theta_n)\| \geq (|u(\theta_n)|/\hat{M})^{1/\hat{\mu}} \geq (\hat{C}_2/\hat{M})^{1/\hat{\mu}}\gamma_n^{-\hat{p}/\hat{\mu}}\phi_\varepsilon(w) \geq \tilde{C}\gamma_n^{-\hat{p}/\hat{\mu}}\phi_\varepsilon(w)$$

(notice that $(\hat{C}_2/\hat{M})^{1/\hat{\mu}} \geq (\hat{C}_2/\hat{M})^{1/2} = \tilde{C}$ owing to $\hat{\mu} \leq 2$). On the other side, if (7.25) is satisfied, then (7.20) yields

$$\|\nabla f(\theta_n)\| \geq \hat{C}_2^{1/2}\gamma_n^{-\hat{p}/2}(\phi_\varepsilon(w))^{\hat{\mu}/2} \geq \tilde{C}\gamma_n^{-\hat{p}/\hat{\mu}}\phi_\varepsilon(w).$$

Thus, as a result of one of (7.24), (7.25), we get

$$\|\nabla f(\theta_n)\| \geq \tilde{C}\gamma_n^{-\hat{p}/\hat{\mu}}\phi_\varepsilon(w).$$

Consequently,

$$\begin{aligned}\hat{t}\|\nabla f(\theta_n)\|^2/8 &\geq (\tilde{C}\hat{t}/8)\gamma_n^{-\hat{p}/\hat{\mu}}\|\nabla f(\theta_n)\|\phi_\varepsilon(w) \geq \gamma_n^{-\hat{p}/\hat{\mu}}\|\nabla f(\theta_n)\|\phi_\varepsilon(w), \\ \hat{t}\|\nabla f(\theta_n)\|^2/8 &\geq (\tilde{C}^2\hat{t}/8)\gamma_n^{-2\hat{p}/\hat{\mu}}(\phi_\varepsilon(w))^2 \geq \hat{C}_1\gamma_n^{-2\hat{p}/\hat{\mu}}(\phi_\varepsilon(w))^2\end{aligned}$$

(notice that $\tilde{C}\hat{t}/8 = \hat{C}_1^{1/2} \geq 1$, $\tilde{C}^2\hat{t}/8 = 8\hat{C}_1/\hat{t} > \hat{C}_1$). Combining this with (7.21), we get

$$u(\theta_{a(n,\hat{t})}) - u(\theta_n) \leq -\hat{t}\|\nabla f(\theta_n)\|^2/4, \quad (7.26)$$

which directly contradicts (7.23). Hence, (7.13) is true for $n > \tau_\varepsilon$. Then, as a result of (7.22) and the fact that $B_{n,\varepsilon} \subseteq A_{n,\varepsilon}$ for $n \geq 0$, we get

$$\begin{aligned}&\left(u(\theta_{a(n,\hat{t})}) - u(\theta_n) + (\hat{t}/\hat{C}_3)u(\theta_n)\right)I_{B_{n,\varepsilon}} \\ &\leq \left(u(\theta_{a(n,\hat{t})}) - u(\theta_n) + (\hat{M}\hat{t}/\hat{C}_3)\|\nabla f(\theta_n)\|^2\right)I_{B_{n,\varepsilon}} \\ &\leq \left(u(\theta_{a(n,\hat{t})}) - u(\theta_n) + \hat{t}\|\nabla f(\theta_n)\|^2/4\right)I_{B_{n,\varepsilon}} \leq 0\end{aligned}$$

for $n > \tau_\varepsilon$ (notice that $u(\theta_n) > 0$ on $B_{n,\varepsilon}$ for each $n \geq 0$; also notice that $\hat{C}_3 \geq 4\hat{M}$). Thus, (7.14) is true for $n > \tau_\varepsilon$.

Now, let us prove (7.15). To do so, we again use contradiction: Suppose that (7.14) does not hold for some $n > \tau_\varepsilon$. Consequently, we have $\hat{\mu} < 2$, $u(\theta_{a(n,\hat{t})}) > 0$ and

$$\gamma_n^{\hat{p}}u(\theta_n) \geq \hat{C}_2(\phi_\varepsilon(w))^{\hat{\mu}} > 0, \quad (7.27)$$

$$v(\theta_{a(n,\hat{t})}) - v(\theta_n) < (\hat{t}/\hat{C}_3)(\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}}. \quad (7.28)$$

Combining (7.27) with (already proved) (7.13), we get (7.26), while $\hat{\mu} < 2$ implies

$$2/\hat{\mu} = 1 + 1/(\hat{\mu}\hat{r}) \leq 1 + 1/\hat{p} \quad (7.29)$$

(notice that $\hat{r} = 1/(2 - \hat{\mu})$ owing to $\hat{\mu} < 2$; also notice that $\hat{p} = \hat{\mu} \min\{r, \hat{r}\} \leq \hat{\mu}\hat{r}$). As $0 < u(\theta_n) \leq \hat{\delta} \leq 1$ (due to (7.27) and the definition of τ_ε), inequalities (7.22), (7.29) yield

$$\|\nabla f(\theta_n)\|^2 \geq \left(u(\theta_n)/\hat{M}\right)^{2/\hat{\mu}} \geq (u(\theta_n))^{1+1/\hat{p}}/\hat{M}^2 \quad (7.30)$$

(notice that $\hat{M}^{2/\hat{\mu}} \leq \hat{M}^2$ due to $\hat{\mu} < 2$, $\hat{M} \geq 1$). Since $\|\nabla f(\theta_n)\| > 0$ and $0 < u(\theta_{a(n,\hat{t})}) < u(\theta_n)$ (due to (7.22), (7.26), (7.27)), inequalities (7.26), (7.30) give

$$\begin{aligned}\frac{\hat{t}}{4} &\leq \frac{u(\theta_n) - u(\theta_{a(n,\hat{t})})}{\|\nabla f(\theta_n)\|^2} \leq \hat{M}^2 \frac{u(\theta_n) - u(\theta_{a(n,\hat{t})})}{(u(\theta_n))^{1+\hat{p}}} \\ &= \hat{M}^2 \int_{u(\theta_{a(n,\hat{t})})}^{u(\theta_n)} \frac{du}{(u(\theta_n))^{1+\hat{p}}} \\ &\leq \hat{M}^2 \int_{u(\theta_{a(n,\hat{t})})}^{u(\theta_n)} \frac{du}{u^{1+\hat{p}}} \\ &= \hat{p}\hat{M}^2 \left(v(\theta_{a(n,\hat{t})}) - v(\theta_n)\right).\end{aligned}$$

Therefore,

$$v(\theta_{a(n,\hat{t})}) - v(\theta_n) \geq \hat{t}/(4\hat{p}\hat{M}^2) \geq (\hat{t}/\hat{C}_3)$$

(notice that $\hat{p} \leq r$, $\hat{C}_3 \geq 4r\hat{M}^2$), which directly contradicts (7.28). Thus, (7.15) is satisfied for $n > \tau_\varepsilon$. \square

LEMMA 7.6. *Suppose that Assumptions 2.1 – 2.3 hold. Moreover, let $\varepsilon \in (0, \infty)$ be an arbitrary positive real number. Then,*

$$u(\theta_n) \geq -\hat{C}_2\gamma_n^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \quad (7.31)$$

on $\Lambda \setminus N_0$ for $n > \tau_\varepsilon$. Furthermore, there exists a random quantity $\hat{C}_4 \in [1, \infty)$ (which is a deterministic function of r, \hat{C}, \hat{M}) such that $1 \leq \hat{C}_4 < \infty$ everywhere and such that

$$\|\nabla f(\theta_n)\|^2 \leq \hat{C}_4 (\varphi(u(\theta_n)) + \gamma_n^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}}) \quad (7.32)$$

on $\Lambda \setminus N_0$ for $n > \tau_\varepsilon$, where function $\varphi(\cdot)$ is defined by $\varphi(x) = x \mathbf{I}_{(0,\infty)}(x)$, $x \in \mathbb{R}$.

Proof. Let $\hat{C}_4 = 4\hat{C}_2/\hat{t}$, while ω is an arbitrary sample from $\Lambda \setminus N_0$ (notice that all formulas which follow in the proof correspond to this ω).

First, we prove (7.31). To do so, we use contradiction: Assume that (7.31) is not satisfied for some $n > \tau_\varepsilon$. Define $\{n_k\}_{k \geq 0}$ recursively by $n_0 = n$ and $n_k = a(n_{k-1}, \hat{t})$ for $k \geq 1$. Let us show by induction that $\{u(\theta_{n_k})\}_{k \geq 0}$ is non-increasing: Suppose that $u(\theta_{n_l}) \leq u(\theta_{n_{l-1}})$ for $0 \leq l \leq k$. Consequently,

$$u(\theta_{n_k}) \leq u(\theta_{n_0}) \leq -\hat{C}_2\gamma_{n_0}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \leq -\hat{C}_2\gamma_{n_k}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}}$$

(notice that $\{\gamma_n\}_{n \geq 0}$ is increasing). Then, Lemma 7.5 (relations (7.13), (7.16)) yields

$$u(\theta_{n_{k+1}}) - u(\theta_{n_k}) \leq -\hat{t}\|\nabla f(\theta_{n_k})\|^2/4 \leq 0,$$

i.e., $u(\theta_{n_{k+1}}) \leq u(\theta_{n_k})$. Thus, $\{u(\theta_{n_k})\}_{k \geq 0}$ is non-increasing. Therefore,

$$\limsup_{n \rightarrow \infty} u(\theta_n) \leq u(\theta_{n_0}) < 0.$$

However, this is not possible, as $\lim_{n \rightarrow \infty} u(\theta_n) = 0$ (due to Lemma 7.4). Hence, (7.31) indeed holds for $n > \tau_\varepsilon$.

Now, (7.32) is demonstrated. Again, we proceed by contradiction: Suppose that (7.32) is violated for some $n > \tau_\varepsilon$. Consequently,

$$\|\nabla f(\theta_n)\|^2 \geq \hat{C}_4\gamma_n^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \geq \hat{C}_2\gamma_n^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}}$$

(notice that $\hat{C}_4 \geq \hat{C}_2$), which, together with Lemma 7.5 (relations (7.13), (7.16)), yields

$$u(\theta_{a(n,\hat{t})}) - u(\theta_n) \leq -\hat{t}\|\nabla f(\theta_n)\|^2/4.$$

Then, (7.31) implies

$$\begin{aligned} \|\nabla f(\theta_n)\|^2 &\leq (4/\hat{t}) \left(u(\theta_n) - u(\theta_{a(n,\hat{t})}) \right) \\ &\leq (4/\hat{t}) \left(\varphi(u(\theta_n)) + \hat{C}_2\gamma_{a(n,\hat{t})}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \right) \\ &\leq \hat{C}_4 (\varphi(u(\theta_n)) + \gamma_n^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}}). \end{aligned}$$

However, this directly contradicts our assumption that n violates (7.32). Thus, (7.32) is indeed satisfied for $n > \tau_\varepsilon$. \square

LEMMA 7.7. *Suppose that Assumptions 2.1 – 2.3 hold. Then, there exists a random quantity \hat{C}_5 (which is a deterministic function of r, \hat{C}, \hat{M}) such that $1 \leq \hat{C}_5 < \infty$ everywhere and such that*

$$\liminf_{n \rightarrow \infty} \gamma_n^{\hat{p}} u(\theta_n) \leq \hat{C}_5 (\phi(w))^{\hat{\mu}} \quad (7.33)$$

on $\Lambda \setminus N_0$.

Proof. Let $\hat{C}_5 = \hat{C}_2 + \hat{C}_3^{2r}$. We prove (7.33) by contradiction: Assume that (7.33) is violated for some sample ω from $\Lambda \setminus N_0$ (notice that the formulas which follow in the proof correspond to this ω). Consequently, there exist $\varepsilon \in (0, \infty)$ and $n_0 > \tau_\varepsilon$ such that

$$u(\theta_n) \geq \hat{C}_5 \gamma_n^{-\hat{p}} (\phi_\varepsilon(w))^{\hat{\mu}} \quad (7.34)$$

for $n \geq n_0$. Let $\{n_k\}_{k \geq 0}$ be defined recursively by $n_k = a(n_{k-1}, \hat{t})$ for $k \geq 1$. In what follows in the proof, we consider separately the cases $\hat{\mu} < 2$ and $\hat{\mu} = 2$.

Case $\hat{\mu} < 2$: Due to (7.34), we have

$$v(\theta_{n_k}) \leq \hat{C}_5^{-1/\hat{p}} \gamma_{n_k} (\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}} \leq \hat{C}_5^{-1/(2r)} \gamma_{n_k} (\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}}$$

(notice that $\hat{p} \leq 2r$). On the other side, Lemma 7.5 (relations (7.15), (7.18)) and (7.34) yield

$$v(\theta_{n_{k+1}}) - v(\theta_{n_k}) \geq (\hat{t}/\hat{C}_3) (\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}} \geq (1/\hat{C}_3) (\gamma_{n_{k+1}} - \gamma_{n_k}) (\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}}$$

for $k \geq 0$ (notice that $\hat{C}_5 \geq \hat{C}_2$; also notice that $\hat{t} \geq \gamma_{n_{k+1}} - \gamma_{n_k}$). Therefore,

$$\begin{aligned} (1/\hat{C}_3) (\gamma_{n_k} - \gamma_{n_0}) (\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}} &\leq \sum_{i=0}^{k-1} (v(\theta_{n_{i+1}}) - v(\theta_{n_i})) \\ &= v(\theta_{n_k}) - v(\theta_{n_0}) \\ &\leq \hat{C}_5^{-1/(2r)} \gamma_{n_k} (\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}} \end{aligned}$$

for $k \geq 1$. Thus,

$$(1 - \gamma_{n_0}/\gamma_{n_k}) \leq \hat{C}_3 \hat{C}_5^{-1/(2r)}$$

for $k \geq 1$. However, this is impossible, since the limit process $k \rightarrow \infty$ (applied to the previous relation) yields $\hat{C}_3 \geq \hat{C}_5^{1/(2r)}$ (notice that $\hat{C}_5 > \hat{C}_3^{2r}$). Hence, (7.33) holds on $\Lambda \setminus N_0$ when $\hat{\mu} < 2$.

Case $\hat{\mu} = 2$: As a result of Lemma 7.5 (relations (7.14), (7.17)) and (7.34), we get

$$u(\theta_{n_{k+1}}) \leq (1 - \hat{t}/\hat{C}_3) u(\theta_{n_k}) \leq \left(1 - (\gamma_{n_{k+1}} - \gamma_{n_k})/\hat{C}_3\right) u(\theta_{n_k})$$

for $k \geq 0$. Consequently,

$$\begin{aligned} u(\theta_{n_k}) &\leq u(\theta_{n_0}) \prod_{i=1}^k \left(1 - (\gamma_{n_i} - \gamma_{n_{i-1}})/\hat{C}_3\right) \\ &\leq u(\theta_{n_0}) \exp \left(- (1/\hat{C}_3) \sum_{i=1}^k (\gamma_{n_i} - \gamma_{n_{i-1}}) \right) \\ &= u(\theta_{n_0}) \exp \left(- (\gamma_{n_k} - \gamma_{n_0})/\hat{C}_3 \right) \end{aligned}$$

for $k \geq 0$. Then, (7.34) yields

$$\hat{C}_5(\phi_\varepsilon(w))^{\hat{\mu}} \leq u(\theta_{n_0})\gamma_{n_k}^{\hat{p}} \exp\left(-(\gamma_{n_k} - \gamma_{n_0})/\hat{C}_3\right)$$

for $k \geq 0$. However, this is not possible, as the limit process $k \rightarrow \infty$ (applied to the previous relation) implies $\hat{C}_5(\phi_\varepsilon(w))^{\hat{\mu}} \leq 0$. Thus, (7.33) holds on $\Lambda \setminus N_0$ also when $\hat{\mu} = 2$. \square

LEMMA 7.8. *Suppose that Assumptions 2.1 – 2.3 hold. Then, there exists a random quantity \hat{C}_6 (which is a deterministic function of r, \hat{C}, \hat{M}) such that $1 \leq \hat{C}_6 < \infty$ everywhere and such that*

$$\limsup_{n \rightarrow \infty} \gamma_n^{\hat{p}} u(\theta_n) \leq \hat{C}_6(\phi(w))^{\hat{\mu}} \quad (7.35)$$

on $\Lambda \setminus N_0$.

Proof. Let $\tilde{C}_1 = \hat{C}_1 + \hat{C}_4 + \hat{C}_5$, $\tilde{C}_2 = 6\tilde{C}_1\hat{C}_2 + \hat{C}_3^{2r}$ and $\hat{C}_6 = 2(\tilde{C}_1 + \tilde{C}_2)^2$. We use contradiction to show (7.35): Suppose that (7.35) is violated for some sample ω from $\Lambda \setminus N_0$ (notice that the formulas which appear in the proof correspond to this ω). Then, it can be deduced from Lemma 7.7 that there exist $\varepsilon \in (0, \infty)$ and $n_0 > m_0 > \tau_\varepsilon$ such that

$$\gamma_{m_0}^{\hat{p}} u(\theta_{m_0}) \leq \tilde{C}_2(\phi_\varepsilon(w))^{\hat{\mu}}, \quad (7.36)$$

$$\gamma_{n_0}^{\hat{p}} u(\theta_{n_0}) \geq \hat{C}_6(\phi_\varepsilon(w))^{\hat{\mu}}, \quad (7.37)$$

$$\min_{m_0 < n \leq n_0} \gamma_n^{\hat{p}} u(\theta_n) > \tilde{C}_2(\phi_\varepsilon(w))^{\hat{\mu}}, \quad (7.38)$$

$$\max_{m_0 \leq n < n_0} \gamma_n^{\hat{p}} u(\theta_n) < \hat{C}_6(\phi_\varepsilon(w))^{\hat{\mu}} \quad (7.39)$$

(notice that $\tilde{C}_2 > \tilde{C}_1 > \hat{C}_5$) and such that

$$(\gamma_{a(m_0, \hat{t})}/\gamma_{m_0})^{\hat{p}} \leq \min\{2, (1 - \hat{t}/\hat{C}_3)^{-1}\}, \quad (7.40)$$

$$\gamma_{m_0}^{-2\hat{p}/\hat{\mu}}(\phi_\varepsilon(w))^2 \leq \gamma_{m_0}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \quad (7.41)$$

(to see that (7.40) holds for all, but finitely many m_0 , notice that $\lim_{n \rightarrow \infty} \gamma_{a(n, \hat{t})}/\gamma_n = 1$; to conclude that (7.41) is true for all, but finitely many m_0 , notice that $2\hat{p}/\hat{\mu} > \hat{p}$ if $\hat{\mu} < 2$ and that the left and right-hand sides of (7.41) are equal when $\hat{\mu} = 2$).

Let $l_0 = a(m_0, \hat{t})$. As a direct consequence of Lemmas 7.2, 7.6 (relations (7.3), (7.32)) and (7.41), we get

$$\begin{aligned} u(\theta_n) - u(\theta_{m_0}) &\leq \gamma_{m_0}^{-\hat{p}/\hat{\mu}} \|\nabla f(\theta_{m_0})\| \phi_\varepsilon(w) + \hat{C}_1 \gamma_{m_0}^{-2\hat{p}/\hat{\mu}} (\phi_\varepsilon(w))^2 \\ &\leq \|\nabla f(\theta_{m_0})\|^2/2 + (\hat{C}_1 + 1/2) \gamma_{m_0}^{-2\hat{p}/\hat{\mu}} (\phi_\varepsilon(w))^2 \\ &\leq \hat{C}_4 \varphi(u(\theta_{m_0})) + (\hat{C}_1 + \hat{C}_4 + 1) \gamma_{m_0}^{-\hat{p}} (\phi_\varepsilon(w))^{\hat{\mu}} \\ &\leq \tilde{C}_1 (\varphi(u(\theta_{m_0})) + \gamma_{m_0}^{-\hat{p}} (\phi_\varepsilon(w))^{\hat{\mu}}) \end{aligned} \quad (7.42)$$

for $m_0 \leq n \leq l_0$ (notice that $\hat{C}_1 + \hat{C}_4 + 1 < \tilde{C}_1$). Then, (7.38), (7.40), (7.42) yield

$$\begin{aligned} u(\theta_{m_0}) + \tilde{C}_1 \varphi(u(\theta_{m_0})) &\geq u(\theta_{m_0+1}) - \tilde{C}_1 \gamma_{m_0}^{-\hat{p}} (\phi_\varepsilon(w))^{\hat{\mu}} \\ &\geq (\tilde{C}_2 \gamma_{m_0+1}^{-\hat{p}} - \tilde{C}_1 \gamma_{m_0}^{-\hat{p}}) (\phi_\varepsilon(w))^{\hat{\mu}} \\ &\geq \left(\tilde{C}_2 (\gamma_{m_0+1}/\gamma_{m_0})^{-\hat{p}} - \tilde{C}_1 \right) \gamma_{m_0}^{-\hat{p}} (\phi_\varepsilon(w))^{\hat{\mu}} \\ &\geq (\tilde{C}_2/2 - \tilde{C}_1) \gamma_{m_0}^{-\hat{p}} (\phi_\varepsilon(w))^{\hat{\mu}} > 0 \end{aligned} \quad (7.43)$$

(notice that $(\gamma_{m_0+1}/\gamma_{m_0})^{\hat{p}} \leq (\gamma_{l_0}/\gamma_{m_0})^{\hat{p}} \leq 2$; also notice that $\tilde{C}_2/2 \geq 3\tilde{C}_1$), while (7.36), (7.40), (7.42) imply

$$\begin{aligned} u(\theta_n) &\leq (1 + \tilde{C}_1)u(\theta_{m_0}) + \tilde{C}_1\gamma_{m_0}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \\ &\leq (\tilde{C}_1 + \tilde{C}_2 + \tilde{C}_1\tilde{C}_2)\gamma_{m_0}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \\ &< (\hat{C}_6/2)(\gamma_n/\gamma_{m_0})^{\hat{p}}\gamma_n^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \\ &\leq \hat{C}_6\gamma_n^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \end{aligned} \quad (7.44)$$

for $m_0 \leq n \leq l_0$ (notice that $(\gamma_n/\gamma_{m_0})^{\hat{p}} \leq (\gamma_{l_0}/\gamma_{m_0})^{\hat{p}} \leq 2$ for $m_0 \leq n \leq l_0$; also notice that $\hat{C}_6/2 = (\tilde{C}_1 + \tilde{C}_2)^2 > \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_1\tilde{C}_2$). Due to (7.37), (7.39), (7.44), we have $l_0 < n_0$. On the other side, as $x + \tilde{C}_1\varphi(x) \geq 0$ only if $x \geq 0$ and $x + \tilde{C}_1\varphi(x) = (1 + \tilde{C}_1)x$ for $x \geq 0$, inequality (7.43) implies

$$u(\theta_{m_0}) \geq (1 + \tilde{C}_1)^{-1}(\tilde{C}_2/2 - \tilde{C}_1)\gamma_{m_0}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \geq \hat{C}_2\gamma_{m_0}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \quad (7.45)$$

(notice that $\tilde{C}_2/2 - \tilde{C}_1 \geq \tilde{C}_1(3\tilde{C}_2 - 1) \geq 2\tilde{C}_1\tilde{C}_2 \geq (1 + \tilde{C}_1)\hat{C}_2$).

In what follows in the proof, we consider separately the cases $\hat{\mu} < 2$ and $\hat{\mu} = 2$.

Case $\hat{\mu} < 2$: Owing to Lemma 7.5 (relations (7.15), (7.18)) and (7.36), (7.45), we have

$$\begin{aligned} v(\theta_{l_0}) &\geq v(\theta_{m_0}) + (\hat{t}/\hat{C}_3)(\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}} \\ &\geq \left(\tilde{C}_2^{-1/\hat{p}}\gamma_{m_0} + \hat{C}_3^{-1}(\gamma_{l_0} - \gamma_{m_0}) \right) (\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}} \\ &> \min\{\hat{C}_2^{-1/\hat{p}}, \hat{C}_3^{-1}\}\gamma_{l_0}(\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}} \\ &= \tilde{C}_2^{-1/\hat{p}}\gamma_{l_0}(\phi_\varepsilon(w))^{-\hat{\mu}/\hat{p}} \end{aligned}$$

(notice that $\hat{t} \geq \gamma_{l_0} - \gamma_{m_0}$; also notice $\tilde{C}_2^{-1/\hat{p}} \leq \tilde{C}_2^{-1/(2r)} < \hat{C}_3^{-1}$). Consequently,

$$u(\theta_{l_0}) = (v(\theta_{l_0}))^{-\hat{p}} < \tilde{C}_2\gamma_{l_0}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}}.$$

However, this directly contradicts (7.38) and the fact that $l_0 < n_0$. Thus, (7.35) holds when $\hat{\mu} < 2$.

Case $\hat{\mu} = 2$: Using Lemma 7.5 (relations (7.14), (7.17)) and (7.45), we get

$$u(\theta_{l_0}) \leq \left(1 - \hat{t}/\hat{C}_3\right) u(\theta_{m_0}).$$

Then, (7.36), (7.40) yield

$$u(\theta_{l_0}) \leq \tilde{C}_2(1 - \hat{t}/\hat{C}_3)(\gamma_{l_0}/\gamma_{m_0})^{\hat{p}}\gamma_{l_0}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}} \leq \tilde{C}_2\gamma_{l_0}^{-\hat{p}}(\phi_\varepsilon(w))^{\hat{\mu}}.$$

However, this is impossible due to (7.38) and the fact that $l_0 < n_0$. Hence, (7.35) also in the case $\hat{\mu} = 2$. \square

PROOF OF THEOREMS 2.1 AND 2.2. Theorem 2.1 is an immediate consequence of Lemmas 7.2, 7.3. To show Theorem 2.2, we use the following notations: $\hat{K} = (\hat{C}_2 + \hat{C}_4 + \hat{C}_6)^2$, $\hat{L} = \hat{K}\hat{N}$. Then, Lemmas 7.5 and 7.7 imply

$$\limsup_{n \rightarrow \infty} \gamma_n^{\hat{p}} |u(\theta_n)| \leq (\hat{C}_2 + \hat{C}_6)(\phi(w))^{\hat{\mu}} \quad (7.46)$$

on $\Lambda \setminus N_0$. On the other side, Lemma 7.5 and (7.46) yield

$$\begin{aligned} \limsup_{n \rightarrow \infty} \gamma_n^{\hat{p}} \|\nabla f(\theta_n)\|^2 &\leq \hat{C}_4 (\phi(w))^{\hat{\mu}} + \hat{C}_4 \limsup_{n \rightarrow \infty} \gamma_n^{\hat{p}} \varphi(u(\theta_n)) \\ &\leq (\hat{C}_2 + \hat{C}_4 + \hat{C}_6)^2 (\phi(w))^{\hat{\mu}} \end{aligned} \quad (7.47)$$

on $\Lambda \setminus N_0$. Combining (7.46), (7.47) with Assumption 2.4, we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \gamma_n^{\hat{q}} d(\theta_n, S) &\leq \hat{N} \limsup_{n \rightarrow \infty} (\gamma_n^{\hat{p}} \|\nabla f(\theta_n)\|^2)^{\nu_{\hat{Q}}/2} \\ &\leq \hat{N} (\hat{C}_2 + \hat{C}_4 + \hat{C}_6)^2 (\phi(w))^{\hat{\nu}} \end{aligned} \quad (7.48)$$

on $\Lambda \setminus N_0$. As a direct consequence of (7.46) – (7.48), we have that (2.8) – (2.10) are satisfied on $\Lambda \setminus N_0$. Hence, Theorem 2.2 holds, too. \square

8. Proof of Theorem 3.1. The following notation is used in this section. For $\theta \in \mathbb{R}^{d_\theta}$, $\xi \in \mathbb{R}^{d_\xi}$, $E_{\theta, \xi}(\cdot)$ denotes $E(\cdot | \theta_0 = \theta, \xi_0 = \xi)$. Moreover, let

$$\begin{aligned} w_n &= F(\theta_n, \xi_{n+1}) - \nabla f(\theta_n), \\ w_{1,n} &= \tilde{F}(\theta_n, \xi_{n+1}) - (\Pi \tilde{F})(\theta_n, \xi_n), \\ w_{2,n} &= (\Pi \tilde{F})(\theta_n, \xi_n) - (\Pi \tilde{F})(\theta_{n-1}, \xi_n), \\ w_{3,n} &= -(\Pi \tilde{F})(\theta_n, \xi_{n+1}) \end{aligned}$$

for $n \geq 1$. Then, it is obvious that algorithm (3.1) admits the form (2.1), while Assumption 3.2 yields

$$\begin{aligned} \sum_{i=n}^k \alpha_i \gamma_i^r w_i &= \sum_{i=n}^k \alpha_i \gamma_i^r w_{1,i} + \sum_{i=n}^k \alpha_i \gamma_i^r w_{2,i} - \sum_{i=n}^k (\alpha_i \gamma_i^r - \alpha_{i+1} \gamma_{i+1}^r) w_{3,i} \\ &\quad - \alpha_{k+1} \gamma_{k+1}^r w_{3,k} + \alpha_n \gamma_n^r w_{3,n-1} \end{aligned} \quad (8.1)$$

for $1 \leq n \leq k$.

LEMMA 8.1. *Let Assumption 3.1 hold. Then, there exists a real number $s \in (0, 1)$ such that $\sum_{n=0}^{\infty} \alpha_n^{1+s} \gamma_n^r < \infty$.*

Proof. Let $p = (2 + 2r)/(2 + r)$, $q = (2 + 2r)/r$, $s = (2 + r)/(2 + 2r)$. Then, using the Hölder inequality, we get

$$\sum_{n=0}^{\infty} \alpha_n^{1+s} \gamma_n^r = \sum_{n=1}^{\infty} (\alpha_n^2 \gamma_n^{2r})^{1/p} \left(\frac{\alpha_n}{\gamma_n^2} \right)^{1/q} \leq \left(\sum_{n=1}^{\infty} \alpha_n^2 \gamma_n^{2r} \right)^{1/p} \left(\sum_{n=1}^{\infty} \frac{\alpha_n}{\gamma_n^2} \right)^{1/q}.$$

Since $\gamma_{n+1}/\gamma_n = 1 + \alpha_n/\gamma_n = O(1)$ for $n \rightarrow \infty$ and

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{\gamma_n^2} = \sum_{n=1}^{\infty} \frac{\gamma_{n+1} - \gamma_n}{\gamma_n^2} \leq \sum_{n=1}^{\infty} \left(\frac{\gamma_{n+1}}{\gamma_n} \right)^2 \int_{\gamma_n}^{\gamma_{n+1}} \frac{dt}{t^2} \leq \frac{1}{\gamma_1} \max_{n \geq 0} \left(\frac{\gamma_{n+1}}{\gamma_n} \right)^2,$$

it is obvious that $\sum_{n=0}^{\infty} \alpha_n^{1+s} \gamma_n^r$ converges. \square

PROOF OF THEOREM 3.1. Let $Q \subset \mathbb{R}^{d_\theta}$ be an arbitrary compact set, while $s \in (0, 1)$ is a real number such that $\sum_{n=0}^{\infty} \alpha_n^{1+s} \gamma_n^r < \infty$. Obviously, it is sufficient to show that $\sum_{n=0}^{\infty} \alpha_n \gamma_n^r w_n$ converges w.p.1 on $\bigcap_{n=0}^{\infty} \{\theta_n \in Q\}$.

Due to Assumption 3.1, we have

$$\begin{aligned}\alpha_{n-1}^s \alpha_n \gamma_n^r &= (1 + \alpha_{n-1}(\alpha_n^{-1} - \alpha_{n-1}^{-1}))^s \alpha_n^{1+s} \gamma_n^r = O(\alpha_n^{1+s} \gamma_n^r), \\ (\alpha_{n-1} - \alpha_n) \gamma_n^r &= (\alpha_n^{-1} - \alpha_{n-1}^{-1}) (1 + \alpha_{n-1}(\alpha_n^{-1} - \alpha_{n-1}^{-1})) \alpha_n^2 \gamma_n^r = O(\alpha_n^2 \gamma_n^r), \\ \alpha_n (\gamma_{n+1}^r - \gamma_n^r) &= \alpha_n \gamma_n^r ((1 + \alpha_n/\gamma_n)^r - 1) = \alpha_n \gamma_n^r (r\alpha_n/\gamma_n + o(\alpha_n/\gamma_n)) = o(\alpha_n^2 \gamma_n^r)\end{aligned}$$

as $n \rightarrow \infty$. Consequently,

$$\sum_{n=0}^{\infty} \alpha_n^s \alpha_{n+1} \gamma_{n+1}^r < \infty, \quad (8.2)$$

$$\sum_{n=0}^{\infty} |\alpha_n \gamma_n^r - \alpha_{n+1} \gamma_{n+1}^r| \leq \sum_{n=0}^{\infty} \alpha_n |\gamma_n^r - \gamma_{n+1}^r| + \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| \gamma_{n+1}^r < \infty. \quad (8.3)$$

On the other side, as a result of Assumption 3.3, we get

$$\begin{aligned}E_{\theta, \xi} (\|w_{1,n}\|^2 I_{\{\tau_Q > n\}}) &\leq 2E_{\theta, \xi} (\varphi_{Q,s}^2(\xi_{n+1}) I_{\{\tau_Q > n\}}) + 2E_{\theta, \xi} (\varphi_{Q,s}^2(\xi_n) I_{\{\tau_Q > n-1\}}), \\ E_{\theta, \xi} (\|w_{2,n}\|^2 I_{\{\tau_Q > n\}}) &\leq E_{\theta, \xi} (\varphi_{Q,s}(\xi_n) \|\theta_n - \theta_{n-1}\|^s I_{\{\tau_Q > n-1\}}) \\ &\leq \alpha_{n-1}^s E_{\theta, \xi} (\varphi_{Q,s}^2(\xi_n) I_{\{\tau_Q > n-1\}}), \\ E_{\theta, \xi} (\|w_{3,n}\|^2 I_{\{\tau_Q > n\}}) &\leq E_{\theta, \xi} (\varphi_{Q,s}^2(\xi_{n+1}) I_{\{\tau_Q > n\}})\end{aligned}$$

for all $\theta \in \mathbb{R}^{d_\theta}$, $\xi \in \mathbb{R}^{d_\xi}$, $n \geq 1$. Then, Assumption 3.1 and (8.2) yield

$$\begin{aligned}E_{\theta, \xi} \left(\sum_{n=1}^{\infty} \alpha_n^2 \gamma_n^{2r} \|w_{1,n}\|^2 I_{\{\tau_Q > n\}} \right) &\leq 4 \left(\sum_{n=1}^{\infty} \alpha_n^2 \gamma_n^{2r} \right) \sup_{n \geq 0} E_{\theta, \xi} (\varphi_{Q,s}^2(\xi_n) I_{\{\tau_Q \geq n\}}) < \infty, \\ E_{\theta, \xi} \left(\sum_{n=1}^{\infty} \alpha_n \gamma_n^r \|w_{2,n}\| I_{\{\tau_Q > n\}} \right) &\leq \left(\sum_{n=1}^{\infty} \alpha_{n-1}^s \alpha_n \gamma_n^r \right) \sup_{n \geq 0} E_{\theta, \xi} (\varphi_{Q,s}^2(\xi_n) I_{\{\tau_Q \geq n\}}) < \infty\end{aligned}$$

for any $\theta \in \mathbb{R}^{d_\theta}$, $\xi \in \mathbb{R}^{d_\xi}$, while (8.3) implies

$$\begin{aligned}E_{\theta, \xi} \left(\sum_{n=1}^{\infty} |\alpha_n \gamma_n^r - \alpha_{n+1} \gamma_{n+1}^r| \|w_{3,n}\| I_{\{\tau_Q > n\}} \right) \\ \leq \left(\sum_{n=1}^{\infty} |\alpha_n \gamma_n^r - \alpha_{n+1} \gamma_{n+1}^r| \right) \sup_{n \geq 0} (E_{\theta, \xi} (\varphi_{Q,s}^2(\xi_n) I_{\{\tau_Q \geq n\}}))^{1/2} < \infty, \\ E_{\theta, \xi} \left(\sum_{n=1}^{\infty} \alpha_{n+1}^2 \gamma_{n+1}^{2r} \|w_{3,n}\|^2 I_{\{\tau_Q > n\}} \right) \\ \leq \left(\sum_{n=1}^{\infty} \alpha_{n+1}^2 \gamma_{n+1}^{2r} \right) \sup_{n \geq 0} E_{\theta, \xi} (\varphi_{Q,s}^2(\xi_n) I_{\{\tau_Q \geq n\}}) < \infty\end{aligned}$$

for each $\theta \in \mathbb{R}^{d_\theta}$, $\xi \in \mathbb{R}^{d_\xi}$. Since

$$E_{\theta, \xi} (w_{1,n} I_{\{\tau_Q > n\}} | \mathcal{F}_n) = \left(E_{\theta, \xi} (\tilde{F}(\theta_n, \xi_{n+1}) | \mathcal{F}_n) - (\Pi \tilde{F})(\theta_n, \xi_n) \right) I_{\{\tau_Q > n\}} = 0$$

w.p.1 for every $\theta \in \mathbb{R}^{d_\theta}$, $\xi \in \mathbb{R}^{d_\xi}$, $n \geq 1$, it can be deduced easily that series

$$\sum_{n=1}^{\infty} \alpha_n \gamma_n^r w_{1,n}, \quad \sum_{n=1}^{\infty} \alpha_n \gamma_n^r w_{2,n}, \quad \sum_{n=1}^{\infty} (\alpha_n \gamma_n^r - \alpha_{n+1} \gamma_{n+1}^r) w_{3,n}$$

converge w.p.1 on $\bigcap_{n=0}^{\infty} \{\theta_n \in Q\}$, as well as that $\lim_{n \rightarrow \infty} \alpha_n \gamma_n^r w_{3,n-1} = 0$ w.p.1 on the same event. Owing to this and (8.1), we have that $\sum_{n=0}^{\infty} \alpha_n \gamma_n^r w_n$ converges w.p.1 on $\bigcap_{n=0}^{\infty} \{\theta_n \in Q\}$. \square

9. Proof of Theorems 4.1 and 4.2. In this section, we use the following notation. For $\theta \in \mathbb{R}^{d_\theta}$, $x \in \mathbb{R}^{d_x}$, $y \in \mathbb{R}$, $\xi = (x, y)$, let

$$F(\theta, \xi) = (y - G_\theta(x))H_\theta(x),$$

while $\xi_n = (x_n, y_n)$ for $n \geq 0$. With this notation, it is obvious that algorithm (4.1) admits the form (3.1).

PROOF OF THEOREM 4.1. Let $\theta = [a'_1 \cdots a'_{N_1} \ a''_{1,1} \cdots a''_{N_1, N_2}]^T \in \mathbb{R}^{d_\theta}$, while

$$\delta_\theta = \frac{\varepsilon}{2KLN_1N_2(1 + \|\theta\|)}$$

and $\hat{U}_\theta = \{\eta \in \mathbb{C}^{d_\theta} : \|\eta - \theta\| < \delta_\theta\}$ (ε is specified in Assumption 4.1). Moreover, for $\eta = [b'_1 \cdots b'_{N_1} \ b''_{1,1} \cdots b''_{N_1, N_2}]^T \in \mathbb{C}^{d_\theta}$, $x \in \mathbb{R}^{d_x}$, let

$$\begin{aligned} \hat{G}_\eta(x) &= \hat{\phi}_1 \left(\sum_{i_1=1}^{N_1} b'_{i_1} \hat{\phi}_2 \left(\sum_{i_2=1}^{N_2} b''_{i_1, i_2} \phi_{i_2}(x) \right) \right), \\ \hat{f}(\eta) &= \frac{1}{2} \int (y - \hat{G}_\eta(x))^2 \pi(dx, dy). \end{aligned}$$

Then, we have

$$\left| \sum_{i_2=1}^{N_2} b''_{i_1, i_2} \psi_{i_2}(x) - \sum_{i_2=1}^{N_2} a''_{i_1, i_2} \psi_{i_2}(x) \right| \leq \sum_{i_2=1}^{N_2} |b''_{i_1, i_2} - a''_{i_1, i_2}| |\psi_{i_2}(x)| \leq \delta_\theta LN_2 < \varepsilon$$

for all $\eta = [b'_1 \cdots b'_{N_1} \ b''_{1,1} \cdots b''_{N_1, N_2}]^T \in \hat{U}_\theta$, $1 \leq i_1 \leq N_1$ and each $x \in \mathbb{R}^{d_x}$ satisfying $\max_{1 \leq k \leq N_2} |\psi_k(x)| \leq L$. Consequently, Assumption 4.1 implies

$$\begin{aligned} & \left| \sum_{i_1=1}^{N_1} b'_{i_1} \hat{\phi}_2 \left(\sum_{i_2=1}^{N_2} b''_{i_1, i_2} \psi_{i_2}(x) \right) - \sum_{i_1=1}^{N_1} a'_{i_1} \phi_2 \left(\sum_{i_2=1}^{N_2} a''_{i_1, i_2} \psi_{i_2}(x) \right) \right| \\ & \leq \sum_{i_1=1}^{N_1} |b'_{i_1} - a'_{i_1}| \left| \hat{\phi}_2 \left(\sum_{i_2=1}^{N_2} b''_{i_1, i_2} \psi_{i_2}(x) \right) \right| \\ & \quad + \sum_{i_1=1}^{N_1} |a'_{i_1}| \left| \hat{\phi}_2 \left(\sum_{i_2=1}^{N_2} b''_{i_1, i_2} \psi_{i_2}(x) \right) - \hat{\phi}_2 \left(\sum_{i_2=1}^{N_2} a''_{i_1, i_2} \psi_{i_2}(x) \right) \right| \\ & \leq \delta_\theta KN_1 + K \sum_{i_1=1}^{N_1} |a'_{i_1}| \left| \sum_{i_2=1}^{N_2} b''_{i_1, i_2} \psi_{i_2}(x) - \sum_{i_2=1}^{N_2} a''_{i_1, i_2} \psi_{i_2}(x) \right| \\ & \leq \delta_\theta KN_1 + \delta_\theta KLN_1N_2 \|\theta\| < \varepsilon \end{aligned}$$

for any $\eta = [b'_1 \cdots b'_{N_1} \ b''_{1,1} \cdots b''_{N_1, N_2}]^T \in \hat{U}_\theta$ and each $x \in \mathbb{R}^{d_x}$ satisfying $\max_{1 \leq k \leq N_2} |\psi_k(x)| \leq L$. Then, it can be deduced that for all $x \in \mathbb{R}^{d_x}$ satisfying $\max_{1 \leq k \leq N_2} |\psi_k(x)| \leq L$,

$\hat{G}_\eta(x)$ is analytical in η on \hat{U}_θ . Moreover, Assumption 4.1 yields

$$\begin{aligned} |\hat{G}_\eta(x)| &\leq K \left(1 + \sum_{i_1=1}^{N_1} |b'_{i_1}| \left| \hat{\phi}_2 \left(\sum_{i_2=1}^{N_2} b''_{i_1, i_2} \psi_{i_2}(x) \right) \right| \right) \leq K^2(1 + \|\eta\|), \\ \left| \frac{\partial}{\partial b'_{k_1}} \hat{G}_\eta(x) \right| &= \left| \hat{\phi}'_1 \left(\sum_{i_1=1}^{N_1} b'_{i_1} \hat{\phi}_2 \left(\sum_{i_2=1}^{N_2} b''_{i_1, i_2} \psi_{i_2}(x) \right) \right) \hat{\phi}_2 \left(\sum_{i_2=1}^{N_2} b''_{k_1, i_2} \psi_{i_2}(x) \right) \right| \leq K^2, \\ \left| \frac{\partial}{\partial b''_{k_1, k_2}} \hat{G}_\eta(x) \right| &= \left| \hat{\phi}'_1 \left(\sum_{i_1=1}^{N_1} b'_{i_1} \hat{\phi}_2 \left(\sum_{i_2=1}^{N_2} b''_{i_1, i_2} \psi_{i_2}(x) \right) \right) \right. \\ &\quad \cdot \left. \hat{\phi}'_2 \left(\sum_{i_2=1}^{N_2} b''_{k_1, i_2} \psi_{i_2}(x) \right) b'_{k_1} b''_{k_1, k_2} \psi_{k_2}(x) \right| \\ &\leq K^2 L \|\eta\|^2 \end{aligned}$$

for all $\eta = [b'_1 \cdots b'_{N_1} \ b''_{1,1} \cdots b''_{N_1, N_2}]^T \in \hat{U}_\theta$, $1 \leq k_1 \leq N_1$, $1 \leq k_2 \leq N_2$ and each $x \in \mathbb{R}^{d_x}$ satisfying $\max_{1 \leq k \leq N_2} |\psi_k(x)| \leq L$. Therefore,

$$\|\nabla_\eta \hat{G}_\eta(x)\| \leq K^2 L N_1 N_2 (1 + \|\eta\|)^2$$

for any $\eta \in \hat{U}_\theta$ and each $x \in \mathbb{R}^{d_x}$ satisfying $\max_{1 \leq k \leq N_2} |\psi_k(x)| \leq L$. Thus,

$$\|\nabla_\eta (y - \hat{G}_\eta(x))^2\| = 2|y - \hat{G}_\eta(x)| \|\nabla_\eta \hat{G}_\eta(x)\| \leq 4K^4 L^2 N_1 N_2 (1 + \|\eta\|)^3$$

for all $\eta \in \hat{U}_\theta$ and each $x \in \mathbb{R}^{d_x}$, $y \in \mathbb{R}$ satisfying $\max_{1 \leq k \leq N_2} |\psi_k(x)| \leq L$, $|y| \leq L$. Then, the dominated convergence theorem and Assumption 4.2 imply that $\hat{f}(\cdot)$ is differentiable on \hat{U}_θ . Consequently, $\hat{f}(\cdot)$ is analytical on \hat{U}_θ . Since $f(\theta) = \hat{f}(\theta)$ for all $\theta \in \mathbb{R}^{d_\theta}$, we conclude that $f(\cdot)$ is real-analytic on entire \mathbb{R}^{d_θ} . \square

PROOF OF THEOREM 4.2. As $\{\xi_n\}_{n \geq 0}$ can be interpreted as a Markov chain whose transition kernel does not depend on $\{\theta_n\}_{n \geq 0}$, it is straightforward to show that Assumptions 3.2 and 3.3 hold. The theorem's assertion then follows directly from Theorem 3.1. \square

10. Proof of Theorems 5.1 and 5.2. In this section, we rely on the following notation. For $n \geq 0$, let $\xi_{n+1} = (x_n, x_{n+1}, y_n)$, while

$$F(\theta, \xi) = -(c(i) + \beta G_\theta(j) - G_\theta(i))y$$

for $\theta, y \in \mathbb{R}^{d_\theta}$, $i, j \in \mathcal{X}$ and $\xi = (i, j, y)$. Moreover, let

$$\begin{aligned} \Pi_\theta((i, j, y), (i', j') \times B) &= P(\xi_1 \in (i', j') \times B | \xi_0 = (i, j, y)) \\ &= I_B(\beta y + H_\theta(j)) P(x_1 = j' | x_0 = j) I_j(i') \end{aligned}$$

for $\theta, y \in \mathbb{R}^{d_\theta}$, $B \in \mathcal{B}^{d_\theta}$, $i, i', j, j' \in \mathcal{X}$. Then, it is straightforward to verify that recursion (5.1), (5.2) admits the form of the algorithm studied in Section 3.

The following notation is also used in this section. e is an N -dimensional column vector whose all components are one. For $1 \leq i \leq N$, $e_i = [e_{i,1} \cdots e_{i,N}]^T$ is an N -dimensional column vector such that $e_{i,i} = 1$ and $e_{i,k} = 0$ for $k \neq i$. P and π denote (respectively) the transition probability matrix and the invariant column probability vector of $\{x_n\}_{n \geq 0}$ (notice that j, i entry of P is $P(x_1 = j | x_0 = i)$).

Furthermore $c = [c(1) \cdots c(N)]$ and $g = c \sum_{n=0}^{\infty} \beta^n P^n$, while $G_\theta = [G_\theta(1) \cdots G_\theta(N)]$, $\tilde{G}_\theta = c + \beta G_\theta P - G_\theta$ and $H_\theta = [H_\theta(1) \cdots H_\theta(N)]$ for $\theta \in \mathbb{R}^{d_\theta}$ (notice that $c, g, G_\theta, \tilde{G}_\theta$ are row vectors).

LEMMA 10.1. *Let Assumption 5.1 and 5.2 hold. Then, there exists a real number $\varepsilon \in (0, 1)$ and for any compact set $Q \subset \mathbb{R}^{d_\theta}$, there exists another real number $C_Q \in [1, \infty)$ such that*

$$\begin{aligned} \|(\Pi^n F)(\theta, \xi) - \nabla f(\theta)\| &\leq C_Q n \varepsilon^n (1 + \|y\|), \\ \|((\Pi^n F)(\theta', \xi) - \nabla f(\theta')) - ((\Pi^n F)(\theta'', \xi) - \nabla f(\theta''))\| &\leq C_Q n \varepsilon^n \|\theta' - \theta''\| (1 + \|y\|), \\ E(\|y_n\|^2 I_{\{\tau_\rho \geq n\}} | \theta_0 = \theta, \xi_0 = \xi) &\leq C_Q (1 + \|y\|)^2 \end{aligned} \quad (10.1)$$

for all $\theta, \theta', \theta'' \in Q$, $y \in \mathbb{R}^{d_\theta}$, $i, j \in \mathcal{X}$ and $\xi = (i, j, y)$.

Proof. Let $Q \subset \mathbb{R}^{d_\theta}$ be an arbitrary compact set, while $\varepsilon \in (0, 1)$, $\tilde{C} \in [1, \infty)$ are real numbers such that $\varepsilon \geq \max\{1/2, \beta\}$, $\|P^n\| \leq \tilde{C}$ and

$$\|P^n - \pi e^T\| \leq \tilde{C} \varepsilon^n$$

for $n \geq 0$ (the existence of ε, \tilde{C} is ensured by Assumption 5.1). Moreover, $\tilde{C}_{1,Q} \in [1, \infty)$ denotes an upper bound of $\|G_\theta\|, \|\tilde{G}_\theta\|, \|H_\theta\|$ on Q , while $\tilde{C}_{2,Q} \in [1, \infty)$ is a Lipschitz constant of $G_\theta, \tilde{G}_\theta, H_\theta$ on the same set. Furthermore, $C_Q = 6\tilde{C}^2(\tilde{C}_{1,Q} + \tilde{C}_{2,Q})^2/(1 - \varepsilon)^2$.

It is straightforward to show $\nabla f(\theta) = H_\theta \text{diag}(G_\theta - g)\pi$ and

$$\begin{aligned} (\Pi^n F)(\theta, \xi) &= -E \left((c(x_n) + \beta G_\theta(x_{n+1}) - G_\theta(x_n)) \right. \\ &\quad \cdot \left. \left(\beta^n y + \sum_{k=0}^{n-1} \beta^k H_\theta(x_{n-k}) \right) \middle| x_1 = j \right) \\ &= -\beta^n y \tilde{G}_\theta P^{n-1} e_j - \sum_{k=0}^{n-1} \beta^k H_\theta \text{diag}(\tilde{G}_\theta P^k) P^{n-k-1} e_j \\ &= \nabla f(\theta) - \beta^n y \tilde{G}_\theta P^{n-1} e_j + H_\theta \text{diag} \left(\tilde{G}_\theta \sum_{k=n}^{\infty} \beta^k P^k \right) \pi \\ &\quad - \sum_{k=0}^{n-1} \beta^k H_\theta \text{diag}(\tilde{G}_\theta P^k) (P^{n-k-1} - \pi e^T) e_j \end{aligned}$$

for $\theta, y \in \mathbb{R}^{d_\theta}$, $i, j \in \mathcal{X}$ and $\xi = (i, j, y)$. Therefore,

$$\begin{aligned} \|(\Pi^n F)(\theta, \xi) - \nabla f(\theta)\| &\leq \tilde{C} \tilde{C}_{1,Q} \beta^n \|y\| + \tilde{C} \tilde{C}_{1,Q}^2 \sum_{k=n}^{\infty} \beta^k + \tilde{C}^2 \tilde{C}_{1,Q}^2 \sum_{k=0}^{n-1} \beta^k \varepsilon^{n-k-1} \\ &\leq C_Q n \varepsilon^n (1 + \|y\|) \end{aligned}$$

for all $\theta \in Q$, $y \in \mathbb{R}^{d_\theta}$, $i, j \in \mathcal{X}$, $n \geq 0$ and $\xi = (i, j, y)$. Moreover,

$$\begin{aligned} & \|((\Pi^n F)(\theta', \xi) - \nabla f(\theta')) - ((\Pi^n F)(\theta'', \xi) - \nabla f(\theta''))\| \\ & \leq \tilde{C} \beta^n \|y\| \|\tilde{G}_{\theta'} - \tilde{G}_{\theta''}\| + \tilde{C} \tilde{C}_{1,Q} (\|\tilde{G}_{\theta'} - \tilde{G}_{\theta''}\| \\ & \quad + \|H_{\theta'} - H_{\theta''}\|) \sum_{k=n}^{\infty} \beta^k + \tilde{C}^2 \tilde{C}_{1,Q} (\|\tilde{G}_{\theta'} - \tilde{G}_{\theta''}\| + \|H_{\theta'} - H_{\theta''}\|) \sum_{k=0}^{n-1} \beta^k \varepsilon^{n-k-1} \\ & \leq C_Q n \varepsilon^n \|\theta' - \theta''\| (1 + \|y\|) \end{aligned}$$

for any $\theta', \theta'' \in Q$, $y \in \mathbb{R}^{d_\theta}$, $i, j \in \mathcal{X}$, $n \geq 0$ and $\xi = (i, j, y)$. On the other side, we have

$$\|y_{n+1}\| I_{\{\tau_Q \geq n+1\}} \leq \beta \|y_n\| I_{\{\tau_Q \geq n\}} + \tilde{C}_{1,Q}$$

for $n \geq 0$. Consequently,

$$\beta \|y_n\| I_{\{\tau_Q \geq n\}} \leq \|y_0\| + \tilde{C}_{1,Q} \sum_{k=0}^{n-1} \beta^k \leq C_Q^{1/2} (1 + \|y_0\|)$$

for $n \geq 0$, wherefrom (10.1) immediately follows. \square

PROOF OF THEOREM 5.1. Since

$$f(\theta) = \frac{1}{2} \sum_{i=1}^{N_x} (g(i) - G_\theta(i))^2 \pi(i)$$

for each $\theta \in \mathbb{R}^{d_\theta}$ ($\pi(i)$ is the i -th component of π), Assumption 5.2 implies that $f(\cdot)$ is analytic on entire \mathbb{R}^{d_θ} . \square

PROOF OF THEOREM 5.2. Using Lemma 10.1, it can be concluded easily that Assumption 3.2 and 3.3 hold. Then, the theorem's assertion directly follows from Theorem 3.1. \square

11. Proof of Theorems 6.1 and 6.2. In this section, we use the following notation. For $n \geq 0$, let

$$z_n = [x_n^T \ y_n \ \cdots \ y_{n-M+1}]^T, \quad \xi_n = [z_n^T \ \varepsilon_n \ \psi_n^T \ \cdots \ \varepsilon_{n-N+1} \ \psi_{n-N+1}^T]^T,$$

while $d_\xi = L + M + N(d_\theta + 1)$. For $\theta \in \Theta$, let $\varepsilon_0^\theta = \cdots = \varepsilon_{-N+1}^\theta = 0$, $\psi_0^\theta = \cdots = \psi_{-N+1}^\theta = 0$, while $\{\varepsilon_n^\theta\}_{n \geq 0}$, $\{\psi_n^\theta\}_{n \geq 0}$ are defined by the following recursion:

$$\begin{aligned} \phi_{n-1}^\theta &= [y_{n-1} \ \cdots \ y_{n-M} \ \varepsilon_{n-1}^\theta \ \cdots \ \varepsilon_{n-N}^\theta]^T, \\ \varepsilon_n^\theta &= y_n - (\phi_{n-1}^\theta)^T \theta, \\ \psi_n^\theta &= \phi_{n-1}^\theta - [\psi_{n-1}^\theta \ \cdots \ \psi_{n-N}^\theta] A_0 \theta, \\ \xi_n^\theta &= [z_n^T \ \varepsilon_n^\theta \ (\psi_n^\theta)^T \ \cdots \ \varepsilon_{n-N+1}^\theta \ (\psi_{n-N+1}^\theta)^T]^T, \quad n \geq 1. \end{aligned}$$

Then, it is straightforward to verify that $\{\varepsilon_n^\theta\}_{n \geq 0}$ satisfies the recursion (6.2), as well as that $\psi_n^\theta = \nabla_\theta \varepsilon_n^\theta$ for $n \geq 0$. Moreover, it can be deduced easily that there exist a matrix valued function $G_\theta : \Theta \rightarrow \mathbb{R}^{d_\xi \times d_\xi}$ and a matrix $H \in \mathbb{R}^{d_\xi \times L}$ with the following properties:

(i) G_θ is linear in θ and its eigenvalues lie outside $\{z \in \mathbb{C} : |z| \leq 1\}$ for each $\theta \in \Theta$.

(ii) Equations

$$\xi_{n+1}^\theta = G_\theta \xi_n^\theta + H w_n, \quad \xi_{n+1} = G_{\theta_n} \xi_n + H w_n$$

hold for all $\theta \in \Theta$, $n \geq 0$.

The following notation is also used in this section. For $\theta \in \Theta$, $z \in \mathbb{R}^{L+M}$, $u_1, \dots, u_N \in \mathbb{R}$, $v_1, \dots, v_N \in \mathbb{R}^{d_\theta}$ and $\xi = [z^T u_1 v_1^T \cdots u_N v_N^T]^T$, let

$$F(\theta, \xi) = v_1 u_1, \quad \phi(\xi) = u_1^2,$$

while

$$\Pi_\theta(\xi, B) = E(I_B(G_\theta \xi + H w_0))$$

for a Borel-measurable set B from \mathbb{R}^{d_ξ} . Then, it can be deduced easily that recursion (6.3) – (6.6) admits the form of the algorithm considered in Section 3. Furthermore, it can be shown that

$$(\Pi^n \phi)(\theta, 0) = E((\varepsilon_n^\theta)^2), \quad (11.1)$$

$$(\Pi^n F)(\theta, 0) = E(\psi_n^\theta \varepsilon_n^\theta) = \nabla_\theta (\Pi^n \phi)(\theta, 0) \quad (11.2)$$

for each $\theta \in \Theta$, $n \geq 0$.

PROOF OF THEOREM 6.1. Let $m = E(y_0)$ and $r_k = r_{-k} = \text{Cov}(y_0, y_k)$ for $k \geq 0$, while

$$\varphi(\omega) = \sum_{k=-\infty}^{\infty} r_k e^{-i\omega k}$$

for $\omega \in [-\pi, \pi]$. Moreover, for $\theta \in \Theta$, $z \in \mathbb{C}$, let $C_\theta(z) = A_\theta(z)/B_\theta(z)$, while

$$\alpha_\theta = 1 + \max_{\omega \in [-\pi, \pi]} |A_\theta(e^{i\omega})|, \quad \beta_\theta = \min_{\omega \in [-\pi, \pi]} |B_\theta(e^{i\omega})|, \quad \delta_\theta = \frac{\beta_\theta}{4d_\theta \alpha_\theta}.$$

Obviously, $1 \leq \alpha_\theta < \infty$, $0 < \beta_\theta, \delta_\theta < \infty$ (notice that the zeros of $B_\theta(\cdot)$ are outside $\{z \in \mathbb{C} : |z| \leq 1\}$).

As $\sum_{k=0}^{\infty} r_k < \infty$, $|\varphi(\cdot)|$ is uniformly bounded. Consequently, the spectral theory for stationary processes (see e.g. [8, Chapter 2]) yields

$$\begin{aligned} \lim_{n \rightarrow \infty} E(\varepsilon_n^\theta) &= C_\theta(1)m, \\ \lim_{n \rightarrow \infty} \text{Cov}(\varepsilon_n^\theta, \varepsilon_{n+k}^\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |C_\theta(e^{i\omega})|^2 \varphi(\omega) e^{i\omega k} d\omega \end{aligned}$$

for all $\theta \in \Theta$, $k \geq 0$ (notice that $\varepsilon_n^\theta = C_\theta(q)y_n$ and the poles of $C_\theta(\cdot)$ are in $\{z \in \mathbb{C} : |z| > 1\}$). Therefore,

$$f(\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} |C_\theta(e^{i\omega})|^2 \varphi(\omega) d\omega + |C_\theta(1)|^2 \frac{m^2}{2} \quad (11.3)$$

for any $\theta \in \Theta$. On the other side, it is straightforward to verify

$$\begin{aligned} \frac{\partial}{\partial a_k} A_\theta(e^{i\omega}) &= -e^{-i\omega k}, \\ \frac{\partial^2}{\partial a_{k_1} \partial a_{k_2}} A_\theta(e^{i\omega}) &= 0, \\ \frac{\partial^{l_1+\dots+l_N}}{\partial b_1^{l_1} \dots \partial b_N^{l_N}} \left(\frac{1}{B_\theta(e^{i\omega})} \right) &= -(l_1 + l_2 + \dots + l_N)! e^{-i\omega(l_1+2l_2+\dots+Nl_N)} \\ &\quad \cdot \left(-\frac{1}{B_\theta(e^{i\omega})} \right)^{l_1+l_2+\dots+l_N+1} \end{aligned}$$

for every $\theta = [a_1 \dots a_M \ b_1 \dots b_N]^T \in \Theta$, $\omega \in [-\pi, \pi]$, $1 \leq k, k_1, k_2 \leq M$, $l_1, \dots, l_N \geq 0$. Thus,

$$\begin{aligned} &\left| \frac{\partial^{k_1+\dots+k_M+l_1+\dots+l_N}}{\partial a_1^{k_1} \dots \partial a_M^{k_M} \partial b_1^{l_1} \dots \partial b_N^{l_N}} C_\theta(e^{i\omega}) \right| \\ &= \left| \frac{\partial^{k_1+\dots+k_M}}{\partial a_1^{k_1} \dots \partial a_M^{k_M}} A_\theta(e^{i\omega}) \right| \left| \frac{\partial^{l_1+\dots+l_N}}{\partial b_1^{l_1} \dots \partial b_N^{l_N}} \left(\frac{1}{B_\theta(e^{i\omega})} \right) \right| \\ &\leq (l_1 + \dots + l_N)! \alpha_\theta (1/\beta_\theta)^{l_1+\dots+l_N+1} \end{aligned}$$

for all $\theta = [a_1 \dots a_M \ b_1 \dots b_N]^T \in \Theta$, $\omega \in [-\pi, \pi]$, $k_1, \dots, k_M \geq 0$, $l_1, \dots, l_N \geq 0$. Then, it can be deduced easily

$$|D_\theta^{k_1, \dots, k_{d_\theta}} C_\theta(e^{i\omega})| \leq (k_1 + \dots + k_{d_\theta})! (\alpha_\theta / \beta_\theta)^{k_1 + \dots + k_{d_\theta} + 1}$$

for all $\theta \in \Theta$, $\omega \in [-\pi, \pi]$, $k_1, \dots, k_{d_\theta} \geq 0$ ($D_\theta^{k_1, \dots, k_{d_\theta}}$ denotes $\partial^{k_1+\dots+k_{d_\theta}} / \partial \vartheta_1^{k_1} \dots \partial \vartheta_{d_\theta}^{k_{d_\theta}}$, where ϑ_i is the i -th component of θ). Since

$$\begin{aligned} D_\theta^{k_1, \dots, k_{d_\theta}} |C_\theta(e^{i\omega})|^2 &= \sum_{j_1=0}^{k_1} \dots \sum_{j_{d_\theta}=0}^{k_{d_\theta}} \binom{k_1}{j_1} \dots \binom{k_{d_\theta}}{j_{d_\theta}} D_\theta^{j_1, \dots, j_{d_\theta}} C_\theta(e^{i\omega}) \\ &\quad \cdot D_\theta^{k_1-j_1, \dots, k_{d_\theta}-j_{d_\theta}} C_\theta(e^{-i\omega}) \end{aligned}$$

for each $\theta \in \Theta$, $\omega \in [-\pi, \pi]$, $k_1, \dots, k_{d_\theta} \geq 0$, we have

$$\begin{aligned} &|D_\theta^{k_1, \dots, k_{d_\theta}} |C_\theta(e^{i\omega})|^2| \\ &\leq (k_1 + \dots + k_{d_\theta})! \left(\frac{\alpha_\theta}{\beta_\theta} \right)^{k_1 + \dots + k_{d_\theta} + 2} \sum_{j_1=0}^{k_1} \dots \sum_{j_{d_\theta}=0}^{k_{d_\theta}} \frac{\binom{k_1}{j_1} \dots \binom{k_{d_\theta}}{j_{d_\theta}}}{\binom{k_1 + \dots + k_{d_\theta}}{j_1 + \dots + j_{d_\theta}}} \\ &\leq (k_1 + \dots + k_{d_\theta})! \left(\frac{\alpha_\theta}{\beta_\theta} \right)^{k_1 + \dots + k_{d_\theta} + 2} \sum_{j_1=0}^{k_1} \dots \sum_{j_{d_\theta}=0}^{k_{d_\theta}} \binom{k_1}{j_1} \dots \binom{k_{d_\theta}}{j_{d_\theta}} \\ &\leq (k_1 + \dots + k_{d_\theta})! \left(\frac{2\alpha_\theta}{\beta_\theta} \right)^{k_1 + \dots + k_{d_\theta} + 2} \end{aligned}$$

for any $\theta \in \Theta$, $\omega \in [-\pi, \pi]$, $k_1, \dots, k_{d_\theta} \geq 0$. Consequently, the multinomial formula (see [14, Theorem 1.3.1]) implies

$$\begin{aligned}
& \sum_{k_1=0}^{\infty} \dots \sum_{k_{d_\theta}=0}^{\infty} \frac{|D_\theta^{k_1, \dots, k_{d_\theta}} |C_\theta(e^{i\omega})|^2|}{k_1! \dots k_{d_\theta}!} \delta_\theta^{k_1 + \dots + k_{d_\theta}} \\
& \leq \left(\frac{2\alpha_\theta}{\beta_\theta} \right)^2 \sum_{k_1=0}^{\infty} \dots \sum_{k_{d_\theta}=0}^{\infty} \frac{(k_1 + \dots + k_{d_\theta})!}{k_1! \dots k_{d_\theta}!} \left(\frac{2\alpha_\theta \delta_\theta}{\beta_\theta} \right)^{k_1 + \dots + k_{d_\theta}} \\
& = \left(\frac{2\alpha_\theta}{\beta_\theta} \right)^2 \sum_{n=0}^{\infty} \sum_{\substack{0 \leq k_1, \dots, k_{d_\theta} \leq n \\ k_1 + \dots + k_{d_\theta} = n}} \frac{(k_1 + \dots + k_{d_\theta})!}{k_1! \dots k_{d_\theta}!} \left(\frac{2\alpha_\theta \delta_\theta}{\beta_\theta} \right)^{k_1 + \dots + k_{d_\theta}} \\
& = \left(\frac{2\alpha_\theta}{\beta_\theta} \right)^2 \sum_{n=0}^{\infty} \left(\frac{2d_\theta \alpha_\theta \delta_\theta}{\beta_\theta} \right)^n \\
& = \left(\frac{2\alpha_\theta}{\beta_\theta} \right)^2 \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n < \infty
\end{aligned}$$

for every $\theta \in \Theta$, $\omega \in [-\pi, \pi]$. Then, the analyticity of $f(\cdot)$ directly follows from (11.3) and the fact that $|\varphi(\cdot)|$ is uniformly bounded (also notice that $C_\theta(1)$ is analytic in θ). \square

PROOF OF THEOREM 6.2. It is straightforward to show

$$\begin{aligned}
& \max\{\|F(\theta, \xi)\|, \phi(\xi)\} \leq \|\xi\|, \\
& \max\{\|F(\theta, \xi') - F(\theta, \xi'')\|, |\phi(\xi') - \phi(\xi'')|\} \leq 2\|\xi' - \xi''\|(\|\xi'\| + \|\xi''\|)
\end{aligned}$$

for all $\theta \in \Theta$, $\xi, \xi', \xi'' \in \mathbb{R}^{d_\xi}$. Moreover, it can be deduced easily that for any compact set $Q \subset \mathbb{R}^{d_\theta}$, there exist real numbers $\delta_{1,Q} \in (0, 1)$, $C_{1,Q} \in [1, \infty)$ such that $\|G_\theta^n\| \leq C_{1,Q} \delta_{1,Q}^n$ and

$$\|G_{\theta'} - G_{\theta''}\| \leq C_{1,Q} \|\theta' - \theta''\|$$

for each $\theta, \theta', \theta'' \in Q$, $n \geq 0$. Then, the results of [2, Section II.2.3] imply that there exist a locally Lipschitz continuous function $g : \Theta \rightarrow \mathbb{R}^{d_\theta}$ and a Borel-measurable function $\tilde{F} : \Theta \times \mathbb{R}^{d_\xi} \rightarrow \mathbb{R}^{d_\theta}$ such that

$$F(\theta, \xi) - g(\theta) = \tilde{F}(\theta, \xi) - (\Pi \tilde{F})(\theta, \xi)$$

for every $\theta \in \Theta$, $\xi \in \mathbb{R}^{d_\xi}$. Due to the same results, there exists a locally Lipschitz continuous function $h : \Theta \rightarrow \mathbb{R}$ and for any compact set $Q \subset \mathbb{R}^{d_\theta}$, there exist real numbers $\delta_{2,Q} \in (0, 1)$, $C_{2,Q} \in [1, \infty)$ such that

$$\begin{aligned}
& \max\{\|(\Pi^n F)(\theta, \xi) - g(\theta)\|, \|(\Pi^n \phi)(\theta, \xi) - h(\theta)\|\} \leq C_{2,Q} \delta_{2,Q}^n (1 + \|\xi\|)^2, \quad (11.4) \\
& \max\{\|\tilde{F}(\theta, \xi)\|, \|(\Pi \tilde{F})(\theta, \xi)\|\} \leq C_{2,Q} (1 + \|\xi\|)^2, \\
& \|\tilde{F}(\theta', \xi) - \tilde{F}(\theta'', \xi)\| \leq C_{2,Q} \|\theta' - \theta''\| (1 + \|\xi\|)^2
\end{aligned}$$

for each $\theta, \theta', \theta'' \in Q$, $\xi, \xi', \xi'' \in \mathbb{R}^{d_\xi}$. Combining (11.1), (11.2), (11.4) with the dominated convergence theorem, we get $h(\cdot) = f(\cdot)$, $g(\cdot) = \nabla f(\cdot)$. On the other side, owing to the fact that $\{x_n\}_{n \geq 0}$ is a geometrically ergodic Markov chain, we have

that $\{y_n\}_{n \geq 0}$ admits a stationary regime for $n \rightarrow \infty$. Consequently, Theorem 6.1 implies that $f(\cdot)$ is analytic on Θ . Then, the theorem's assertion directly follows from Theorem 3.1. \square

Appendix. In this section, we study certain aspects of Assumption 2.3. More specifically, we show that Assumption 2.3 is true if its 'local version', Assumption 2.3' (below) holds. We also demonstrate that (Lojasiewicz coefficients) $\delta_{Q,a}$, $\mu_{Q,a}$ and $M_{Q,a}$ have 'measurable versions' for which $\hat{\delta}$, $\hat{\mu}$ and \hat{M} (defined in Section 2) are random variables in probability space (Ω, \mathcal{F}, P) (i.e., measurable with respect to \mathcal{F}). We study these aspects of Assumption 2.3 under the following condition:

ASSUMPTION 2.3'. There exists an open vicinity U of S with the following property: For any compact set $Q \subset U$ and any real number $a \in f(Q)$, there exist real numbers $\delta'_{Q,a} \in (0, 1)$, $\mu'_{Q,a} \in (1, 2]$, $M'_{Q,a} \in [1, \infty)$ such that

$$|f(\theta) - a| \leq M'_{Q,a} \|\nabla f(\theta)\|^{\mu'_{Q,a}}$$

for all $\theta \in Q$ satisfying $|f(\theta) - a| \leq \delta'_{Q,a}$.

Throughout this section, we rely on the following notation. $\varepsilon \in (0, 1)$ is a fixed constant. For a compact set $Q \subset \mathbb{R}^{d_\theta}$, $a \in f(Q)$ and $\delta \in (0, 1)$, let

$$\phi_{Q,a}(\delta) = \sup \left\{ \frac{1}{2}, \frac{\log \|\nabla f(\theta)\|}{\log |f(\theta) - a|} : \theta \in Q \setminus S, 0 < |f(\theta) - a| \leq \delta \right\},$$

while

$$\delta_{Q,a} = \sup \{ \varepsilon \delta : \delta \in (0, 1), \phi_{Q,a}(\delta) < 1 \}$$

and $\mu_{Q,a} = 1/\phi_{Q,a}(\delta_{Q,a})$, $M_{Q,a} = 1$.

LEMMA A.1. *Let Assumption 2.3' hold. Moreover, let $Q \subset \mathbb{R}^{d_\theta}$ be an arbitrary compact set, while $a \in f(Q)$ is an arbitrary real number. Then, $\delta_{Q,a}$, $\mu_{Q,a}$, $M_{Q,a}$ specified in this section satisfy all requirements of Assumption 2.3.*

Proof. First, we show $\delta_{Q,a} > 0$. To do so, we consider separately the following cases:

Case $Q \cap S = \emptyset$: Let

$$\tilde{\delta}_{Q,a} = \inf \{ \exp(-2|\log \|\nabla f(\theta)\||) : \theta \in Q \}.$$

Obviously, $0 < \tilde{\delta}_{Q,a} < 1$ (notice that $\inf_{\theta \in Q} \|\nabla f(\theta)\| > 0$). We also have

$$2|\log \|\nabla f(\theta)\|| \leq \log(1/\tilde{\delta}_{Q,a}) \tag{A.1}$$

for all $\theta \in Q$. Consequently,

$$\left| \frac{\log \|\nabla f(\theta)\|}{\log |f(\theta) - a|} \right| \leq \frac{|\log \|\nabla f(\theta)\||}{\log(1/\tilde{\delta}_{Q,a})} \leq 1/2 \tag{A.2}$$

for any $\theta \in Q$ satisfying $0 < |f(\theta) - a| \leq \tilde{\delta}_{Q,a}$. Thus, $\phi_{Q,a}(\delta) \leq 1/2$ for each $\delta \in (0, \tilde{\delta}_{Q,a}]$, and hence, $\delta_{Q,a} \geq \varepsilon \tilde{\delta}_{Q,a} > 0$.

Case $Q \cap S \neq \emptyset$, $a \notin f(Q \cap S)$: Let

$$\begin{aligned} \tilde{\delta}'_{Q,a} &= \frac{1}{2} \inf \{ 1, |f(\theta) - a| : \theta \in Q \cap S \}, \\ \tilde{\delta}''_{Q,a} &= \inf \left\{ \exp(-2|\log \|\nabla f(\theta)\||) : \theta \in Q, |f(\theta) - a| \leq \tilde{\delta}'_{Q,a} \right\}, \end{aligned}$$

while $\tilde{\delta}_{Q,a} = \min\{\tilde{\delta}'_{Q,a}, \tilde{\delta}''_{Q,a}\}$. Obviously, $0 < \tilde{\delta}_{Q,a} \leq 1/2$ (notice that $0 < \tilde{\delta}'_{Q,a} \leq 1/2$ and that $\theta \notin Q \cap S$ if $|f(\theta) - a| \leq \tilde{\delta}'_{Q,a}$; also notice that $0 < \inf\{\|\nabla f(\theta)\| : \theta \in Q, |f(\theta) - a| \leq \tilde{\delta}'_{Q,a}\}$). Moreover, (A.1) holds for all $\theta \in Q$ satisfying $0 < |f(\theta) - a| \leq \tilde{\delta}_{Q,a}$. Then, (A.2) is true for any $\theta \in Q$ fulfilling $0 < |f(\theta) - a| \leq \tilde{\delta}_{Q,a}$. Hence, $\phi_{Q,a}(\delta) \leq 1/2$ for all $\delta \in (0, \tilde{\delta}_{Q,a}]$, and consequently, $\delta_{Q,a} \geq \varepsilon \tilde{\delta}_{Q,a} > 0$.

Case $Q \cap S \neq \emptyset$, $a \in f(Q \cap S)$: Let $\rho_Q = d(Q \cap S, U^c)/2$ and $\tilde{Q} = \{\theta \in \mathbb{R}^{d_\theta} : d(\theta, Q \cap S) \leq \rho_Q\}$, while $\tilde{\delta}'_{Q,a} = \delta'_{\tilde{Q},a}$, $\tilde{\mu}_{Q,a} = \mu'_{\tilde{Q},a}$, $\tilde{M}_{Q,a} = M'_{\tilde{Q},a}$ ($\delta'_{\tilde{Q},a}$, $\mu'_{\tilde{Q},a}$, $M'_{\tilde{Q},a}$ are introduced in Assumption 2.3'). Moreover, let

$$\tilde{\delta}''_{Q,a} = \inf \left\{ \frac{1}{2}, \exp(-2|\log \|\nabla f(\theta)\||) : \theta \in Q \setminus \tilde{Q} \right\}$$

and $\tilde{\delta}_{Q,a} = \min \left\{ \tilde{\delta}'_{Q,a}, \tilde{\delta}''_{Q,a}, \tilde{M}_{Q,a}^{-2/(\tilde{\mu}_{Q,a}-1)} \right\}$. Obviously, $\tilde{Q} \subset U$ and $0 < \tilde{\delta}_{Q,a} \leq 1/2$. Moreover, (A.1) is true for all $\theta \in Q \setminus \tilde{Q}$. Therefore, (A.2) holds for all $\theta \in Q \setminus \tilde{Q}$ satisfying $0 < |f(\theta) - a| \leq \tilde{\delta}_{Q,a}$. On the other side, Assumption 2.3' implies

$$\log |f(\theta) - a| \leq \log \tilde{M}_{Q,a} + \tilde{\mu}_{Q,a} \log \|\nabla f(\theta)\|$$

for all $\theta \in \tilde{Q} \setminus S$ satisfying $0 < |f(\theta) - a| \leq \tilde{\delta}_{Q,a}$ (notice that $\tilde{\delta}_{Q,a} \leq \delta'_{\tilde{Q},a}$). Consequently,

$$\begin{aligned} \frac{\log \|\nabla f(\theta)\|}{\log |f(\theta) - a|} &\leq \frac{1}{\tilde{\mu}_{Q,a}} \left(1 - \frac{\log \tilde{M}_{Q,a}}{\log |f(\theta) - a|} \right) \\ &\leq \frac{1}{\tilde{\mu}_{Q,a}} \left(1 + \frac{\log \tilde{M}_{Q,a}}{\log(1/\tilde{\delta}_{Q,a})} \right) \\ &\leq \frac{\tilde{\mu}_{Q,a} + 1}{2\tilde{\mu}_{Q,a}} < 1 \end{aligned} \quad (\text{A.3})$$

for all $\theta \in \tilde{Q} \setminus S$ satisfying $0 < |f(\theta) - a| \leq \tilde{\delta}_{Q,a}$ (notice that $\log(1/\tilde{\delta}_{Q,a}) \geq 2 \log \tilde{M}_{Q,a}/(\tilde{\mu}_{Q,a} - 1)$). Thus, as a result of (A.2), (A.3), we have $\phi_{Q,a}(\delta) < 1$ for all $\delta \in (0, \tilde{\delta}_{Q,a}]$, and consequently, $\delta_{Q,a} \geq \varepsilon \tilde{\delta}_{Q,a} > 0$.

Now, we prove that $\delta_{Q,a}$, $\mu_{Q,a}$, $M_{Q,a}$ fulfill all other requirements of Assumption 2.3. By the definition of $\phi_{Q,a}(\cdot)$ and $\delta_{Q,a}$, we have $0 < \delta_{Q,a} < 1$, $1/2 \leq \phi_{Q,a}(\delta_{Q,a}) < 1$ and

$$\frac{\log \|\nabla f(\theta)\|}{\log |f(\theta) - a|} \leq \phi_{Q,a}(\delta_{Q,a})$$

for all $\theta \in Q \setminus S$ satisfying $0 < |f(\theta) - a| \leq \delta_{Q,a}$. Therefore, $1 < \mu_{Q,a} = 1/\phi_{Q,a}(\delta_{Q,a}) \leq 2$ and

$$\mu_{Q,a} \log \|\nabla f(\theta)\| = \frac{\log \|\nabla f(\theta)\|}{\phi_{Q,a}(\delta_{Q,a})} \geq \log |f(\theta) - a|$$

for each $\theta \in Q \setminus S$ fulfilling $0 < |f(\theta) - a| \leq \delta_{Q,a}$. Hence, (2.2) holds for all $\theta \in Q$ satisfying $0 < |f(\theta) - a| \leq \delta_{Q,a}$. \square

LEMMA A.2. *Let $\hat{\delta}$, $\hat{\mu}$, \hat{M} be defined using (2.4), (2.5) and $\delta_{Q,a}$, $\mu_{Q,a}$, $M_{Q,a}$ specified in this section. Then, $\hat{\delta}$, $\hat{\mu}$, \hat{M} are random variables in probability space (Ω, \mathcal{F}, P) .*

Proof. For $\theta \in \mathbb{R}^{d_\theta}$, $\delta \in (0, 1)$, let

$$\hat{\Phi}(\theta, \delta) = \frac{\log \|\nabla f(\theta)\|}{\log |f(\theta) - \hat{f}|} I_{S^c}(\theta) I_{(0, \delta]} \left(|f(\theta) - \hat{f}| \right) I_{[0, \rho]} \left(\liminf_{n \rightarrow \infty} \|\theta - \theta_n\| \right)$$

(ρ is specified in the definition of \hat{Q} , Section 2), while

$$\hat{\phi}(\delta) = \sup \left\{ 1/2, \hat{\Phi}(\theta, \delta) : \theta \in \mathbb{R}^{d_\theta} \right\} I_\Lambda$$

(Λ is defined in Section 7). Obviously, $\hat{\Phi}(\theta, \delta)$ and $\hat{\phi}(\delta)$ are measurable random functions of (θ, δ) and δ (i.e., $\hat{\Phi}(\theta, \delta)$ and $\hat{\phi}(\delta)$ are measurable with respect to σ -algebras $\mathcal{B}(\mathbb{R}^{d_\theta}) \times \mathcal{B}((0, 1)) \times \mathcal{F}$ and $\mathcal{B}((0, 1)) \times \mathcal{F}$). On the other side, it is straightforward to verify that

$$\hat{\delta} = \sup \{ \varepsilon \delta : \delta \in (0, 1), \hat{\phi}(\delta) < 1 \}$$

and $\hat{\mu} = 1/\hat{\phi}(\hat{\delta})$ on Λ . Then, it is clear that $\hat{\delta}$, $\hat{\mu}$, \hat{M} are random variables in probability space (Ω, \mathcal{F}, P) . \square

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