Continuity of set-valued maps
revisited in the light of tame geometry

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Abstract Continuity of set-valued maps is hereby revisited: after recalling some basic concepts of variational analysis and a short description of the State-of-the-Art, we obtain as by-product two Sard type results concerning local minima of scalar and vector valued functions. Our main result though, is inscribed in the framework of tame geometry, stating that a closed-valued semialgebraic set-valued map is almost everywhere continuous (in both topological and measure-theoretic sense). The result –depending on stratification techniques– holds true in a more general setting of o-minimal (or tame) set-valued maps. Some applications are briefly discussed at the end.

Key words Set-valued map, (strict, outer, inner) continuity, Aubin property, semialgebraic, piecewise polyhedral, tame optimization.

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1 Introduction

We say that S is a *set-valued map* (we also use the term *multivalued function* or simply *multifunction*) from X to Y, denoted by $S : X \rightrightarrows Y$, if for every $x \in X$, $S(x)$ is a subset of Y. All single-valued maps in classical analysis can be seen as set-valued maps, while many problems in applied mathematics are set-valued in nature. For instance, problems of stability (parametric optimization) and controllability are often best treated with set-valued maps, while gradients of (differentiable) functions, tangents and normals of sets (with a structure of differentiable manifold) have natural set-valued generalizations in the nonsmooth case, by means of variational analysis techniques. The inclusion $y \in S(x)$ is the heart of modern variational analysis. We refer the reader to [1, 22] for more details.

Continuity properties of set-valued maps are crucial in many applications. A typical set-valued map arising from some construction or variational problem will not be continuous. Nonetheless, one often expects a kind of semicontinuity (inner or outer) to hold. (We refer to Section 2 for relevant definitions.)

A standard application of a Baire argument entails that closed-valued set-valued maps are generically continuous, provided they are either inner or outer semicontinuous. Recalling briefly these results, as well as other concepts of continuity for set-valued maps, we illustrate their sharpness by means of appropriate examples. We also mention an interesting consequence of these results by establishing a Sard-type result for the image of local minima.

Moving forward, we limit ourselves to semialgebraic maps [3, 8] or more generally, to maps whose graph is a definable set in some o-minimal structure [11, 9]. This setting aims at eliminating most pathologies that pervade analysis which, aside from their indisputable theoretical interest, do not appear in most practical applications. The definition of a definable set might appear reluctant at the first sight (in particular for researchers in applied mathematics), but it determines a large class of objects (sets, functions, maps) encompassing for instance the well-known class of semialgebraic sets [3, 8], that is, the class of Boolean combinations of subsets of $\mathbb{R}^n$ defined by finite polynomials and inequalities. All these classes enjoy an important stability property —in the case of semialgebraic sets this is expressed by the Tarski-Seidenberg (or quantifier elimination) principle— and share the important property of stratification: every definable set (so in particular, every semialgebraic set) can be written as a disjoint union of smooth manifolds which fit each other in a regular way (see Theorem 21 for a precise statement). This tame behaviour has been already exploited in various ways in variational analysis, see for instance [2] (convergence of proximal algorithm), [4] (Łojasiewicz gradient inequality), [5] (semismoothness), [14] (Sard-Smale type result for critical values) or [15] for a recent survey of what is nowadays called *tame optimization.*

The main result of this work is to establish that every semialgebraic (more generally, definable) closed-valued set-valued map is generically continuous. Let us point out that in this semialgebraic context, genericity implies that possible failures can only arise in a set of lower dimension, and thus is equivalent to the measure-theoretical notion of *almost-everywhere* (see Proposition 23 for a precise statement). The proof uses properties of stratification, some technical lemmas of variational analysis and a recent result of Ioffe [14].

The paper is organized as follows. In Section 2 we recall basic notions of variational analysis and revisit results on the continuity of set-valued maps. As by-product of our development we obtain, in Section 3 two Sard-type results: the first one concerns minimum values of (scalar) functions, while the second one concerns Pareto minimum values of set-valued maps. We also grind our tools by adapting the Mordukhovich criterion to set-valued maps with domain a smooth submanifold $\mathscr{X}$ of $\mathbb{R}^n$. In Section 4 we move into the semialgebraic case. Adapting a recent result of Ioffe [14, Theorem 7] to our needs, we prove an intermediate result concerning generic strict continuity of set-valued maps with a closed semialgebraic graph. Then, relating the failure of continuity of the mapping with the failure of its trace on a stratum of its graph, and using two technical lemmas we establish our main result. Section 5 contains some applications of the main result.

Notation. Denote $\mathbb{B}^n(x,\delta)$ to be the closed ball of center x and radius r in $\mathbb{R}^n$, and $\mathbb{S}^{n-1}(x,r)$ to be the sphere of center x and radius r in $\mathbb{R}^n$. When there is no confusion of the dimensions of $\mathbb{B}^n(x,r)$ and $\mathbb{S}^{n-1}(x,r)$, we omit the superscript. The unit ball $\mathbb{B}(\mathbf{0},1)$ is denoted by $\mathbb{B}$. We denote by $\mathbf{0}_n$ the neutral element of $\mathbb{R}^n$. As before, if there is no confusion on the dimension we shall omit the subscript. Given a subset A of $\mathbb{R}^n$ we denote by $\mathrm{cl}(A)$, $\mathrm{int}(A)$ and ∂A respectively, its topological closure, interior and boundary. For $A_1, A_2 \subset \mathbb{R}^n$ and $r \in \mathbb{R}$ we set

$$A_1 + rA_2 := \{a_1 + ra_2 : a_1 \in A_1, a_2 \in A_2\}.$$

We recall that the Hausdorff distance $\mathscr{D}(A_1, A_2)$ between two bounded subsets A_1, A_2 of $\mathbb{R}^n$ is defined as the infimum of all $\delta > 0$ such that both inclusions $A_1 \subset A_2 + \delta\mathbb{B}$ and $A_2 \subset A_1 + \delta\mathbb{B}$ hold (see [22, Section 9C] for example). Finally, we denote by

$$\mathrm{Graph}(S) = \{(x,y) \in X \times Y : y \in S(x)\},$$

the graph of the set-valued map $S : X \rightrightarrows Y$.

2 Basic notions in set-valued analysis

In this section we recall the definitions of continuity (outer, inner, strict) for set-valued maps, and other related notions from variational analysis. We refer to [1, 22] for more details.

2.1 Continuity concepts for set-valued maps

We start this section by recalling the definitions of continuity for set-valued maps.

(Kuratowski limits of sequences) We first recall basic notions about (Kuratowski) limits of sets. Given a sequence $\{C_\nu\}_{\nu\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ we define:

- the *outer limit* $\limsup_{\nu\to\infty} C_\nu$, as the set of all accumulation points of sequences $\{x_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^n$ with $x_\nu \in C_\nu$ for all $\nu \in \mathbb{N}$. In other words, $x \in \limsup_{\nu\to\infty} C_\nu$ if and only if for every $\varepsilon > 0$ and $N \geq 1$ there exists $\nu \geq N$ with $C_\nu \cap B(x,\varepsilon) \neq \emptyset$;
- the *inner limit* $\liminf_{\nu\to\infty} C_\nu$, as the set of all limits of sequences $\{x_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{R}^n$ with $x_\nu \in C_\nu$ for all $\nu \in \mathbb{N}$. In other words, $x \in \liminf_{\nu\to\infty} C_\nu$ if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $\nu \geq N$ we have $C_\nu \cap B(x,\varepsilon) \neq \emptyset$.

Furthermore, we say that the *limit* of the sequence $\{C_\nu\}_{\nu\in\mathbb{N}}$ exists if the outer and inner limit sets are equal. In this case we write:

$$\lim_{\nu\to\infty} C_\nu := \limsup_{\nu\to\infty} C_\nu = \liminf_{\nu\to\infty} C_\nu.$$

(Outer/inner continuity of a set-valued map) Given a set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, we define the outer (respectively, inner) limit of S at $\bar{x} \in \mathbb{R}^n$ as the union of all upper limits $\limsup_{\nu\to\infty} S(x_\nu)$ (respectively, intersection of all lower limits $\liminf_{\nu\to\infty} S(x_\nu)$) over all sequences $\{x_\nu\}_{\nu\in\mathbb{N}}$ converging to $\bar{x}$. In other words:

$$\limsup_{x\to\bar{x}} S(x) := \bigcup_{x_\nu\to\bar{x}} \limsup_{\nu\to\infty} S(x_\nu) \qquad \text{and} \qquad \liminf_{x\to\bar{x}} S(x) := \bigcap_{x_\nu\to\bar{x}} \liminf_{\nu\to\infty} S(x_\nu).$$

We are now ready to recall the following definition.

Definition 1. [22, Definition 5.4] A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called *outer semicontinuous* at $\bar{x}$ if

$$\limsup_{x\to\bar{x}} S(x) \subset S(\bar{x}),$$

or equivalently, $\limsup_{x\to\bar{x}} S(x) = S(\bar{x})$, and *inner semicontinuous* at $\bar{x}$ if

$$\liminf_{x\to\bar{x}} S(x) \supset S(\bar{x}),$$

or equivalently when S is closed-valued, $\liminf_{x\to\bar{x}} S(x) = S(\bar{x})$. It is called *continuous* at $\bar{x}$ if both conditions hold, i.e., if $S(x) \to S(\bar{x})$ as $x \to \bar{x}$.

If these terms are invoked relative to X, a subset of $\mathbb{R}^n$ containing $\bar{x}$, then the properties hold in restriction to convergence $x \to \bar{x}$ with $x \in X$ (in which case the sequences $x_\nu \to \bar{x}$ in the limit formulations are required to lie in X).

Notice that every outer semicontinuous set-valued map has closed values. In particular, it is well known that

- S is outer semicontinuous if and only if S has a closed graph.

When S is a single-valued function, both outer and inner semicontinuity reduce to the standard notion of continuity. The standard example of the mapping

$$S(x) := \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases} \tag{2.1}$$

shows that it is possible for a set-valued map to be nowhere outer and nowhere inner semicontinuous. Nonetheless, the following genericity result holds. (We recall that a set is *nowhere dense* if its closure has empty interior, and *meager* if it is the union of countably many sets that are nowhere dense in X.) The following result appears in [22, Theorem 5.55] and [1, Theorem 1.4.13] and is attributed to [17, 7, 24]. The domain of S below can be taken to be a complete metric space, while the range can be taken to be a complete separable metric space, but we shall only state the result in the finite dimensional case.

Theorem 2. *Let $X \subset \mathbb{R}^n$ and $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a closed-valued set-valued map. Assume S is either outer semicontinuous or inner semicontinuous relative to X. Then the set of points $x \in X$ where S fails to be continuous relative to X is meager in X.*

The following example shows the sharpness of the result, if we move to incomplete spaces.

Example 3. Let $c_{00}(\mathbb{N})$ denote the vector space of all real sequences $x = \{x_n\}_{n\in\mathbb{N}}$ with finite support $\mathrm{supp}(x) := \{i \in \mathbb{N} : x_i \neq 0\}$. Then the operator $S_1(x) = \mathrm{supp}(x)$ is everywhere inner semicontinuous and nowhere outer semicontinuous, while the operator $S_2(x) = \mathbb{Z} \setminus S_1(x)$ is everywhere outer semicontinuous and nowhere inner semicontinuous. □

(Strict continuity of set-valued maps) A stronger concept of continuity for set-valued maps is that of *strict continuity* [22, Definition 9.28], which is equivalent to Lipschitz continuity when the map is single-valued. For set-valued maps $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ with bounded values, strict continuity is quantified by the Hausdorff distance. Namely, a set-valued map S is strictly continuous at $\bar{x}$ (relative to X) if the quantity

$$\mathrm{lip}_X S(\bar{x}) := \limsup_{\substack{x,x' \to \bar{x} \\ x \neq x'}} \frac{\mathscr{D}(S(x), S(x'))}{|x - x'|}$$

is bounded. In the general case (that is, when S maps to unbounded sets), we say that S is strictly continuous, whenever the truncated map $S_r : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ defined by

$$S_r(x) := S(x) \cap r\mathbb{B},$$

is Lipschitz continuous for every $r > 0$.

2.2 Normal cones, coderivatives and the Aubin property

Before we consider other concepts of continuity of set-valued maps we need to recall some basic concepts from variational analysis. We first recall the definition of the Hadamard and limiting normal cones.

Definition 4. (Normal cones) [22, Definition 6.3] For a closed set $D \subset \mathbb{R}^n$ and a point $\bar{z} \in D$, we recall that the *Hadamard normal cone* $\hat{N}_D(\bar{z})$ and the *limiting normal cone* $N_D(\bar{z})$ are defined by

$$\begin{aligned}
\hat{N}_D(\bar{z}) &:= \{v \mid \langle v, z-\bar{z}\rangle \leq o(|z-\bar{z}|) \text{ for } z \in D\}, \\
N_D(\bar{z}) &:= \{v \mid \exists \{z_i, v_i\}_{i=1}^{\infty} \subset \mathrm{Graph}(\hat{N}_D),\ v_i \to v \text{ and } z_i \to \bar{z}\} \\
&= \limsup_{z \to \bar{z}, z \in D} \hat{N}_D(z).
\end{aligned}$$

When D is a smooth manifold, both notions of normal cone coincide and define the same subspace of $\mathbb{R}^n$. A dual concept to the normal cone is the *tangent cone* $T_D(\bar{z})$. While tangent cones can be defined for nonsmooth sets, our use here shall be restricted only to tangent cones of manifolds, that is, tangent spaces in the sense of differential geometry, in which case $T_D(\bar{z}) = (N_D(\bar{z}))^{\perp}$.

As is well-known, the generalization of the adjoint of a linear operator for set-valued maps is derived from the normal cones of its graph.

Definition 5. (Coderivatives) [22, Definition 8.33] For $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $(\bar{x}, \bar{y}) \in \mathrm{Graph}(F)$, the *limiting coderivative* $D^*F(\bar{x} \mid \bar{y}) : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is defined by

$$D^*F(\bar{x} \mid \bar{y})(y^*) = \left\{x^* \mid (x^*, -y^*) \in N_{\mathrm{Graph}(F)}(\bar{x}, \bar{y})\right\}.$$

It is clear from the definitions that the coderivative is a positively homogeneous map, which can be measured with the outer norm below.

Definition 6. [22, Section 9D] The *outer norm* $|\cdot|^+$ of a positively homogeneous map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined by

$$|H|^+ := \sup_{w \in \mathbb{B}^n(\mathbf{0},1)} \sup_{z \in H(w)} |z| = \sup\left\{\frac{|z|}{|w|} \mid (w,z) \in \mathrm{Graph}(H)\right\}.$$

(Aubin property and Mordukhovich criterion) We now recall the Aubin Property and the graphical modulus, which are important to study local Lipschitz continuity properties of a set-valued map.

Definition 7. (Aubin property and graphical modulus) [22, Definition 9.36] A map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ has the *Aubin property relative to X at $\bar{x}$ for $\bar{u}$*, where $\bar{x} \in X \subset \mathbb{R}^n$ and $\bar{u} \in S(\bar{x})$, if $\mathrm{Graph}(S)$ is locally closed at $(\bar{x}, \bar{u})$ and there are neighborhoods V of $\bar{x}$ and W of $\bar{u}$, and a constant $\kappa \in \mathbb{R}_+$ such that

$$S(x') \cap W \subset S(x) + \kappa |x' - x| \mathbb{B} \text{ for all } x, x' \in X \cap V.$$

This condition with V in place of $X \cap V$ is simply the *Aubin property at $\bar{x}$ for $\bar{u}$*. The *graphical modulus of S relative to X at $\bar{x}$ for $\bar{u}$* is then

$$\begin{aligned}
\mathrm{lip}_X S(\bar{x} \mid \bar{u}) := \inf\{\kappa \mid \exists &\text{ neighborhoods } V \text{ of } \bar{x} \text{ and } W \text{ of } \bar{u} \text{ s.t.} \\
&S(x') \cap W \subset S(x) + \kappa |x' - x| \mathbb{B} \text{ for all } x, x' \in X \cap V\}.
\end{aligned}$$

In the case where $X = \mathbb{R}^n$, the subscript X is omitted.

The following result (known as Mordukhovich criterion [22, Theorem 9.40]) characterizes the Aubin property by means of the corresponding coderivative. (For a primal characterization using the graphical derivative see [12, Theorem 1.2].)

Proposition 8 (Mordukhovich criterion). *Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map whose graph* $\mathrm{Graph}\,(S)$ *is locally closed at $(\bar{x},\bar{u}) \in \mathrm{Graph}\,(S)$. Then S has the Aubin property at $\bar{x}$ with respect to $\bar{u}$ if and only if $D^*S(\bar{x} \mid \bar{u})(\mathbf{0}) = \{\mathbf{0}\}$ or equivalently $|D^*S(\bar{x} \mid \bar{u})|^+ < \infty$. In this case,* $\mathrm{lip}\, S\,(\bar{x} \mid \bar{u}) = |D^*S(\bar{x} \mid \bar{u})|^+$.

Using the above criterion we show that an everywhere continuous strictly increasing single-valued map from the reals to the reals could be nowhere Lipschitz continuous.

Example 9. Let $A \subset \mathbb{R}$ be a measurable set with the property that for every $a, b \in \mathbb{R}$, $a < b$, the Lebesgue measure of $A \cap (a,b)$ satisfies $0 < m(A \cap [a,b]) < |b-a|$. Consider the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = m(A \cap (0,x))$. Note that the derivative $f'(x)$ exists almost everywhere and is equal to $\chi_A(x)$, the characteristic function of A (equal to 1 if $x \in A$ and 0 if not). This means that every point $\bar{x} \in [0,1]$ is arbitrarily close to a point x where $f'(x)$ is well-defined and equals zero. Thus $(0,1) \in N_{\mathrm{Graph}(f)}(\bar{x}, f(\bar{x}))$. The function f is strictly increasing and continuous, so it has a continuous inverse $g : [0, f(1)] \to [0,1]$. Applying the Mordukhovich criterion (Proposition 8) we obtain that g does not have the Aubin property at $f(\bar{x})$. It follows that g is not strictly continuous at $f(\bar{x})$ and in fact neither is so at any $y \in [0, f(1)]$. □

3 Preliminary results in Variational Analysis

In this section we establish a Sard type result for the image of the set of local minima (respectively, local Pareto minima) in case of single–valued scalar (respectively, vector–valued) functions. We also obtain several auxiliary results that will be used in Section 4.

3.1 Sard result for local (Pareto) minima

In this subsection we use simple properties on the continuity of set-valued maps to obtain a Sard type result for local minima for both scalar and vector-valued functions. Let us recall that a (single-valued) function $f : X \to \mathbb{R}$ is called *lower semicontinuous at $\bar{x}$* if

$$\liminf_{x \to \bar{x}} f(x) \geq f(\bar{x}).$$

The function f is called *lower semicontinuous*, if it is lower semicontinuous at every $x \in X$. It is well-known that a function f is lower semicontinuous if and only if its sublevel sets

$$[f \leq r] := \{x \in X : f(x) \leq r\}$$

are closed for all $r \in \mathbb{R}$.

Proposition 10 (Sublevel map). *Let D be a closed subset of a complete metric space X and $f : D \to \mathbb{R}$ be a lower semicontinuous function. Then the (sublevel) set-valued map*

$$\begin{cases} L_f : \mathbb{R} \rightrightarrows D \\ L_f(r) = [f \leq r] \cup \partial D \end{cases}$$

is outer semicontinuous. Moreover, L_f is continuous at $\bar{r} \in f(D)$ if and only if there is no $x \in \mathrm{int}\,(D)$ such that $f(x) = \bar{r}$ and x is a local minimizer of f.

Proof. The map $L'_f : \mathbb{R} \rightrightarrows D$ defined by $L'_f(r) = f^{-1}((-\infty, r])$ is outer semicontinuous since f is lower semicontinuous (see [22, Example 5.5] for example), so L_f is easily seen to be outer semicontinuous.

We now prove that L_f is inner semicontinuous at $\bar{r}$ under the additional conditions mentioned in the statement. For any $r_i \to \bar{r}$, we want to show that if $\bar{x} \in L_f(\bar{r})$, then there exists $x_i \to \bar{x}$ such that $x_i \in L_f(r_i)$.

We can assume that $f(\bar{x}) = \bar{r}$ and $r_i < \bar{r}$ for all i, otherwise we can take $x_i = \bar{x}$ for i large enough. Since $\bar{x}$ is not a local minimum, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|\bar{r} - r_i| < \delta$, there exists an x_i such that $f(x_i) \leq r_i$ and $|x_i - \bar{x}| < \varepsilon$.

For the converse, assume now that L_f is inner semicontinuous at $\bar{r}$. Then taking $r_i \nearrow \bar{r}$ we obtain that for every $x \in \mathrm{int}(D) \cap f^{-1}(\bar{r})$, there exists $x_i \in f^{-1}(r_i)$ with $x_i \to x$. Since $f(x_i) = r_i < \bar{r} = f(x)$, x cannot be a local minimum. $\square$

According to the above result, if f has no local minima, then the set-valued map L_f is continuous everywhere. The above result has the following interesting consequence.

Corollary 11 (Local minimum values)**.** *Let M_f denote the set of local minima of a lower semicontinuous function $f : D \to \mathbb{R}$ (where D is a closed subset of a complete space X). Then the set $f(M_f \cap int(D))$ is meager in $\mathbb{R}$.*

Proof. Since the set-valued map L_f (defined in Proposition 10) is outer semicontinuous (with closed-values), it is generically continuous by Theorem 2. The second part of Proposition 10 yields the result on f. $\square$

It is interesting to compare the above result with the classical Sard theorem. We recall that the Sard theorem asserts that the image of critical points (derivative not surjective) of a C^k function $f : \mathbb{R}^n \to \mathbb{R}^m$ is of measure zero provided $k > n - m$. (See [23]; the case $m = 1$ is known as the Sard-Brown theorem [6].) Corollary 11 asserts the topological sparsity of the (smaller) set of minimum values for scalar functions ($m = 1$), without assuming anything but lower semicontinuity (and completeness of the domain).

We shall now extend Corollary 11 in the vectorial case. We recall that a set $K \subset \mathbb{R}^m$ is a *cone*, if $\lambda K \subset K$ for all $\lambda \geq 0$. A cone K is called *pointed* if $K \cap (-K) = \{\mathbf{0}_m\}$ (or equivalently, if K contains no full lines). It is well-known that there is a one-to-one correspondence between pointed convex cones of $\mathbb{R}^m$ and partial orderings in $\mathbb{R}^m$. In particular, given such a cone K of $\mathbb{R}^m$ we set $y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$ (see for example, [22, Section 3E]). Further, given a set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ we say that

- $\bar{x}$ is a *(local) Pareto minimum* of S with *(local) Pareto minimum value* $\bar{y}$ if there is a neighborhood U of $\bar{x}$ such that if $x \in U$ and $y \in S(x)$, then $y \not\leq_K \bar{y}$, i.e., $S(U) \cap (\bar{y} - K) = \bar{y}$.

For $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, define the map $S_K : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ by $S_K(x) = S(x) + K$. The graph of S_K is also known as the *epigraph* [13, 16] of S. One easily checks that $y \in S_K(x)$ implies $y + K \subset S_K(x)$. Here is our result on local Pareto minimum values.

Proposition 12 (Pareto minimum values)**.** *Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be an outer semicontinuous map such that $y \in S(x)$ implies $y + K \subset S(x)$ (that is, $S = S_K$). Then the set of local Pareto minimum values is meager.*

Proof. Since S is outer semicontinuous, then S^{-1} is outer semicontinuous as well by [22, Theorem 5.7(a)], so S^{-1} is generically continuous by Theorem 2. Suppose that $\bar{y}$ is a local Pareto minimum of a local Pareto minimizer $\bar{x}$.

By the definition of local Pareto minimum, there is a neighborhood U of $\bar{x}$ such that if $y \leq_K \bar{y}$ and $y \neq \bar{y}$, then $S^{-1}(y) \cap U = \emptyset$. (We can assume that y is arbitrarily close to $\bar{y}$ since $S^{-1}(y) \subset S^{-1}(\lambda y + (1-\lambda)\bar{y})$ for all $0 \leq \lambda \leq 1$.) Therefore, $\bar{x} \notin \liminf_{y \to \bar{y}} S^{-1}(y)$. In other words, S^{-1} is not continuous at $\bar{y}$. Therefore, the set of local Pareto minimum values is meager. $\square$

We show how the above result compares to critical point results. Let us recall from [14] the definition of critical points of a set-valued map. Given a metric space X (equipped with a distance ρ) we denote by $B_\rho(x, \lambda)$ the set of all $x' \in X$ such that $\rho(x, x') \leq \lambda$.

Definition 13. Let (X,ρ_1) and (Y,ρ_2) be metric spaces, and let $S : X \rightrightarrows Y$. For $(x,y) \in \mathrm{Graph}\,(S)$, we set

$$\mathrm{Sur}\, S\,(x \mid y)\,(\lambda) = \sup\left\{r \geq 0 \mid B_{\rho_2}(y,r) \subset S\left(B_{\rho_1}(x,\lambda)\right)\right\}$$

and then for $(\bar{x},\bar{y}) \in \mathrm{Graph}\,(S)$ define the *rate of surjection* of S at $(\bar{x},\bar{y})$ by

$$\mathrm{sur}\, S\,(\bar{x} \mid \bar{y}) = \liminf_{(x,y,\lambda)\to(\bar{x},\bar{y},+0)} \frac{1}{\lambda}\,\mathrm{Sur}\, S\,(x \mid y)\,(\lambda)\,.$$

We say that S is *critical* at $(\bar{x},\bar{y}) \in \mathrm{Graph}\,(S)$ if $\mathrm{sur} S\,(\bar{x} \mid \bar{y}) = 0$, and regular otherwise. Also, $\bar{y}$ is a *(proper) critical value* of S if there exists $\bar{x}$ such that $\bar{y} \in S\,(\bar{x})$ and S is critical at $(\bar{x},\bar{y})$.

This definition of critical values characterizes the values at which metric regularity is absent. In the particular case where $S : \mathbb{R}^n \to \mathbb{R}^m$ is a $\mathscr{C}^1$ function, critical points correspond exactly to where the Jacobian has rank less than m. We refer to [14] for more details.

One easily sees that if y is a Pareto minimum value of S, then there exists $x \in X$ such that $(x,y) \in \mathrm{Graph}\,(S)$, and $\mathrm{Sur}\, S\,(x \mid y)\,(\lambda) = 0$ for all small $\lambda > 0$. This readily implies that y is a critical value.

3.2 Extending the Mordukhovich criterion and a critical value result

The two results of this subsection are important ingredients of the forthcoming proof of our main theorem. The first result we need is an adaptation of the Mordukhovich criterion (Proposition 8) to the case where the domain of a set-valued function S is (included in) a smooth submanifold $\mathscr{X}$ of $\mathbb{R}^n$. (Note that this new statement recovers the Mordukhovich criterion if $\mathscr{X} = \mathbb{R}^n$.)

Proposition 14. *(Extended Mordukhovich criterion) Let $\mathscr{X} \subset \mathbb{R}^n$ be a $\mathscr{C}^1$ smooth submanifold of dimension d and $S : \mathscr{X} \rightrightarrows \mathbb{R}^m$ be a set-valued map whose graph is locally closed at $(\bar{x},\bar{y}) \in \mathrm{Graph}\,(S)$. Consider the mapping*

$$\begin{cases} H : \mathbb{R}^m \rightrightarrows \mathbb{R}^n \\ H\,(y^*) = D^*S\,(\bar{x} \mid \bar{y})\,(y^*) \cap T_{\mathscr{X}}\,(\bar{x})\,. \end{cases}$$

If $H\,(\mathbf{0}_m) = \{\mathbf{0}_n\}$, or equivalently

$$N_{\mathrm{Graph}(S)}\,(\bar{x},\bar{y}) \cap (T_{\mathscr{X}}\,(\bar{x}) \times \{\mathbf{0}_m\}) = \{\mathbf{0}_{n+m}\}\,,$$

then S has the Aubin property at $\bar{x}$ for $\bar{y}$ relative to $\mathscr{X}$. Furthermore,

$$\mathrm{lip}_{\mathscr{X}} S\,(\bar{x} \mid \bar{y}) = |H|^+ = \sup\left\{\frac{|u|}{|v|} \mid (u,v) \in N_{\mathrm{Graph}(S)}\,(\bar{x},\bar{y}) \cap (T_{\mathscr{X}}\,(\bar{x}) \times \mathbb{R}^m)\right\}.$$

Proof. Fix $(\bar{x},\bar{y}) \in \mathrm{Graph}\,(S)$ and denote by $N_{\mathscr{X}}\,(\bar{x})$ the normal space of $\mathscr{X}$ at $\bar{x}$ (seing as subspace of $\mathbb{R}^n$, that is, $T_{\mathscr{X}}\,(\bar{x}) \oplus N_{\mathscr{X}}\,(\bar{x}) = \mathbb{R}^n$). Given a closed neighborhood U of $(\bar{x},\bar{y})$, we define the function

$$\begin{cases} \tilde{S} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m \\ \mathrm{Graph}\left(\tilde{S}\right) = (\mathrm{Graph}\,(S) \cap U) + (N_{\mathscr{X}}\,(\bar{x}) \times \{\mathbf{0}_m\})\,. \end{cases}$$

Shrinking the neighborhood U around $(\bar{x},\bar{y})$ if necessary, we may assume that every $(x,y) \in U$ can be represented uniquely as a sum of elements in $(\mathscr{X} \times \mathbb{R}^m) \cap U$ and $N_{\mathscr{X}}\,(\bar{x}) \times \{\mathbf{0}_m\}$. Since $\mathrm{Graph}\,(S)$ is locally closed, we can choose U small enough so that $\mathrm{Graph}\,(S) \cap U$ is closed. Further, since $\mathrm{Graph}\left(\tilde{S}\right)$ is homeomorphic to $(\mathrm{Graph}\,(S) \cap U) \times \mathbb{R}^{n-d}$, it is also closed.

Step 1: (Relating $\tilde{S}$ to H) By applying a result on the normal cones under set addition [22, Exercise 6.44], we have $N_{\mathrm{Graph}(\tilde{S})}\,(\bar{x},\bar{y}) \subset N_{\mathrm{Graph}(S)}\,(\bar{x},\bar{y}) \cap (T_{\mathscr{X}}\,(\bar{x}) \times \mathbb{R}^m)$. To prove the reverse inclusion, note that for every $(x,y) \in \mathrm{Graph}\left(\tilde{S}\right)$ near $(\bar{x},\bar{y})$ with $(x,y) = (x_1,y)+(x_2,\mathbf{0}_m)$, where $(x_1,y) \in \mathrm{Graph}(S)$ and $x_2 \in N_{\mathscr{X}}(\bar{x})$, one easily sees that $\hat{N}_{\mathrm{Graph}(\tilde{S})}(x,y) \supset \hat{N}_{\mathrm{Graph}(S)}(x_1,y) \cap (T_{\mathscr{X}}(\bar{x}) \times \mathbb{R}^m)$. The extension of this inclusion to limiting normal cones is immediate. Therefore we obtain

$$N_{\mathrm{Graph}(\tilde{S})}(\bar{x},\bar{y}) = N_{\mathrm{Graph}(S)}(\bar{x},\bar{y}) \cap (T_{\mathscr{X}}(\bar{x}) \times \mathbb{R}^m),$$

and so $D^*\tilde{S}(\bar{x} \mid \bar{y})$ equals the set-valued map H described in the statement. Thus

$$\begin{aligned} D^*\tilde{S}(\bar{x} \mid \bar{y})(\mathbf{0}_m) &= \left\{x^* \mid (x^*,\mathbf{0}_m) \in N_{\mathrm{Graph}(\tilde{S})}(\bar{x},\bar{y})\right\} \\ &= \left\{x^* \mid (x^*,\mathbf{0}_m) \in N_{\mathrm{Graph}(S)}(\bar{x},\bar{y}) \cap (T_{\mathscr{X}}(\bar{x}) \times \mathbb{R}^m)\right\} \\ &= \{\mathbf{0}_n\}, \end{aligned}$$

and by the Mordukhovich criterion, the map $\tilde{S}$ has the Aubin property at $\bar{x}$ for $\bar{y}$.

Taking neighborhoods V of $\bar{x}$ and W of $\bar{y}$ so that $S(x) \cap W = \tilde{S}(x) \cap W$ for all $x \in V \cap \mathscr{X}$, we deduce that S has the Aubin property at $\bar{x}$ for $\bar{y}$ relative to $\mathscr{X}$ as asserted.

Step 2: ($\mathrm{lip}_{\mathscr{X}} S(\bar{x} \mid \bar{y}) = |H|^+$) The Mordukhovich criterion on $\tilde{S}$ yields

$$|H|^+ = \mathrm{lip}\,\tilde{S}(\bar{x} \mid \bar{y}) \geq \mathrm{lip}_{\mathscr{X}} S(\bar{x} \mid \bar{y}).$$

Our task is thus to prove that the above inequality is actually an equality. Since $\mathrm{lip}\,\tilde{S}(\bar{x} \mid \bar{y}) = |H|^+$, for any $\kappa < |H|^+$ and neighborhoods V of $\bar{x}$ and W of $\bar{y}$, there exist $x_1, x_2 \in V$ such that

$$\tilde{S}(x_2) \cap W \not\subset \tilde{S}(x_1) + \kappa |x_1 - x_2| \mathbb{B}.$$

Note that $\tilde{S}(x_1) = \tilde{S}(P(x_1))$, $\tilde{S}(x_2) = \tilde{S}(P(x_2))$ and $|P(x_1) - P(x_2)| \leq |x_1 - x_2|$, where P stands for the projection of $\mathbb{R}^n$ onto $\bar{x} + T_{\mathscr{X}}(\bar{x})$. We may choose V to be a ball containing $\bar{x}$, and define the projection parametrization $L : (\bar{x} + T_{\mathscr{X}}(\bar{x})) \cap V \to \mathscr{X}$ of the manifold $\mathscr{X}$ by the relation $x - L(x) \in N_{\mathscr{X}}(\bar{x})$. Shrinking V if needed, the map L becomes single-valued and smooth (in fact, it is a local chart of $\mathscr{X}$ at $\bar{x}$ provided we identify $\bar{x} + T_{\mathscr{X}}(\bar{x})$ with $\mathbb{R}^d$). Furthermore, L has Lipschitz constant 1 at $\bar{x}$. Therefore, for any $\varepsilon > 0$, we can reduce V as needed so that L is Lipschitz continuous in its domain with Lipschitz constant at most $(1+\varepsilon)$ using standard arguments (*e.g.* [22, Thms 9.7, 9.2]). This means that

$$S(L(x_2)) \cap W = \tilde{S}(x_2) \cap W \not\subset \tilde{S}(x_1) + \kappa |x_1 - x_2| \mathbb{B} = S(L(x_1)) + \kappa |x_1 - x_2| \mathbb{B}.$$

By the Lipschitz continuity of L, we have $|L(x_1) - L(x_2)| \leq (1+\varepsilon)|x_1 - x_2|$, which gives

$$S(L(x_2)) \cap W \not\subset S(L(x_2)) + \frac{\kappa}{(1+\varepsilon)} |L(x_1) - L(x_2)| \mathbb{B},$$

yielding

$$\frac{\kappa}{1+\varepsilon} \leq \mathrm{lip}_{\mathscr{X}} S(\bar{x} \mid \bar{y}).$$

Since κ and ε are arbitrary, we conclude that $|H|^+ = \mathrm{lip}_{\mathscr{X}} S(\bar{x} \mid \bar{y})$ as asserted.

The proof is complete. □

The second result is an adaptation of part of [14, Theorem 6]. Recall that for a smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$, $\bar{x} \in \mathbb{R}^n$ is a *critical point* if the derivative $\nabla f(\bar{x})$ is not surjective, while $\bar{y} \in \mathbb{R}^m$ is a *critical value* if there is a critical point $\bar{x}$ for which $f(\bar{x}) = \bar{y}$. (Note this is a particular case of the general definition given in Definition 13.)

Lemma 15. *Let $\mathscr{X}$ be a $\mathscr{C}^k$ smooth manifold in $\mathbb{R}^n$ of dimension d, and $\mathscr{M}$ be a $\mathscr{C}^k$ manifold in $\mathbb{R}^{n+m}$ such that $\mathscr{M} \subset \mathscr{X} \times \mathbb{R}^m$, with $k > \dim \mathscr{M} - \dim \mathscr{X}$. Then the set of points $x \in \mathscr{X}$ such that there exists some y satisfying $(x,y) \in \mathscr{M}$ and $N_{\mathscr{M}}(x,y) \cap (T_{\mathscr{X}}(x) \times \{\mathbf{0}_m\}) \supsetneq \{\mathbf{0}_{n+m}\}$ is of Lebesgue measure zero in $\mathscr{X}$.*

Proof. Let $\mathrm{Proj}_{\mathscr{M}}$ denote the restriction to the manifold $\mathscr{M}$ of the projection of $\mathscr{X} \times \mathbb{R}^m$ onto $\mathscr{X}$. As $k > \dim \mathscr{M} - \dim \mathscr{X}$, the set of critical values of $\mathrm{Proj}_{\mathscr{M}}$ is a set of measure zero by the classical Sard theorem [23]. Let $(x,y) \in \mathscr{M}$ and assume $(x^*, \mathbf{0}_m) \in N_{\mathscr{M}}(x,y) \cap (T_{\mathscr{X}}(x) \times \{\mathbf{0}_m\})$ with $x^* \neq \mathbf{0}_n$. This gives

$$T_{\mathscr{M}}(x,y) = (N_{\mathscr{M}}(x,y))^{\perp} \subset \{x^*\}^{\perp} \times \mathbb{R}^m,$$

where $\{x^*\}^{\perp} = \{x' \in \mathbb{R}^n \mid \langle x^*, x' \rangle = 0\}$. Since $T_{\mathscr{M}}(x,y) \subset T_{\mathscr{X}}(x) \times \mathbb{R}^m$ we obtain

$$T_{\mathscr{M}}(x,y) \subset \Big(\{x^*\}^{\perp} \cap T_{\mathscr{X}}(x)\Big) \times \mathbb{R}^m.$$

Let Z stand for the subspace on the right hand side. Then the projection of Z onto $T_{\mathscr{X}}(x)$ is a proper subspace of $T_{\mathscr{X}}(x)$. All the more, this applies to $T_{\mathscr{M}}(x,y)$. By [14, Corollary 3], this implies that (x,y) is a singular point of $\mathrm{Proj}_{\mathscr{M}}$, so x is a critical value of $\mathrm{Proj}_{\mathscr{M}}$. The conclusion of the lemma follows. □

3.3 Linking sets

We introduce the notion of *linking* that is commonly used in critical point theory. Let us fix some terminology: if $B \subset \mathbb{R}^n$ is homeomorphic to a subset of $\mathbb{R}^d$ with nonempty interior, we say that the set ∂B is the *relative boundary* of B if it is a homeomorphic image of the boundary of the set in $\mathbb{R}^d$.

Definition 16. [25, Section II.8] Let A be a subset of $\mathbb{R}^{n+m}$ and let B be a submanifold of $\mathbb{R}^{n+m}$ with relative boundary ∂B. Then we say that A and $\Gamma = \partial B$ *link* if

(i) $A \cap \Gamma = \emptyset$

(ii) for any continuous map $h \in \mathscr{C}^0(\mathbb{R}^{n+m}, \mathbb{R}^{n+m})$ such that $h|_{\Gamma} = id$ we have $h(B) \cap A \neq \emptyset$.

In particular, the following result holds. This result will be used in Section 4.

Theorem 17 (Linking sets)**.** *Let $\mathscr{K}_1$ and $\mathscr{K}_2$ be linear subspaces such that $\mathscr{K}_1 \oplus \mathscr{K}_2 = \mathbb{R}^{n+m}$, and take any $\bar{v} \in \mathscr{K}_1 \setminus \{\mathbf{0}\}$. Then for $0 < r < R$, the sets*

$$A := \mathbb{S}(\mathbf{0}, r) \cap \mathscr{K}_1 \qquad \text{and} \qquad \Gamma := (\mathbb{B}(\mathbf{0}, R) \cap \mathscr{K}_2) \cup (\mathbb{S}(\mathbf{0}, R) \cap (\mathscr{K}_2 + \mathbb{R}_+ \{\bar{v}\}))$$

link.

Proof. Use methods in [25, Section II.8], or infer from Example 3 there. □

We finish this section with two useful results. The first one is well-known (with elementary proof) and is mentioned for completeness.

Proposition 18. *If $\mathscr{K}_1$ and $\mathscr{K}_2$ are subspaces of $\mathbb{R}^{n+m}$, then $\mathscr{K}_1^{\perp} \cap \mathscr{K}_2^{\perp} = \{\mathbf{0}\}$ if and only if $\mathscr{K}_1 + \mathscr{K}_2 = \mathbb{R}^{n+m}$.*

The following lemma will be needed in the proof of forthcoming Lemma 26 (Section 4).

Lemma 19. *If the sets $\mathbb{B}(\mathbf{0}, 1)$ and D are homeomorphic, then any homeomorphism f between $\mathbb{S}(\mathbf{0}, 1)$ and ∂D can be extended to a homeomorphism $F : \mathbb{B}(\mathbf{0}, 1) \to D$ so that $F|_{\mathbb{S}(\mathbf{0},1)} = f$.*

Proof. Let $H : \mathbb{B}(\mathbf{0},1) \to D$ be a homeomorphism between $\mathbb{B}(\mathbf{0},1)$ and D and denote $h : \mathbb{S}(\mathbf{0},1) \to \partial D$ by $h = H\mid_{\mathbb{S}(\mathbf{0},1)}$. We define the (continuous) function $F : \mathbb{B}(\mathbf{0},1) \to D$ by

$$F(x) = \begin{cases} H\left(|x|\, h^{-1}(f(x/|x|)\right) & \text{if } x \neq \mathbf{0} \\ H(\mathbf{0}) & \text{if } x = \mathbf{0}. \end{cases}$$

It is straightforward to check that $F\mid_{\mathbb{S}(\mathbf{0},1)} = f$. Let us show that F is injective: indeed, if $F(x_1) = F(x_2)$, then $|x_1|\, h^{-1}(f(x_1/|x_1|)) = |x_2|\, h^{-1}(f(x_2/|x_2|))$. If both sides are zero, then $x_1 = x_2 = \mathbf{0}$. Otherwise $|x_1| = |x_2|$ and $x_1/|x_1| = x_2/|x_2|$, which implies that $x_1 = x_2$.

To see that F is a bijection, fix any $y \in D$, and let $x' \in \mathbb{B}(\mathbf{0},1)$ be such that $y = H(x')$. If $x' = \mathbf{0}$, then $y = F(\mathbf{0})$. Otherwise,

$$y = H\left(|x'|\left(\frac{x'}{|x'|}\right)\right) = H(|x'|\, h^{-1} \circ f\left(f^{-1} \circ h(\frac{x'}{|x'|})\right) = F\left(|x'|\, f^{-1} \circ h(\frac{x'}{|x'|})\right).$$

This shows that F is also surjective, thus a continuous bijection. Since $\mathbb{B}(\mathbf{0},1)$ is compact, it follows that F is a homeomorphism. $\square$

4 Generic continuity of tame set-valued maps

From now on we limit our attention to the class of semialgebraic (or more generally, o-minimal) set-valued maps. In this setting our main result eventually asserts that every such set-valued map is generically strictly continuous (see Section 4.3). To prove this, we shall need several technical lemmas, given in Section 4.2. In Section 4.1 we give preliminary definitions and results of our setting.

4.1 Semialgebraic and definable mappings

In this section we recall basic notions from semialgebraic and o-minimal geometry. Let us define properly the notion of a semialgebraic set ([3], [8]). (We denote by $\mathbb{R}[x_1,\ldots,x_n]$ the ring of real polynomials of n variables.)

Definition 20 (Semialgebraic set)**.** A subset A of $\mathbb{R}^n$ is called *semialgebraic* if it has the form

$$A = \bigcup_{i=1}^{k} \{x \in \mathbb{R}^n : p_i(x) = 0, q_{i1}(x) > 0, \ldots, q_{i\ell}(x) > 0\},$$

where $p_i, q_{ij} \in \mathbb{R}[x_1,\ldots,x_n]$ for all $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,\ell\}$.

In other words, a set is semialgebraic if it is a finite union of sets that are defined by means of a finite number of polynomial equalities and inequalities. A set-valued map $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called *semialgebraic*, if its graph $\mathrm{Graph}(S)$ is a semialgebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$.

An important property of semialgebraic sets is that of Whitney stratification ([11, §4.2], [8, Theorem 6.6]).

Theorem 21. *($\mathscr{C}^k$ stratification) For any $k \in \mathbb{N}$ and any semialgebraic subsets $X_1,\ldots,X_l$ of $\mathbb{R}^n$, we can write $\mathbb{R}^n$ as a disjoint union of finitely many semialgebraic $\mathscr{C}^k$ manifolds $\{\mathscr{M}_i\}_i$ (that is, $\mathbb{R}^n = \dot{\bigcup}_{i=1}^{I} \mathscr{M}_i$) so that each X_j is a finite union of some of the $\mathscr{M}_i$'s. Moreover, the induced stratification $\{\mathscr{M}_i^j\}_i$ of X_j has the Whitney property, that is, for any sequence $\{x_\nu\}_\nu \subset \mathscr{M}_i^j$ converging to $x \in \mathscr{M}_{i_0}^j$ we have*

$$\limsup_{\nu\to\infty} N_{\mathscr{M}_i^j}(x_\nu) \subset N_{\mathscr{M}_{i_0}^j}(x_\nu).$$

In particular, every semi-algebraic set can be written as a finite disjoint union of manifolds ("strata") that fit together in a regular way ("Whitney stratification"). (The Whitney property is also called *normal regularity* of the stratification, see [14, Definition 5].) The *dimension* $\dim(X)$ of a semialgebraic set X can thus be defined as the dimension of the manifold of highest dimension of its stratification. This dimension is well defined and independent of the stratification of X [8, Section 3.3].

As a matter of the fact, semialgebraic sets constitute an *o-minimal structure*. Let us recall the definitions of the latter (see for instance [9], [11]).

Definition 22 (o-minimal structure)**.** An o-minimal structure on $(\mathbb{R},+,.)$ is a sequence of Boolean algebras $\mathscr{O} = \{\mathscr{O}_n\}$, where each algebra $\mathscr{O}_n$ consists of subsets of $\mathbb{R}^n$, called *definable* (in $\mathscr{O}$), and such that for every dimension $n \in \mathbb{N}$ the following properties hold.

(i) For any set A belonging to $\mathscr{O}_n$, both $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belong to $\mathscr{O}_{n+1}$.

(ii) If $\Pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denotes the canonical projection, then for any set A belonging to $\mathscr{O}_{n+1}$, the set $\Pi(A)$ belongs to $\mathscr{O}_n$.

(iii) $\mathscr{O}_n$ contains every set of the form $\{x \in \mathbb{R}^n : p(x) = 0\}$, for polynomials $p : \mathbb{R}^n \to \mathbb{R}$.

(iv) The elements of $\mathscr{O}_1$ are exactly the finite unions of intervals and points.

When $\mathscr{O}$ is a given o-minimal structure, a function $f : \mathbb{R}^n \to \mathbb{R}^m$ (or a set-valued mapping $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$) is called *definable* (in $\mathscr{O}$) if its graph is definable as a subset of $\mathbb{R}^n \times \mathbb{R}^m$.

It is obvious by definition that semialgebraic sets are stable under Boolean operations. As a consequence of the Tarski-Seidenberg principle, they are also stable under projections, thus they satisfy the above properties. Nonetheless, broader o-minimal structures also exist. In particular, the Gabrielov theorem implies that "globally subanalytic" sets are o-minimal. These two structures in particular provide rich practical tools, because checking semi-algebraicity or subanalyticity of sets in concrete problems of variational analysis is often easy. We refer to [4], [5], and [15] for details. Let us mention that Theorem 21 still holds in an arbitrary o-minimal structure (it is sufficient to replace the word "semialgebraic" by "definable" in the statement). As a matter of the fact, the statement of Theorem 21 can be reinforced for definable sets (namely, the stratification can be taken analytic), but this is not necessary for our purposes.

Remark. Besides formulating our results and main theorem for semialgebraic sets (the reason being their simple definition), the validity of these results is not confined to this class. In fact, all forthcoming statements will still hold for "definable" sets (replace "semialgebraic" by "definable in an o-minimal structure") with an identical proof. Moreover, since our key arguments are essentially of a local nature, one can go even further and formulate the results for the so-called *tame* sets (e.g. [5], [15]), that is, sets whose intersection with every ball is definable in some o-minimal structure. (In the latter case though, slight technical details should be taken into consideration.)

We close this section by mentioning an important property of semialgebraic (more generally, o-minimal) sets. Let us recall that (topological) genericity and full measure (*i.e.,* almost everywhere) are different ways to affirm that a given property holds in a large set. However, these notions are often complementary. In particular, it is possible for a (topologically) generic subset of $\mathbb{R}^n$ to be of null measure, or for a full measure set to be meager (see [20] for example). Nonetheless, this situation disappears in our setting.

Proposition 23 (Genericity in a semialgebraic setting)**.** *Let U,V be semialgebraic subsets of $\mathbb{R}^n$, and assume $V \subset U$. Then the following properties are equivalent:*

(i) V is dense in U ;
(ii) V is (topologically) generic in U ;
(iii) $U \setminus V$ is of null (Lebesgue) measure ;
(iv) the dimension of $U \setminus V$ is strictly smaller than that of U.

4.2 Some technical results

In the sequel we shall always consider a set-valued map $S : \mathscr{X} \rightrightarrows \mathbb{R}^m$, where $\mathscr{X} \subset \mathbb{R}^n$, and we shall assume that S is semialgebraic.

Theorem 24. *Assume that $S : \mathscr{X} \rightrightarrows \mathbb{R}^m$ is outer semicontinuous, and the sets $\mathscr{X} \subset \mathbb{R}^n$ and* $\operatorname{Graph}(S)$ *are semi-algebraic. Then S is strictly continuous with respect to $\mathscr{X}$ everywhere except on a set of dimension at most* $(\dim \mathscr{X} - 1)$.

Proof. Using Theorem 21 we stratify $\mathscr{X}$ into a disjoint union of manifolds (strata) $\{\mathscr{X}_j\}_j$ and study how S behaves on the strata $\mathscr{X}_j$ of full dimension (that is, $\dim(\mathscr{X}_j) = \dim(\mathscr{X}) = d \leq n$). For each such stratum $\mathscr{X}_j$, if S is not strictly continuous at $\bar{x} \in \mathscr{X}_j$ relative to $\mathscr{X}_j$, then by [22, Theorem 9.38], there is some $\bar{y} \in S(\bar{x})$ such that $\operatorname{lip}_{\mathscr{X}_j} S(\bar{x} \mid \bar{y}) = \infty$. Since S is outer semicontinuous, we deduce from Proposition 14 that there is a nonzero vector $v \in N_{\operatorname{Graph}(S)}(\bar{x},\bar{y}) \cap \left(T_{\mathscr{X}_j}(\bar{x}) \times \{\mathbf{0}_m\}\right)$.

We now stratify the semialgebraic set $\operatorname{Graph}(S) \cap (\mathscr{X}_j \times \mathbb{R}^m)$ into a finite union of disjoint manifolds $\{\mathscr{M}_k\}_k$. Since $v \in N_{\operatorname{Graph}(S)}(\bar{x},\bar{y}) \setminus \{\mathbf{0}_{n+m}\}$, it can be written as a limit of Hadamard normal vectors $v_i \in \hat{N}_{\operatorname{Graph}(S)}(x_i,y_i)$ with $(x_i,y_i) \to (\bar{x},\bar{y})$. Passing to a subsequence if necessary, we may assume that the sequence $\{(x_i,y_i)\}_i$ belongs to the same stratum, say $\mathscr{M}_{k^*}$ and $v_i \in \hat{N}_{\mathscr{M}_{k^*}}(x_i,y_i)$ (note that $\mathscr{M}_{k^*} \subset \operatorname{Graph}(S)$). Since $\mathscr{M}_{k^*}$ is a smooth manifold, we have $\hat{N}_{\mathscr{M}_{k^*}}(x_i,y_i) = N_{\mathscr{M}_{k^*}}(x_i,y_i) = [T_{\mathscr{M}_{k^*}}(x_i,y_i)]^{\perp}$. Using the Whitney property (normal regularity) of the stratification, we deduce that v must lie in some $N_{\mathscr{M}}(\bar{x},\bar{y}) \cap \left(T_{\mathscr{X}_j}(\bar{x}) \times \{\mathbf{0}_m\}\right)$, where $\mathscr{M}$ is the stratum that contains $(\bar{x},\bar{y})$. Lemma 15 then tells us that the set of all possible $\bar{x}$ is of lower dimension than that of $\mathscr{X}_j$ (or $\mathscr{X}$). Since there are finitely many strata $\mathscr{X}_j$, the result follows. $\square$

Remark. Note that the domain of S

$$\operatorname{dom}(S) := \{x \in \mathscr{X} : S(x) \neq \emptyset\},$$

being the projection to $\mathbb{R}^n$ of the semialgebraic set $\operatorname{Graph}(S)$, is always semialgebraic. Thus, if S has nonempty values, the above assumption "$\mathscr{X}$ semialgebraic" becomes superfluous. In any case, one can eliminate this assumption from the statement and replace $\mathscr{X}$ by $\mathscr{X}' := \operatorname{dom}(S)$ the domain of S.

The next lemma will be crucial in the sequel. We shall first need some notation. In the sequel we denote by

$$\mathscr{L} := \{\mathbf{0}_n\} \times \mathbb{R}^m \tag{4.1}$$

as a subspace of $\mathbb{R}^n \times \mathbb{R}^m$ and we denote by $\bar{S} : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ the set-valued map whose graph is the closure of the graph of S, that is,

$$\operatorname{Graph}\left(\bar{S}\right) = \operatorname{cl}\left(\operatorname{Graph}(S)\right).$$

Lemma 25. *Let $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a closed-valued semialgebraic set-valued map. For any $k > 0$, there is a $\mathscr{C}^k$ stratification $\{\mathscr{M}_i\}_i$ of* Graph(S) *such that if $S(\bar{x}) \neq \bar{S}(\bar{x})$ for some $\bar{x} \in \mathbb{R}^n$, then there exist $\bar{y} \in \mathbb{R}^m$, a stratum $\mathscr{M}_i$ of the stratification of* Graph(S) *and a neighborhood U of $(\bar{x}, \bar{y})$ such that $(\bar{x}, \bar{y}) \in \mathrm{cl}(\mathscr{M}_i)$ and*

$$((\bar{x}, \bar{y}) + \mathscr{L}) \cap \mathscr{M}_i \cap U = \emptyset.$$

Proof. By Theorem 21 we stratify Graph(S) into a disjoint union of finitely many manifolds, that is Graph$(S) = \cup_i \mathscr{M}_i$. Consider the set-valued map $S_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ whose graph consists of the manifold $\mathscr{M}_i$. Let further $\dot{S}_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be the map such that $\dot{S}_i(x) = \mathrm{cl}(S_i(x))$ for all x, and $\bar{S}_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be the map whose graph is cl (Graph(S_i)), also equal to $\mathrm{cl}\left(\mathrm{Graph}\left(\dot{S}_i\right)\right)$. Both $\dot{S}_i$ and $\bar{S}_i$ are semialgebraic (for example, [8]), and there exists a stratification of cl(Graph(S)) such that the graphs of S_i, $\dot{S}_i$ and $\bar{S}_i$ can be represented as a finite union of strata of that stratification, by Theorem 21 again.

We now prove that if $S(\bar{x}) \neq \bar{S}(\bar{x})$, then there is some i such that $\dot{S}_i$ is not outer semicontinuous at $\bar{x}$. Indeed, in this case there exists $\bar{y}$ such that $(\bar{x}, \bar{y}) \in \mathrm{cl}(\mathrm{Graph}(S)) \setminus \mathrm{Graph}(S)$. Note that $\mathrm{cl}(\mathrm{Graph}(S)) = \cup_i \mathrm{Graph}\left(\bar{S}_i\right)$. This means that $(\bar{x}, \bar{y})$ must lie in $\mathrm{Graph}\left(\bar{S}_i\right) \setminus \mathrm{Graph}\left(\dot{S}_i\right)$ for some i, which means that $\dot{S}_i$ is not outer semicontinuous at $\bar{x}$ as claimed.

Obviously $(\bar{x}, \bar{y}) \in \mathrm{cl}(\mathscr{M}_i)$. Suppose that $((\bar{x}, \bar{y}) + \mathscr{L}) \cap \mathscr{M}_i \cap U \neq \emptyset$ for all neighborhoods U containing $(\bar{x}, \bar{y})$. Then there is a sequence $y_j \to \bar{y}$ such that $(\bar{x}, y_j) \in \mathscr{M}_i$. Since $\dot{S}_i$ is closed-valued, this would yield $(\bar{x}, \bar{y}) \in \mathrm{Graph}\left(\dot{S}_i\right)$, which contradicts $(\bar{x}, \bar{y}) \notin \mathrm{Graph}\left(\dot{S}_i\right)$ earlier. □

Keeping now the notation of the proof of the previous lemma, let us set $\bar{z} := (\bar{x}, \bar{y})$. Let further $\mathscr{M}_i, \mathscr{M}'$ be the strata of cl(Graph(S)) such that $z \in \mathscr{M}' \subset \mathrm{cl}(\mathscr{M}_i)$. In the next lemma we are working with normals on manifolds, so it does not matter which kind of normal in Definition 4 we consider.

Lemma 26. *Suppose there is a neighborhood U of $\bar{z}$ such that $\bar{z} \in \mathscr{M}'$, $\mathscr{M}' \subset \mathrm{cl}(\mathscr{M}_i)$ and $(\bar{z} + \mathscr{L}) \cap \mathscr{M}_i \cap U = \emptyset$, where $\mathscr{L}$ is defined in* (4.1). *Then $N_{\mathscr{M}'}(\bar{z}) \cap \mathscr{L}^{\perp} \supsetneq \{\mathbf{0}_{n+m}\}$.*

Proof. We prove the result by contradiction. Suppose that $N_{\mathscr{M}'}(\bar{z}) \cap \mathscr{L}^{\perp} = \{\mathbf{0}_{n+m}\}$. Then $T_{\mathscr{M}'}(\bar{z}) + \mathscr{L} = \mathbb{R}^{n+m}$ by Proposition 18. We may assume, by taking a submanifold of $\mathscr{M}'$ if necessary, that $\dim \mathscr{M}' = n$ so that $\dim \mathscr{M}' + \dim \mathscr{L} = n + m$ and $T_{\mathscr{M}'}(\bar{z}) \oplus \mathscr{L} = \mathbb{R}^{n+m}$. Owing to the so-called wink lemma (see [10, Proposition 5.10] *e.g.*) we may assume that $\dim \mathscr{M}_i = n + 1$.

(Case $m = 1$**)** We first consider the case where $m = 1$. In this case, the subspace $\mathscr{L}$ is a line whose spanning vector $v = (\mathbf{0}, 1)$ is not in $T_{\mathscr{M}'}(\bar{z})$. There is a neighborhood U' of $\bar{z}$ such that $U' \subset U$, $\mathscr{M}' \cap U'$ equals $f^{-1}(0)$ for some smooth function $f : U' \to \mathbb{R}$ (local equation of $\mathscr{M}'$), and $\mathscr{M}_i \cap U' = f^{-1}((0, \infty))$. The gradient $\nabla f(\bar{z})$ is nonzero and is not orthogonal to v since $T_{\mathscr{M}'}(\bar{z})$ is the set of vectors orthogonal to $\nabla f(\bar{z})$ and $T_{\mathscr{M}'}(\bar{z}) \oplus \mathscr{L} = \mathbb{R}^{n+1}$. There are points in $(\bar{z} + \mathscr{L}) \cap U'$ such that f is positive, which means that $(\bar{z} + \mathscr{L}) \cap \mathscr{M}_i \cap U' \neq \emptyset$, contradicting the stipulated conditions. Therefore, we assume that $m > 1$ for the rest of the proof.

(Case $m > 1$**)** As in the previous case, we shall eventually prove that $(\bar{z} + \mathscr{L}) \cap \mathscr{M}_i \cap U' \neq \emptyset$ reaching to a contradiction. To this end, let us denote by h_0 the (semialgebraic) homeomorphism of $\mathbb{R}^{n+m}$ to $\mathbb{R}^{n+m}$ which, for some neighborhood $V \subset U$ of $\bar{z}$, maps homeomorphically $V \cap (\mathscr{M}_i \cup \mathscr{M}')$ to $\mathbb{R}^n \times (\mathbb{R}_+ \times \{\mathbf{0}_{m-1}\}) \subset \mathbb{R}^{n+m}$ and $V \cap \mathscr{M}'$ to $\mathbb{R}^n \times \{\mathbf{0}_m\}$ (see [8, Theorem 3.12] *e.g.*).

Claim. We first show that there exists a closed neighborhood $W \subset V$ of $\bar{z}$ such that $W \cap \mathscr{M}'$ and $\partial W \cap \mathscr{M}_i$ are both homeomorphic to $\mathbb{B}^n$ and $W \cap \mathscr{M}' = \mathbb{B}^{n+m}(\bar{z}, R_1) \cap \mathscr{M}'$ for some $R_1 > 0$.

Since $\mathscr{M}'$ is a smooth manifold, there exists $R_1 > 0$ such that $\mathbb{B}^{n+m}(\bar{z}, R_1) \cap \mathscr{M}'$ is homeomorphic (in fact, diffeomorphic) to $(T_{\mathscr{M}'}(\bar{z}) + \bar{z}) \cap \mathbb{B}^{n+m}(\bar{z}, R_1)$, which in turn is homeomorphic to $\mathbb{B}^n$, as is shown by the homeomorphism:

$$z \mapsto \left(\frac{|z - \bar{z}|}{|P(z) - \bar{z}|} (P(z) - \bar{z}) \right) + \bar{z},$$

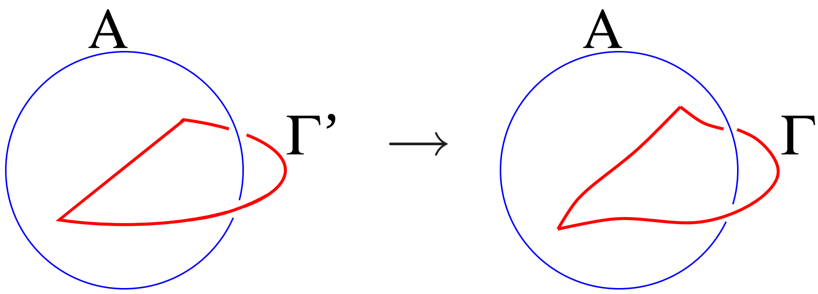

Figure 4.1: Linking sets (A,Γ') and (A,Γ).

where P denotes the projection onto the tangent space $\bar{z}+T_{\mathscr{M}'}(\bar{z})$. Consider the image of $\mathbb{B}^{n+m}(\bar{z},R_1)\cap\mathscr{M}'$ under the map h_0. This image lies in the set $\mathbb{R}^n\times\{\mathbf{0}_m\}$. Therefore, for $r_1>0$ sufficiently small, the set $W=h_0^{-1}\left(h_0\left(\mathbb{B}^{n+m}(\bar{z},R_1)\cap\mathscr{M}'\right)+[-r_1,r_1]^m\right)$ satisfies the required properties, concluding the proof of our claim.

Let us further fix $v\in\mathscr{L}\setminus\{\mathbf{0}\}$ and consider the set

$$\Gamma' := \underbrace{\left(\mathbb{B}^{n+m}(\bar{z},R)\cap\left(\bar{z}+T_{\mathscr{M}'}(\bar{z})\right)\right)}_{\Gamma_1'}\cup\underbrace{\left(\mathbb{S}^{n+m-1}(\bar{z},R)\cap\left(\bar{z}+T_{\mathscr{M}'}(\bar{z})+\mathbb{R}_+\{v\}\right)\right)}_{\Gamma_2'}.$$

Setting

$$A:=\mathbb{S}^{n+m-1}(\bar{z},r)\cap\mathscr{L},\ \text{where } 0<r<R,$$

we immediately get that the sets A and Γ' link (*c.f.* Theorem 17). Based on this, our objective is to prove that the sets A and Γ also link, where Γ is defined by

$$\Gamma=\underbrace{\left(W\cap\mathscr{M}'\right)}_{\Gamma_1}\cup\underbrace{\left(\partial W\cap\left(\mathscr{M}_i\cup\mathscr{M}'\right)\right)}_{\Gamma_2},$$

provided $r>0$ is chosen appropriately. Once we succeed in doing so, we apply Definition 16 (for $h=id$) to deduce that $(\bar{z}+\mathscr{L})\cap\mathscr{M}_i\cap U\neq\emptyset$, which contradicts our initial assumptions.
Figure 4.1 illustrates the sets A, Γ and Γ' for $n=1$ and $m=2$.

For the sequel, we introduce the notation “$\xrightarrow{\simeq}$” in $f:D_1\xrightarrow{\simeq}D_2$ to mean that f is a homeomorphism between the sets D_1 and D_2. In Step 1 and Step 2, we define a continuous function $H:\left(\mathbb{B}^{n+1}(\mathbf{0},1)\times\{0\}\right)\cup\left(\mathbb{S}^n(\mathbf{0},1)\times[0,2]\right)\to\mathbb{B}^{n+m}(\bar{z},R)$ that will be used in Step 3.

Step 1: Determine H on $\left(\mathbb{B}^{n+1}(\mathbf{0},1)\times\{0\}\right)\cup\left(\mathbb{S}^n(\mathbf{0},1)\times[0,2]\right)$.

In Steps 1 (a) to 1 (c), we define a continuous function H on $\mathbb{S}^n(\mathbf{0},1)\times[0,2]$ so that $H|_{\mathbb{S}^n(\mathbf{0},1)\times[0,2]}$ is a homotopy between Γ and Γ'. More precisely, denoting by

$$\mathbb{S}^n_+(\mathbf{0},1):=\mathbb{S}^n(\mathbf{0},1)\cap\left(\mathbb{R}^n\times[0,\infty)\right),$$
$$\mathbb{S}^n_-(\mathbf{0},1):=\mathbb{S}^n(\mathbf{0},1)\cap\left(\mathbb{R}^n\times(-\infty,0]\right),$$

we want to define H in such a way that its restrictions

$$H(\cdot,0)|_{\mathbb{S}^n_+(\mathbf{0},1)}:\mathbb{S}^n_+(\mathbf{0},1)\xrightarrow{\simeq}\Gamma_1\subset\mathbb{R}^{n+m},$$
$$H(\cdot,0)|_{\mathbb{S}^n_-(\mathbf{0},1)}:\mathbb{S}^n_-(\mathbf{0},1)\xrightarrow{\simeq}\Gamma_2\subset\mathbb{R}^{n+m},$$
$$H(\cdot,2)|_{\mathbb{S}^n_+(\mathbf{0},1)}:\mathbb{S}^n_+(\mathbf{0},1)\xrightarrow{\simeq}\Gamma_1'\subset\mathbb{R}^{n+m},$$
$$H(\cdot,2)|_{\mathbb{S}^n_-(\mathbf{0},1)}:\mathbb{S}^n_-(\mathbf{0},1)\xrightarrow{\simeq}\Gamma_2'\subset\mathbb{R}^{n+m},$$

are homeomorphisms between the respective spaces. Note that both $\mathbb{S}^n_+(\mathbf{0},1)$ and $\mathbb{S}^n_-(\mathbf{0},1)$ are homeomorphic to $\mathbb{B}^n(\mathbf{0},1)$. For notational convenience, we denote by $\mathbb{S}^n_=(\mathbf{0},1)$ the set $\mathbb{S}^n(\mathbf{0},1)\cap(\mathbb{R}^n\times\{0\}) = \mathbb{S}^{n-1}(\mathbf{0},1)\times\{0\}$.

Step 1 (a). **Determine H on $\mathbb{S}(\mathbf{0},1)\times[0,1]$.**
Since $\partial W\cap\operatorname{cl}\mathscr{M}_i$ is a closed set that does not contain $\bar{z}$, there is some $R>0$ such that $(\partial W\cap\mathscr{M}_i)\cap\mathbb{B}^{n+m}(\bar{z},R)=\emptyset$ and $\mathbb{B}^{n+m}(\bar{z},R)\subset U$. We proceed to create the homotopy H so that

$$H(\cdot,1)\mid_{\mathbb{S}^n_+(\mathbf{0},1)}:\mathbb{S}^n_+(\mathbf{0},1)\xrightarrow{\simeq}\Gamma_1''\subset\mathbb{R}^{n+m},$$
$$H(\cdot,1)\mid_{\mathbb{S}^n_-(\mathbf{0},1)}:\mathbb{S}^n_-(\mathbf{0},1)\xrightarrow{\simeq}\Gamma_2''\subset\mathbb{R}^{n+m},$$

where

$$\begin{aligned}&\Gamma_1''=\mathbb{B}^{n+m}(\bar{z},R)\cap\mathscr{M}',\\ \text{and}\qquad&\Gamma_2''\subset\mathbb{S}^{n+m-1}(\bar{z},R)\text{ is homeomorphic to }\Gamma_2.\end{aligned}$$

The first homotopy between Γ_1 and Γ_1'' can be chosen such that $H(s,t)\in\mathscr{M}'$ for all $s\in\mathbb{S}^n_+(\mathbf{0},1)$ and $t\in[0,1]$. We also require that $d(\bar{z},H(s,t))\geq R$ for all $s\in\mathbb{S}^n_=(\mathbf{0},1)$ and $t\in[0,1]$, which does not present any difficulties.

For the second homotopy between Γ_2 and Γ_2'', we first extend $H(\cdot,1)$ so that $H(\cdot,1)\mid_{\mathbb{S}^n(\mathbf{0},1)}:\mathbb{S}^n(\mathbf{0},1)\xrightarrow{\simeq}\Gamma_1''\cup\Gamma_2''$ is a homeomorphism between the corresponding spaces. This is achieved by showing that there is a homeomorphism $H(\cdot,1)\mid_{\mathbb{S}^n_-(\mathbf{0},1)}$ between $\mathbb{S}^n_-(\mathbf{0},1)$ and Γ_2''. Let $h_2:\mathbb{B}^n(\mathbf{0},1)\xrightarrow{\simeq}\mathbb{S}^n_-(\mathbf{0},1)$ be a homeomorphism between $\mathbb{B}^n(\mathbf{0},1)$ and $\mathbb{S}^n_-(\mathbf{0},1)$. Then $H(\cdot,1)\mid_{\mathbb{S}^n_=(\mathbf{0},1)}\circ h_2\mid_{\mathbb{S}^{n-1}(\mathbf{0},1)}:\mathbb{S}^{n-1}(\mathbf{0},1)\xrightarrow{\simeq}\partial\Gamma_2''$. By Lemma 19 this can be extended to a homeomorphism $G:\mathbb{B}^n(\mathbf{0},1)\xrightarrow{\simeq}\Gamma_2''$. Define $H(\cdot,1)\mid_{\mathbb{S}^n_-(\mathbf{0},1)}:\mathbb{S}^n_-(\mathbf{0},1)\xrightarrow{\simeq}\Gamma_2''$ by $H(\cdot,1)\mid_{\mathbb{S}^n_-(\mathbf{0},1)}=G\circ h_2^{-1}$.

It remains to resolve H on $\mathbb{S}^n_-(\mathbf{0},1)\times(0,1)$. Note that the sets

$$H(\mathbb{S}^n_=(\mathbf{0},1)\times[0,1]),\quad H\left(\mathbb{S}^n_-(\mathbf{0},1)\times\{0\}\right)=\Gamma_2\quad\text{and}\quad H\left(\mathbb{S}^n_-(\mathbf{0},1)\times\{1\}\right)=\Gamma_2''$$

are all of dimension at most n, so the radial projection of these sets onto $\mathbb{S}^{n+m-1}(\bar{z},R)$ is of dimension at most n. Since $\mathbb{S}^{n+m-1}(\bar{z},R)$ is of dimension at least $n+1$, we can find some point $p\in\mathbb{S}^{n+m-1}(\bar{z},R)$ not lying in the radial projections of these sets. The set

$$D:=\mathbb{R}^{n+m}\backslash(((\mathbb{R}_+\{p-\bar{z}\})+\{\bar{z}\})\cup\mathbb{B}^{n+m}(\bar{z},R))$$

is homeomorphic to $\mathbb{R}^{n+m}$, so by the Tietze extension theorem (see for example [19]), we can extend H continuously to $\mathbb{S}^n_-(\mathbf{0},1)\times[0,1]$ so that $H(\mathbb{S}^n_-(\mathbf{0},1)\times[0,1])\subset D$.

Step 1 (b). **Determine H on $\mathbb{S}^n_+(\mathbf{0},1)\times[1,2]$.**
We next define $H\mid_{\mathbb{S}^n_+(\mathbf{0},1)\times[1,2]}$, the homotopy between Γ_1'' and Γ_1'. Since $\mathscr{M}'$ is a manifold, for any $\delta>0$, we can find R small enough such that for any $z\in\mathbb{B}^{n+m}(\bar{z},R)\cap\mathscr{M}'$, the distance from z to $\bar{z}+T_{\mathscr{M}'}(\bar{z})$ is at most δR. The value R can be reduced if necessary so that the mapping P, which projects a point $z\in\mathbb{B}^{n+m}(\bar{z},R)\cap\mathscr{M}'$ to the closest point in $\bar{z}+T_{\mathscr{M}'}(\bar{z})$, is a homeomorphism of $\mathbb{B}^{n+m}(\bar{z},R)\cap\mathscr{M}'$ to its image.

Define the map $H_1:(\mathbb{B}^{n+m}(\bar{z},R)\cap\mathscr{M}')\times[1,2]\to\mathbb{B}^{n+m}(\bar{z},R)$ by

$$H_1(z,t):=\left(\frac{|z-\bar{z}|}{|(2-t)z+(t-1)P(z)-\bar{z}|}((2-t)z+(t-1)P(z)-\bar{z})\right)+\bar{z}.$$

This is a homotopy from Γ_1 to Γ_1'. For any homeomorphism $h_1:\mathbb{B}^{n+m}(\bar{z},R)\cap\mathscr{M}'\xrightarrow{\simeq}\mathbb{S}^n_+(\mathbf{0},1)$, we define $H\mid_{\mathbb{S}^n_+(\mathbf{0},1)\times[0,1]}$ via $H(s,t)=H_1\left(h_1^{-1}(s),t\right)$.

Step 1 (c). **Determine** H **on** $\mathbb{S}^n_-(\mathbf{0},1)\times[1,2]$**.** We now define $H\mid_{\mathbb{S}^n_-(\mathbf{0},1)\times[1,2]}$, the homotopy between Γ_2'' and Γ_2' that respects the boundary conditions stipulated by $H\mid_{\mathbb{S}^n_=(\mathbf{0},1)\times[1,2]}$. We extend $H(\cdot,1)\mid_{\mathbb{S}^n(\mathbf{0},1)}$ so that it is a homeomorphism between $\mathbb{S}^n(\mathbf{0},1)$ and $\Gamma_1'\cup\Gamma_2'$ by using methods similar to that used in Step 1(a).

We now use the Tietze extension theorem to establish a continuous extension of H to $\mathbb{S}^n(\mathbf{0},1)\times[1,2]$. We are left only to resolve H on $\mathbb{S}^n_-(\mathbf{0},1)\times(1,2)$. Much of this is now similar to the end of step 1(a). The dimension of $\mathbb{S}^{n+m-1}(\bar{z},R)$ is $n+m-1$, while the dimensions of Γ_2'', Γ_2' and $H(\mathbb{S}^n_=(\mathbf{0},1)\times[1,2])$ are all at most n. Therefore, there is one point in $\mathbb{S}^{n+m-1}(\bar{z},R)$ outside these three sets, say p. Since $\mathbb{S}^{n+m-1}(\bar{z},R)\setminus\{p\}$ is homeomorphic to $\mathbb{R}^{n+m-1}$, the Tietze extension theorem again implies that we can extend H continuously in $\mathbb{S}^n(\mathbf{0},1)\times[1,2]$.

Step 1 (d). **Determine** H **on** $\mathbb{B}^{n+1}(\mathbf{0},1)\times\{0\}$. We use Lemma 19 to extend the domain of the function

$$H(\cdot,0):\mathbb{S}^n(\mathbf{0},1)\xrightarrow{\simeq}\left(\mathscr{M}'\cap W\right)\cup\left(\mathscr{M}_i\cap\partial W\right)$$

to $\mathbb{B}^{n+1}(\mathbf{0},1)$ so that

$$H(\cdot,0):\mathbb{B}^{n+1}(\mathbf{0},1)\xrightarrow{\simeq}\left(\mathscr{M}'\cup\mathscr{M}_i\right)\cap W$$

is a homeomorphism.

Step 2: Choice of R **and** r**.** We now choose R and r so that $H(\mathbb{S}^n(\mathbf{0},1)\times[0,2])$ does not intersect $A=\mathbb{S}^{n+m-1}(\bar{z},r)\cap(\bar{z}+\mathscr{L})$. To this end, consider the minimization problem

$$\min\left\{\operatorname{dist}\left(z,T_{\mathscr{M}'}(\bar{z})+\bar{z}\right):z\in\mathbb{S}^{n+m-1}(\bar{z},r)\cap(\bar{z}+\mathscr{L})\right\}.$$

Since $\mathbb{S}^{n+m-1}(\bar{z},r)\cap(\bar{z}+\mathscr{L})$ is compact, the above minimum is attained at some point z_r and its value is not zero (otherwise $z_r-\bar{z}$ would be a nonzero element in $T_{\mathscr{M}'}(\bar{z})\cap\mathscr{L}$, contradicting $T_{\mathscr{M}'}(\bar{z})\cap\mathscr{L}=\{\mathbf{0}\}$). Therefore, for some constant $\varepsilon\in(0,1]$ independent of r, it holds $\operatorname{dist}(z_r,T_{\mathscr{M}'}(\bar{z})+\bar{z})=\varepsilon r$.

Given $\delta>0$, we can shrink R if necessary to get $\mathrm{d}(z,T_{\mathscr{M}'}(\bar{z})+\bar{z})\leq\delta R$ for all $z\in H\left(\mathbb{S}^n_+(\mathbf{0},1)\times[0,1]\right)$. If $\delta<\varepsilon$, we can find some r satisfying $\delta R<\varepsilon r\leq r<R$. Since $\delta R<\varepsilon r$, $H\left(\mathbb{S}^n_+(\mathbf{0},1)\times[1,2]\right)$ does not intersect $\mathbb{S}^{n+m-1}(\bar{z},r)\cap(\bar{z}+\mathscr{L})$. From $r<R$, it is clear that $H\left(\mathbb{S}^n_-(\mathbf{0},1)\times[0,2]\right)$, being a subset of $\operatorname{cl}(\mathbb{R}^{n+m}\backslash\mathbb{B}^{n+m}(\bar{z},R))$, does not intersect $\mathbb{S}^{n+m-1}(\bar{z},r)\cap(\bar{z}+\mathscr{L})$. Elements in $H\left(\mathbb{S}^n_+(\mathbf{0},1)\times[0,1]\right)$ are either in $\mathbb{B}^{n+m}(\bar{z},R)\cap\mathscr{M}'$ or outside $\mathbb{B}^{n+m}(\bar{z},R)$, so $H(\mathbb{S}^n(\mathbf{0},1)\times[0,2])$ does not intersect A as needed.

Step 3: "Set-up" for linking theorem. Let

$$h_3:\mathbb{B}^{n+1}(\mathbf{0},1)\xrightarrow{\simeq}\left(\mathbb{B}^{n+1}(\mathbf{0},1)\times\{0\}\right)\cup\left(\mathbb{S}^n(\mathbf{0},1)\times[0,2]\right)$$

be a homeomorphism between the respective spaces. We can extend the homeomorphism

$$H\mid_{\mathbb{S}^n(\mathbf{0},1)\times\{2\}}\circ h_3\mid_{\mathbb{S}^n(\mathbf{0},1)}:\mathbb{S}^n(\mathbf{0},1)\xrightarrow{\simeq}\Gamma'$$

to

$$h_4:\mathbb{B}^{n+1}(\mathbf{0},1)\xrightarrow{\simeq}\left(T_{\mathscr{M}'}(\bar{z})+\mathbb{R}_+\{v\}+\bar{z}\right)\cap\mathbb{B}^{n+m}(\bar{z},R).$$

Define the map

$$g:\left(T_{\mathscr{M}'}(\bar{z})+\mathbb{R}_+\{v\}+\bar{z}\right)\cap\mathbb{B}^{n+m}(\bar{z},R)\to\mathbb{B}^{n+m}(\bar{z},R)$$

by $g=H\circ h_3\circ h_4^{-1}$. By construction, the map $g\mid_{\Gamma'}$ is the identity map there. Furthermore, g can be extended continuously to the domain $\mathbb{R}^{n+m}$ by the Tietze extension theorem.

Step 4: Apply linking theorem. Recall that $A:=\mathbb{B}^{n+m}(\bar{z},r)\cap(\bar{z}+\mathscr{L})$ and Γ' link by Theorem 17. This means that there is a nonempty intersection of $g\left(\left(T_{\mathscr{M}'}(\bar{z})+\mathbb{R}_+\{v\}+\bar{z}\right)\cap\mathbb{B}^{n+m}(\bar{z},R)\right)$ with A. Step 2 asserts that the intersection is not in $H(\mathbb{S}^n(\mathbf{0},1)\times[0,2])$, so the intersection lies in $H\left(\mathbb{B}^{n+1}(\mathbf{0},1)\times\{0\}\right)$. In other words, A and Γ link. This means that $W\cap\mathscr{M}_i$ intersects $\mathbb{B}^{n+m}(\bar{z},r)\cap(\bar{z}+\mathscr{L})$, which means that $(\bar{z}+\mathscr{L})\cap\mathscr{M}_i\cap U\neq\emptyset$, contradicting our assumption. $\square$

4.3 Main result

In this section we put together all previous results to obtain the following theorem. Recall that $\bar{S}$ is the set-valued map whose graph is the closure of the graph of S (thus, $\bar{S}$ is outer semicontinuous by definition).

Theorem 27. *If $S : \mathscr{X} \rightrightarrows \mathbb{R}^m$ is a closed-valued semialgebraic set-valued map, where $\mathscr{X} \subset \mathbb{R}^n$ is semi-algebraic, then S and $\bar{S}$ differ outside a set of dimension at most* $(\dim \mathscr{X} - 1)$.

Proof. We first consider the case where $\mathscr{X} = \mathbb{R}^n$ and a $\mathscr{C}^k$ stratification of $\operatorname{cl}(\operatorname{Graph}(S))$. If $S(\bar{x}) \neq \bar{S}(\bar{x})$, then Lemma 25 and Lemma 26 yield that there exists some $\bar{y}$ and stratum $\mathscr{M}'$ containing $\bar{z} := (\bar{x}, \bar{y})$ such that $N_{\mathscr{M}'}(\bar{z}) \cap \mathscr{L}^{\perp} \supsetneq \{\mathbf{0}_{n+m}\}$. Finally, since there are only finitely many strata, Lemma 15 tells us that $S(x)$ and $\bar{S}(x)$ may differ only on a set of dimension at most $n-1$. This proves the result in this particular case.

We now consider the case where $\mathscr{X} \neq \mathbb{R}^n$. Let $\mathscr{X} = \dot{\cup} \mathscr{X}_j$ be a stratification of $\mathscr{X}$, and let $\mathscr{D}$ be the union of strata of full dimension in $\mathscr{X}$. Each stratum in $\mathscr{D}$ is semialgebraically homeomorphic to $\mathbb{R}^d$, where $d := \dim \mathscr{X}$ and let $h_j : \mathbb{R}^d \to \mathscr{X}_j$ denote such a homeomorphism. By considering the set-valued maps $S \circ h_j$ for all j, we reduce the problem to the aforementioned case. Since the set of strata (*a fortiori* the set of full-dimensional strata) is finite, we deduce that $S(x) \neq \bar{S}(x)$ can only occur in a set of dimension at most $d-1$. □

The following result is now an easy consequence of the above.

Theorem 28 (Main result)**.** *A closed-valued semialgebraic set-valued map $S : \mathscr{X} \rightrightarrows \mathbb{R}^m$, where $\mathscr{X} \subset \mathbb{R}^n$ is semialgebraic, is strictly continuous outside a set of dimension at most* $(\dim \mathscr{X} - 1)$.

Proof. By Theorem 27 the map S differs from the outer semicontinuous map $\bar{S}$ on a set of dimension at most $(\dim \mathscr{X} - 1)$. Apply Theorem 24. □

Remark. Our main result (Theorem 28) as well as all previous preliminary results (Lemmas 25, 26, Theorems 24, 27) can be restated for the case where S is definable in an o-minimal structure. With slightly more effort we can further extend these results in case where S is tame, noting that one performs a locally finite stratification in the tame case as opposed to a finite stratification.

5 Applications in tame variational analysis

A standard application of Theorem 2 is to take first the closure of the graph of S, and then deduce generic continuity for the obtained set-valued map. While this operation is convenient, this new set-valued map no longer reflects the same local properties. For example, for a set $C \subset \mathbb{R}^n$, consider the Hadamard normal cone mapping $\hat{N}_C : \partial C \rightrightarrows \mathbb{R}^n$ and the limiting normal cone mapping $N_C : \partial C \rightrightarrows \mathbb{R}^n$, where $\operatorname{cl}(\operatorname{Graph}(\hat{N}_C)) = \operatorname{Graph}(N_C)$. The Hadamard normal cone $\hat{N}_C(\bar{z})$ for $\bar{z} \in \partial C$ depends on how C behaves at $\bar{z}$, whereas the normal cone $N_C(\bar{z})$ offers instead an aggregate information from points around $\bar{z}$. The following result is comparable with [22, Proposition 6.49], and is a straightforward application of Theorem 28.

Corollary 29 (Generic regularity)**.** *Given closed semi-algebraic sets C and D with $D \subset C$, the set-valued map $\hat{N}_C : C \rightrightarrows \mathbb{R}^n$ is continuous on $D \backslash D'$, where D' is semialgebraic and $\dim(D') < \dim(D)$. When $D = \partial C$, we deduce that $\hat{N}_C(z) = N_C(z)$ for all z in $(\partial C) \backslash C'$, with $\dim(C') < \dim(\partial C)$.*

An analogous statement of the above corollary can be made for (nonsmooth) tangent cones $\hat{T}_C$ and T_C as well.

Remark. From the definition of subdifferential of a lower semicontinuous function [22, Definition 8.3], we can deduce that the regular (Fréchet) subdifferentials are continuous outside a set of smaller dimension. This result is comparable with [22, Exercise 8.54]. Therefore nonsmoothness in tame functions and sets is structured.

Let us finally make another connection to functions whose graph is a finite union of polyhedra, hereafter referred to as *piecewise polyhedral functions*. Robinson [21] proved that a piecewise polyhedral function is calm (outer-Lipschitz) everywhere [22, Example 9.57], and a uniform Lipschitz constant suffices over the whole domain of the function (although this latter is not explicitly stated therein). A straightforward application of Theorem 28 yields that piecewise polyhedral functions are set-valued continuous outside a set of small dimension. We now show that a uniform Lipschitz constant for strict continuity applies.

Proposition 30 (Uniformity of graphical modulus)**.** *Let $S : X \rightrightarrows \mathbb{R}^m$ be a piecewise polyhedral set-valued map, where $X \subset \mathbb{R}^n$. Then S is strictly continuous outside a set X', with* $\dim(X') < \dim(X)$. *Moreover, there exists some $\bar{\kappa} > 0$ such that if S is strictly continuous at $\bar{x}$, then the graphical modulus* $\mathrm{lip}_X S(\bar{x} \mid \bar{y})$ *is a nonnegative real number smaller than $\bar{\kappa}$.*

Proof. The first part of the proposition of strict continuity is a direct consequence of Theorem 28 since S is clearly semialgebraic. We proceed to prove the statement on the graphical modulus. We first consider the case where the graph of S is a convex polyhedron. The graph of S can be written as a finitely constrained set $\mathrm{Graph}(S) = \{z \in \mathbb{R}^{n+m} \mid Az = b, Cz \leq d\}$ for some matrices A, C with finitely many rows. The projection of $\mathrm{Graph}(S)$ onto $\mathbb{R}^n$ is the domain of S, which we can again write as $\mathrm{dom}(S) = X = \{x \in \mathbb{R}^n \mid A'x = b', C'x \leq d'\}$. Let $\mathscr{L}$ be the lineality space of $\mathrm{dom}(S)$, which is the set of vectors orthogonal to the rows of A'. We seek to find a constant $\bar{\kappa} > 0$ such that if x lies in the relative interior (in the sense of convex analysis) of X and $y \in S(x)$, then $\mathrm{lip}_X S(x \mid y) \leq \bar{\kappa}$. By Proposition 14, we have

$$\bar{\kappa} = \sup_{(x,y) \in \text{r-int}(X)} \left\{ \frac{|a|}{|b|} \mid (a,b) \in N_{\mathrm{Graph}(S)}(x,y) \cap (\mathscr{L} \times \mathbb{R}^m) \right\},$$

where "r-int" stands for the relative interior. The above value is finite because of two reasons. Firstly, if $(a, \mathbf{0}) \in N_{\mathrm{Graph}(S)}(x,y) \cap (\mathscr{L} \times \mathbb{R}^m)$, then by the convexity of $\mathrm{Graph}(S)$, x lies on the relative boundary of X. Secondly, the "sup" in the formula is attained and can be replaced by "max". This is because the normal cones of $\mathrm{Graph}(S)$ at $z = (x,y)$ can be deduced from the rows of C in which $Cz \leq d$ is actually an equation, of which there are only finitely many possibilities. In the case where S is a union of finitely many polyhedra, we consider the set-valued maps denoted by each of these polyhedra. The maximum of the Lipschitz constants for strict continuity on each polyhedral domain gives us the required $\bar{\kappa}$. $\square$

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