Stability for solution of Differential Variational Inequalitiy

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Abstract

In this paper we study the class of differential variational inequality(DVI) in a finite-dimension Euclidean space $\Re^n$, which is the following form

$$\dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t) \,, \quad x(0) = x_0 \in \Re^n$$
$$0 \leq (\tilde{u} - u(t))^T [G(t, x(t)) + F(u(t))] \text{ for almost all } \tilde{u} \in K$$
$$u(t) \in K$$

We study stability and perturbation of the DVI under the OSL condition. Besides, we establish a Prior Bound Theorem, which is a useful tool to prove stability of DVI. In this paper, we replace the classical Lipshitz continuity by one-sided Lipschitz condition, which is important improvement of the conventional Lipschitz continuity.

Keywords Differential variational inequality; One-sided Lipschitz continuity; Stability.

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1 Introduction

Differential Variational Inequality(DVI) is a new entrant to the field of differential algebraic system, which combine differential equations with variational inequalities. This concept was firstly defined in reference [1] and the authors have explicitly introduced the development of DVI and some important theories. The DVI provides a powerful modeling paradigm for many applied problems in which dynamics, differential Nash game and hybrid engineering system. As much,these system extend the notion of Differential Complementarity Problem(DCPs) [9], Linear Complementarity Systems(LCSs)[10, 11, 12], Projected Dynamical Systems(PDSs)[13, 14, 15] and a variety of other systems such as [16, 17].

In this paper DVI of form

$$\dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t)\ , \quad x(0) = x_0 \in \Re^n \tag{1.1}$$

$$0 \leq (\tilde{u} - u(t))^T [G(t, x(t)) + F(u(t))] \ \text{ for almost all } t \in [0, T] \tag{1.2}$$

$$u(t) \in K \tag{1.3}$$

are considered.

The DVI consists of an Ordinary Differential Equation(ODE), a Variational Inequality(VI) and a boundary condition. The basic conditions of the DVI are as follows: Let $T > 0$ be a terminal time. Given functions $f : [0, T] \times \Re^n \longrightarrow \Re^n$, $B : [0, T] \times \Re^n \longrightarrow \Re^{n\times m}$, $x_0 \in \Re^n$, $G : [0, T] \times \Re^n \longrightarrow \Re^m$, $F : \Re^m \longrightarrow \Re^m$ and K is a nonempty closed convex set in $\Re^m$. We need to seek a time-dependent trajectories $x : [0, T] \longrightarrow \Re^n$ and $u : [0, T] \longrightarrow K$ that satisfy DVI.

The classic applications of this model are the multi-rigid-body dynamics problem with contact and friction and the differential Nash game, which are introduced in details in reference[1].

The primary purpose of this paper is to study the stability of DVI in finite-dimensional space. Stability of the solutions of DVI with respect to various perturbations are of greatly important for its qualitative and quantitative analysis, which are closely related to existence and to sensitivity and approximations analysis.

We denote by $U(t, x(t))$ the solution set of DVI(1.2), which is through out the whole paper. That is to say, if $u(t) \in U(t, x(t))$ then $u(t) \in K$, and

$$(\tilde{u} - u(t))^T [G(t, x(t)) + H(u(t))] \geq 0 \quad \text{for all } \tilde{u} \in K$$

Let $F(t, x(t)) = f(t, x(t)) + B(t, x(t))U(t, x(t))$. It is well known that DVI is turned into differential inclusions (DI), i.e. $\dot{x}(t) \in F(t, x(t))$. We can use theories of DI to study the DVI. The most theorems in this direction impose the classical Lipschitz continuity of F in x. In this paper we replace the classical Lipschitz continuity by one-sided Lipschitz(OSL) condition, which is important improvement. Differential inclusions under the OSL condition have been already studied in references [3, 4, 5].

2 Preliminary discussion

We use the $< \cdot, \cdot >$ for the scalar product in $\Re^n$, i.e. $< x, y >= x^T y$ for all $x, y \in \Re^n$, and $\| \cdot \|$ for the norm, that is $< x, x >= x^T x = \|x\|^2$.

Now we recall the main definitions and lemmas used in this paper.

Definition 2.1 . *A function $g : [0, T] \times \Re^n \longrightarrow \Re^n$ is called Lipschitz continuous if there exists an integrable function $\xi : [0, T] \to \Re^+$ such that for any $x, y \in \Re^n$*

$$\| g(t, x) - g(t, y) \| \leq \xi(t) \| x - y \|$$

where $\xi(t)$ is called Lipschitz function of g.

Definition 2.2 . *Let Ω is a subset of $\Re^n$. The set-valued mapping $F : [0, T] \times \Re^n \longrightarrow 2^{\Re^n}$ is called one-sided Lipschitz (OSL) continuous (with respect to x) in Ω if there is an integrable function $L : [0, T] \to \Re$ such that for every $x, y \in \Omega$, and $v \in F(t, x)$, there exists $w \in F(t, y)$ such that*

$$< x - y, v - w > \leq L(t) \| x - y \|^2$$

Remark 2.1 *If $f(t, x)$ is a single-valued mapping and satisfies*

$$< x - y, f(t, x) - f(t, y) > \leq L(t) \| x - y \|^2 \quad \text{for any } x, y \in \Omega$$

where $L(t)$ is an integrable function in $[0, T]$, then f is called one-sided Lipschitz continuous with respect to x in Ω.

Definition 2.3 . *(integrably bounded) Assume that* $f : [0,T] \times \Re^n \longrightarrow \Re^n$. *An integral bound for* f *is a Lebesgue integrable function* $\zeta : [0,T] \longrightarrow \Re^+$ *such that*

$$\| f(t,x) \| \leq \zeta(t) \quad \text{for almost all } (t,x) \in [0,T] \times \Re^n$$

We say that f *is integrably bounded in* $[0,T] \times \Re^n$ *if there exists an integral bound for* f.

Definition 2.4 . *A function* $H : \Re^m \longrightarrow \Re^m$ *is called monotone if for any* $x, y \in \Re^n$, $(x-y)^T(H(x) - H(y)) \geq 0$.

We make the following assumptions which are throughout the whole paper.

(A) $f : [0,T] \times \Re^n \longrightarrow \Re^n$ is a measurable function for t and satisfies one-sided Lipschitz continuous respect to x , i.e. there exists an integrable function $L_f(t)$ in $[0,T]$ such that

$$< x - y, f(t,x) - f(t,y) > \leq L_f(t) \|x - y\|^2$$

holds for any $x, y \in \Re^n$.

(B) $B : [0,T] \times \Re^n \longrightarrow \Re^{n \times m}$ is a matric-valued function which satisfies Lipschitz continuous respect to variable x with Lipschitz function $\xi_B(t)$:

$$\| B(t,x) - B(t,y) \| \leq \xi_B(t) \| x - y \|, \quad \text{for any } x, y \in \Re^n .$$

(C) $G : [0,T] \times \Re^n \longrightarrow \Re^m$ is a Lipschitz continuous respect to variable x with Lipschitz function $\xi_G(t)$: $\| G(t,x) - G(t,y) \| \leq \xi_G(t) \| x - y \|$, for any $x, y \in \Re^n$. And satisfies $J_x G(t,x) = B(t,x)^T$, where $J_x G$ is the Jacobian matrix of $G(t,x)$ for variable x.

(D) $H : \Re^m \longrightarrow \Re^m$ is a continuous and monotone function which satisfies: there exists a $\bar{u} \in K$ such that $\liminf_{u \in K, \|u\| \to \infty} \frac{(u-\bar{u})^T H(u)}{\|u\|^2} > 0$.

The followings are some lemmas used in this paper:

Lemma 2.1 .*(see[1] Proposition 2) Let* $K \subseteq \Re^n$ *be nonempty closed convex. If* H *satisfies the assumption(D), then for every* $r \in \Re^m$, *the variational inequality*

$$(\tilde{u} - u)^T (r + H(u)) \geq 0 , \quad \text{for any} \quad \tilde{u} \in K$$

has a nonempty compact and convex solution set. Moreover, there exists a constant $c > 0$ such that

$$\| u \| \leq c(1+ \| r \|) \tag{2.1}$$

where u is a solution of the variational inequality.

Remark 2.2 . *Since $G(t,x)$ satisfies $\| G(t,x_2) - G(t,x_1) \| \leq \xi_G(t) \| x_2 - x_1 \|$ for any x_1 , $x_2 \in \Re^n$, it implies*

$$\| G(t,x) - G(0,0) \| \leq \xi_G(t) \| x \|$$

$$\begin{aligned} \| G(t,x) \| &\leq \| G(0,0) \| + \xi_G(t) \| x \| + \xi_G(t) + \| G(0,0) \| \| x \| \\ &\leq \| G(0,0) \| (1+ \| x \|) + \xi_G(t)(1+ \| x \|) \\ &= \eta(t)(1+ \| x \|) \end{aligned}$$

where $\eta(t) = \| G(0,0) \| + \xi_G(t)$. If $u(t)$ is a solution of DVI(1.2), according to (2.1),

$$\begin{aligned} \| u(t) \| &\leq c(1+ \| G(t,x(t)) \|) \\ &\leq c(1 + \eta(t)(1+ \| x(t) \|)) \\ &\leq c + c\eta(t)(1+ \| x(t) \|) + c \| x(t) \| \\ &\leq (c + c\eta(t))(1+ \| x(t) \|) \\ &\leq \rho(t)(1+ \| x(t) \|) \end{aligned} \tag{2.2}$$

Where $\rho(t) = c + c\eta(t)$. So the solution of DVI(1.2) satisfies linear growth condition. Note that $\rho(t)$ is an integrable function in $[0,T]$.

Lemma 2.2 *Let $F(t,x) = f(t,x) + B(t,x)U(t,x)$. If the assumption (A)–(D) hold, the graph of $F(t,x)$ is closed.*

Proof. Please see the proof of Lemma2 in [1].

Lemma 2.3 *(Grownwall inequality) Let $P, Q : [a,b] \longrightarrow \Re$ are both two continuous functions. Suppose $m(t)$ is a solution of the following inequality:*

$$\dot{m}(t) \leq P(t)m(t) + Q(t) \ , \quad m(a) = m_a$$

then for all $t \in [a,b]$

$$m(t) \leq e^{\int_a^t P(s)ds}[m_a + \int_a^t Q(\tau)e^{-\int_a^\tau P(s)ds} d\tau].$$

3 Main results

Before introducing our primary results, we give some explanations.

Let $E(0, M)$ be an open ball of radius $M > 0$ which is centered in the origin and $\bar{E}(0, M)$ be the closed ball.

We call a subset $\Omega \in \Re^n$ is bounded, that is, there exist a positive scalar M such that $\| u \| \leq M$ for any $u \in \Omega$.

We denote the distance from a point $\xi \in \Re^n$ to a set $A \subset \Re^n$ by $dist(\xi, A)$.

Theorem 3.1 *(Prior Bound Theorem) If the assumptions(A)–(D) hold, and $f(t, x)$ and $B(t, x)$ are both integrably bounded with integral bounds respectively $\varphi(t)$ and $\psi(t)$. Then, the DVI(1.1)-(1.3) dose not admit a solution in $\Re^n \times \Re^m \setminus \bar{E}(0, M_1) \times E(0, M_2)$, where*

$$M_1 = \max_{t \in [0,T]} \{e^{k(t)} \| x_0 \| + \int_0^t [\varphi(s) + \rho(s)\psi(s)] e^{k(t)-k(s)} ds\} \ \ and \ \ k(t) = \int_0^t \rho(s)\psi(s) ds$$

and $M_2 = \max_{t \in [0,T]} [\rho(t) + \rho(t) M_1] + 1$.

Proof. We assume for sake of contradiction that $(x(t), u(t))$ is a solution of DVI(1.1)-(1.3) in $\Re^n \times \Re^m \setminus \bar{E}(0, M_1) \times E(0, M_2)$. That is: $\| x(t) \| > M_1$ and $\| u(t) \| \geq M_2$. By (2.2), we know that there exists an integrable function $\rho(t)$ such that $\| u(t) \| \leq \rho(t)(1 + \| x(t) \|)$. Since $x(t)$ satisfies:

$$\dot{x}(t) = f(t, x(t)) + B(t, x(t))u(t)$$

so we have

$$\| \dot{x}(t) \| \leq \varphi(t) + \rho(t)\psi(t)(1 + \| x(t) \|)$$

Thus,

$$\frac{d}{dt} \| x(t) \| \leq \rho(t)\psi(t) \| x(t) \| + (\varphi(t) + \rho(t)\psi(t))$$

We can use Lemma(2.3) to deduce: $\| x(t) \| \leq M_1$. Therefore, we can also know that $\| u(t) \| < M_2$. So we obtain a contradiction to the assumption $(x(t), u(t)) \in \Re^n \times \Re^m \setminus \bar{E}(0, M_1) \times E(0, M_2)$. □

Remark. From Theorem3.1, we can come a conclusion that the solution of DVI is bounded if it admits a solution. The pair (M_1, M_2) is called a prior bound for solutions of DVI. It is sufficient to consider existence of solution in the bounded domain $\bar{E}(0, M_1) \times E(0, M_2)$.

Lemma 3.1 *Suppose $\Omega \subset \Re^n$ is a bounded set with the bound $M > 0$. Let $F(t,x) = f(t,x) + B(t,x)U(t,x)$. If the assumptions in Theorem3.1 hold, then for every $x, y \in \Omega$, $F(t,x)$ satisfies OSL continuous in Ω.*

Proof. We should show that for every $x, y \in \Omega$, and every $v \in F(t,x)$, there exists $w \in F(t,y)$ and an integrable function $L(t)$ in $[0,T]$ such that

$$< x - y, v - w > \leq L(t) \parallel x - y \parallel^2$$

Since $v \in F(t,x)$, we can find an element $u_x \in U(t,x)$ such that $v = f(t,x) + B(t,x)u_x$, By Lemma 2.1, there exists at least an element $u_y \in U(t,y)$. Let $w = f(t,y) + B(t,y)u_y$. Then we consider

$$\begin{aligned} < x - y, v - w > &= < x - y, f(t,x) - f(t,y) + B(t,x)u_x - B(t,y)u_y > \\ &= < x - y, f(t,x) - f(t,y) > + < x - y, B(t,x)u_x - B(t,y)u_y > \end{aligned}$$

The first item of right side $< x - y, f(t,x) - f(t,y) > \leq L_f(t) \| x - y \|^2$ according to the assumption(A).

The following proof is shown that the second item also satisfies OSL condition. Since $u_x \in U(t,x)$, $u_y \in U(t,y)$ and u_x , $u_y \in K$, then

$$(u_y - u_x)^T (G(t,x) + H(u_x)) \geq 0$$
$$(u_x - u_y)^T (G(t,y) + H(u_y)) \geq 0$$

Adding gives (after some arrangement)

$$(u_x - u_y)^T (G(t,x) - G(t,y) + H(u_x) - H(u_y)) \leq 0$$

Since H is monotone, $(u_x - u_y)^T (H(u_x) - H(u_y)) \geq 0$, so

$$(u_x - u_y)^T (G(t,x) - G(t,y)) \leq 0 \tag{3.1}$$

Now $G(t,x) - G(t,y) = J_x G(t,x)(x - y) + h$ where $\parallel h \parallel \leq \xi_B \| x - y \|^2$. Since $x, y \in \Omega$ then $\parallel u_x \parallel \leq M$ and $\parallel u_y \parallel \leq M$. And $J_x G(t,x) = B(t,x)^T$. Then (3.1) can be re-written as

$$(u_x - u_y)^T [J_x G(t,x)(x - y) + h] \leq 0$$

thus

$$(u_x - u_y)^T B(t,x)^T (x-y) \leq 2M\xi_B(t) \| x - y \|^2$$

i.e.

$$(B(t,x)u_x - B(t,x)u_y)^T (x-y) \leq 2M\xi_B(t) \| x - y \|^2$$

$$(B(t,x)u_x - B(t,y)u_y + B(t,y)u_y - B(t,x)u_y)^T (x-y) \leq 2M\xi_B(t) \| x - y \|^2$$

$$(B(t,x)u_x - B(t,y)u_y)^T (x-y) \leq 2M\xi_B(t) \| x - y \|^2 + M \| B(t,x) - B(t,y) \| \| x - y \|$$

then

$$< x - y, B(t,x)u_x - B(t,y)u_y > \leq 3M\xi_B(t) \| x - y \|^2 \tag{3.2}$$

Let $L(t) = L_f(t) + 3M\xi_B(t)$ and it is easy to show that $L(t)$ is integrable in $[0,T]$. Thus we can conclude that

$$\begin{aligned} < x - y, v - w > &= < x - y, f(t,x) - f(t,y) > + < x - y, B(t,x)u_x - B(t,y)u_y > \\ &\leq L(t) \| x - y \|^2 \end{aligned}$$

So one can know that $F(t,x)$ satisfies OSL condition. □

Remark. In this Lemma, it is necessary that Ω is bounded. This essential condition ensures that u_x and u_y are both bounded, which is independent condition in this proof.

Theorem 3.2 *(boundedness of solution of the perturbed DVI) Let $g : [0,T] \to \Re_+$ be an integrable function. If the assumptions in Theorem3.1 hold, then all the solutions of the differential inclusion*

$$\dot{x} \in f(t,x) + B(t,x)U(t,x) + g(t)E(0,1), \quad x(0) \in x_0 + d_0 E(0,1) \tag{3.3}$$

are contained in a ball of radius M which is centered in the origin, where

$$M = \max_{t \in [0,T]} \{ e^{k(t)} [\| x_0 \| + d_0 + \int_0^t [\varphi(s) + g(s) + \psi(s)\rho(s)] e^{k(s)}] \}$$

and $k(t) = \int_0^t \psi(s)\rho(s)ds$.

Proof. This proof is the same process as the one in Theorem3.1. Suppose $x(t)$ is a solution of (3.3), then there exist a function $u_x(t) \in U(t,x)$ and $v \in E(0,1)$ such that

$$\dot{x} = f(t,x) + B(t,x)u_x(t) + g(t)v \,, \quad x(0) \in x_0 + d_0 E(0,1)$$

By Lemma2.1, there exist an integrable function $\rho(t)$ in $[0,T]$ such that $\| u_x(t) \| \leq \rho(t)(1+ \| x(t) \|)$.
Thus

$$\| \dot{x}(t) \| \leq \psi(t)\rho(t) \| x(t) \| +[\varphi(t) + g(t) + \psi(t)\rho(t)]$$

According to Gronwall inequality, we have $\| x(t) \| \leq M$. So we complete the proof of Theorem3.2. □

Theorem 3.3 *(Stability of solution of DVI) If the assumptions in Theorem3.1 hold and* $y : [0,T] \to \Re^n$ *is an absolutely continuous function satisfying* $dist(\dot{y},\ f(t,y) + B(t,y)U(t,y)) \leq g(t)$ *for a.e.* $t \in [0,T]$, *where* $g(t)$ *is an integrable function in* $[0,T]$. *Then there exists a solution* $(x(t), u(t))$ *of DVI(1.1)-(1.3). Moreover, there exists an integrable function* $L(t)$ *in* $[0,T]$ *such that*

$$\| x(t) - y(t) \| \leq de^{m(t)} + \int_0^t e^{m(t)-m(s)} g(s)ds \tag{3.4}$$

where $d = \| y(0) - x_0 \|$ *and* $m(t) = \int_0^t L(s)ds$.

Proof. Let $F(t,x) = f(t,x) + B(t,x)U(t,x)$. By theorem3.2, $y(t)$ is bounded. Then for any $x \in \bar{E}(0, M_1)$ where M_1 is the same scalar as the one in theorem3.1 and for any $w \in F(t,y)$, there exists an integrable function $L(t)$ in $[0,T]$ and $v \in F(t,x)$ such that

$$< y(t) - x, w - v > \leq L(t) \| y(t) - x \|^2$$

Let $H(t,x) = \{v \in \Re^n | < y(t) - x, \dot{y}(t) - v > \leq L(t) \| y(t) - x \|^2 + g(t) \| y(t) - x \|\}$
And let $G(t,x) = F(t,x) \bigcap H(t,x)$. We consider the ordinary differential inclusion

$$\dot{x}(t) \in G(t,x), \quad x(0) = x_0 \tag{3.5}$$

The first step. In the following, it was shown that differential inclusion (3.5) has a solution on $[0,T]$. According to the Prior Bound Theorem, it is sufficient to consider (3.5) in just the domain $x \in \bar{E}(0, M_1)$.

We can verify that $G(t,x)$ satisfies that following properties:
(1) $G(t,x)$ is nonempty, convex, and closed-valued, measurable in t.
We firstly prove that $G(t,x) \neq \phi$ for each t and every $x \in E(0, M_1)$. Let $w \in F(t,x)$ be

such that $\| \dot{y}(t)-w \|= dist(\dot{y}(t), F(t, y(t)))$. Then $< y(t)-x, w-v >\leq L(t) \| y(t)-x \|^2$. Thus

$$\begin{aligned} < y(t) - x, \dot{y}(t) - v > &= < y(t) - x, \dot{y}(t) - w > + < y(t) - x, w - v > \\ &\leq g(t) \| y(t) - x \| + L(t) \| y(t) - x \|^2 \end{aligned} \tag{3.6}$$

i.e. $v \in G(t, x)$.

By Lemma2.1, we can know $U(t, x)$ is convex, which results in $F(t, x)$ is convex, Hence, it is easy to verify that $G(t, x)$ is convex. And obviously $G(t, x)$ is closed-valued, measurable in t.

(2) for very t, $G(t, x)$ has a closed graph, which implies that $G(t, x)$ is USC.

Let

$$GraphG(t, \cdot) = GraphF(t, \cdot) \bigcap GraphH(t, \cdot)$$

It is closed because the graph of $F(t, x)$ and $H(t, x)$ are closed.

(3) $G(\cdot, \cdot)$ satisfies the linear growth condition.

If let $\lambda(t) = \max_{t\in[0,T]}\{\varphi(t), c\psi(t)\}$ where c is the constant in Lemma2.1, then by (2.1), we can know that $F(t, x)$ satisfies linear growth condition: $\sup\{v(t)|v(t) \in F(t, x)\} \leq 2\lambda(t)(1+ \| x \|)$, so dose $G(t, x)$ do.

From Theorem 5.2 of [6], there exists a solution $x(t)$ of inclusion(3.5) in $[0, T]$. Hence,

$$\dot{x}(t) \in f(t, x) + B(t, x)U(t, x), \quad x(0) = x_0$$

i.e., there exists a function $u(t) \in U(t, x)$ such that

$$\dot{x}(t) = f(t, x) + B(t, x)u(t)\ , \quad x(0) = x_0$$

The second step. We should show that (3.4) hold.

By (3.6),

$$\begin{aligned} < y(t) - x(t), \dot{y}(t) - \dot{x}(t) > &\leq L(t) \| y(t) - x(t) \|^2 + g(t) \| y(t) - x(t) \| \\ \| y(t) - x(t) \| \frac{d}{dt} \| y(t) - x(t) \| &\leq L(t) \| y(t) - x(t) \|^2 + g(t) \| y(t) - x(t) \| \end{aligned}$$

Let $s(t) =\| y(t) - x(t) \|$, one gets $s(t)$ is an absolutely function. Then

$$s(t)\dot{s}(t) \leq L(t)s^2(t) + g(t)s(t)$$

Let $\mathcal{T} = \{t \in [0, T] : s(t) = 0\}$ and $\mathcal{T}'$ be the set of all density points of $\mathcal{T}$. It is well known that $meas\mathcal{T}' = meas\mathcal{T}$. if $t \notin \mathcal{T}$, then $\dot{s}(t) \leq L(t)s(t) + g(t))$, since $s(t) > 0$. If

$t \in \mathcal{T}'$ and if $\dot{s}(t)$ exists, then $\dot{s}(t) = 0$. Hence $\dot{s}(t) \leq L(t)s(t) + g(t))$ for a.e. $t \in [0, T]$. By Grownwall inequality, we have

$$\| y(t) - x(t) \| \leq de^{m(t)} + \int_0^t e^{m(t)-m(s)} g(s)ds$$

So we complete the proof of this theorem. □

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