DIPARTIMENTO DI MATEMATICA
Complesso universitario, Via Vivaldi - 81100 Caserta


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Marco D'Apuzzo[1], Valentina De Simone[1], Daniela di Serafino[1]

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[1] Dipartimento di Matematica, Seconda Università degli Studi di Napoli, via Vivaldi 43, I-81100 Caserta, Italy.

# Starting-Point Strategies for an Infeasible Potential Reduction Method*

Marco D'Apuzzo, Valentina De Simone, Daniela di Serafino

Department of Mathematics, Second University of Naples,
via Vivaldi, 43, I-81100 Caserta, Italy.
{marco.dapuzzo, valentina.desimone, daniela.diserafino}@unina2.it

**Abstract**

We present two strategies for choosing a "hot" starting-point in the context of an infeasible Potential Reduction (PR) method for convex Quadratic Programming. The basic idea of both strategies is to select a preliminary point and to suitably scale it in order to obtain a starting point such that its nonnegative entries are sufficiently bounded away from zero, and the ratio between the duality gap and a suitable measure of the infeasibility is small. One of the two strategies is naturally suggested by the convergence theory of the PR method; the other has been devised to reduce the initial values of the duality gap and the infeasibility measure, with the objective of decreasing the number of PR iterations. Numerical experiments show that the second strategy generally performs better than the first, and both outperform a starting-point strategy based on the affine-scaling step.

**Keywords:** Starting Point, Potential Reduction, Quadratic Programming.

## 1 Introduction

The theory of infeasible Interior Point (IP) methods generally ensures convergence under loose restrictions on the starting point. In practice, IP methods may be very sensitive to the initial values of the primal and dual variables: the choice of "poor" starting points may drastically reduce the steps toward the solution, thus making the methods unacceptably slow or even stalling. Therefore, the choice of "hot" initial values is an important issue in the implementation of IP methods. A common way to face up with this issue is to use heuristic-based strategies that try to keep the infeasibility small while ensuring that nonnegative variables are far enough from zero. Different strategies based on these principles have been implemented in optimization packages, where they have demonstrated effectiveness.

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We focus on the development of starting-point strategies to be used with an infeasible Potential Reduction (PR) method for the solution of the following primal-dual Quadratic Programming (QP) problem:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & q(x) \equiv \dfrac{1}{2} x^T Q x + c^T x \\ \text{s. t.} & Ax - z = b, \\ & Gx = d, \\ & x + v = u, \\ & (x, z, v) \geq 0, \end{array} \tag{1}$$

$$\begin{array}{ll} \underset{x,y,\lambda,t}{\text{maximize}} & p(x, y, \lambda, t) \equiv -\dfrac{1}{2} x^T Q x + b^T y + d^T \lambda - v^T t \\ \text{s. t.} & Qx - A^T y - G^T \lambda - s + t = -c, \\ & (x, y, t, s) \geq 0, \end{array} \tag{2}$$

where $Q \in \Re^{n \times n}$ is symmetric positive semidefinite, $A \in \Re^{m \times n}$, $G \in \Re^{p \times n}$, $x, s, t, v \in \Re^n$, $y, z \in \Re^m$ and $\lambda \in \Re^p$. Note that $s, v$ and $z$ are slack variables, introduced to transform inequaliy constraints into equality ones; furthermore, the constraints $x \geq 0$ and $x + v$ in (1) correspond to $0 \leq x \leq u$. This formulation is quite general and we assume that some of the constraints may be missing.

The PR methods are based on the idea of generating a sequence of approximations to an optimal solution by driving toward $-\infty$ a suitable merit function, i.e. the so-called potential function. We consider the Infeasible Constrained PR (ICPR) method by Mizuno et al. [13], based on the following primal-dual potential function:

$$\Phi(w) = \rho \log \left( x^T s + y^T z + t^T v \right) - \sum_{i=1}^{n} \log (x_i s_i) - \sum_{i=1}^{m} \log (y_i z_i) - \sum_{i=1}^{n} \log (t_i v_i), \tag{3}$$

where $\rho \geq 2n + m$. Given the current iterate $w = (x, y, \lambda, s, t, v, z)$, the generic iteration of the ICPR method computes a search direction by applying a Newton step to the KKT equations with the complementarity conditions perturbed by an amount $\Delta/\rho$, where

$$\Delta = x^T s + y^T z + t^T v$$

is the current duality gap. The search direction $\delta w = (\delta x, \delta y, \delta \lambda, \delta s, \delta t, \delta v, \delta z)$ is therefore obtained by solving the linear system

$$\begin{array}{rcl} Q\delta x - A^T \delta y - G^T \delta \lambda - \delta s + \delta t &=& -r_d, \\ A\delta x - \delta z &=& -r_{p_1}, \\ G\delta x &=& -r_{p_2}, \\ \delta x + \delta v &=& -r_{p_3}, \\ S\delta x + X\delta s &=& -XSe_n + (\Delta/\rho)e_n, \\ Z\delta y + Y\delta z &=& -YZe_n + (\Delta/\rho)e_m, \\ V\delta t + T\delta v &=& -TVe_n + (\Delta/\rho)e_n, \end{array} \tag{4}$$

where

$$\begin{aligned} r_d &= Qx - A^T y - G^T \lambda - s + t + c, \\ r_{p_1} &= Ax - z - b, \\ r_{p_2} &= Gx - d, \\ r_{p_3} &= x + v - u \end{aligned}$$

are the current dual and primal infeasibilities, $H = diag(h)$ for any vector $h$, and $e_j$ is the vector of all ones of dimension $j$ (in the following, the subscript $j$ is omitted if the dimension is clear from the context). Once the direction has been computed, the approximation of the solution is updated as

$$w' = w + \vartheta \delta w \tag{5}$$

where $\vartheta$ is a suitable step length, ensuring the decrease of the potential function and the positivity of all the variables but $\lambda$. An inexact version of this method, in which the linear system (4) is solved approximately, has been proposed in [3, 4].

We note that the ICPR method does not make any explicit requirement on the closeness of the iterate to the central path, i.e. it does not impose that the complementarity products are close, in some sense, to the duality measure $\mu = \Delta/(2n + m)$. However, the perturbation parameter $\Delta/\rho$ can be regarded as the product $\xi\mu$, where $\xi = (2n + m)/\rho$ is a centering parameter, and hence, unless $\rho >> 2n + m$, system (4) implicitly places some centrality requirement. On the other hand, as discussed in Section 3, poorly centered points lead to small step lengths and hence slow the progress of the method toward the solution.

We present two starting-point strategies for the ICPR method. We first consider a strategy naturally deriving from the ICPR convergence theory, which provides a well-centered point such that the ratio between the corresponding duality gap and 2-norm of the infeasibility is below a suitable threshold. Then we present a different strategy, aimed at preserving the previous properties while obtaining smaller values of the duality gap and the infeasibility measure.

This paper is organized as follows. In Section 2 we outline basic strategies used to choose the initial point in IP methods. In Section 3, we describe the two starting-points strategies mentioned above. Numerical experiments performed to analyze the effectiveness of the two strategies, as well as to compare them with a starting-point strategy based on the affine-scaling step, are reported in Section 4. Concluding remarks are given in Section 5.

## 2 Starting-point strategies: basic ideas

Most of IP methods for Quadratic (and Linear) Programming explicitly or implicitly attempt to follow the central path. In the light of this, a very simple choice for the starting point would be to set the nonnegative variables to one and the others to zero. Such a choice guarantees that the point is perfectly centered, i.e. all the pairwise complementarity products are equal to $\mu$, and the nonnegative variables are sufficiently bounded away from zero; however, this point may be far from satisfying the constraints.

A strategy aimed at overcoming the previous drawback is proposed in [14] for QP problems. In this case, the starting point is chosen by attempting to satisfy "at least some of the constraints" in the primal-dual problem; furthermore, variables that are too close to zero are adjusted by setting them to large positive values. Strategies aimed at satisfying the requirement that the starting point is both "well centered" and "not too infeasible" are presented in [12] and [11], for Linear and Quadratic Programming, respectively. In [12] the starting point is obtained by first computing least squares solutions to the equations corresponding to the linear constraints of the optimization problem and then shifting the values of the variables involved in the duality gap to ensure they are kept far from the boundary of the nonnegative orthant. In [11], the unbounded variables are initially set to zero, while the nonnegative ones are set to a suitable constant value related to the magnitude of the problem data. This preliminary point is then shifted by using the affine-scaling step, i.e. the step obtained by solving system (4) without the perturbation terms $(\Delta/\rho)e$; this step should be able to capture the scale of the variables. A further shift is applied to avoid the violation of the nonnegativity requirement and to make the initial point entries large enough.

As observed in [9], the previous strategies may be regarded as attempts to find a point such that the complementarity products $x_i s_i$, $y_i z_i$ and $t_i v_i$ are greater than a suitable fraction of $\mu$ and the ratio $\sigma/\mu$ is not too large, where

$$\sigma = ||(r_d, r_{p_1}, r_{p_2}, r_{p_3})||_2$$

is a measure of the infeasibility. The rationale for this choice is that many path following methods are able to perform long steps toward the solution, and to drive the infeasibility to zero at least at the same rate as the duality gap, by keeping the iterates in the central path neighbourhood

$$\begin{aligned} \mathcal{N}_{-\infty}(\beta,\zeta) \;\; = \;\; & \left\{ w : \; \frac{\sigma}{\mu} \leq \beta \frac{\sigma^0}{\mu^0}, \;\; (x,y,s,t,v,z) > 0, \right. \\ & \left. x_i s_i, \; y_j z_j, \; t_i v_i \geq \zeta\mu, \; i = 1,\ldots,n, \; j = 1,\ldots,m \right\}, \end{aligned}$$

where $\beta \geq 1$, $0 < \zeta < 1$ and the superscript 0 refers to the starting point [15]. An initial value of $\sigma/\mu$ smaller than 1 keeps the infeasibility below the duality gap during the execution of the method.

We note that strategies based on the $\mathcal{N}_{-\infty}$ neighbourhood, using the affine-scaling step to correct a preliminary point, have been devised also for IP methods for nonlinear programming [1, 9]. The basic idea is that these methods consider at each iteration a quadratic model of the optimization problem; hence, a good starting point for a the quadratic model corresponding to a preliminary point should be looked for. However, in this case user-provided initial values for the primal variable $x$ are left unchanged; furthermore, initial values of the Lagrange multipliers of the equality constraints, which are estimated by the method itself, are not modified to avoid introducing unnecessary non-convexities in the problem [1].

## 3 A starting-point strategy for ICPR

The convergence theory of the ICPR method suggests a simple way to choose the starting-point. More precisely, indications about the starting point come from the following theorem, which implies that the inexact ICPR method has a polynomial-time convergence.

**Theorem 1** *Let $\rho \geq 2n + m + \sqrt{2n+m}$ and $\eta > 0$ such that an optimal solution $w^*$ of problem (1)-(2) exists with*

$$||w^*||_\infty \leq \eta. \tag{6}$$

*If*

$$w^0 = \gamma\eta(e, e, 0, e, e, e, e), \tag{7}$$

*with $\gamma \in (0, 1]$, and the residual vector $r_{KKT}$ in the inexact solution of system (4) satisfies*

$$||r_{KKT}||_2 < (\sqrt{3}/4)(\Delta/\rho),$$

*then a step length $\vartheta$ exists,*

$$\vartheta < (v_{min}\gamma^4)/(200\rho\Delta), \tag{8}$$

*with $v_{min} = \min\{x_i s_i,\ y_j z_j,\ t_i v_i;\ i = 1, \ldots, n,\ j = 1, ..., m\}$, such that*

$$\Phi(w') - \Phi(w) \leq -\delta, \quad \delta > 0$$

*and*

$$\Delta' \geq (1 - \vartheta)\Delta, \tag{9}$$

*where $w'$ is the iterate defined in (5) and $\Delta'$ is the corresponding duality gap.*

Theorem 1 is a reformulation of the theorem proved in [4] for convex QP problems with inequality constraints only; it includes also the exact version of ICPR as a special case ($r_{\text{KKT}} = 0$).

Since $||w^*||_\infty$ is usually not known a priori, large values of $\eta$ should be considered to satisfy (6). Furthermore, from (8) we see that poorly centered points, i.e. points for which some of the complementarity products are much smaller than $\mu$, drastically reduce $\vartheta$. However, a step length much larger than the one prescribed in (8) can be used in practice. Numerical experiments reported in [3] showed that a step length defined as

$$\vartheta = \chi\vartheta_{max}, \quad \vartheta_{max} = \max\{\vartheta > 0 : w' > 0\}, \tag{10}$$

where $\chi \approx 1$, works effectively. Such a step length is not explicitly dependent on $\gamma$, therefore we may assume $\gamma = 1$ in (7). Finally, condition (9) implies that $\Delta/\Delta^0 \geq \sigma/\sigma^0$ [4], i.e. the infeasibility reduces at least at the same rate as the duality gap; therefore, by choosing a starting point such that $\sigma^0 \leq \Delta^0$, the infeasibility does not exceed the duality gap at each step of ICPR.

In the light of the previous considerations, the starting point can be chosen as:

$$w^0 = \eta\tilde{w},$$

where

$$\tilde{w} = (e, e, 0, e, e, e, e)$$

and $\eta$ is such that

$$\frac{\sigma^0}{\Delta^0} \leq \tau, \tag{11}$$

with $0 < \tau \leq 1$. It is easy to verify that, since $\Delta^0$ increases quadratically with $\eta$ and $\sigma^0$ increases linearly, inequality (11) (approximately) holds if $\eta$ satisfies

$$\eta \geq \frac{\tilde{\sigma}}{\tau\tilde{\Delta}},$$

where $\tilde{\sigma}$ and $\tilde{\Delta}$ are the infeasibility and the duality gap associated to $\tilde{w}$. Therefore, we may set $\eta = 1$ if $\tilde{\sigma}/\tilde{\Delta} \leq \tau$ and $\eta = \tilde{\sigma}/(\tau\tilde{\Delta})$ otherwise. In the following, this strategy is referred to as STP1.

We note that, although STP1 ensures that $w^0$ is perfectly centered, that all its components with lower bound 0 are "sufficiently positive", and that the infeasibility is bounded by the duality gap throughout the computation, it has a drawback: even if $\sigma^0/\Delta^0$ is small, the initial values of the duality gap and the infeasibility may be very large, thus possibly slowing the progress toward the solution.

Therefore, we devised a different starting-point strategy, aimed at obtaining smaller values of $\sigma^0$ and $\Delta^0$. This strategy is referred to as STP2. The basic idea is to build a preliminary point $\bar{w}$ by assigning positive values to all the variables but $\lambda$, so that the box constraints are satisfied, the infeasibility is kept small and the perfect centrality is achieved. If (11) does not hold, some variables are scaled to satisfy it, trying to preserve the previous properties.

Let $\mathcal{I} = \{1, ..., n\}$ and $\mathcal{J} = \{1, ..., m\}$, and let us partition $\mathcal{I}$ as follows:

$$\mathcal{I} = \mathcal{I}_b \cup \mathcal{I}_{ub}, \;\; \mathcal{I}_b = \{i \in \mathcal{I} \;:\; u_i < +\infty\}, \;\; \mathcal{I}_{ub} = \{i \in \mathcal{I} \;:\; u_i = +\infty\},$$

i.e. $\mathcal{I}_b$ identifies the entries of $x$ that are bounded from above, and $\mathcal{I}_{ub}$ the remaining ones. Accordingly, $n$ can be written as $n = n_b + n_{ub}$, with $n_b = |\mathcal{I}_b|$ and $n_{ub} = |\mathcal{I}_{ub}|$. The preliminary vector $\bar{x}$ is defined as

$$\bar{x}_i = \begin{cases} u_i/2 & \text{if } u_i \leq 2\epsilon, \\ \epsilon & \text{otherwise,} \end{cases} \qquad i \in \mathcal{I},$$

with $\epsilon \geq 1$, in order to strictly satisfy the box constraints. The slack vectors $\bar{z}$ and $\bar{v}$ are set as

$$\bar{z}_j = \max\left\{(A\bar{x} - b)_j, \epsilon\right\}, \;\; j \in \mathcal{J},$$

$$\bar{v}_i = \begin{cases} u_i - \bar{x}_i & \text{if } i \in \mathcal{I}_b, \\ 0 & \text{otherwise,} \end{cases}$$

in order to annhilate infeasibility components whenever possible. The Lagrange multipliers associated to the linear equality constraints are set as $\bar{\lambda} = 0$, and the remaining variables are initialized in the following way:

$$\bar{s}_i = 1/\bar{x}_i, \;\; i \in \mathcal{I}, \quad \bar{y}_j = 1/\bar{z}_j, \;\; j \in \mathcal{J},$$

$$\bar{t}_i = \begin{cases} 1/\bar{v}_i & \text{if } i \in \mathcal{I}_b, \\ 0 & \text{otherwise,} \end{cases}$$

so that $\bar{w}$ is perfectly centered. We note that $\bar{\Delta} =( n + m + n_b)$.

If $\bar{\sigma}/\bar{\Delta} \leq \tau$, then we set $w^0 = \bar{w}$. Otherwise, we suitably modify the preliminary point $\bar{w}$. This modification involves only the variables $\bar{s}$, $\bar{t}$ and $\bar{v}$, as follows:

$$w^0 = \bar{w} + (0, 0, 0, (\alpha - 1)\bar{s}^{ub}, (\alpha - 1)\bar{t}, (\alpha - 1)\bar{v}, 0),$$

where

$$\bar{s}_i^{ub} = \begin{cases} \bar{s}_i & \text{if } i \in \mathcal{I}_{ub}, \\ 0 & \text{otherwise,} \end{cases}$$

and $\alpha > 1$ is a suitable constant to be determined in such a way that (11) holds. Note that $w^0$ is obtained from $\bar{w}$ by scaling the entries of $\bar{s}$ corresponding to the entries of $x$ with no upper bounds, and the entries of $\bar{t}$ and $\bar{v}$ corresponding to the entries of $x$ bounded from above. In this way the duality gap and the infeasibility change to

$$\begin{array}{l} \Delta^0 = \bar{\Delta} + (\alpha^2 - 1)n_b + (\alpha - 1)n_{ub}; \\ \sigma^0 \leq \bar{\sigma} + (\alpha - 1)\omega, \end{array} \tag{12}$$

with

$$\omega = ||\bar{s}^{ub}||_2 + ||\bar{v}||_2 + ||\bar{t}||_2.$$

From (12), we see that the previous modification of $\bar{w}$ tends to increase the duality gap faster than the infeasibility. Furthermore, scaling only $\bar{s}$, $\bar{v}$ and $\bar{t}$ produces a duality gap smaller than that obtained by scaling all the variables.

Concerning the choice of $\alpha$, (12) implies that condition (11) holds if

$$h_2\alpha^2 + h_1\alpha + h_0 \geq 0, \tag{13}$$

where:

$$h_2 = \tau n_b; \;\; h_1 = \tau n_{ub} - \omega; \quad h_0 = -\tau n + \omega + \tau\bar{\Delta} - \bar{\sigma}.$$

We consider first the case $h_2 > 0$, corresponding to $n_b > 0$. Let $\alpha_1$ and $\alpha_2$, with $\alpha_1 \leq \alpha_2$ be the zeros of the quadratic polynomial in (13). It is easy to verify that $\alpha_2 > 0$. Then, $\alpha$ can be set as

$$\alpha = \begin{cases} \alpha_1 & \text{if } \alpha_{min} \leq \alpha_1, \\ \alpha_2 & \text{if } \alpha_1 \leq \alpha_{min} \leq \alpha_2, \\ \alpha_{min} & \text{otherwise,} \end{cases} \tag{14}$$

where $\alpha_{min} > 1$ is a suitable threshold. If $h_2 = 0$, i.e. $n_b = 0$, inequality (13) reduces to

$$h_1\alpha + h_0 \geq 0, \tag{15}$$

where the coefficients $h_1$ and $h_2$ have the following expressions:

$$h_1 = \tau n - ||\bar{s}||_2; \quad h_0 = -\tau n + ||\bar{s}||_2 + \tau\bar{\Delta} - \bar{\sigma}.$$

If $h_1 > 0$, $\alpha$ can be chosen as:

$$\alpha = \max(\alpha_{min}, \alpha^*), \tag{16}$$

where $\alpha^*$ is the zero of the linear function in (15); otherwise, a value of $\alpha$ such that (11) holds may not exist, hence we set $w^0$ as in STP1.

We observe that STP2 can be implemented at a modest computational cost, since it does not require the solution of a KKT linear system or a linear least squares problem, which are performed by other strategies (see Section 2). We note also that if the primal QP problem has only box constraints, we can use the starting-point strategy described in [6], which has the advantage of providing a point that is feasible for both the primal and the dual problem.

# 4 Numerical experiments

We tested the starting-point strategies STP1 and STP2 within the PRQP optimization solver [3, 7], which implements the inexact ICPR method for the solution of the primal-dual problem (1)-(2). For comparison purposes, we implemented also a strategy based on the affine-scaling step, called STP3. In STP3 the same preliminary point $\bar{w}$ as in STP2 is chosen. If condition (11) is satisfied, then the starting point is set as $w^0 = \bar{w}$; otherwise, an affine-scaling step $\delta w^a$ is computed using the preliminary point, and all entries of $w^0$, except those of $\lambda^0$, are set as

$$w_i^0 = \max\left\{\bar{w}_i + \delta w_i^a, \epsilon\right\},$$

while $\lambda^0 = \bar{\lambda} = 0$.

Extensive numerical experiments showed that an effective choice of the value of $\tau$ in (11) should be dependent on $n$. Our experiments led us to set $\tau$ to the following value:

$$\tau_1 = \begin{cases} 1 & \text{if } n < 1000, \\ 1/10 & \text{if } 1000 \leq n < 10000, \\ 1/100 & \text{if } n \geq 10000. \end{cases} \tag{17}$$

Numerical experiments showed also that, if the preliminary point $\bar{w}$ gives a large value of $\bar{\sigma}/\bar{\Delta}$, then STP2 may benefit from using a starting point $w^0$ that satisfies (11) with a smaller value of $\tau$, depending on $\bar{\sigma}/\bar{\Delta}$. Therefore, we implemented also a variant of STP2 that uses $\tau_1$ to decide if the prelinary point can be accepted as starting point, and, if this is not the case, uses

$$\tau_2 = \max\left\{\min\left\{\tau_1, \frac{\bar{\Delta}}{\bar{\sigma}}\right\}, \tau_{min}\right\} \tag{18}$$

to satisfy (11). The threshold $\tau_{min}$ was set to 1/1000. The two versions of STP2 using (17) and (17)-(18) are denoted as STP2-A and STP2-B, respectively. The same choices of $\tau$ were implemented also within STP1; the corresponding versions are called STP1-A and STP1-B. The remaining parameters of STP2 were set as $\epsilon = 1$ and $\alpha_{min} = 1.7$.

In the PRQP solver, the parameter $\rho$ in (3) was set to $6(2n+m)$ if $2n+m \leq 15000$, and to $10(2n+m)$ otherwise. At each iteration of PRQP, the search direction was computed by solving system (4), reduced to the usual KKT system form, through the Conjugate Gradient (CG) method, coupled with a Constraint Preconditioner (see [2, 7]

for details). This preconditioner was applied through a sparse $LDL^T$ factorization. The step length was computed as in (10), with $\chi = 0.99$. The ICPR iterations were stopped requiring that the relative duality gap $\Delta/(1+q(x))$ and suitable measures of the primal and dual infeasibilities were bounded $10^{-7}$ and $10^{-8}$, respectively (see [3]). A maximum numer of 1000 ICPR iterations was set too. The preconditioned CG iterations were stopped using the following adaptive criterion:

$$||\tilde{r}||_2 \leq \frac{\Delta}{\rho}, \tag{19}$$

where $\tilde{r}$ is the residual corresponding to the approximate solution of the KKT system; a smaller tolerance was used in (19) when the relative duality gap got very small (see [3]). Note that, in order to compute a highly accurate affine-scaling step, we solved the related KKT system by requiring $||\tilde{r}||_2 \leq 10^{-10}$.

The previous starting-point strategies were tested on 50 convex QP problems from the CUTEr [10] collection; different dimensions, such that $n + m + p \geq 1000$, were chosen for the variable-size problems, thus obtaining 72 problems in total. We did not consider problems with only box constraints, since the "ad hoc" strategy mentioned at the end of Section 3 can be used in this case. We observed that for 28 problems the preliminary points $\tilde{w}$ and $\bar{w}$, used by STP1 and by STP2 and STP3, respectively, coincide and satisfy inequality (11). There are 12 more problems for which all the starting-point strategies produce the same number of ICPR iterations (and about the same number of CG ones); therefore, we do not consider them in our analysis. We neglect also 8 more problems that are very small, i.e. such that $n + m + p \leq 100$. The remaining 24 problems are listed in Table 1 along with their dimensions.

A comparison of the starting-point strategies, using the performance profiles by Dolan and Moré [8], is shown in Figure 1. The performance measure considered here is the number of ICPR iterations to solve the problem with the required accuracy. Thus, the performance ratio $p_s(\psi)$ gives the fraction of problems for which the number of iterations of ICPR with the strategy $s$ is within a factor $\psi$ of the smallest number of iterations achieved. For $\psi = 1$ the performance ratio measures the fraction of problems for which the strategy $s$ leads to the smallest number of iterations. We see that the best performance is obtained by using the two variants of the STP2 strategy, which produce very similar behaviours of ICPR; STP2-A leads to the smallest number of iterations for 71% of the problems, while STP2-B for 67% (the two variants generate the same starting point for 15 problems). The number of iterations obtained with STP1-B is generally close to that obtained with STP2, while a slight increase can be observed with STP1-B. The worst performance is achieved by STP3, even if it produces the smallest number of iterations for 50% of the problems (actually, STP3 has the same iteration count as the STP2 variants for 10 problems). It is worth noting that even a small difference in the number of iterations may affect the overall cost of the ICPR method, since each ICPR step requires the $LDL^T$ factorization of the Constraint Preconditioner, which, for large-scale problems, may account for a significant execution time [5]. More details on the performance of the starting-point strategies can be obtained by looking at Table 2, where the number of ICPR and CG iterations are reported for each test

Table 1: Test problems used for comparing the starting-point strategies.

| *Problem* | $n$ | $m$ | $p$ | $n_b$ |
|---|---|---|---|---|
| CVXQP1-a | 1000 | 0 | 500 | 1000 |
| CVXQP1-b | 10000 | 0 | 5000 | 10000 |
| CVXQP2-a | 1000 | 0 | 250 | 1000 |
| CVXQP2-b | 10000 | 0 | 2500 | 10000 |
| CVXQP3-a | 1000 | 0 | 750 | 1000 |
| CVXQP3-b | 10000 | 0 | 7500 | 10000 |
| DUALC1 | 9 | 214 | 1 | 9 |
| DUALC2 | 7 | 228 | 1 | 7 |
| DUALC5 | 8 | 277 | 1 | 8 |
| GOULDQP2-a | 1999 | 0 | 999 | 1999 |
| GOULDQP2-b | 19999 | 0 | 9999 | 19999 |
| GOULDQP3-a | 1999 | 0 | 999 | 1999 |
| GOULDQP3-b | 19999 | 0 | 9999 | 19999 |
| HUEST-MOD | 5000 | 0 | 2 | 0 |
| HUESTIS | 5000 | 0 | 2 | 0 |
| HAGER1-a | 5001 | 0 | 2500 | 0 |
| HAGER1-b | 10001 | 0 | 5000 | 0 |
| HAGER3 | 10001 | 0 | 5000 | 0 |
| MOSARQP1 | 2500 | 700 | 0 | 0 |
| MOSARQP2 | 2500 | 700 | 0 | 0 |
| STCQP2-a | 1025 | 0 | 510 | 1025 |
| STCQP2-b | 4097 | 0 | 2052 | 4097 |
| STCQP2-c | 8193 | 0 | 4095 | 8193 |
| STCQP2-d | 16385 | 0 | 8190 | 16385 |

problem. We see that STP1 and STP2 largely outperform STP3 for all the CVXQP problems; furthermore, ICPR with STP3 is not able to solve DUALC1 and DUALC2 within 1000 iterations. On the other hand, STP3 results much more efficient than the other strategies on HUESTIS-MOD. We observe that a decrease in the number of ICPR iterations may not correspond to a decrease in the number of CG ones; however, the reduction of the ICPR iteration count is generally convenient for large-scale problems, since in this case the factorization of the preconditioner often dominates the cost of CG. We note also that the starting point used for the GOULDQP and HAGER1-a problems (marked with *) is always the preliminary one; this confirms that the preliminary point of STP2 and STP3 leads to better results. Finally, STP2 reduces to STP1 for HAGER1-b and HAGER3, since in this case $h_1 < 0$.

We analyzed also the duality gaps and infeasibility measures corresponding to the starting points used by the various strategies (these data are not repoted here for the

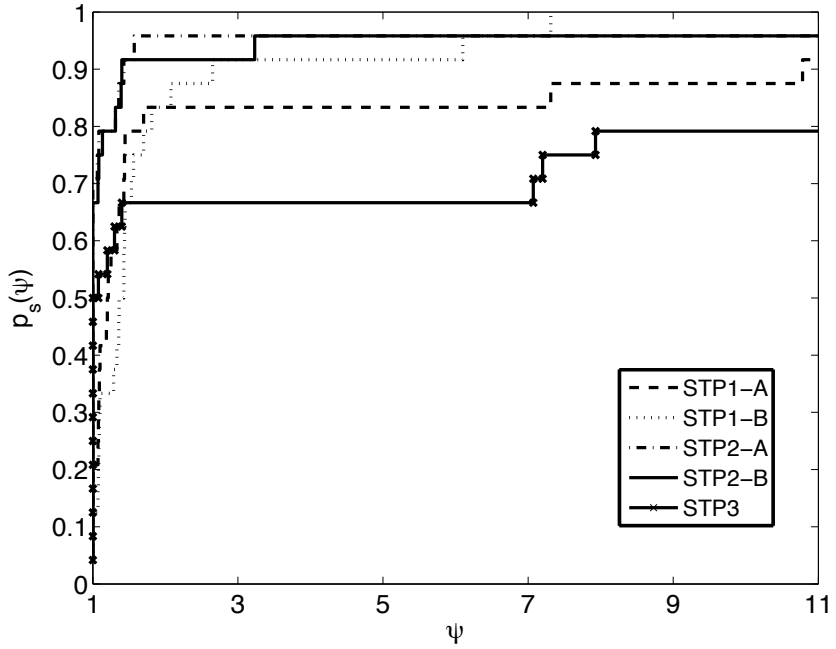


Figure 1: Performance profiles of the starting-point strategies (number of ICPR iterations).

sake of space). We found that the values of $\sigma^0$ and $\Delta^0$ obtained with STP2 are generally smaller than those obtained by the other strategies (a reduction of several orders of magnitude can be observed in a few cases), thus confirming the effectiveness of STP2 in reducing the ratio $\sigma^0/\Delta^0$ without increasing $\sigma^0$ and $\Delta^0$ too much.

# 5 Conclusions

We focused on starting-point strategies for the ICPR method for convex QP. We first considered a strategy naturally deriving from the convergence theory of the ICPR method, which provides a sufficiently positive point for which the ratio between the initial duality gap and infeasibility measure is small. However, this point may be far from satisfying the constraints and may produce a too large duality gap. Then, we devised a different strategy aimed at overcoming these drawbacks, while preserving the favourable properties of the previous point. Numerical experiments on a set of convex QP problems from the CUTEr collection showed that the second strategy generally performs better than the first, and that both approaches are more effective than a strategy based on the affine-scaling step. We believe that starting-point strategies similar to that proposed here can be successfully applied in the context of other IP methods.

Table 2: Number of ICPR and CG iterations with the different starting-point strategies (* identifies the problems using the preliminary point as starting point; — indicates that ICPR reaches the maximum number of iterations without satisfying the accuracy requirement).

| ICPR and CG iterations | | | | | |
|---|---|---|---|---|---|
| *Problem* | STP1-A | STP1-B | STP2-A | STP2-B | STP3 |
| CVXQP1-a | 14 (144) | 18 (142) | 19 (330) | 14 (160) | 111 (4023) |
| CVXQP1-b | 20 (227) | 25 (257) | 16 (273) | 16 (204) | 387 (21079) |
| CVXQP2-a | 14 (175) | 19 (163) | 20 (503) | 15 (202) | 99 (5124) |
| CVXQP2-b | 19 (267) | 29 (330) | 17 (515) | 16 (383) | 303 (35458) |
| CVXQP3-a | 18 (159) | 23 (168) | 15 (196) | 17 (137) | 108 (2028) |
| CVXQP3-b | 17 (202) | 20 (192) | 22 (229) | 14 (159) | 334 (13215) |
| DUALC1 | 248 (10096) | 61 (2587) | 23 (305) | 32 (621) | — |
| DUALC2 | 968 (14127) | 54 (1513) | 26 (236) | 84 (1453) | — |
| DUALC5 | 132 (5324) | 61 (1739) | 10 (67) | 14 (73) | 13 (89) |
| GOULDQP2-a* | 10 (34) | 10 (34) | 7 (15) | 7 (15) | 7 (7) |
| GOULDQP2-b* | 13 (106) | 13 (106) | 9 (49) | 9 (49) | 9 (49) |
| GOULDQP3-a* | 12 (60) | 12 (60) | 9 (50) | 9 (50) | 9 (50) |
| GOULDQP3-b* | 15 (86) | 15 (86) | 11 (79) | 11 (79) | 11 (79) |
| HUEST-MOD | 117 (117) | 117 (117) | 247 (247) | 247 (247) | 16 (16) |
| HUESTIS | 23 (23) | 23 (23) | 21 (21) | 21 (21) | 16 (17) |
| HAGER1-a* | 17 (17) | 17 (17) | 10 (10) | 10 (10) | 10 (10) |
| HAGER1-b | 15 (15) | 15 (15) | 15 (15) | 15 (15) | 21 (21) |
| HAGER3 | 15 (41) | 15 (41) | 15 (41) | 15 (41) | 18 (58) |
| MOSARQP1 | 15 (65) | 15 (65) | 14 (59) | 14 (59) | 14 (62) |
| MOSARQP2 | 12 (52) | 12 (52) | 11 (48) | 11 (48) | 11 (50) |
| STCQP2-a | 11 (141) | 11 (141) | 10 (127) | 10 (127) | 10 (132) |
| STCQP2-b | 12 (205) | 12 (205) | 13 (232) | 13 (232) | 12 (212) |
| STCQP2-c | 13 (223) | 13 (223) | 12 (209) | 12 (209) | 12 (203) |
| STCQP2-d | 14 (258) | 14 (258) | 13 (242) | 13 (242) | 14 (277) |

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