On Cone of Nonsymmetric Positive Semidefinite Matrices

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Abstract

In this paper, we analyze and characterize the cone of nonsymmetric positive semidefinite matrices (NS-psd). Firstly, we study basic properties of the geometry of the NS-psd cone and show that it is a hyperbolic but not homogeneous cone. Secondly, we prove that the NS-psd cone is a maximal convex subcone of P_0-matrix cone which is not convex. But the interior of the NS-psd cone is not a maximal convex subcone of P-matrix cone. As the byproducts, some new sufficient and necessary conditions for a nonsymmetric matrix to be positive semidefinite are given. Finally, we present some properties of metric projection onto the NS-psd cone.

Keywords: Nonsymmetric positive semidefinite matrix; hyperbolic cone; facial structure; maximal convex subcone; P_0-matrix; projection.

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1 Introduction

We consider the space of $n \times n$ real matrices, denoted by $\mathcal{M}^n$, with the trace inner product $\langle X, Y\rangle := tr(X^T Y)$ for $X, Y \in \mathcal{M}^n$ and the induced Frobenius matrix norm $\|X\| = \sqrt{tr(X^T X)}$. A nonsymmetric matrix $X \in \mathcal{M}^n$ is called positive semidefinite (NS-psd for short) if $u^T X u \geq 0$ for all $u \in \mathbb{R}^n$, and called positive definite if $u^T X u > 0$ for all $0 \neq u \in \mathbb{R}^n$. We use $\mathcal{M}^n_+$ to denote the set of all nonsymmetric positive semidefinite matrices in $\mathcal{M}^n$, and $\mathcal{M}^n_{++}$ to denote the set of all nonsymmetric positive definite matrices in $\mathcal{M}^n$. Then $\mathcal{M}^n_+$ is a closed convex cone and $\mathcal{M}^n_{++}$ is the interior of $\mathcal{M}^n_+$.

Let $\mathcal{S}^n$ be the subspace of $n \times n$ symmetric matrices in $\mathcal{M}^n$. Correspondingly, let $\mathcal{S}^n_+$ denote the cone of positive semidefinite matrices in $\mathcal{S}^n$, and $\mathcal{S}^n_{++}$ denote the cone of positive definite matrices in $\mathcal{S}^n$. $\mathcal{S}^n_+$ is a closed convex cone and its interior is $\mathcal{S}^n_{++}$.

It's well known that $\mathcal{S}^n_+$, as an very important non-polyhedral convex cone, has nice geometric properties and arises in many areas, including engineering, statistics, and system and

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control theory, etc. Linear optimization problem over $\mathcal{S}^n_+$, known as semidefinite programming (SDP), plays a fundamental role in mathematical programming, see, e.g., [21, 2, 7]. However, compared with the $\mathcal{S}^n_+$, the NS-psd cone $\mathcal{M}^n_+$ hasn't been well studied on convex analysis. Actually, $\mathcal{S}^n_+$ and $\mathcal{M}^n_+$ are very different in many aspects. Let us observe the following four examples:

- $\mathcal{S}^n_+$ is a self-dual homogenous cone in $\mathcal{S}^n$, but $\mathcal{M}^n_+$ is a hyperbolic cone and not a homogeneous cone in $\mathcal{M}^n$ (see Theorem 3.2).

- A matrix $X \in \mathcal{S}^n_+$ is invertible if and only if it belongs to the interior of S^n_+, whereas an invertible matrix $X \in \mathcal{M}^n_+$ doesn't imply that X is in the interior of $\mathcal{M}^n_+$. In fact, if we take $X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{M}^2_+$, then for any $\epsilon > 0$, we have
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \epsilon & 1 \\ -1 & \epsilon \end{pmatrix} \in \mathcal{M}^2_{++},$$
meanwhile
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \epsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\epsilon & 1 \\ -1 & -\epsilon \end{pmatrix} \notin \mathcal{M}^2_+.$$
This means that $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ belongs to the boundary of $\mathcal{M}^2_+$ but it is invertible.

- It's well known that a *symmetric* matrix belongs to $\mathcal{S}^n_+$ (resp. $\mathcal{S}^n_{++}$) if and only if it is a P_0 *(resp. P)-matrix*, but the equivalence fails when symmetry assumption is dropped (see Example 3.3.2 in [5]).

- $\mathcal{S}^n_+$ and $\mathcal{M}^n_+$ are subsets of P_0-matrix cone ($\mathcal{P}_0$ for short). For $\mathcal{S}^n_+$, $bd(\mathcal{S}^n_+) \subseteq bd(\mathcal{P}_0)$, which implies $bd(\mathcal{S}^n_+) \cap int(\mathcal{P}_0) = \emptyset$. While, for $\mathcal{M}^n_+$, $bd(\mathcal{M}^n_+) \cap int(\mathcal{P}_0) \neq \emptyset$ (see Proposition 4.1 (ii)).

In this paper, we will take a close look at the NS-psd cone in the view of convex analysis. First, we study the facial structure of $\mathcal{M}^n_+$ in Section 3, which is a representative property for closed convex cones. We'll show that $\mathcal{M}^n_+$ is a hyperbolic cone but not a homogeneous cone in $\mathcal{M}^n$, while $\mathcal{S}^n_+$ is a self-dual homogenous cone in $\mathcal{S}^n$. In Section 4, we study the relationship between the NS-psd cone and the P_0-matrix cone, where the latter one is a very important class of matrices in linear complementarity theory and it contains $\mathcal{M}^n_+$ as a proper subclass. By proving some fundamental and interesting results about matrix determinant and the boundary properties of $\mathcal{P}_0$ and $\mathcal{M}^n_+$, we obtain that $\mathcal{M}^n_+$ is a maximal convex subcone of $\mathcal{P}_0$, however, $\mathcal{M}^n_{++}$ is not a maximal convex subcone of $\mathcal{P}$. Some necessary and sufficient conditions for a matrix to be NS-psd are also presented. Finally, we study the metric projection onto $\mathcal{M}^n_+$ in Section 5, including the strong semismoothness, explicit formulas of directional derivative and Clarke's generalized Jacobian of the projection, which extend a series of results in [12, 17, 18].

2 Preliminaries

In this section, we review some concepts and properties about convex cones in a finite-dimensional real vector space $\mathcal{E}$ equipped with a inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$.

Convex cone A *convex cone* $\mathcal{K} \subseteq \mathcal{E}$ is a nonempty set that is closed under nonnegative linear combination of all its members, i.e., $\lambda\mathcal{K} \subseteq \mathcal{K}, \mathcal{K} + \mathcal{K} \subseteq \mathcal{K}, \forall \lambda \geq 0$. The biggest subspace contained in $\mathcal{K}$ is called the *linearlity space* of $\mathcal{K}$, denoted by $L(\mathcal{K})$. If $L(\mathcal{K}) = \{0\}$, we call the convex cone $\mathcal{K}$ is *pointed*. In other words, a convex cone $\mathcal{K}$ is pointed iff it has no lines. If a closed convex pointed cone has nonempty interior, we call it a *proper cone*.

Given two nonempty convex cones $\mathcal{K}_1$ and $\mathcal{K}_2$ in $\mathcal{E}$, let $\mathcal{K}_1 \oplus \mathcal{K}_2$ denote the direct sum of $\mathcal{K}_1$ and $\mathcal{K}_2$, i.e., each vector $x \in \mathcal{K}_1 + \mathcal{K}_2$ can be expressed uniquely in the form $x = y + z$ where $y \in \mathcal{K}_1, z \in \mathcal{K}_2$. One special case of $\mathcal{K}_1 \oplus \mathcal{K}_2$ is when $\langle \mathcal{K}_1, \mathcal{K}_2 \rangle = 0$.

Given a set $\mathcal{C} \subseteq \mathcal{E}$, the dual cone of $\mathcal{C}$ is defined as $\mathcal{C}^* = \{x : \langle x, y \rangle \geq 0, \forall y \in \mathcal{C}\}$. The convex hull of $\mathcal{C}$ is denoted by $conv(\mathcal{C})$, $cone(\mathcal{C})$ denotes the convex cone generated by $\mathcal{C}$. We let $int(\mathcal{C})$, $cl(\mathcal{C})$, $bd(\mathcal{C})$, $ri(\mathcal{C})$ and $rb(\mathcal{C})$ denote the interior, closure, boundary, relative interior and relative boundary of $\mathcal{C}$, respectively. And we use $\mathcal{C}_1 \subset \mathcal{C}_2$ denote a proper subset, i.e., $\mathcal{C}_1 \subsetneq \mathcal{C}_2$.

Typical closed convex cones A closed pointed convex cone $\mathcal{K} \subseteq \mathcal{E}$ with nonempty interior is *homogeneous* if for any $x, y \in int(\mathcal{K})$ there exists an invertible linear mapping g such that $g(\mathcal{K}) = \mathcal{K}$ and $g(x) = y$, i.e., the group of automorphisms of $\mathcal{K}$ acts transitively on the interior of $\mathcal{K}$. If $\mathcal{K}$ is homogenous and $\mathcal{K}^* = \mathcal{K}$ (self dual), we call $\mathcal{K}$ a *symmetric cone*. It's well known that $\mathcal{S}_+^n$ is a symmetric cone in $\mathcal{S}^n$. And there are several other common symmetric cones, such as the nonnegative orthant (R_+^n), the Lorentz cone (i.e., second order cone), and so on. For more details about homogeneous and symmetric cones, see [6, 8, 20, 19], etc.

Besides the homogenous cone, there exists a more general closed convex cone, called hyperbolic cone. It is defined as follows. Given a homogeneous polynomial p of degree n on $\mathcal{E}$, p is called to be *hyperbolic* with respect to the direction $d \in \mathcal{E}$, if $p(d) \neq 0$ and the polynomial $t \mapsto p(td + x)$ has only real roots for every $x \in \mathcal{E}$. The associated *hyperbolic cone* of the hyperbolic polynomial p with direction d is defined as the set of all such x that the univariate polynomial $\lambda \mapsto p(\lambda d - x)$ has only nonnegative roots, where $\lambda \mapsto p(\lambda d - x)$ is called the characteristic polynomial of x. Hyperbolic cones contain homogeneous cones as a subclass. For more details see [1, 14, 9] and references therein.

Faces of a closed convex cone Let $\mathcal{K} \subseteq \mathcal{E}$ be a closed convex cone. A convex subset $\mathcal{F} \subseteq \mathcal{K}$ is *a face* of $\mathcal{K}$, denoted by $\mathcal{F} \trianglelefteq \mathcal{K}$, if

$$x \in \mathcal{F},\ 0 \preceq_{\mathcal{K}} y \preceq_{\mathcal{K}} x \quad \Rightarrow \quad cone(\{y\}) \subseteq \mathcal{F},$$

where $\preceq_{\mathcal{K}}$ denotes the partial order with respect to $\mathcal{K}$, that is, $x_1 \preceq_{\mathcal{K}} x_2$ means that $x_2 - x_1 \in \mathcal{K}$. Equivalently, $\mathcal{F} \trianglelefteq \mathcal{K}$ if $x + y \in \mathcal{F}, x \in \mathcal{K}$, and $y \in \mathcal{K}$ implies that $x \in \mathcal{F}$ and $y \in \mathcal{F}$. If $\mathcal{F} \trianglelefteq \mathcal{K}$ but $\mathcal{F} \neq \mathcal{K}$, we write $\mathcal{F} \triangleleft \mathcal{K}$. If $\emptyset \neq \mathcal{F} \triangleleft \mathcal{K}$, then $\mathcal{F}$ is a *proper face* of $\mathcal{K}$. Every proper face of $\mathcal{K}$ belongs to $rb(\mathcal{K})$. The *complementary or conjugate face* of $\mathcal{F} \trianglelefteq \mathcal{K}$, denoted by $\mathcal{F}^c$, is defined as $\mathcal{F}^c = \mathcal{K}^* \cap \mathcal{F}^\perp$. For $\mathcal{C} \subseteq \mathcal{K}$, we let $\mathcal{F}(\mathcal{C}, \mathcal{K})$ denote the smallest face that contains $\mathcal{C}$, i.e., $\mathcal{F}(\mathcal{C}, \mathcal{K})$ is the intersection of all faces containing $\mathcal{C}$. Followings are two important properties about facial structure of the closed convex cone $\mathcal{K}$ (see [15] or [16]):

$$\mathcal{F} = \mathcal{F}(\mathcal{C}, \mathcal{K}) \text{ if and only if } ri(\mathcal{C}) \subseteq ri(\mathcal{F}); \tag{2.1}$$

$$\mathcal{U} := \{ri(\mathcal{F}),\ \mathcal{F} \lhd \mathcal{K}\} \text{ is a partition of } rb(\mathcal{K}). \tag{2.2}$$

Here, (2.2) means that the elements of $\mathcal{U}$ are pairwise disjoint and cover $bd(\mathcal{K})$. Due to (2.1), it holds for any $\bar{x} \in ri(\mathcal{C})$ that

$$\mathcal{F}(\mathcal{C}, \mathcal{K}) = \{y \in \mathcal{K} : \alpha\bar{x} - y \in \mathcal{K}, \exists \alpha > 0\}. \tag{2.3}$$

The ray generated by $0 \neq x \in \mathcal{K}$ is called an *extreme ray* if $cone\{x\} \unlhd \mathcal{K}$. Every extreme ray is a one-dimensional face. And a zero-dimensional face is called a *extreme point*, or a vertex. We use $Exe(\mathcal{K})$ denote the set of extreme rays of $\mathcal{K}$. For $\mathcal{S}^n_+$,

$$Exe(\mathcal{S}^n_+) = \{uu^T : 0 \neq u \in \mathbb{R}^n\}. \tag{2.4}$$

If the closed convex cone $\mathcal{K}$ is not pointed, then it has no extreme ray and no extreme point. Conversely (see Section 2.8, [7]),

Every proper cone is equivalent to the convex hull of its extreme points and extreme rays. (2.5)

A face $\mathcal{F} \unlhd \mathcal{K}$ is an *exposed face* if it is the intersection of $\mathcal{K}$ with a hyperplane. If every face of $\mathcal{K}$ is exposed, we call $\mathcal{K}$ the *facially exposed.* Further, $\mathcal{K}$ is called a *nice cone* if $\mathcal{F}^* = \mathcal{K}^* + \mathcal{F}^\perp$ for all $\mathcal{F} \unlhd \mathcal{K}$. All nice cones are facially exposed (see [11]). And all proper faces of hyperbolic cones are exposed (Theorem 23, [14]), so do homogeneous cones.

Basic notations

$X \succeq 0$: X is a nonsymmetric positive semidefinite matrix.

$\mathcal{K}_1 \setminus \mathcal{K}_2$: difference of two sets $\mathcal{K}_1$ and $\mathcal{K}_2$, i.e., $\{x \in \mathcal{K}_1 : x \notin \mathcal{K}_2\}$.

$A\mathcal{S}^n$: the subspace of antisymmetric matrices.

$N(A)$: the null space of a linear operator or a matrix A.

$\mathcal{K}^\perp$: the orthogonal complement of $\mathcal{K}$ in $\mathcal{M}^n$.

$\mathcal{K}^{\perp_s}$: the orthogonal complement of $\mathcal{K}$ in $\mathcal{S}^n$.

$span(\mathcal{K})$: linear space spanned by set $\mathcal{K}$.

E^{ij}: the matrix in $\mathcal{M}^n$ with (i, j)th element being 1, all else being zeros.

$X_{\alpha\beta}$: $(X_{ij})_{i\in\alpha, j\in\beta}$, where α, β are subsets of $\{1, \cdots, n\}$.

I : identity matrix of size depending on the context.

$diag(X)$: a vector generated by the diagonal elements of $X \in \mathcal{M}^n$.

X^* : the classic adjoint matrix of $X \in \mathcal{M}^n$, i.e., the transpose of the matrix formed by taking the cofactor of each element of X.

3 The geometry of NS-psd cone

In this section, we study some basic properties of $\mathcal{M}^n_+$, mainly on its facial structure of it.

Since any real square matrix $A \in \mathcal{M}^n$ has a representation in terms of its symmetric and antisymmetric parts by

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}, \tag{3.1}$$

the antisymmetric part vanishes under quadratic form, i.e., $u^T \frac{A - A^T}{2} u = 0 \ \forall u \in \mathbb{R}^n$, and the symmetric part has a role determining positive semidefiniteness, we easily obtain the following basic facts and proposition.

Fact 1 $X \succeq 0 \Leftrightarrow X^T \succeq 0 \Leftrightarrow \frac{X+X^T}{2} \succeq 0$.

Fact 2 $\mathcal{M}^n = \mathcal{S}^n \oplus A\mathcal{S}^n$.

Fact 3 $\{X \in \mathcal{M}^n : X + X^T \in \mathcal{C}\} = \mathcal{C} \oplus A\mathcal{S}^n$ for any subset $\mathcal{C} \subseteq \mathcal{S}^n$.

Proposition 3.1 *In space $\mathcal{M}^n$, the following statements are true:*

(i) $\mathcal{M}^n_+ = \mathcal{S}^n_+ \oplus A\mathcal{S}^n$.

(ii) $L(\mathcal{M}^n_+) = A\mathcal{S}^n$.

(iii) $(\mathcal{M}^n_+)^* = \mathcal{S}^n_+$.

Proof. (i) Direct result of Fact 3 by taking $\mathcal{C} = \mathcal{S}^n_+$.

(ii) Since $\mathcal{S}^n_+$ is a pointed cone, i.e., $\mathcal{S}^n_+$ contains no line. The biggest subspace in $\mathcal{M}^n_+$ is just $A\mathcal{S}^n$ by (i).

(iii) It's known that $\mathcal{S}^n_+$ is self dual in $\mathcal{S}^n$. Then using the property $(\mathcal{K}_1 + \mathcal{K}_2)^* = \mathcal{K}_1^* \cap \mathcal{K}_2^*$ for any cones $\mathcal{K}_1$ and $\mathcal{K}_2$ ([16]), and by item (i), we have

$$(\mathcal{M}^n_+)^* = (\mathcal{S}^n_+ \oplus A\mathcal{S}^n)^* = (\mathcal{S}^n_+)^* \cap (A\mathcal{S}^n)^* = (\mathcal{S}^n_+)^* \cap (A\mathcal{S}^n)^\perp = (\mathcal{S}^n_+)^* \cap \mathcal{S}^n = \mathcal{S}^n_+.$$

This completes the proof. □

Clearly, $\mathcal{M}^n_+$ is not a symmetric cone since $\mathcal{M}^n_+ \neq (\mathcal{M}^n_+)^*$. Also, due to the above statement (ii), $\mathcal{M}^n_+$ is not a pointed cone. This implies that $\mathcal{M}^n_+$ is not a homogeneous cone. Actually, it is a hyperbolic cone.

Theorem 3.2 *$\mathcal{M}^n_+$ is a* ***hyperbolic*** *cone and not a homogeneous cone.*

Proof. Let $P(X) = \det(\frac{X+X^T}{2})$, $X \in \mathcal{M}^n$. Then $P(X)$ is a homogeneous polynomial of degree n on $\mathcal{M}^n$. Since a real symmetric matrix has only real eigenvalues,

$$P(X + tI) = \det(tI + \frac{X + X^T}{2}) = 0$$

has only real roots for all $X \in \mathcal{M}^n$. Thus, $P(X)$ is a hyperbolic polynomial with respect to the identity matrix I. Let $\Lambda_1(X), \Lambda_2(X), \cdots, \Lambda_n(X)$ denote n roots of $P(\lambda I - X) = \det(\lambda I - \frac{X+X^T}{2}) = 0$. Due to the fact that

$$X \in \mathcal{M}^n_+ \ \Leftrightarrow \ \frac{X + X^T}{2} \in \mathcal{S}^n_+,$$

we conclude that $\mathcal{M}^n_+ = \{X \in \mathcal{M}^n : \Lambda_i(X) \geq 0, i = 1, 2, \cdots, n\}$, i.e., $\mathcal{M}^n_+$ is a hyperbolic cone of P with direction I. □

From (2.5), Proposition 3.1 and Theorem 3.2, we know that all proper faces of $\mathcal{M}_+^n$ are exposed and $\mathcal{M}_+^n$ has no extreme ray and no extreme point.

To establish the facial structure of $\mathcal{M}_+^n$, we first present the following two lemmas.

Lemma 3.3 *Given closed convex cones $\mathcal{K}_1 \subset \mathcal{E}$ and $\mathcal{K}_2 \subset \mathcal{E}$, $\langle \mathcal{K}_1, \mathcal{K}_2 \rangle = 0$. If $x \in \mathcal{K}_1, y \in \mathcal{K}_2$, $\mathcal{C}_1 \subseteq \mathcal{K}_1, \mathcal{C}_2 \subseteq \mathcal{K}_2$, and $x + y \in \mathcal{C}_1 \oplus \mathcal{C}_2$, then there holds $x \in \mathcal{C}_1, y \in \mathcal{C}_2$.*

Proof. By the given, there exist $x_1 \in \mathcal{C}_1, y_1 \in \mathcal{C}_2$ such that $x + y = x_1 + y_2$. From $\langle span(\mathcal{K}_1), y\rangle = \langle span(\mathcal{K}_1), y_2\rangle = 0$, we obtain

$$\langle span(\mathcal{K}_1), x\rangle = \langle span(\mathcal{K}_1), x + y\rangle = \langle span(\mathcal{K}_1), x_1 + y_2\rangle = \langle span(\mathcal{K}_1), x_1\rangle,$$

i.e., $\langle span(\mathcal{K}_1), x - x_1\rangle = 0$. As $x - x_1 \in span(\mathcal{K}_1)$, we have

$$\langle x - x_1, x - x_1\rangle = 0,$$

which implies that $x = x_1 \in \mathcal{C}_1$. Similarly, we have $y \in \mathcal{C}_2$. This finishes the proof. $\square$

Lemma 3.4 *If $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, where $\mathcal{K}_1, \mathcal{K}_2$ are two closed convex cones in $\mathcal{E}$, and $\mathcal{K} \ni x = x_1 + x_2, x_1 \in \mathcal{K}_1, x_2 \in \mathcal{K}_2$, then $\mathcal{F}(x, \mathcal{K}) = \mathcal{F}(x_1, \mathcal{K}_1) \oplus \mathcal{F}(x_2, \mathcal{K}_2)$.*

Proof. We first show that $\big(\mathcal{F}(x_1, \mathcal{K}_1) \oplus \mathcal{F}(x_2, \mathcal{K}_2)\big) \trianglelefteq \mathcal{K}$. Let $\mathcal{F} := \mathcal{F}(x_1, \mathcal{K}_1) \oplus \mathcal{F}(x_2, \mathcal{K}_2)$ and $y, z \in \mathcal{K}$ satisfying $y + z \in \mathcal{F}$. Then there exist $y_1, z_1 \in \mathcal{K}_1,\ y_2, z_2 \in \mathcal{K}_2$ such that $y = y_1 + y_2,\ z = z_1 + z_2$. So, $(y_1 + z_1) + (y_2 + z_2) = y + z \in \mathcal{F} = \mathcal{F}(x_1, \mathcal{K}_1) \oplus \mathcal{F}(x_2, \mathcal{K}_2)$. Noting that $y_1 + z_1 \in \mathcal{K}_1,\ y_2 + z_2 \in \mathcal{K}_2$, by Lemma 3.3 we have $y_1 + z_1 \in \mathcal{F}(x_1, \mathcal{K}_1),\ y_2 + z_2 \in \mathcal{F}(x_2, \mathcal{K}_2)$. Using the definition of face, we immediately imply that

$$y_1, z_1 \in \mathcal{F}(x_1, \mathcal{K}_1),\ y_2, z_2 \in \mathcal{F}(x_2, \mathcal{K}_2).$$

Thus,

$$y = y_1 + y_2 \in \mathcal{F},\ z = z_1 + z_2 \in \mathcal{F},$$

which means that $\mathcal{F} = \mathcal{F}(x_1, \mathcal{K}_1) \oplus \mathcal{F}(x_2, \mathcal{K}_2)$ is a face of $\mathcal{K}$.

Now we prove the desired result.

(1) "$\subseteq$" : Noting that $x = x_1 + x_2 \in \big(\mathcal{F}(x_1, \mathcal{K}_1) \oplus \mathcal{F}(x_2, \mathcal{K}_2)\big)$, we have $\mathcal{F}(x, \mathcal{K}) \subseteq \mathcal{F}(x_1, \mathcal{K}_1) \oplus \mathcal{F}(x_2, \mathcal{K}_2)$ due to the definition of minimal face.

(2) "$\supseteq$" : By (2.3), we have

$$\begin{aligned} \mathcal{F}(x_1, \mathcal{K}_1) &= \{y \in \mathcal{K}_1 : \alpha x_1 - y \in \mathcal{K}_1, \exists \alpha > 0\}, \\ \mathcal{F}(x_2, \mathcal{K}_2) &= \{y \in \mathcal{K}_2 : \alpha x_2 - y \in \mathcal{K}_2, \exists \alpha > 0\}. \end{aligned}$$

Let $y_1 \in \mathcal{F}(x_1, \mathcal{K}_1), y_2 \in \mathcal{F}(x_2, \mathcal{K}_2)$. Then there exist $\alpha_1 > 0, \alpha_2 > 0$ such that $\alpha_1 x_1 - y_1 \in \mathcal{K}_1,\ \alpha_2 x_2 - y_2 \in \mathcal{K}_2$. Let $\alpha := \max\{\alpha_1, \alpha_2\}$. Thus

$$\alpha(x_1 + x_2) - (y_1 + y_2) = (\alpha_1 x_1 - y_1) + (\alpha - \alpha_1)x_1 + (\alpha_2 x_2 - y_2) + (\alpha - \alpha_2)x_2 \in \mathcal{K}_1 \oplus \mathcal{K}_2,$$

i.e.,

$$\alpha x - (y_1 + y_2) \in \mathcal{K},$$

which yields $y_1 + y_2 \in \mathcal{F}(x, \mathcal{K})$. So $\mathcal{F}(x_1, \mathcal{K}_1) \oplus \mathcal{F}(x_2, \mathcal{K}_2) \subseteq \mathcal{F}(x, \mathcal{K})$. The proof is complete. $\square$

Utilizing Lemma 3.4, we give out the following results.

Theorem 3.5 *In space $\mathcal{M}^n$, the following statements are true:*
(i) $(\mathcal{F} \oplus A\mathcal{S}^n) \trianglelefteq \mathcal{M}_+^n$, $\forall \mathcal{F} \trianglelefteq \mathcal{S}_+^n$; adversely, $\forall \mathcal{F} \trianglelefteq \mathcal{M}_+^n$, $\exists \mathcal{F}_1 \trianglelefteq \mathcal{S}_+^n$ s.t. $\mathcal{F} = \mathcal{F}_1 \oplus A\mathcal{S}^n$.
(ii) $\mathcal{F}(X, \mathcal{M}_+^n) = \{Y \succeq 0 : N(Y + Y^T) \supseteq N(X + X^T)\}$, $X \in \mathcal{M}_+^n$.
(iii) $\mathcal{F}^* = (\mathcal{M}_+^n)^* + \mathcal{F}^\perp$, for any $\mathcal{F} \trianglelefteq \mathcal{M}_+^n$.
(iv) $bd(\mathcal{M}_+^n) = bd(\mathcal{S}_+^n) \oplus A\mathcal{S}^n$.

Proof. We present two existing results in $\mathcal{S}^n$ (see [7] or [21]):

$$\mathcal{F}(X, \mathcal{S}_+^n) = \{Y \in \mathcal{S}_+^n : N(Y) \supseteq N(X)\}, \tag{3.2}$$
$$\mathcal{F}^* = \mathcal{S}_+^n + \mathcal{F}^{\perp_s}, \forall \mathcal{F} \lhd \mathcal{S}_+^n. \tag{3.3}$$

Then we prove (i)-(iv):

(i) Let $\mathcal{F} \trianglelefteq \mathcal{S}_+^n$ and take $X \in \mathcal{M}^n$ such that $\frac{X+X^T}{2} \in ri(\mathcal{F})$. Due to (2.1), we have $\mathcal{F} = \mathcal{F}(\frac{X+X^T}{2}, \mathcal{S}_+^n)$. Note that for any subspace $\mathcal{L} \subset \mathcal{M}^n$, $\mathcal{F}(X, \mathcal{L}) \equiv \mathcal{L}, \forall X \in \mathcal{L}$. From Lemma 3.4, we obtain

$$\begin{aligned} \mathcal{F}(X, \mathcal{M}_+^n) &= \mathcal{F}(\tfrac{X+X^T}{2} + \tfrac{X-X^T}{2}, \mathcal{M}_+^n) \\ &= \mathcal{F}(\tfrac{X+X^T}{2}, \mathcal{S}_+^n) \oplus \mathcal{F}(\tfrac{X-X^T}{2}, A\mathcal{S}^n) \\ &= \mathcal{F} \oplus A\mathcal{S}^n. \end{aligned}$$

That is, $\mathcal{F} \oplus A\mathcal{S}^n \trianglelefteq \mathcal{M}_+^n$.

Adversely, for each $\mathcal{F} \trianglelefteq \mathcal{M}_+^n$, taking $Y \in ri(\mathcal{F})$, we have $\mathcal{F} = \mathcal{F}(Y, \mathcal{M}_+^n) = \mathcal{F}(\frac{Y+Y^T}{2}, \mathcal{S}_+^n) \oplus A\mathcal{S}^n = \mathcal{F}_1 \oplus A\mathcal{S}^n$ where $\mathcal{F}_1 = \mathcal{F}(\frac{Y+Y^T}{2}, \mathcal{S}_+^n)$.

(ii) By Lemma 3.4, Fact 3 and (3.2), we have

$$\begin{aligned} \mathcal{F}(X, \mathcal{M}_+^n) &= \mathcal{F}(\tfrac{X+X^T}{2}, \mathcal{S}_+^n) \oplus \mathcal{F}(\tfrac{X-X^T}{2}, A\mathcal{S}^n) \\ &= \mathcal{F}(X + X^T, \mathcal{S}_+^n) \oplus A\mathcal{S}^n \\ &= \{Y \in \mathcal{M}^n : Y + Y^T \in \mathcal{F}(X + X^T, \mathcal{S}_+^n)\} \\ &= \{Y \succeq 0 : N(Y + Y^T) \supseteq N(X + X^T)\}. \end{aligned}$$

(iii) Let $\mathcal{F} \trianglelefteq \mathcal{M}_+^n$. Due to above result (i), there exists $\mathcal{F}_1 \trianglelefteq \mathcal{S}_+^n$ such that $\mathcal{F} = \mathcal{F}_1 \oplus A\mathcal{S}^n$. By (3.3), we have $\mathcal{F}^* = (\mathcal{F}_1 \oplus A\mathcal{S}^n)^* = \mathcal{F}_1^* \cap \mathcal{S}^n = \mathcal{S}_+^n + \mathcal{F}_1^{\perp_s}$. Moreover, $(\mathcal{M}_+^n)^* + \mathcal{F}^\perp = \mathcal{S}_+^n + (\mathcal{F}_1 \oplus A\mathcal{S}^n)^\perp = \mathcal{S}_+^n + (\mathcal{F}_1^\perp \cap \mathcal{S}^n) = \mathcal{S}_+^n + \mathcal{F}_1^{\perp_s}$. Hence, $\mathcal{F}^* = (\mathcal{M}_+^n)^* + \mathcal{F}^\perp$.

(iv) By (2.2), the boundary $bd(\mathcal{M}_+^n)$ consists of all the relative interior of proper faces in $\mathcal{M}_+^n$. Using the result (i), we immediately get the result (iv). □

Theorem 3.5 (iii) tells us that $\mathcal{M}_+^n$ is a nice cone in $\mathcal{M}^n$. And the fact $bd(\mathcal{M}_+^n) = bd(\mathcal{S}_+^n) \oplus A\mathcal{S}^n$ from Theorem 3.5 (iv) implies that $int(\mathcal{M}_+^n) = int(\mathcal{S}_+^n) \oplus A\mathcal{S}^n$, which further means that for any given matrix $X \in \mathcal{M}_+^n$, X belongs to $\mathcal{M}_{++}^n$ if and only if $(X + X^T)$ is invertible.

From Proposition 3.1, Theorems 3.2 and 3.5, we can see the difference between the psd cone and the NS-psd cone in geometry.

4 Relation with P_0-matrix cone

A matrix $X \in \mathcal{M}^n$ is said to be a P_0 (resp. P)-matrix if all its principal minors are nonnegative (resp. positive). Let $\mathcal{P}_0$ and $\mathcal{P}$ denote the sets of P_0-matrices and P-matrices respectively [10], i.e.,

$$\begin{array}{lcl} \mathcal{P}_0 & := & \{X \in \mathcal{M}^n : \det(X_{\alpha\alpha}) \geq 0, \ \forall\, \alpha \subseteq \{1, \dots, n\}\}, \\ \mathcal{P} & := & \{X \in \mathcal{M}^n : \det(X_{\alpha\alpha}) > 0, \ \forall\, \alpha \subseteq \{1, \dots, n\}\}. \end{array}$$

Then, they have following properties ([10], or see Section 3, [5]):

- $X \in \mathcal{P}_0 \ \Leftrightarrow \ \forall\, \alpha \subseteq \{1, \dots, n\}$, all real eigenvalues of $X_{\alpha\alpha}$ are nonnegative.
- $X \in \mathcal{P} \ \Leftrightarrow \ \forall\, \alpha \subseteq \{1, \dots, n\}$, all real eigenvalues of $X_{\alpha\alpha}$ are positive.
- $X \in \mathcal{P}_0 \ \Leftrightarrow \ \forall \varepsilon > 0, \ X + \varepsilon I \in \mathcal{P}$.

Further, there exist

$$\begin{array}{rl} (i) & \mathcal{P}_0 \cap \mathcal{S}^n \ = \ \mathcal{S}^n_+, \ \ \mathcal{P} \cap \mathcal{S}^n \ = \ \mathcal{S}^n_{++}, \\ (ii) & \mathcal{P}_0 \supset \mathcal{M}^n_+, \ \ \mathcal{P} \supset \mathcal{M}^n_{++}, \\ (iii) & \mathcal{P} = int(\mathcal{P}_0). \end{array}$$

Obviously, $\mathcal{P}_0$ is a cone in $\mathcal{M}^n$, but it's not convex. For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are P_0 matrices, but their sum $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ doesn't belong to $\mathcal{P}_0$.

Followings are some basic facts about geometry of $\mathcal{P}_0$.

Proposition 4.1 *The following statements are right:*
(i) $conv(\mathcal{P}_0) = cone(\mathcal{P}_0) = \{X \in \mathcal{M}^n : diag(X) \geq 0\}$.
(ii) $bd(\mathcal{M}^n_+) \nsubseteq \mathcal{P}$, $bd(\mathcal{M}^n_+) \cap \mathcal{P} \neq \emptyset$.

Proof. (i) Given matrices $E^{ij}, 1 \leq i, j \leq n$ and $-E^{kl}, 1 \leq k \neq l \leq n$. It's trivial that they are all P_0-matrices. Thus

$$\begin{array}{rcl} conv(\mathcal{P}_0) & = & cone(\mathcal{P}_0) \supseteq cone(\{E^{ij}, 1 \leq i, j \leq n\} \cup \{-E^{kl}, 1 \leq k \neq l \leq n\}) \\ & = & \{\sum\limits_{1 \leq i,j \leq n} \lambda_{ij} E^{ij} + \sum\limits_{1 \leq k \neq l \leq n} \mu_{kl}(-E^{kl}) : \lambda_{ij}, \mu_{kl} \geq 0, \forall i, j, k, l\} \\ & = & \{\sum\limits_{1 \leq i \leq n} \lambda_{ii} E^{ii} + \sum\limits_{1 \leq i \neq j \leq n} \eta_{ij} E^{ij} : \eta_{ij} \in \mathbb{R}, \lambda_{ii} \geq 0, \forall i, j\} \\ & = & \{X \in \mathcal{M}^n : diag(X) \geq 0\}. \end{array}$$

Since every P_0-matrix must be with nonnegative diagonal elements by its definition, i.e., $\mathcal{P}_0 \subseteq \{X \in \mathcal{M}^n : diag(X) \geq 0\}$, there holds

$$conv(\mathcal{P}_0) \subseteq conv(\{X \in \mathcal{M}^n : diag(X) \geq 0\}) = \{X \in \mathcal{M}^n : diag(X) \geq 0\}.$$

Thus $conv(\mathcal{P}_0) = \{X \in \mathcal{M}^n : diag(X) \geq 0\}$. The proof of statement (i) is complete.

(ii) By Theorem 3.5 (iv), $bd(\mathcal{M}^n_+) = bd(\mathcal{S}^n_+) \oplus A\mathcal{S}^n_+ \supset bd(\mathcal{S}^n_+)$. Note that

$$bd(\mathcal{S}^n_+) = \{X \in \mathcal{S}^n_+ : \det X = 0\} \nsubseteq \mathcal{P}.$$

We know $bd(\mathcal{M}^n_+) \nsubseteq \mathcal{P}$. To prove $bd(\mathcal{M}^n_+) \cap \mathcal{P} \neq \emptyset$, we just need to find an element X in $\mathcal{P}$ and $bd(\mathcal{M}^n_+)$. For example, take an upper triangular matrix $X = \begin{pmatrix} 1 & 2 & \cdots & 2 \\ & \ddots & & \vdots \\ & & \ddots & 2 \\ & & & 1 \end{pmatrix} \in \mathcal{P}$. Then $X + X^T = \begin{pmatrix} 2 & 2 & \cdots & 2 \\ 2 & \ddots & & \vdots \\ \vdots & & \ddots & 2 \\ 2 & \cdots & \cdots & 2 \end{pmatrix}$ belongs to $bd(\mathcal{S}^n_+)$, which means $X \in bd(\mathcal{M}^n_+)$. This completes the proof. $\square$

Since $\mathcal{P}_0$ is not a convex cone, we are interested in the *maximal convex subcone* contained in $\mathcal{P}_0$ whose definition is introduced as below.

Definition 4.2 *Given a cone $\mathcal{C} \subset \mathcal{E}$. A subset $\mathcal{D} \subseteq \mathcal{C}$ is said to be a maximal convex subcone of $\mathcal{C}$ if it is a convex cone and there are no other convex cones in $\mathcal{C}$ containing $\mathcal{D}$. In other words, there isn't such $x \in \mathcal{C} \setminus \mathcal{D}$ that $cone(x \cup \mathcal{D}) \subseteq \mathcal{C}$.*

Because the convexity for a cone is equivalent to the closedness under nonnegative linear combination of any two elements in it, by the above definition, a convex cone $\mathcal{D}$ is a maximal convex subcone of $\mathcal{C}$ if and only if

$$\forall\ x \in (\mathcal{C} \setminus \mathcal{D}),\ \exists\ y \in \mathcal{D} \text{ such that } x + y \notin \mathcal{C}. \tag{4.1}$$

In other words, a convex cone $\mathcal{D}$ is a maximal convex subcone of $\mathcal{C}$ if and only if

$$x + y \in \mathcal{C},\ \forall y \in \mathcal{D} \ \Rightarrow\ x \in \mathcal{D}. \tag{4.2}$$

The implication (4.2) tells us that $\mathcal{D}$ can't be expanded to a larger convex cone than itself in $\mathcal{C}$.

Obviously, if cone $\mathcal{C}$ is not empty, the maximal convex subcone of $\mathcal{C}$ must exist. And, for a convex cone, its maximal convex subcone is just itself. For a general nonconvex cone, its maximal convex subcones are not always unique.

Now, we investigate the relationship between the NS-psd cone and P_0-matrix cone in low-dimensional space $\mathcal{M}^2$.

Proposition 4.3 *Let $X \in \mathcal{M}^2$. The following statements are true:*
(i) $\det(X + dd^T) = \det X + d^T X^* d,\ \forall d \in \mathbb{R}^2$.
(ii) $X \succeq 0 \Leftrightarrow X^* \succeq 0 \Leftrightarrow \det(X + dd^T) \geq 0, \forall d \in \mathbb{R}^2$.
(iii) $\mathcal{M}^2_+$ is a maximal convex subcone of $\mathcal{P}_0$.

Proof. (i) For any $d \in \mathbb{R}^2$, expanding $\det(X + dd^T)$, we have

$$
\begin{aligned}
& \det(X + dd^T) = \det \begin{pmatrix} X_{11} + d_1 d_1 & X_{12} + d_1 d_2 \\ X_{21} + d_2 d_1 & X_{22} + d_2 d_2 \end{pmatrix} \\
= \; & (X_{11}X_{22} - X_{12}X_{21}) + X_{11}d_2^2 - (X_{12} + X_{21})d_1 d_2 + X_{22}d_1^2 \\
= \; & \det X + d^T \begin{pmatrix} X_{22} & -X_{12} \\ -X_{21} & X_{11} \end{pmatrix} d \\
= \; & \det X + d^T X^* d.
\end{aligned}
$$

(ii) The first "⇔" is due to the following fact:

$$
\begin{aligned}
& X^* = \begin{pmatrix} X_{22} & -X_{12} \\ -X_{21} & X_{11} \end{pmatrix} \succeq 0 \\
\Leftrightarrow \; & \begin{pmatrix} X_{22} & -\frac{X_{12}+X_{21}}{2} \\ -\frac{X_{12}+X_{21}}{2} & X_{11} \end{pmatrix} \succeq 0 \\
\Leftrightarrow \; & \begin{pmatrix} X_{11} & \frac{X_{12}+X_{21}}{2} \\ \frac{X_{12}+X_{21}}{2} & X_{22} \end{pmatrix} \succeq 0 \\
\Leftrightarrow \; & \frac{X+X^T}{2} \succeq 0 \Leftrightarrow X \succeq 0.
\end{aligned}
$$

For the second "⇔", the necessity is due to

$$X^* \succeq 0 \Rightarrow X \succeq 0 \Rightarrow X + dd^T \succeq 0, \forall d \in \mathbb{R}^2 \Rightarrow \det(x + dd^T) \geq 0, \forall d \in \mathbb{R}^2.$$

For the sufficiency, by (i), we have

$$\det X + d^T X^* d \geq 0, \ \forall d \in \mathbb{R}^2.$$

which implies

$$d^T X^* d \geq 0, \ \forall d \in \mathbb{R}^2,$$

otherwise, if $\hat{d}^T X^* \hat{d} < 0$ for some $\hat{d} \in \mathbb{R}^2$, then

$$(\lambda \hat{d})^T X^* (\lambda \hat{d}) = \lambda^2 \hat{d}^T X^* \hat{d} \to -\infty, \text{ when } \lambda \to \infty.$$

Thus $X^* \succeq 0$. This completes the proof of statement (ii).

(iii) In $\mathcal{M}^2$, suppose that $\mathcal{M}^2_+$ is not a maximal convex subcone of $\mathcal{P}_0$. Then by (4.1),

$$\exists X \in (\mathcal{P}_0 \setminus \mathcal{M}^n_+), \ s.t. \ X + Y \in \mathcal{P}_0, \ \forall \ Y \in \mathcal{M}^n_+.$$

Hence, for all $d \in \mathbb{R}^2$, $X + dd^T \in \mathcal{P}_0$, i.e., $\det(X + dd^T) \geq 0$. By (ii), we get $X \succeq 0$, which contradicts the known fact $X \in (\mathcal{P}_0 \setminus \mathcal{M}^n_+)$. So we conclude that $\mathcal{M}^2_+$ is a maximal convex subcone of $\mathcal{P}_0$ in $\mathcal{M}^2$. □

We'll try to generalize the above results to high-dimensional space $\mathcal{M}^n$ ($n > 2$) in the rest of this section.

Proposition 4.4 *Let $X \in \mathcal{M}^n$ with $n > 2$. The the following statements hold:*
(i) $\det(X + dd^T) = \det X + d^T X^* d, \ \forall d \in \mathbb{R}^n$.
(ii) If $\det X > 0$*, then*

$$X \succeq 0 \Leftrightarrow X^* \succeq 0 \Leftrightarrow \det(X + dd^T) \geq 0, \forall d \in \mathbb{R}^n.$$

Proof. (i) Clearly, $(X + dd^T)_{ij} = X_{ij} + d_i d_j, \ \forall i, j = 1, \cdots, n.$ Then

$$\det(X + dd^T) = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} (X_{1j_1} + d_1 d_{j_1})(X_{2j_2} + d_2 d_{j_2}) \cdots (X_{nj_n} + d_n d_{j_n})$$

$$= \sum_{j_1 j_2 \cdots j_n} \big[(-1)^{\tau(j_1 j_2 \cdots j_n)} \sum_{\substack{k=0,\cdots,n \\ 1 \leq i_1 < i_2 < \cdots < i_k \leq n}} \big(X_{i_1 j_{i_1}} X_{i_2 j_{i_2}} \cdots X_{i_k j_{i_k}} \cdot d_{i_{k+1}} d_{j_{i_{k+1}}} \cdots d_{i_n} d_{j_{i_n}}\big)\big]$$

$$= \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdots j_n)} \Big[\prod_{i=1}^{n} X_{i j_i} + \prod_{i=1}^{n} d_i d_{j_i} + \sum_{k=1}^{n-2} \sum_{\substack{\{l_1,\cdots,l_k\} \\ \subseteq \{1,\cdots,n\}}} \Big(\prod_{t \in \mathcal{N} \setminus \{l_1,\cdots,l_k\}} d_t d_{j_t}\Big)\Big(\prod_{t \in \{l_1,\cdots,l_k\}} X_{t j_t}\Big)$$

$$+ \sum_{i=1}^{n} d_i d_{j_i} \prod_{t \neq i} X_{t j_t}\Big]$$

$$= \sum_{j_1 j_2 \cdots j_n} \Big[(-1)^{\tau(j_1 j_2 \cdots j_n)} \prod_{i=1}^{n} X_{i j_i}\Big] + \sum_{j_1 j_2 \cdots j_n} \Big[(-1)^{\tau(j_1 j_2 \cdots j_n)} \prod_{i=1}^{n} d_i d_{j_i}\Big]$$

$$+ \sum_{j_1 j_2 \cdots j_n} \Big[(-1)^{\tau(j_1 j_2 \cdots j_n)} \sum_{k=1}^{n-2} \sum_{\substack{\{l_1,\cdots,l_k\} \\ \subseteq \{1,\cdots,n\}}} \Big(\prod_{t \in \mathcal{N} \setminus \{l_1,\cdots,l_k\}} d_t d_{j_t}\Big)\Big(\prod_{t \in \{l_1,\cdots,l_k\}} X_{t j_t}\Big)\Big]$$

$$+ \sum_{j_1 j_2 \cdots j_n} \Big[(-1)^{\tau(j_1 j_2 \cdots j_n)} \sum_{i=1}^{n} d_i d_{j_i} \prod_{t \neq i} X_{t j_t}\Big]$$

$$= \det X + \det dd^T + \sum_{k=1}^{n-2} \sum_{\substack{\{l_1,\cdots,l_k\} \\ \subseteq \{1,\cdots,n\}}} \sum_{j_1 j_2 \cdots j_n} \Big[(-1)^{\tau(j_1 j_2 \cdots j_n)} \Big(\prod_{t \in \mathcal{N} \setminus \{l_1,\cdots,l_k\}} d_t d_{j_t}\Big)\Big(\prod_{t \in \{l_1,\cdots,l_k\}} X_{t j_t}\Big)\Big]$$

$$+ \sum_{i=1}^{n} \sum_{j_1 j_2 \cdots j_n} \Big[(-1)^{\tau(j_1 j_2 \cdots j_n)} d_i d_{j_i} \prod_{t \neq i} X_{t j_t}\Big]$$

$$= \det X + \det dd^T + \sum_{k=1}^{n-2} \sum_{\substack{\{l_1,\cdots,l_k\} \\ \subseteq \{1,\cdots,n\}}} A_{\{l_1,\cdots,l_k\}} + \sum_{i=1}^{n} B_i, \tag{4.3}$$

where, $j_1j_2\cdots j_n$ is a permutation of $12\cdots n$, $\tau(j_1j_2\cdots j_n)$ denotes the inverse ordinal number of this permutation, $\mathcal{N}$ denotes the set $\{1,2,\cdots,n\}$, and

$$A_{\{l_1,\cdots,l_k\}} := \sum_{j_1j_2\cdots j_n}\Big[(-1)^{\tau(j_1j_2\cdots j_n)}\Big(\prod_{t\in\mathcal{N}\setminus\{l_1,\cdots,l_k\}} d_td_{j_t}\Big)\Big(\prod_{t\in\{l_1,\cdots,l_k\}} X_{tj_t}\Big)\Big],$$

$$B_i := \sum_{j_1j_2\cdots j_n}\Big[(-1)^{\tau(j_1j_2\cdots j_n)}d_id_{j_i}\prod_{t\neq i} X_{tj_t}\Big].$$

Note that, $A_{\{l_1,\cdots,l_k\}} = \det Y$, where $Y=(Y_{ij})_{n\times n}$, and $Y_{ij} = \begin{cases} X_{tj_t}, & t\in\{l_1,\cdots,l_k\} \\ d_td_{j_t}, & t\in\mathcal{N}\setminus\{l_1,\cdots,l_k\}\end{cases}$.

When $1\le k\le n-2$, Y has at least two rows whose components are proportional. Thus

$$A_{\{l_1,\cdots,l_k\}} = \det Y = 0,\ \forall\ 1\le k\le n-2.$$

Meanwhile,

$$\begin{aligned} B_i &= \sum_{j_1j_2\cdots j_n}(-1)^{\tau(j_1j_2\cdots j_n)}X_{1j_1}\cdots X_{i-1j_{i-1}}d_id_{j_i}X_{i+1j_{i+1}}\cdots X_{nj_n} \\ &= \det\begin{pmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ d_id_1 & \cdots & d_id_n \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn}\end{pmatrix} \\ &= d_id_1(X^*)_{1i}+\cdots+d_id_n(X^*)_{ni}\ \ \text{(expanding by the } i\text{-th row)},\end{aligned}$$

from which we conclude that

$$\sum_{i=1}^n B_i = \sum_{1\le i,j\le n}(X^*)_{ij}d_id_j = d^TX^*d.$$

Above all, due to (4.3), we obtain $\det(X+dd^T) = \det X + d^TX^*d$.

(ii) The first "$\Leftrightarrow$": If $X\succeq 0$, then $X+dd^T\succeq 0$ for all $d\in\mathbb{R}^n$. Thus $\det(X+dd^T)\ge 0$, for all $d\in\mathbb{R}^n$. By statement (i), we have

$$\det X + d^TX^*d\ge 0,\forall d\in\mathbb{R}^n \Rightarrow d^TX^*d\ge 0,\forall d\in\mathbb{R}^n\Rightarrow X^*\succeq 0.$$

If $X^*\succeq 0$, applying the above implication again, we have $(X^*)^*\succeq 0$. Since $\det X>0$ and

$$(X^*)^* = \det X^*\cdot(X^*)^{-1} = \det[\det X\cdot X^{-1}]\cdot(\det X\cdot X^{-1})^{-1} = (\det X)^{n-2}X,$$

we have $X\succeq 0$.

The second "$\Leftrightarrow$": By (i) and the first "$\Leftrightarrow$", we easily obtain the desired result. $\square$

Proposition 4.4 can be generalized to any principal submatrix of X by replacing X with $X_{\alpha\alpha}$ ($\alpha \subset \{1, \cdots, n\}$). And Proposition 4.4 (i) is the generalization of Proposition 4.3 (i) to case $n > 2$. However, Proposition 4.3 (ii) is no longer correct for $n > 2$, because $rank(X^*) \equiv 0$ whenever $rank(X) \leq n-2$, which means " $X^* \succeq 0 \nRightarrow X \succeq 0$ " for $n > 2$. Coming with it, is it still true that $\mathcal{M}_+^n$ is a maximal convex subcone of $\mathcal{P}_0$ for general n? The answer is affirmative. In order to prove it, we present several basic facts on theory of maximal convex subcones.

Let $\mathcal{M}cs(l, \mathcal{C})$ denote the collection of maximal convex subcones of $\mathcal{C}$ that contain l, where $l \subset \mathcal{C}$, $\mathcal{C}$ is a cone in $\mathcal{E}$. And we call $\mathcal{D}$ *a maximal convex cone generated by* l *in* $\mathcal{C}$ if $\mathcal{D} \in \mathcal{M}cs(l, \mathcal{C})$. Since $cone(l)$ is the smallest convex cone containing l, every maximal convex cone generated by l in $\mathcal{C}$ (if it exists) contains $cone(l)$ as a subset, the maximal convex cones generated by l in $\mathcal{C}$ equal the maximal convex cones generated by $cone(l)$ in $\mathcal{C}$, i.e.,

$$\mathcal{M}cs(l, \mathcal{C}) = \mathcal{M}cs(cone(l), \mathcal{C}). \tag{4.4}$$

Apparently, $\mathcal{M}cs(l, \mathcal{C}) = \{\emptyset\}$ if $cone(l) \nsubseteq \mathcal{C}$. Two evident facts can be directly derived from Definition 4.2:

$$(i) \quad \text{If } l_1 \subseteq l_2, \text{ then } \mathcal{M}cs(l_1, \mathcal{C}) \supseteq \mathcal{M}cs(l_2, \mathcal{C}). \tag{4.5}$$

$$(ii) \quad \text{If } \mathcal{D} \in \mathcal{M}cs(l, \mathcal{C}),\ \mathcal{D} \subseteq \mathcal{K} \subseteq \mathcal{C}, \text{ and } \mathcal{K} \text{ is a convex cone, then } \mathcal{K} = \mathcal{D}. \tag{4.6}$$

Lemma 4.5 *Suppose that* $\mathcal{M}cs(l, \mathcal{C}) = \{\mathcal{D}\}$. *Then the following statements hold:*
(i) $\mathcal{M}cs(\mathcal{D}_1, \mathcal{C}) = \{\mathcal{D}\}$, *for any subset* $\mathcal{D}_1 \subset \mathcal{C}$ *such that* $l \subseteq \mathcal{D}_1 \subseteq \mathcal{D}$.
(ii) If l *is a convex cone, then*

$$x + y \in \mathcal{C},\ \forall\ y \in l \Rightarrow x \in \mathcal{D}.$$

Proof. (i) By the fact $\mathcal{M}cs(l, \mathcal{C}) = \{\mathcal{D}\}$ and (4.5), we have

$$\{\mathcal{D}\} = \mathcal{M}cs(\mathcal{D}, \mathcal{C}) \subseteq \mathcal{M}cs(\mathcal{D}_1, \mathcal{C}) \subseteq \mathcal{M}cs(l, \mathcal{C}) = \{\mathcal{D}\}.$$

So we get the proof of statement (i).

(ii) By contradiction, suppose that $x \notin \mathcal{D}$ and $x + y \in \mathcal{C}$ for any $y \in l$. Since l is a convex cone and $\mathcal{C}$ is a cone, it follows that, for any nonnegative integer k,

$$\{\mu x + \sum_{i=1}^{k} \lambda_i y_i : y_i \in l, \mu \geq 0, \lambda_i \geq 0, i = 1, \ldots, k\} \subseteq \mathcal{C}.$$

In other words,

$$cone(\{x\} \cup l) \subseteq \mathcal{C}.$$

Since $cone(\{x\} \cup l)$ is a convex cone and $cone(\{x\} \cup l) \supset l$, there exists $\hat{\mathcal{D}}_1 \in \mathcal{M}cs(l, \mathcal{C})$ such that $\hat{\mathcal{D}}_1 \supseteq cone(\{x\} \cup l)$. Noting that $x \notin \mathcal{D}$, we know $\hat{\mathcal{D}}_1 \neq \mathcal{D}$, which contradicts $\mathcal{M}cs(l, \mathcal{C}) = \{\mathcal{D}\}$. So we get the proof of statement (ii). $\square$

Lemma 4.6 *Suppose that* $\mathcal{M}cs(l,\mathcal{C}) \neq \{\emptyset\}$, *where* $ri(\mathcal{C}) = ri\big(cl(\mathcal{C})\big)$ *and* $l = ri(l) \subseteq ri(\mathcal{C})$. *Let* $\mathcal{K}$ *be a subset of* $\mathcal{E}$. *If* $\mathcal{D}_1 \subseteq \mathcal{K}$ *holds for any* $\mathcal{D}_1 \in \mathcal{M}cs(l,\mathcal{C})$, *then* $\mathcal{D}_2 \subseteq cl(\mathcal{K})$ *holds for any* $\mathcal{D}_2 \in \mathcal{M}cs(l, cl(\mathcal{C}))$.

Proof. By contradiction, we assume that there exists $\hat{\mathcal{D}}_2 \in \mathcal{M}cs(l, cl(\mathcal{C}))$ such that $\hat{\mathcal{D}}_2 \nsubseteq cl(\mathcal{K})$. We'll show that

$$ri(\hat{\mathcal{D}}_2) \nsubseteq \mathcal{K}.$$

If $ri(\hat{\mathcal{D}}_2) \subseteq \mathcal{K}$, by convexity of $\hat{\mathcal{D}}_2$, we have

$$\hat{\mathcal{D}}_2 \subseteq cl(\hat{\mathcal{D}}_2) = cl\big(ri(\hat{\mathcal{D}}_2)\big) \subseteq cl(\mathcal{K}),$$

which contradicts $\hat{\mathcal{D}}_2 \nsubseteq cl(\mathcal{K})$. So $ri(\hat{\mathcal{D}}_2) \nsubseteq \mathcal{K}$.

Note that $ri(\hat{\mathcal{D}}_2) \subseteq ri\big(cl(\mathcal{C})\big) = ri(\mathcal{C}) \subseteq \mathcal{C}$ and $ri(\hat{\mathcal{D}}_2)$ is a convex cone which contains $ri(l) = l$. So there exists $\hat{\mathcal{D}}_1 \in \mathcal{M}cs(l,\mathcal{C})$ such that $ri(\hat{\mathcal{D}}_2) \subseteq \hat{\mathcal{D}}_1$. Since $ri(\hat{\mathcal{D}}_2) \nsubseteq \mathcal{K}$, we have $\hat{\mathcal{D}}_1 \nsubseteq \mathcal{K}$. This contradicts the precondition that $\hat{\mathcal{D}}_1 \subseteq \mathcal{K}$ since $\hat{\mathcal{D}}_1 \in \mathcal{M}cs(l,\mathcal{C})$. The proof is complete. □

Utilizing the Lemma 4.6 and Proposition 4.4, we are ready to prove the main result of this section.

Theorem 4.7 $\mathcal{M}_+^n$ *is the unique maximal convex subcone generated by* $\mathcal{S}_{++}^n$ *in* $\mathcal{P}_0$.

Proof. First we show that

$$\mathcal{D} \subseteq \mathcal{M}_+^n \text{ for any } \mathcal{D} \in \mathcal{M}cs(\mathcal{S}_{++}^n, \mathcal{P}). \tag{4.8}$$

For any $\mathcal{D} \in \mathcal{M}sc(\mathcal{S}_{++}^n, \mathcal{P})$, take $0 \neq d \in \mathbb{R}^n$. There exists $n-1$ vectors $v_1, \cdots, v_{n-1} \in \mathbb{R}^n$ such that

$$\left[\ v_1,\ \ \cdots,\ v_{n-1},\ \frac{d}{\|d\|_2}\ \right]$$

forms an orthogonal matrix. In this case, for any $\lambda > 0, \lambda_i > 0, i = 1, \cdots, n-1$, $\sum_{i=1}^{n-1} \lambda_i v_i v_i^T + \lambda dd^T$ is a symmetric matrix whose eigenvalues are all positive, i.e.,

$$\{\sum_{i=1}^{n-1} \lambda_i v_i v_i^T + \lambda dd^T : \lambda_i > 0, \lambda > 0,\ i = 1, \cdots, n-1\} \subset \mathcal{S}_{++}^n.$$

Taking any $X \in \mathcal{D}$, by the convexity of $\mathcal{D}$ and $\mathcal{S}_{++}^n \subseteq \mathcal{D}$, we have

$$\{X + \sum_{i=1}^{n-1} \lambda_i v_i v_i^T + \lambda dd^T : \lambda_i > 0, \lambda > 0,\ i = 1, \cdots, n-1\} \subset \mathcal{D},$$

which implies that

$$\det(X + \sum_{i=1}^{n-1} \lambda_i v_i v_i^T + \lambda dd^T) > 0,\ \forall \lambda_i > 0, \lambda > 0,\ i = 1, \cdots, n-1.$$

Taking limit when $\lambda_i \to 0, i = 1, \dots, n-1$, and fixing $\lambda = 1$, we have

$$\det(X + dd^T) \geq 0.$$

By the arbitrariness of $d \in \mathbb{R}^n$, we obtain

$$\det(X + dd^T) \geq 0, \text{ for all } d \in \mathbb{R}^n.$$

For $n = 2$, this implies $X \succeq 0$ by Proposition 4.3. For $n > 2$, noting that $X \in \mathcal{P}$ means $\det X > 0$, by Proposition 4.4 (ii), we also get $X \succeq 0$. Hence, $\mathcal{D} \subseteq \mathcal{M}^n_+$.

Seeing that $ri(\mathcal{P}) = ri\big(cl(\mathcal{P})\big)$ and $\mathcal{S}^n_{++} = ri(\mathcal{S}^n_{++}) \subset ri(\mathcal{P}_0) = \mathcal{P}$, then applying Lemma 4.6 to (4.8), we have

$$\mathcal{D} \subseteq cl(\mathcal{M}^n_+) \text{ for any } \mathcal{D} \in \mathcal{M}cs(\mathcal{S}^n_{++}, cl(\mathcal{P})),$$

i.e.,

$$\mathcal{D} \subseteq \mathcal{M}^n_+ \text{ for any } \mathcal{D} \in \mathcal{M}cs(\mathcal{S}^n_{++}, \mathcal{P}_0). \tag{4.9}$$

Since $\mathcal{M}^n_+$ is a convex cone and $\mathcal{M}^n_+ \subseteq \mathcal{P}_0$, applying the fact (4.6) to (4.9), it holds that

$$\mathcal{D} = \mathcal{M}^n_+ \text{ for any } \mathcal{D} \in \mathcal{M}cs(\mathcal{S}^n_{++}, \mathcal{P}_0).$$

That is to say

$$\mathcal{M}cs(\mathcal{S}^n_{++}, \mathcal{P}_0) \equiv \{\mathcal{M}^n_+\}. \tag{4.10}$$

This completes the proof. □

Consequently, we obtain the following corollary.

Corollary 4.8 *Let $X \in \mathcal{M}^n$. The following statements are true:*
(i) $X \in \mathcal{M}^n_+$ if and only if $X + Y \in \mathcal{P}_0$ for all $Y \in \mathcal{S}^n_{++}$, i.e.,

$$X \succeq 0 \;\Leftrightarrow\; \det(X+Y)_{\alpha\alpha} \geq 0, \forall\, Y \in \mathcal{S}^n_{++}, \forall \alpha \subseteq \{1, \cdots, n\}.$$

(ii) $\mathcal{M}cs(Exe(\mathcal{S}^n_+), \mathcal{P}_0) = \{\mathcal{M}^n_+\}$.

Proof. (i) The necessity of statement (i) is trivial. The sufficiency is the straight result of (4.10) and Lemma 4.5 (ii).

(ii) Combining with (4.10) and Lemma 4.5 (i), we have

$$\mathcal{M}cs(\mathcal{S}^n_+, \mathcal{P}_0) \equiv \{\mathcal{M}^n_+\}.$$

And due to the fact that $cone(Exe(\mathcal{S}^n_+)) = cone(\{dd^T : d \in \mathbb{R}^n\}) = \mathcal{S}^n_+$ and by (4.4), we have

$$\mathcal{M}cs(\mathcal{S}^n_+, \mathcal{P}_0) = \mathcal{M}cs(Exe(\mathcal{S}^n_+), \mathcal{P}_0).$$

Hence

$$\mathcal{M}cs(Exe(\mathcal{S}^n_+), \mathcal{P}_0) = \{\mathcal{M}^n_+\}.$$

The proof is complete. □

However, $\mathcal{M}^n_{++}$ is not a maximal convex subcone of $\mathcal{P}$.

Theorem 4.9 *$\mathcal{M}^n_{++}$ is not a maximal convex subcone of $\mathcal{P}$. Therefore, if $X \in \mathcal{P}$, then*

$$\det(X+Y)_{\alpha\alpha} > 0, \forall\, Y \in \mathcal{S}^n_{++}, \forall \alpha \subseteq \{1, \cdots, n\} \;\; \not\Rightarrow \;\; X \succ 0.$$

Proof. By Proposition 4.1 (ii), $bd(\mathcal{M}^n_+) \cap \mathcal{P} \neq \emptyset$. Let $X \in bd(\mathcal{M}^n_+) \cap \mathcal{P}$. Then $cone(\{X\}) \subset bd(\mathcal{M}^n_+) \cap \mathcal{P}$. Moreover,

$$\mathcal{M}^n_{++} \subset \big(\mathcal{M}^n_{++} \cup cone(\{X\})\big) \subset \mathcal{P},$$

where $\big(\mathcal{M}^n_{++} \cup cone(\{X\})\big)$ is a convex cone since for any $Y \in \mathcal{M}^n_{++}, Z \in bd(\mathcal{M}^n_+)$, $Y + Z \in \mathcal{M}^n_{++}$. Thus, $\mathcal{M}^n_{++}$ is not a maximal convex subcone of $\mathcal{P}$. □

We end this section by stating some other maximal convex subcones in $\mathcal{P}_0$.

Theorem 4.10 *Let $\mathcal{M}_u = \{X \in \mathcal{M}^n : diag(X) \geq 0, X_{ij} = 0, i > j\}$, $\mathcal{M}_l = \{X \in \mathcal{M}^n : diag(X) \geq 0, X_{ij} = 0, i < j\}$. Then both $\mathcal{M}_u$ and $\mathcal{M}_l$ are maximal convex subcones of $\mathcal{P}_0$.*

Proof. It's clear that $\mathcal{M}_u, \mathcal{M}_l$ are two convex cones in $\mathcal{M}^n$. And for any $X \in \mathcal{M}_u$ or $\mathcal{M}_l$, any $\alpha \subseteq \{1, \cdots, n\}$, the eigenvalues of $X_{\alpha\alpha}$ are exactly the diagonal elements of $X_{\alpha\alpha}$. So all eigenvalues of $X_{\alpha\alpha}$ with $\alpha \subseteq \{1, \cdots, n\}$ are nonnegative, which means $X \in \mathcal{P}_0$. This further implies that $\mathcal{M}_u, \mathcal{M}_l$ are two convex subcones of $\mathcal{P}_0$.

Next, we just prove the maximal convexity of $\mathcal{M}_u$ in $\mathcal{P}_0$. The proof for $\mathcal{M}_l$ is in the similar way. As we know, the maximal convexity of $\mathcal{M}_u$ in $\mathcal{P}_0$ is equivalent to

$$\forall\, X \in \mathcal{P}_0 \setminus \mathcal{M}_u,\ \exists Y \in \mathcal{M}_u, \text{ s.t. } (X+Y) \notin \mathcal{P}_0.$$

Take any $X \in \mathcal{P}_0 \setminus \mathcal{M}_u$, which implies that $X_{kl} \neq 0$ for some $k > l$. Choose $Y \in \mathcal{M}_u$ satisfying $X_{kl}Y_{lk} > 0$ and $(X_{ll} + Y_{ll})(X_{kk} + Y_{kk}) < X_{kl}X_{lk} + X_{kl}Y_{lk}$. Such Y always exists because of the arbitrariness of Y_{lk} (one just needs to make $|Y_{lk}|$ big enough such that the right hand side of the above inequality is bigger enough than the left). Let $\alpha = \{l, k\}$, it follows

$$\begin{aligned} \det(X+Y)_{\alpha\alpha} &= \det \begin{pmatrix} X_{ll} + Y_{ll} & X_{lk} + Y_{lk} \\ X_{kl} & X_{kk} + Y_{kk} \end{pmatrix} \\ &= (X_{ll} + Y_{ll})(X_{kk} + Y_{kk}) - (X_{kl}X_{lk} + X_{kl}Y_{lk}) < 0, \end{aligned}$$

which implies $X + Y \notin \mathcal{P}_0$. So $\mathcal{M}_u$ is a maximal convex subcones of $\mathcal{P}_0$. □

Above all, the sets $\mathcal{M}_u$, $\mathcal{M}_l$ and $\mathcal{M}^n_+$ are members of $\mathcal{M}cs(\mathcal{I}_+, \mathcal{P}_0)$, where

$$\mathcal{I}_+ := \{X \in \mathcal{M}^n : diag(X) \geq 0, X_{ij} = 0\ \forall i \neq j\}$$

is the intersection of $\mathcal{M}_u$, $\mathcal{M}_l$ and $\mathcal{M}^n_+$. Therefore,

$$\mathcal{M}cs(\mathcal{I}_+, \mathcal{P}_0) \supseteq \{\mathcal{M}_u,\ \mathcal{M}_l,\ \mathcal{M}^n_+\} \supset \{\mathcal{M}^n_+\} = \mathcal{M}cs(\mathcal{S}^n_+, \mathcal{P}_0).$$

This inclusion is consistent with the fact (4.5), which says that the smaller l is, the larger $\mathcal{M}cs(l, \mathcal{C})$ is.

5 Projection onto NS-psd cone

Let $\Pi_{\mathcal{C}} : \mathcal{E} \to \mathcal{E}$ denote the metric projection of x onto $\mathcal{C}$, where $\mathcal{C} \subseteq \mathcal{E}$ is a nonempty closed convex set. Then, for any $x \in \mathcal{E}$,

$$\Pi_{\mathcal{C}}(x) = arg\min\{\frac{1}{2}\|x-y\|^2 : y \in \mathcal{C}\}.$$

Equivalently,

$$\langle \Pi_{\mathcal{C}}(x) - y, \Pi_{\mathcal{C}}(x) - x\rangle \leq 0, \ \forall y \in \mathcal{C}. \tag{5.1}$$

It's well known that $\Pi_{\mathcal{C}}(\cdot)$ is unique and contractive, i.e., $\|\Pi_{\mathcal{C}}(x) - \Pi_{\mathcal{C}}(y)\| \leq \|x-y\|$, $\forall x, y \in \mathcal{E}$. Let $\mathrm{dist}(x, \mathcal{C}) := \min\{\|x-y\| : y \in \mathcal{C}\}$. Then $\mathrm{dist}(x, \mathcal{C}) = \|x - \Pi_{\mathcal{C}}(x)\|$.

For the projection onto $\mathcal{M}^n_+$, Qi and Sun have already given its expression as follows (see Section 4.3, [13])

$$\Pi_{\mathcal{M}^n_+}(X) = \Pi_{\mathcal{S}^n_+}(\tfrac{X+X^T}{2}) + \tfrac{X-X^T}{2}. \tag{5.2}$$

From the positive homogeneity of $\Pi_{\mathcal{K}}(\cdot)$ for any closed convex cone $\mathcal{K}$, we immediately get

$$\mathrm{dist}(X, \mathcal{M}^n_+) = \frac{1}{2}\mathrm{dist}(X + X^T, \mathcal{S}^n_+). \tag{5.3}$$

We now discuss the tangent cone and second order tangent set of $\mathcal{M}^n_+$. For the closed convex set $\mathcal{C} \subseteq \mathcal{E}$, the *tangent cone* of $\mathcal{C}$ at $x \in \mathcal{C}$ is defined as (see Section 2.2.4, [4])

$$T_{\mathcal{C}}(x) := \{y \in \mathcal{E} : dist(x + ty, \mathcal{C}) = o(t), t \geq 0\}.$$

And the *inner and outer second order tangent sets* of $\mathcal{C}$ at $x \in \mathcal{C}$ in direction $h \in \mathcal{E}$ are respectively defined by (see Section 3.2.1, [4])

$$T^{i,2}_{\mathcal{C}}(x, h) := \{y \in \mathcal{E} : \mathrm{dist}(x + th + \frac{1}{2}t^2 y, \mathcal{C}) = o(t^2), t \geq 0\}$$

and

$$T^2_{\mathcal{C}}(x, h) := \{y \in \mathcal{E} : \exists t_k \downarrow 0, \mathrm{dist}(x + t_k h + \frac{1}{2}t_k^2 y, C) = o(t_k^2), t_k \geq 0\}.$$

Obviously $T^{i,2}_{\mathcal{C}}(x, h) \subseteq T^2_{\mathcal{C}}(x, h)$.

From Example 3.40 of [4], we know that in space $\mathcal{S}^n$, the inner and outer second order tangent sets for $\mathcal{S}^n_+$ are the same at any point and in any direction, and their explicit formulas are presented therein. For convenience, let $T^s_{\mathcal{S}^n_+}(\cdot)$, $T^{s,2}_{\mathcal{S}^n_+}(\cdot\,,\cdot)$ and $T^{s,i,2}_{\mathcal{S}^n_+}(\cdot\,,\cdot)$ denote the tangent cone, inner and outer second order tangent sets of $\mathcal{S}^n_+$ restricted in space $\mathcal{S}^n$, respectively. Then $T^{s,2}_{\mathcal{S}^n_+}(X, H) = T^{s,i,2}_{\mathcal{S}^n_+}(X, H)$ for any $X \in \mathcal{S}^n$ and any $H \in \mathcal{S}^n$. For the tangent cone and second order tangent sets of $\mathcal{M}^n_+$ in space $\mathcal{M}^n$, they have the following forms.

Theorem 5.1 *For any $X \in \mathcal{M}^n$, the following statements hold:*
(i) $T_{\mathcal{M}^n_+}(X) = T^s_{\mathcal{S}^n_+}(X + X^T) \oplus A\mathcal{S}^n$.
(ii) $T^2_{\mathcal{M}^n_+}(X, H) = T^{i,2}_{\mathcal{M}^n_+}(X, H) = T^{s,2}_{\mathcal{S}^n_+}(X + X^T, H + H^T) \oplus A\mathcal{S}^n$.

Proof. From (5.3) and Fact 3, we have

$$\begin{aligned} T_{\mathcal{M}_+^n}(X) &= \{Y : \text{dist}(X+tY, \mathcal{M}_+^n) = o(t), t \geq 0\} \\ &= \{Y : \tfrac{1}{2}\text{dist}(X+X^T+t(Y+Y^T), \mathcal{S}_+^n) = o(t), t \geq 0\} \\ &= \{Y : Y+Y^T \in T^s_{\mathcal{S}_+^n}(X+X^T)\} = T^s_{\mathcal{S}_+^n}(X+X^T) \oplus A\mathcal{S}^n. \end{aligned}$$

Similarly,

$$\begin{aligned} T^2_{\mathcal{M}_+^n}(X,H) &= \{Y : Y+Y^T \in T^{s,2}_{\mathcal{S}_+^n}(X+X^T, H+H^T)\} \\ &= T^{s,2}_{\mathcal{S}_+^n}(X+X^T, H+H^T) \oplus A\mathcal{S}^n, \\ T^{i,2}_{\mathcal{M}_+^n}(X,H) &= \{Y : Y+Y^T \in T^{s,i,2}_{\mathcal{S}_+^n}(X+X^T, H+H^T)\} \\ &= T^{s,i,2}_{\mathcal{S}_+^n}(X+X^T, H+H^T) \oplus A\mathcal{S}^n. \end{aligned}$$

Then the proof is complete. □

Now, we discuss the differentiability of $\Pi_{\mathcal{M}_+^n}(\cdot)$. Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite-dimensional real vector spaces equipped with a inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. We say that $f : \mathcal{X} \to \mathcal{Y}$ is *directionally differentiable at x in the Fréchet sense* if f is directionally differentiable at x, and $f(x+h) = f(x) + f'(x;h) + o(\|h\|)$, $h \in \mathcal{X}$. In addition, if $f'(x;\cdot)$ is linear and continuous, then f is said to be *F(réchet)-differentiable* at x. Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a locally Lipschitz function. Thus f is F-differentiable almost everywhere in $\mathcal{X}$ from the well-known Rademacher's theroem that every locally Lipschitz continuous function is F-differentiable almost everywhere. Let D_f denote the set of points where f is F-differentiable in $\mathcal{X}$. Then the *Clarke's generalized Jacobian* of f at x is defined as

$$\partial f(x) := conv\{\partial_B f(x)\}$$

with

$$\partial_B f(x) := \{\lim_{k\to\infty} \mathcal{J}f(y^k) : y^k \in D_f, y^k \to x\},$$

where $\mathcal{J}f(y^k)$ denotes the F-derivative of f at y^k. Bonnans et al. [3] showed that $\Pi_{\mathcal{S}_+^n}(\cdot)$ is directionally differentiable everywhere in $\mathcal{S}^n$. Sun and Sun [17] proved that $\Pi_{\mathcal{S}_+^n}(\cdot)$ is strongly semismooth on $\mathcal{S}^n$. Qi and Sun [13] showed the strong semismoothness of $\Pi_{\mathcal{M}_+^n}(\cdot)$ over $\mathcal{M}^n$. Furthermore, we have the following conclusions.

Theorem 5.2 *The following statements hold:*
(i) $\Pi_{\mathcal{M}_+^n}(\cdot)$ is directional differentiable everywhere in $\mathcal{M}^n$, and the directional derivative

$$\Pi'_{\mathcal{M}_+^n}(X;H) = \Pi'_{\mathcal{S}_+^n}(\tfrac{X+X^T}{2}; \tfrac{H+H^T}{2}) + \tfrac{H-H^T}{2}, \ H \in \mathcal{M}^n. \tag{5.4}$$

(ii) For any $V_1 \in \partial_B \Pi_{\mathcal{M}_+^n}(X)$ (resp. $\partial \Pi_{\mathcal{M}_+^n}(X)$), there exists $V_2 \in \partial_B \Pi_{\mathcal{S}_+^n}(\frac{X+X^T}{2})$ (resp. $\partial \Pi_{\mathcal{S}_+^n}(\frac{X+X^T}{2})$), such that

$$V_1(H) = V_2(\tfrac{H+H^T}{2}) + \tfrac{H-H^T}{2}, \ H \in \mathcal{M}^n.$$

Proof. (i) Taking $X \in \mathcal{M}^n$, by the definition of directional derivative and from (5.2), we have

$$\begin{aligned}\Pi'_{\mathcal{M}^n_+}(X;H) &= \lim_{t\downarrow 0}\tfrac{\Pi_{\mathcal{M}^n_+}(X+tH)-\Pi_{\mathcal{M}^n_+}(X)}{t} = \lim_{t\downarrow 0}\tfrac{\Pi_{\mathcal{S}^n_+}(\frac{X+X^T}{2}+t\frac{H+H^T}{2})-\Pi_{\mathcal{S}^n_+}(\frac{X+X^T}{2})}{t} + \tfrac{H-H^T}{2}\\ &= \Pi'_{\mathcal{S}^n_+}(\tfrac{X+X^T}{2};\tfrac{H+H^T}{2}) + \tfrac{H-H^T}{2}.\end{aligned}$$

The result (i) is proved.

(ii) Let $D_{\mathcal{M}}$ and $D_{\mathcal{S}}$, respectively, denote the sets of points in $\mathcal{M}^n$ and $\mathcal{S}^n$ where $\Pi_{\mathcal{M}^n_+}(\cdot)$ and $\Pi_{\mathcal{S}^n_+}(\cdot)$ are F-differentiable. Since $\Pi_{\mathcal{M}^n_+}(\cdot)$ and $\Pi_{\mathcal{S}^n_+}(\cdot)$ are almost everywhere F-differentiable over $\mathcal{M}^n$, $D_{\mathcal{M}}$ and $D_{\mathcal{S}}$ are nonempty. From (5.4), it's clear that

$$X \in D_{\mathcal{M}} \quad \Leftrightarrow \quad \tfrac{X+X^T}{2} \in D_{\mathcal{S}}. \tag{5.5}$$

Taking any $V_1 \in \partial_B \Pi_{\mathcal{M}^n_+}(X)$, there exists a sequence $\{X^k\}$ in $D_{\mathcal{M}}$ converging to X such that $V_1 = \lim_{k\to\infty} \mathcal{J}\Pi_{\mathcal{M}^n_+}(X^k)$. Then by (5.5), it follows that for any $H \in \mathcal{M}^n$,

$$\begin{aligned}V_1(H) &= \lim_{k\to\infty} \mathcal{J}\Pi_{\mathcal{M}^n_+}(X^k)(H)\\ &= \lim_{k\to\infty} \Pi'_{\mathcal{M}^n_+}(X^k;H)\\ &= \lim_{k\to\infty} \Pi'_{\mathcal{S}^n_+}(\tfrac{X^k+X^{kT}}{2};\tfrac{H+H^T}{2}) + \tfrac{H-H^T}{2}\\ &= [\lim_{k\to\infty} \mathcal{J}\Pi_{\mathcal{S}^n_+}(\tfrac{X^k+X^{kT}}{2})](\tfrac{H+H^T}{2}) + \tfrac{H-H^T}{2}.\end{aligned}$$

Letting $V_2 = \lim_{k\to\infty} \mathcal{J}\Pi_{\mathcal{S}^n_+}(\frac{X^k+X^{kT}}{2})$, we obtain the desired result. Taking the convex hull of the above limit points will yield the corresponding result for $\partial\Pi_{\mathcal{M}^n_+}(X)$. Here, we omit the proof. □

Utilizing Theorem 5.2 and combining with the results in [17, 18, 12], we can obtain the explicit formulas for the directional derivative and Clarke's generalized Jacobian of $\Pi_{\mathcal{M}^n_+}(\cdot)$.

Let $X \in \mathcal{M}^n$ with the spectral decomposition $\frac{X+X^T}{2} = P\Lambda P^T$, where Λ is the diagonal matrix of eigenvalues of $\frac{X+X^T}{2}$ and P is a corresponding orthogonal matrix of orthonormal eigenvectors. Denote three index sets of positive, zero and negative eigenvalues of $\frac{X+X^T}{2}$, respectively, by

$$\alpha := \{i : \lambda_i(\tfrac{X+X^T}{2}) > 0\}, \quad \beta := \{i : \lambda_i(\tfrac{X+X^T}{2}) = 0\}, \quad \gamma := \{i : \lambda_i(\tfrac{X+X^T}{2}) < 0\}.$$

Rearrange Λ as $\begin{bmatrix} \Lambda_\alpha & 0 & 0 \\ 0 & \Lambda_\beta & 0 \\ 0 & 0 & \Lambda_\gamma \end{bmatrix}$ and P as $[P_\alpha, P_\beta, P_\gamma]$, where $P_\alpha \in \mathbb{R}^{n\times|\alpha|}$, $P_\beta \in \mathbb{R}^{n\times|\beta|}$, $P_\gamma \in \mathbb{R}^{n\times|\gamma|}$. And define the matrix $U \in \mathcal{S}^n$ with entries

$$U_{ij} = \frac{\max\{\lambda_i(\frac{X+X^T}{2}),0\} + \max\{\lambda_j(\frac{X+X^T}{2}),0\}}{|\lambda_i(\frac{X+X^T}{2})| + |\lambda_j(\frac{X+X^T}{2})|}, \quad i,j = 1,\ldots,n,$$

where $0/0$ is also defined to be 1. Using Sun and Sun's results (Corollary 10 and Lemma 11,

[12]; Proposition 4, [18]) we have

$$\Pi'_{\mathcal{S}^n_+}(\tfrac{X+X^T}{2}; \tfrac{H+H^T}{2}) = P \begin{bmatrix} \tilde{H}_{\alpha\alpha} & \tilde{H}_{\alpha\beta} & U_{\alpha\gamma} \circ \tilde{H}_{\alpha\gamma} \\ \tilde{H}^T_{\alpha\beta} & \Pi_{\mathcal{S}^{|\beta|}_+}(\tilde{H}_{\beta\beta}) & 0 \\ \tilde{H}^T_{\alpha\gamma} \circ U^T_{\alpha\gamma} & 0 & 0 \end{bmatrix} P^T, \tag{5.6}$$

where $\tilde{H} := P^T(\frac{H+H^T}{2})P$, and $\circ$ denotes the Hadamard product. Clearly, from the above expression (5.6) and Theorem 5.2 (i), we claim that

$$\Pi'_{\mathcal{M}^n_+}(X; H) = P \begin{bmatrix} \tilde{H}_{\alpha\alpha} & \tilde{H}_{\alpha\beta} & U_{\alpha\gamma} \circ \tilde{H}_{\alpha\gamma} \\ \tilde{H}^T_{\alpha\beta} & \Pi_{\mathcal{S}^{|\beta|}_+}(\tilde{H}_{\beta\beta}) & 0 \\ \tilde{H}^T_{\alpha\gamma} \circ U^T_{\alpha\gamma} & 0 & 0 \end{bmatrix} P^T + \frac{H - H^T}{2}.$$

Moreover we obtain:

- $\Pi_{\mathcal{M}^n_+}(\cdot)$ is F-differentiable at $X \in \mathcal{M}^n$ if and only if $X + X^T$ is nonsingular.
- The directional derivative $\Pi'_{\mathcal{M}^n_+}(X; \cdot)$ is F-differentiable at $H \in \mathcal{M}^n$ if and only if $\tilde{H}_{\beta\beta}$ is nonsingular.
- For any $V \in \partial_B \Pi_{\mathcal{M}^n_+}(X)$ (*resp.* $\partial \Pi_{\mathcal{M}^n_+}(X)$), there exists a $W \in \partial_B \Pi_{\mathcal{S}^{|\beta|}_+}(0)$ (*resp.* $\partial \Pi_{\mathcal{S}^{|\beta|}_+}(0)$) such that

$$V(H) = P \begin{bmatrix} \tilde{H}_{\alpha\alpha} & \tilde{H}_{\alpha\beta} & U_{\alpha\gamma} \circ \tilde{H}_{\alpha\gamma} \\ \tilde{H}^T_{\alpha\beta} & W(\tilde{H}_{\beta\beta}) & 0 \\ \tilde{H}^T_{\alpha\gamma} \circ U^T_{\alpha\gamma} & 0 & 0 \end{bmatrix} P^T + \frac{H - H^T}{2}.$$

  Conversely, for any $W \in \partial_B \Pi_{\mathcal{S}^{|\beta|}_+}(0)$ (*resp.* $\partial \Pi_{\mathcal{S}^{|\beta|}_+}(0)$), there exists a $V \in \partial_B \Pi_{\mathcal{M}^n_+}(X)$ (*resp.* $\partial \Pi_{\mathcal{M}^n_+}(X)$) such that the above equation holds.

6 Conclusions

In this paper, we mainly have given a convex analysis on the NS-psd cone from three aspects that are presented in Sections 3, 4 and 5. Especially, we have given some conditions on determining the positive semidefiniteness of a nonsymmetric matrix in Propositions 4.3, 4.4 and Corollary 4.8. The research results are useful for us to study nonlinear optimization problems over the NS-psd cone, and may help us to open a typical instance for hyperbolic cone programming, where the hyperbolic cone is a research-worthy and more general kind of convex cones so far as we've known.

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