

Basis Reduction, and the Complexity of Branch-and-Bound

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Abstract

The classical branch-and-bound algorithm for the integer feasibility problem

$$\text{Find } x \in Q \cap \mathbb{Z}^n, \text{ with } Q = \left\{ x \mid \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \leq \begin{pmatrix} A \\ I \end{pmatrix} x \leq \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\} \quad (1)$$

has exponential worst case complexity.

We prove that it is surprisingly efficient on reformulations of (1), in which the columns of the constraint matrix are short, and near orthogonal, i.e. a reduced basis of the generated lattice; when the entries of A are from $\{1, \dots, M\}$ for a large enough M , branch-and-bound solves almost all reformulated instances at the rootnode. For all A matrices we prove an upper bound on the width of the reformulations along the last unit vector.

The analysis builds on the ideas of Furst and Kannan to bound the number of integral matrices for which the shortest nonzero vectors of certain lattices are long, and also uses a bound on the size of the branch-and-bound tree based on the norms of the Gram-Schmidt vectors of the constraint matrix.

We explore practical aspects of these results. First, we compute numerical values of M which guarantee that 90, and 99 percent of the reformulated problems solve at the root: these turn out to be surprisingly small when the problem size is moderate. Second, we confirm with a computational study that random integer programs become easier, as the coefficients grow.

1 Introduction and main results

The Integer Programming (IP) feasibility problem asks whether a polyhedron Q contains an integral point. Branch-and-bound, which we abbreviate

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as B&B is a classical solution method, first proposed by Land and Doig in [20]. It starts with Q as the sole subproblem (node). In a general step, one chooses a subproblem Q' , a variable x_i , and creates nodes $Q' \cap \{x|x_i = \gamma\}$, where γ ranges over all possible integer values of x_i . We repeat this until all subproblems are shown to be empty, or we find an integral point in one of them.

B&B (and its version used to solve optimization problems) enhanced by cutting planes is a dependable algorithm implemented in most commercial software packages. However, instances in [14, 8, 13, 17, 3, 4] show that it is theoretically inefficient: it can take an exponential number of subproblems to prove the infeasibility of simple knapsack problems. While B&B is inefficient in the worst case, Cornuéjols et al. in [10] developed useful computational tools to give an early estimate on the size of the B&B tree in practice.

Since IP feasibility is NP-complete, one can ask for polynomiality of a solution method only in fixed dimension. All algorithms that achieve such complexity rely on advanced techniques. The algorithms of Lenstra [22] and Kannan [15] first round the polyhedron (i.e. apply a transformation to make it have a spherical appearance), then use basis reduction to reduce the problem to a provably small number of smaller dimensional subproblems. On the subproblems the algorithms are applied recursively, e.g. rounding is done again. Generalized basis reduction, proposed by Lovász and Scarf in [23] avoids rounding, but needs to solve a sequence of linear programs to create the subproblems.

There is a simpler way to use basis reduction in integer programming: preprocessing (1) to create an instance with short and near orthogonal columns in the constraint matrix, then simply feeding it to an IP solver. We describe two such methods that were proposed recently. We assume that A is an integral matrix with m rows, and n columns, and the w_i and ℓ_i are integral vectors.

The rangespace reformulation of (1) proposed by Krishnamoorthy and Pataki in [17] is

$$\text{Find } y \in Q_R \cap \mathbb{Z}^n, \text{ with } Q_R = \left\{ y \mid \begin{pmatrix} \ell_1 \\ \ell_2 \end{pmatrix} \leq \begin{pmatrix} A \\ I \end{pmatrix} U y \leq \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\}, \quad (2)$$

where U is a unimodular matrix computed to make the columns of the constraint matrix a reduced basis of the generated lattice.

The nullspace reformulation of Aardal, Hurkens, and Lenstra proposed in [2], and further studied in [1] is applicable, when the rows of A are linearly

independent, and $w_1 = \ell_1$. It is

$$\text{Find } y \in Q_N \cap \mathbb{Z}^{n-m}, \text{ with } Q_N = \{ y \mid \ell_2 - x_0 \leq By \leq w_2 - x_0 \}, \quad (3)$$

where $x_0 \in \mathbb{Z}^n$ satisfies $Ax_0 = \ell_1$, and the columns of B are a reduced basis of the lattice $\{ x \in \mathbb{Z}^n \mid Ax = 0 \}$.

We analyze the use of Lenstra-Lenstra-Lovász (LLL) [21], and reciprocal Korkhine-Zolotarev (RKZ) reduced bases [18] in the reformulations, and use Korkhine-Zolotarev (KZ) reduced bases [15], [16] in our computational study. The definitions of these reducedness concepts are given in Section 2. When Q_R is computed using LLL reduction, we call it the LLL-rangespace reformulation of Q , and abusing notation we also call (2) the LLL-rangespace reformulation of (1). Similarly we talk about LLL-nullspace, RKZ-rangespace, and RKZ-nullspace reformulations.

Example 1. *The polyhedron*

$$\begin{aligned} 207 &\leq 41x_1 + 38x_2 &\leq 217 \\ 0 &\leq x_1, x_2 &\leq 10 \end{aligned} \quad (4)$$

is shown on the first picture of Figure 1. It is long and thin, and defines an infeasible, and relatively difficult integer feasibility problem for $B\mathcal{E}B$, as branching on either x_1 or x_2 yields 6 subproblems. Lenstra's and Kannan's algorithms would first transform this polyhedron to make it more spherical; generalized basis reduction would solve a sequence of linear programs to find the direction $x_1 + x_2$ along which the polyhedron is thin.

The LLL-rangespace reformulation is

$$\begin{aligned} 207 &\leq -3x_1 + 8x_2 &\leq 217 \\ 0 &\leq -x_1 - 10x_2 &\leq 10 \\ 0 &\leq x_1 + 11x_2 &\leq 10 \end{aligned} \quad (5)$$

shown on the second picture of Figure 1: now branching on y_2 proves integer infeasibility. (A similar example was given in [17]).

The reformulation methods are easier to describe, than, say Lenstra's algorithm, and are also successful in practice in solving several classes of hard integer programs: see [2, 1, 17]. For instance, the original formulations of the marketshare problems of Cornuéjols and Dawande in [9] are notoriously difficult for commercial solvers, while the nullspace reformulations are much easier to solve as shown by Aardal et al in [1].

However, they seem difficult to analyze in general. Aardal and Lenstra in [3, 4] studied knapsack problems with a nonnegativity constraint, and

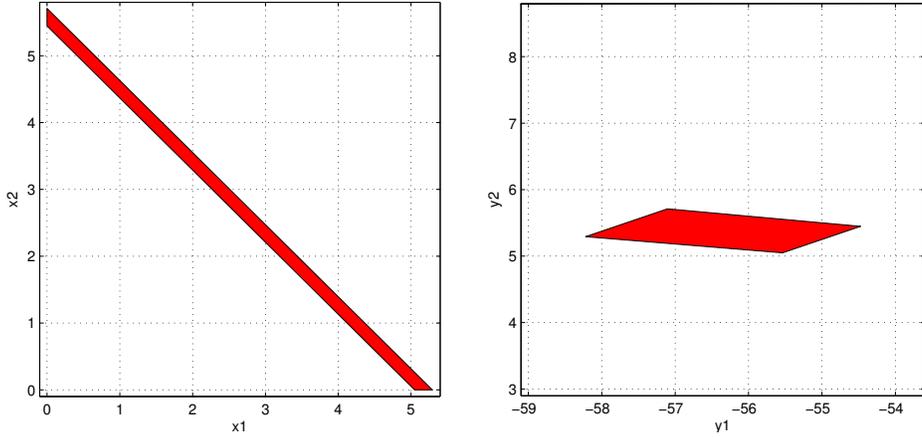


Figure 1: The polyhedron of Example 1 before and after the reformulation

the constraint vector a having a given decomposition $a = \lambda p + r$, with p and r integral vectors, and λ an integer, large compared to $\|p\|$ and $\|r\|$. They proved a lower bound on the norm of the last vector in the nullspace reformulation, and argued that branching on the corresponding variable will create a small number of B&B nodes. Krishnamoorthy and Pataki in [17] pointed out a gap in this argument, and showed that branching on the constraint px in Q (which creates a small number of subproblems, as λ is large), is equivalent to branching on the last variable in Q_R and Q_N .

A result one may hope for is proving polynomiality of B&B on the reformulations of (1) when the dimension is fixed. While this seems difficult, we give a different, and perhaps even more surprising complexity analysis. It is in the spirit of Furst and Kannan’s work in [12] on subset sum problems and builds on a generalization of their Lemma 1 to bound the fraction of integral matrices for which the shortest nonzero vectors of certain corresponding lattices are short. We also use an upper bound on the size of the B&B tree, which depends on the norms of the Gram-Schmidt vectors of the constraint matrix. We introduce necessary notation, and state our results, then give a comparison with [12].

When a statement is true for all, but at most a fraction of $1/2^n$ of the elements of a set S , we say that it is true for *almost all* elements. The value of n will be clear from the context. *Reverse B&B* is B&B branching on the variables in reverse order starting with the one of highest index. We assume $w_2 > \ell_2$, and for simplicity of stating the results we also assume $n \geq 5$. For positive integers m , n and M we denote by $G_{m,n}(M)$ the set of matrices with m rows, and n columns, and the entries drawn from $\{1, \dots, M\}$. We denote by $G'_{m,n}(M)$ the subset of $G_{m,n}(M)$ consisting of matrices with linearly

independent rows, and let

$$\chi(m, n, M) = \frac{|G'_{m,n}(M)|}{|G_{m,n}(M)|}. \quad (6)$$

It is shown by Bourgain et. al in [6] that $\chi(m, m, M)$ (and therefore also $\chi(m, n, M)$ for $m \leq n$) are of the order $1 - o(1)$. In this paper we will use $\chi(m, n, M) \geq 1/2$ for simplicity.

For matrices (and vectors) A and B , we write $(A; B)$ for $\begin{pmatrix} A \\ B \end{pmatrix}$. For an m by n integral matrix A with independent rows we write $\gcd(A)$ for the greatest common divisor of the m by m subdeterminants of A . If B&B generates at most one node at each level of the tree, we say that it solves an integer feasibility problem at the rootnode.

If Q is a polyhedron, and z is an integral vector, then the width of Q along z is

$$\text{width}(z, Q) = \max \{ \langle z, x \rangle \mid x \in Q \} - \min \{ \langle z, x \rangle \mid x \in Q \}. \quad (7)$$

The main results of the paper follow.

Theorem 1. *There are positive constants $d_1 \leq 2$, and $d_2 \leq 12$ such that the following hold.*

(1) If

$$M > (d_1 n \|(w_1; w_2) - (\ell_1; \ell_2)\|)^{n/m+1}, \quad (8)$$

then for almost all $A \in G_{m,n}(M)$ reverse B&B solves the RKZ-rangespace reformulation of (1) at the rootnode.

(2) If

$$M > (d_2(n - m) \|w_2 - \ell_2\|)^{n/m}, \quad (9)$$

then for almost all $A \in G'_{m,n}(M)$ reverse B&B solves the RKZ-nullspace reformulation of (1) at the rootnode. \square

The proofs also show that when M obeys the above bounds, then Q has at most one element for almost all $A \in G_{m,n}(M)$. When n/m is fixed, and the problems are binary, the magnitude of M required is a polynomial in n .

Theorem 2. *The conclusions of Theorem 1 hold for the LLL-reformulations, if the bounds on M are*

$$(2^{(n+4)/2} \|(w_1; w_2) - (\ell_1; \ell_2)\|)^{n/m+1}, \text{ and } (2^{(n-m+4)/2} \|w_2 - \ell_2\|)^{n/m},$$

respectively. \square

Furst and Kannan, based on Lagarias' and Odlyzko's [19] and Frieze's [11] work show that the subset sum problem is solvable in polynomial time using a simple iterative method for almost all weight vectors in $\{1, \dots, M\}^n$, and all right hand sides, when M is sufficiently large, and a reduced basis of the orthogonal lattice of the weight vector is available. The lower bound on M is $2^{cn \log n}$, when the basis is RKZ reduced, and 2^{dn^2} , when it is LLL reduced. Here c and d are positive constants.

Theorems 1 and 2 generalize the solvability results from subset sum problems to bounded integer programs; also, we prove them via branch-and-bound, an algorithm considered inefficient from the theoretical point of view.

Proposition 1 gives another indication why the reformulations are relatively easy. One can observe that $\det(AA^T)$ can be quite large even for moderate values of M , if $A \in G_{m,n}(M)$ is a random matrix with $m \leq n$, although we could not find any theoretical studies on the subject. For instance, for a random $A \in G_{4,30}(100)$ we found $\det(AA^T)$ to be of the order 10^{18} .

While we cannot give a tight upper bound on the size of the B&B tree in terms of this determinant, we are able to bound the width of the reformulations along the last unit vector for any A (i.e. not just almost all).

Proposition 1. *If Q_R and Q_N are computed using RKZ reduction, then*

$$\text{width}(e_n, Q_R) \leq \frac{\sqrt{n} \|(w_1; w_2) - (\ell_1; \ell_2)\|}{\det(AA^T + I)^{1/(2n)}}. \quad (10)$$

Also, if A has independent rows, then

$$\text{width}(e_{n-m}, Q_N) \leq \frac{\gcd(A) \sqrt{n-m} \|w_2 - \ell_2\|}{\det(AA^T)^{1/(2(n-m))}}. \quad (11)$$

The same results hold for the LLL-reformulations, if \sqrt{n} and $\sqrt{n-m}$ are replaced by $2^{(n-1)/4}$ and $2^{(n-m-1)/4}$, respectively. \square

Remark 1. *As described in Section 5 of [17], and in [25] for the nullspace reformulation, directions achieving the same widths exist in Q , and they can be quickly computed. For instance, if p is the last row of U^{-1} , then $\text{width}(e_n, Q_R) = \text{width}(p, Q)$.*

A practitioner of integer programming may ask for the value of Theorems 1 and 2. Proposition 2 and a computational study put these results into a more practical perspective. Proposition 2 shows that when m and n are not

too large, already fairly small values of M guarantee that the RKZ-nullspace reformulation (which has the smallest bound on M) of the majority of binary integer programs get solved at the rootnode.

Proposition 2. *Suppose that m and n are chosen according to Table 1, and M is as shown in the third column. Then for at least 90% of $A \in G'_{m,n}(M)$,*

n	m	M for 90 %	M for 99 %
30	20	33	37
50	20	1912	2145
50	30	96	103
60	30	420	454
70	40	197	209

Table 1: Values of M to make sure that the RKZ-nullspace reformulation of 90 or 99 % of the instances of type (12) solve at the rootnode

and all b right hand sides, reverse B&B solves the RKZ-nullspace reformulation of

$$\begin{aligned} Ax &= b \\ x &\in \{0, 1\}^n \end{aligned} \tag{12}$$

at the rootnode. The same is true for 99% of $A \in G'_{m,n}(M)$, if M is as shown in the fourth column. \square

Note that 2^{n-m} is the best upper bound one can give on the number of nodes when B&B is run on the original formulation (12); also, randomly generated IPs with $n - m = 30$ are nontrivial even for commercial solvers.

According to Theorems 1 and 2, random integer programs with coefficients drawn from $\{1, \dots, M\}$ should get easier, as M grows. Our computational study confirms this somewhat counterintuitive hypothesis on the family of marketshare problems of Cornuéjols and Dawande in [9].

We generated twelve 5 by 40 matrices with entries drawn from $\{1, \dots, M\}$ with $M = 100, 1000$, and 10000 (this is 36 matrices overall), set $b = \lfloor Ae/2 \rfloor$, where e is the vector of all ones, and constructed the instances of type (12), and

$$\begin{aligned} b - e &\leq Ax \leq b \\ x &\in \{0, 1\}^n. \end{aligned} \tag{13}$$

The latter of these are a relaxed version, which correspond to trying to find an almost-equal market split.

Table 2 shows the average number of nodes that the commercial IP solver CPLEX 9.0 took to solve the rangespace reformulation of the inequality- and the nullspace reformulation of the equality constrained problems.

M	EQUALITY	INEQUALITY
100	17531.92	38884.92
1000	1254.42	22899.67
10000	200.83	1975.67

Table 2: Average number of B&B nodes to solve the inequality- and equality-constrained marketshare problems

Since RKZ reformulation is not implemented in any software that we know of, we used the Korkhine-Zolotarev (KZ) reduction routine from the NTL library [26]. For brevity we only report the number of B&B nodes, and not the actual computing times.

All equality constrained instances turned out to be infeasible, except two, corresponding to $M = 100$. Among the inequality constrained problems there were fifteen feasible ones: all twelve with $M = 100$, and three with $M = 1000$. Since infeasible problems tend to be harder, this explains the more moderate decrease in difficulty as we go from $M = 100$ to $M = 1000$.

Table 2 confirms the theoretical findings of the paper: the reformulations of random integer programs become easier as the size of the coefficients grow.

In Section 2 we introduce further necessary notation, and give the proof of Theorems 1 and 2.

2 Further notation, and proofs

A lattice is a set of the form

$$L = \mathbb{L}(B) = \{ Bx \mid x \in \mathbb{Z}^r \}, \quad (14)$$

where B is a real matrix with r independent columns, called a *basis* of L , and r is called the *rank* of L .

The euclidean norm of a shortest nonzero vector in L is denoted by $\lambda_1(L)$, and Hermite's constant is

$$C_j = \sup \left\{ \lambda_1(L)^2 / (\det L)^{2/j} \mid L \text{ is a lattice of rank } j \right\}. \quad (15)$$

We define

$$\gamma_i = \max \{C_1, \dots, C_i\}. \quad (16)$$

A matrix A defines two lattices that we are interested in:

$$L_R(A) = \mathbb{L}(A; I), \quad L_N(A) = \{x \in \mathbb{Z}^n \mid Ax = 0\}, \quad (17)$$

where we recall that $(A; I)$ is the matrix obtained by stacking A on top of I .

Given independent vectors b_1, \dots, b_r , the vectors b_1^*, \dots, b_r^* form the Gram-Schmidt orthogonalization of b_1, \dots, b_r , if $b_1^* = b_1$, and b_i^* is the projection of b_i onto the orthogonal complement of the subspace spanned by b_1, \dots, b_{i-1} for $i \geq 2$. We have

$$b_i = b_i^* + \sum_{j=1}^{i-1} \mu_{ij} b_j^*, \quad (18)$$

with

$$\mu_{ij} = \langle b_i, b_j^* \rangle / \|b_j^*\|^2 \quad (1 \leq j < i \leq r). \quad (19)$$

We call b_1, \dots, b_r *LLL-reduced* if

$$|\mu_{ij}| \leq \frac{1}{2} \quad (1 \leq j < i \leq r), \quad (20)$$

$$\|\mu_{i,i-1} b_{i-1}^* + b_i^*\|^2 \geq \frac{3}{4} \|b_{i-1}^*\|^2 \quad (1 < i \leq r). \quad (21)$$

An LLL-reduced basis can be computed in polynomial time for varying n .

Let

$$b_i(k) = b_i^* + \sum_{j=k}^{i-1} \mu_{ij} b_j^* \quad (1 \leq k \leq i \leq r), \quad (22)$$

and for $i = 1, \dots, r$ let L_i be the lattice generated by

$$b_i(i), b_{i+1}(i), \dots, b_r(i).$$

We call b_1, \dots, b_r *Korkhine-Zolotarev reduced* (KZ-reduced for short) if $b_i(i)$ is the shortest nonzero vector in L_i for all i . Since $L_1 = L$ and $b_1(1) = b_1$, in a KZ-reduced basis the first vector is the shortest nonzero vector of L . Computing the shortest nonzero vector in a lattice is expected to be hard, though it is not known to be NP-hard. It can be done in polynomial time when the dimension is fixed, and so can be computing a KZ reduced basis.

Given a lattice L its reciprocal lattice L' is defined as

$$L' = \{z \in \text{lin } L \mid \langle z, x \rangle \in \mathbb{Z} \forall x \in L\}.$$

For a basis b_1, \dots, b_r of the lattice L , there is a unique basis b'_1, \dots, b'_r of L' called the *reciprocal basis* of b_1, \dots, b_r , with

$$\langle b_i, b'_j \rangle = \begin{cases} 1 & \text{if } i + j = r + 1 \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

We call a basis b_1, \dots, b_r a reciprocal Korkhine Zolotarev (RKZ) basis of L , if its reciprocal basis is a KZ reduced basis of L' . Below we collect the important properties of RKZ and LLL reduced bases.

Lemma 1. *Suppose that b_1, \dots, b_r is a basis of the lattice L with Gram-Schmidt orthogonalization b_1^*, \dots, b_r^* . Then*

(1) *if b_1, \dots, b_r is RKZ reduced, then*

$$\|b_i^*\| \geq \lambda_1(L)/C_i, \quad (24)$$

and

$$\|b_r^*\| \geq (\det L)^{1/r}/\sqrt{r}. \quad (25)$$

(2) *if b_1, \dots, b_r is LLL reduced, then*

$$\|b_i^*\| \geq \lambda_1(L)/2^{(i-1)/2}, \quad (26)$$

and

$$\|b_r^*\| \geq (\det L)^{1/r}/2^{(r-1)/4}. \quad (27)$$

Proof Statement (24) is proven in [18]. Let b'_1, \dots, b'_r be the reciprocal basis. Since b'_1 is the shortest nonzero vector of L' , Minkowski's theorem implies

$$\|b'_1\| \leq \sqrt{r}(\det L')^{1/r}. \quad (28)$$

Combining this with $\|b'_1\| = 1/\|b_r^*\|$, and $\det L' = 1/\det L$ prove (25). Statement (26) was proven in [21]. Multiplying the inequalities

$$\|b_i^*\| \leq 2^{(r-i)/2} \|b_r^*\| \quad (i = 1, \dots, r), \quad (29)$$

and using $\|b_1^*\| \dots \|b_r^*\| = \det L$ gives (27). \square

Lemma 2. *Let P be a polyhedron*

$$P = \{y \in \mathbb{R}^r \mid \ell \leq By \leq w\}, \quad (30)$$

and b_1^*, \dots, b_r^* the Gram-Schmidt orthogonalization of the columns of B . When reverse $B\mathcal{E}B$ is applied to P , the number of nodes on the level of y_i is at most

$$\left(\left\lfloor \frac{\|w - \ell\|}{\|b_i^*\|} \right\rfloor + 1 \right) \dots \left(\left\lfloor \frac{\|w - \ell\|}{\|b_r^*\|} \right\rfloor + 1 \right). \quad (31)$$

Proof First we show

$$\text{width}(e_r, P) \leq \|w - \ell\| / \|b_r^*\|. \quad (32)$$

Let $x_{r,1}$ and $x_{r,2}$ denote the maximum and the minimum of x_r over P . Writing \bar{B} for the matrix composed of the first $r - 1$ columns of B , and b_r for the last column, it holds that there is $x_1, x_2 \in \mathbb{R}^{r-1}$ such that $\bar{B}x_1 + b_r x_{r,1}$ and $\bar{B}x_2 + b_r x_{r,2}$ are in P . So

$$\begin{aligned} \|w - \ell\| &\geq \|(\bar{B}x_1 + b_r x_{r,1}) - (\bar{B}x_2 + b_r x_{r,2})\| = \|\bar{B}(x_1 - x_2) + b_r(x_{r,1} - x_{r,2})\| \\ &\geq \|b_r^*\| |x_{r,1} - x_{r,2}| = \|b_r^*\| \text{width}(e_r, P) \end{aligned}$$

holds, and so does (32).

After branching on e_r, \dots, e_{i+1} , each subproblem is defined by a matrix formed of the first i columns of B , and bound vectors ℓ_i and w_i , which are translates of ℓ and w by the same vector. Hence the above proof implies that the width along e_i in each of these subproblems is at most

$$\|w - \ell\| / \|b_i^*\|, \quad (33)$$

and this completes the proof. \square

Our Lemma 3 builds on Furst and Kannan's Lemma 1 in [12], with inequality (35) also being a direct generalization.

Lemma 3. *For a positive integer k , let ϵ_R be the fraction of $A \in G_{m,n}(M)$ with $\lambda_1(L_R(A)) \leq k$, and ϵ_N be the fraction of $A \in G'_{m,n}(M)$ with $\lambda_1(L_N(A)) \leq k$. Then*

$$\epsilon_R \leq \frac{(2k+1)^{n+m}}{M^m}, \quad (34)$$

and

$$\epsilon_N \leq \frac{(2k+1)^n}{M^m \chi_{m,n}(M)}. \quad (35)$$

Proof We first prove (35). For v , a fixed nonzero vector in \mathbb{Z}^n , consider the equation

$$Av = 0. \quad (36)$$

There are at most $M^{m(n-1)}$ matrices in $G'_{m,n}(M)$ that satisfy (36): if the components of $n - 1$ columns of A are fixed, then the components of the column corresponding to a nonzero entry of v are determined from (36). The number of vectors v in \mathbb{Z}^n with $\|v\| \leq k$ is at most $(2k+1)^n$, and the number of matrices in $G'_{m,n}(M)$ is $M^{mn} \chi_{m,n}(M)$. Therefore

$$\epsilon_N \leq \frac{(2k+1)^n M^{m(n-1)}}{M^{mn} \chi_{m,n}(M)} = \frac{(2k+1)^n}{M^m \chi_{m,n}(M)}.$$

For (34), note that $(v_1; v_2) \in \mathbb{Z}^{m+n}$ is a nonzero vector in $L_R(A)$, iff $v_2 \neq 0$, and

$$Av_2 = v_1. \quad (37)$$

An argument like the one in the proof of (35) shows that for fixed $(v_1; v_2) \in \mathbb{Z}^{m+n}$ with $v_2 \neq 0$, there are at most $M^{m(n-1)}$ matrices in $G_{m,n}(M)$ that satisfy (37). The number of vectors in \mathbb{Z}^{n+m} with norm at most k is at most $(2k+1)^{n+m}$, so

$$\epsilon_R \leq \frac{(2k+1)^{n+m} M^{m(n-1)}}{M^{mn}} = \frac{(2k+1)^{n+m}}{M^m}.$$

□

Proof of Theorems 1 and 2 For part (1) in Theorem 1, let b_1^*, \dots, b_n^*

be the Gram-Schmidt orthogonalization of the columns of $(A; I)U$. Lemma 2 implies that the number of nodes generated by reverse B&B applied to Q_R is at most one, if

$$\|b_i^*\| > \|(w_1; w_2) - (\ell_1; \ell_2)\| \quad (38)$$

for $i = 1, \dots, n$. Since the columns of $(A; I)U$ form an RKZ reduced basis of $L_R(A)$, (24) implies

$$\|b_i^*\| \geq \lambda_1(L_R(A))/C_i, \quad (39)$$

so (38) holds, when

$$\lambda_1(L_R(A)) > C_i \|(w_1; w_2) - (\ell_1; \ell_2)\| \quad (40)$$

does for $i = 1, \dots, n$, which is in turn implied by

$$\lambda_1(L_R(A)) > \gamma_n \|(w_1; w_2) - (\ell_1; \ell_2)\|. \quad (41)$$

By Lemma 3 (41) is true for all, but at most a fraction of ϵ_R of $A \in G_{m,n}(M)$ if

$$M > \frac{(\lfloor 2\gamma_n \|(w_1; w_2) - (\ell_1; \ell_2)\| + 1 \rfloor)^{(m+n)/m}}{\epsilon_R^{1/m}}, \quad (42)$$

and using the known estimate $\gamma_n \leq 1 + n/4$ (see for instance [24]), setting $\epsilon_R = 1/2^n$, and doing some algebra yields the required result.

The proof of part (2) of Theorem 1 is along the same lines: now b_1^*, \dots, b_{n-m}^* is the Gram-Schmidt orthogonalization of the columns of B , which is an RKZ reduced basis of $L_N(A)$. Lemma 2, and the reducedness of B implies that the number of nodes generated by reverse B&B applied to Q_N is at most one, if

$$\lambda_1(L_N(A)) > \gamma_{n-m} \|w_2 - \ell_2\|, \quad (43)$$

and by Lemma 3 (43) is true for all, but at most a fraction of ϵ_N of $A \in G_{m,n}(M)$ if

$$M > \frac{(\lfloor 2\gamma_{n-m} \|w_2 - \ell_2\| + 1 \rfloor)^{n/m}}{\epsilon_N^{1/m} \chi_{m,n}(M)^{1/m}}. \quad (44)$$

Then simple algebra, and using $\chi_{m,n}(M) \geq 1/2$ completes the proof.

The proof of Theorem 2 is an almost verbatim copy, now using the estimate (26) to lower bound $\|b_i^*\|$. \square

Proof of Proposition 1 To see (10), we start with

$$\text{width}(e_n, P) \leq \|w - \ell\| / \|b_n^*\| \quad (45)$$

from (32), combine it with the lower bound on $\|b_n^*\|$ from (25), and the fact that

$$\det L_R(A) = \det(AA^T + I)^{1/2}, \quad (46)$$

which follows from the definition of $L_R(A)$. The proof of (11) is analogous, but now we need to use

$$\det L_R(A) = \det(AA^T)^{1/2} / \gcd(A), \quad (47)$$

whose proof can be found in [7] for instance. To prove the claims about the LLL-reformulations, we need to use (27) in place of (25). \square

Proof of Proposition 2 Let $N(n, k)$ denote the number of integral points in the n -dimensional ball of radius k . In the previous proofs we used $(2k+1)^n$ as an upper bound for $N(n, k)$. The proof of Part (2) of Theorem 1 actually implies that when

$$M > \frac{(N(n, \lceil \gamma_{n-m} \|w_2 - \ell_2\| \rceil))^{1/m}}{\epsilon_N^{1/m} \chi_{m,n}(M)}, \quad (48)$$

then for all, but at most a fraction of ϵ_N of $A \in G_{m,n}(M)$ reverse B&B solves the nullspace reformulation of (12) at the rootnode.

We use $\chi_{m,n}(M) \geq 1/2$, Blichfeldt's upper bound

$$C_i \leq \frac{2}{\pi} \Gamma\left(\frac{i+4}{2}\right)^{2/i}, \quad (49)$$

from [5] to bound γ_{n-m} in (48), dynamic programming to exactly find the values of $N(n, k)$, and the values $\epsilon_N = 0.1$, and $\epsilon_N = 0.01$ to obtain Table 1.

We note that in general $N(n, k)$ is hard to compute, or find good upper bounds for; however for small values of n and k a simple dynamic programming algorithm finds the exact value quickly. \square

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