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Facial reduction algorithms for conic optimization problems

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Abstract

To obtain a primal-dual pair of conic programming problems having zero duality gap, two methods have been proposed: the facial reduction algorithm due to Borwein and Wolkowicz [1, 2] and the conic expansion method due to Luo, Sturm, and Zhang [5]. We establish a clear relationship between them. Our results show that although the two methods can be regarded as dual to each other, the facial reduction algorithm can produce a finer sequence of faces including the feasible region. We illustrate the facial reduction algorithm in LP, SOCP and an example of SDP. A simple proof of the convergence of the facial reduction algorithm for conic programming is also presented.

Key words: Facial Reduction, Conic Programming.

1. Introduction

We consider the Conic Programming (CP) problem:

$$\theta_D = \sup \left\{ b^T y \;\middle|\; c - A^T y \in \mathcal{K} \right\} \tag{1}$$

where $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times n}$ and $\mathcal{K} \subseteq \mathbb{R}^n$ is a closed convex cone. For CP (1), its dual problem can be formulated as follows:

$$\theta_P = \inf \left\{ c^T x \mid Ax = b, x \in \mathcal{K}^* \right\}, \tag{2}$$

where $\mathcal{K}^* = \{ s \in \mathbb{R}^n \mid x^T s \geq 0 \ (\forall x \in \mathcal{K}) \}$ is the dual cone of $\mathcal{K}$. We denote:

$$\mathcal{A} = \left\{ c - A^T y \mid y \in \mathbb{R}^m \right\} \tag{3}$$

$$\mathcal{F}_D = \mathcal{A} \cap \mathcal{K} \tag{4}$$

$$\mathcal{F}_P = \{ x \mid Ax = b \} \cap \mathcal{K}^*. \tag{5}$$

It is easy to see that for any pair of feasible solutions of CPs (1) and (2), it holds that

$$c^T x - b^T y = c^T x - (Ax)^T y = x^T (c - A^T y) \geq 0,$$

where the last inequality is due to x is contained in $\mathcal{K}^*$. This means that $\theta_P \geq \theta_D$. In general, however, the equality does not hold. See [7, 13] for such examples. In the numerical computation of CPs, problems having positive dualty gap are very difficult to solve by the primal-dual interior-point methods; the primal-dual interior-point methods try to reduce the duality gap to zero, which is impossible in this case. In addition, even if the duality gap between CP and its dual is zero, the CP and/or its dual may not have optimal solutions. The lack of optimal solutions also makes the numerical computation for CP problems difficult.

For the case where a positive duality gap exists between (1) and (2), two approaches have been proposed to close the duality gap by finding a new primal-dual pair of a given CP problem to compute θ_D. The first one, called the Facial Reduction Algorithm (FRA), was proposed by Borwein and Wolkowicz [1, 2], and later simplified by Pataki [6].

Below we breifly explain FRA. A detailed description of FRA together with its convergence proof will be givein in Section 2.

A closed subset $\mathcal{F}$ of $\mathcal{K}$ is a *face* of $\mathcal{K}$, denoted $\mathcal{F} \trianglelefteq \mathcal{K}$, if for any $x, y \in \mathcal{K}$, $x + y \in \mathcal{F}$ implies $x, y \in \mathcal{F}$. The sets $\emptyset$ and $\mathcal{K}$ are faces of $\mathcal{K}$ and the other faces are called *proper faces.* For $C \subseteq \mathcal{K}$, we denote the smallest face of $\mathcal{K}$ including C by $\mathrm{face}(C, \mathcal{K})$. It is easy to see that any face of a closed convex cone is also a closed convex cone, which is the case we deal with throughout this paper. For a given face $\mathcal{F}$ of $\mathcal{K}$, we define the following CP:

$$\theta_D(\mathcal{F}) = \sup \left\{ b^T y \;\middle|\; c - A^T y \in \mathcal{F} \right\}. \tag{6}$$

The minimal cone $\mathcal{K}_{\min}$ of CP (1) is defined by

$$\mathcal{K}_{\min} := \mathrm{face}(\mathcal{F}_D, \mathcal{K}).$$

Note that $\mathcal{K}_{\min}$ could be empty, when CP (1) is infeasible, or equivalently, $\theta_D = -\infty$. It is easy to see that if $\mathcal{K}_{\min} \subseteq \mathcal{F} \subseteq \mathcal{K}$, then $\theta_D(\mathcal{K}_{\min}) = \theta_D(\mathcal{F}) = \theta_D(\mathcal{K}) = \theta_D$ (see Lemma 2.1).

Beginnig with $\mathcal{K}$, FRA repeatedly finds smaller faces of $\mathcal{K}$ until it finds $\mathcal{K}_{\min}$ when CP (1) is feasible, or detects infeasibility of CP (1). Once $\mathcal{K}_{\min}$ is found, then $\mathrm{relint}(\mathcal{K}_{\min} \cap \mathcal{A}) \neq \emptyset$, which means that the duality gap between $\theta(\mathcal{K}_{\min})$ and its dual is zero, and that the dual has an optimal solution. This may enhance the numerical stability of primal-dual interior-point methods applied to $\theta(\mathcal{K}_{\min})$([14]).

The other approach was proposed by Luo *et al.* [5] and Sturm [11, 12], which is called the dual regularization approach. In this paper, to make this approach clear, we call this the *conic expansion approach.* The approach tries to close a duality gap between CP (1) and its dual by expanding the cone $\mathcal{K}^*$ in θ_P, and terminates in a finite many iterations. We will present the detail and some results proved by Luo *et al.* [5] and Sturm [11, 12] in Section 3.

A contribution of this paper is that we establish a clear relationship between FRA and the conic expansion approach. Specifically, we can apply FRA to $\mathcal{K}$ in such a way that each reduced cone is the dual of the cone generated by the conic expansion approach (Theorem 3.4). Note that we can apply FRA in a different way from the conic expansion approach; in fact, FRA will produce a finer sequence of optimization problems than the conic expansion approach. We will show such an example in Section 3.

Another contribution is that we propose a variant of FRA which can be applied to a general conic programming (1). Below we point out the difference of our FRA and FRAs proposed so far.

Borwein and Wolkowicz [1, 2] discussed FRA in a different setting. Their works were done in early 80's, and a conic programming problem seemed to be not poplular as it is now. Their argument is confined to establish duality theorem without any constraint qualification. Note that our algorithm is not identical to their FRA, because their FRA needs an initial feasible solution, which we do not need in our variant. Their FRA is closely related to the Extended Lagrange-Slater Dual (ELSD) derived from a given SDP by Ramana [8]. The ELSD is an SDP and has polynomially many variables. Ramana [8] first showed that the duality gap between the given SDP and its ELSD is zero without assuming any constraint qualification. In Ramana *et al.* [9], they showed that the ELSD can be reformulated by the minimal cone obtained by their FRA.

Pataki [6] proposed an FRA for a conic programming where the cone is *nice*. His interest seemed to describe the primal-dual pair of problems having no duality gap by using FRA. For this purpose, he needed the notion of niceness. In contrast, we propose to apply our FRA iteratively to find a primal-dual pair of problems having no duality gap. As a result, we can deal with several smaller problems, instead of one huge problem. In addition, we do not need the niceness assumption on our cones any more. To show that our FRA works well for some basic conic programming, we provide some examples in Section 4

The remaining of this paper is constructed as follows. In Section 2, we introduce our FRA for general closed convex cones, and prove the finite convergence of FRA. Section 3 is devoted to establish the relationship between FRA and the conic expansion approach. Section 4 shows FRA working on Linear Programming (LP), Second-Order Cone Programming (SOCP), and Semi-definite programming (SDP). For SDP, we deal with an example of FRA for an SDP problem obtained by Lasserre's SDP relaxation [4]. In Section 5, we give some concluding remarks.

In this paper, we use the following formulas for convex sets extensively. The proofs of these formulas are given in textbooks of convexity, e.g., Rockafellar [10]. For convex sets C_1 and $C_2 \subset \mathbb{R}^n$,

$$\begin{aligned} \operatorname{relint}(\operatorname{cl}(C_1)) &= \operatorname{relint}(C_1), && (7)\\ \operatorname{relint}(C_1 \oplus C_2) &= \operatorname{relint}(C_1) \oplus \operatorname{relint}(C_2), && (8) \end{aligned}$$

where relint, cl, and $\oplus$ stand for the relative interior, the closure, and the Minkowski sum, respectively. If $\operatorname{relint}(C_1) \cap \operatorname{relint}(C_2)$ is non-empty, then

$$\operatorname{relint}(C_1 \cap C_2) = \operatorname{relint}(C_1) \cap \operatorname{relint}(C_2). \tag{9}$$

If K_1 and $K_2 \subseteq \mathbb{R}^n$ are convex cones, then

$$\begin{aligned} K_1^{**} &= \operatorname{cl}(K_1), && (10)\\ K_1^* \cap K_2^* &= (K_1 \oplus K_2)^*. && (11) \end{aligned}$$

2. Facial reduction algorithms for general closed convex cones

We give a lemma on the feasible region of CP $\theta_D(\mathcal{K}_{\min})$.

Lemma 2.1. *If CP (1) is feasible, then $\mathcal{F}_D = \mathcal{A} \cap \mathcal{F}$ for any face $\mathcal{F}$ of $\mathcal{K}$ containing $\mathcal{K}_{\min}$.*

Proof : We consider the case where CP (1) is feasible. By definition, $\mathcal{F}_D \subseteq \mathcal{K}_{\min} \subseteq \mathcal{F}$, which means the right-hand side includes the left-hand side. The other inclusion is also obvious because $\mathcal{F}_D = \mathcal{A} \cap \mathcal{K} \supseteq \mathcal{A} \cap \mathcal{F}$. □

From this Lemma, it follows that $\theta_D(\mathcal{K}_{\min}) = \theta_D(\mathcal{F}) = \theta_D$ for each face $\mathcal{F}$ of $\mathcal{K}$ including $\mathcal{K}_{\min}$. This also holds for the case where CP (1) is infeasible. If $\mathcal{F}_D = \emptyset$, then $\mathcal{F}_D = \mathcal{A} \cap \mathcal{K} \supseteq \mathcal{A} \cap \mathcal{F}$ in the case, which implies that $\theta_D(\mathcal{F}) = \theta_D = -\infty$.

We denote $H_c^- = \left\{ x \mid c^T x \leq 0 \right\}$. This is the half space defined by c when $c \neq 0$ which we do not assume in general.

The key idea of FRA is to consider the following system for a face $\mathcal{F} \subseteq \mathcal{K}$:

$$w \in \ker A \cap H_c^- \cap \mathcal{F}^*. \tag{12}$$

FRA can be stated as follows.

Algorithm 2.2. ***FRA*** *(Facial Reduction Algorithm)*

Step 1: *Set* $i = 0$ *and* $\mathcal{F}_0 = \mathcal{K}$.
Step 2: *If* $\ker A \cap H_c^- \cap \mathcal{F}_i^* \subseteq span(w_1, \dots, w_i)$, *then stop.* $\mathcal{F}_i = \mathcal{K}_{\min}$.
Step 3: *Find* $w_{i+1} \in (\ker A \cap H_c^- \cap \mathcal{F}_i^*) \setminus span(w_1, \dots, w_i)$.
Step 4: *If* $c^T w_{i+1} < 0$, *then stop. CP* (1) *is infeasible.*
Step 5: *Set* $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{w_{i+1}\}^\perp$ *and* $i = i + 1$, *and go back to Step 2.*

When $\ker A \cap H_c^- \cap \mathcal{F}_0^* = \{0\}$ at the initial iteration, the algorithm stops because $\text{span}(\emptyset) = \{0\}$, and $\mathcal{F}_0$ is the minimal cone for CP (1).

The main effort of FRA is Step 3 where we find a nonzero solution w_{i+1} of (12) which is not written as a linear combination of $w_1, \dots, w_i$. Finding such a solution is sometimes as difficult as solving the original problem. However, there are several cases where we can find such solutions easily and/or directly. In those cases, FRA efficiently shrinks the cone into the minimal cone, and as a result, we get robustness of the problem. We will give such examples where FRA works well later in this paper. The forthcoming paper [14] also shows that Lasserre's SDP relaxation for polynomial optimization problems is such an example where we can perform FRA systematically to reduce the size of SDP.

Below we show that the above algorithm is correct.

Lemma 2.3. *Let* $\mathcal{F}$ *be a closed convex cone such that relint*$(\mathcal{F}) \cap \mathcal{A}$ *is empty. Then there exists a nonzero* $w \in \ker A \cap H_c^- \cap \mathcal{F}^*$ *indicating*

(i) *if* $c^T w < 0$, *then* $\theta_D(\mathcal{F}) = -\infty$.
(ii) *if* $c^T w = 0$, *then* $\mathcal{F} \cap \{w\}^\perp \subsetneq \mathcal{F}$ *and* $\mathcal{F} \cap \{w\}^\perp \cap \mathcal{A} = \mathcal{F} \cap \mathcal{A}$.

In proving the above lemma, the following theorem by Rockafellar [10] plays a crucial role.

Theorem 2.4. *(Theorem 20.2 of* [10]*) Assume that* C_1 *and* C_2 *are nonempty convex sets and* C_1 *is polyhedral. Then the followings are equivalent:*

(i) *There exists a hyperplane* $H \not\supseteq C_2$ *which separates* C_1 *and* C_2.
(ii) $C_1 \cap$ *relint*$(C_2) = \emptyset$.

Proof of Lemma 2.3 : Because $\mathcal{A} \cap \text{relint}(\mathcal{F}) = \emptyset$, Theorem 2.4 with $C_1 = \mathcal{A}$ and $C_2 = \mathcal{F}$ implies the existence of a separating hyperplane H which does not contain $\mathcal{F}$, i.e., there exist a nonzero vector $\bar{w}$ and a real number δ satisfying:

$$\bar{w}^T s \leq \delta \leq \bar{w}^T f \ (\forall s \in \mathcal{A},\ \forall f \in \mathcal{F}), \text{ and } \exists \bar{f} \in \mathcal{F} \text{ such that } \bar{w}^T \bar{f} > \delta. \tag{13}$$

The left inequality of the left expression gives $\bar{w}^T(c - A^T y) = c^T \bar{w} - y^T A \bar{w} \leq \delta$ for any $y \in \mathbb{R}^m$, from which $A\bar{w} = 0$ and $c^T \bar{w} \leq \delta$ follow. Because $0 \in \mathcal{F}$, we have $\delta \leq 0$ thus $\bar{w} \in \ker A \cap H_c^-$.

We divide the proof of $\bar{w} \in \mathcal{F}^*$ into two cases. If $\mathcal{F} = \{0\}$, then $\mathcal{F}^* = \mathbb{R}^n$, thus the relation is obvious. When $\mathcal{F} \neq \{0\}$, we claim that δ can be chosen to be 0. If this is true, looking at the right inequalty of the left relation of (13) with $\delta = 0$, we immediately see that $\bar{w} \in \mathcal{F}^*$. To the contrary, suppose that there exists $f \in \mathcal{F}$ such that $\bar{w}^T f < 0$. Since $\mathcal{F}$ is a nontrivial cone, we have $\delta = -\infty$, which contradicts the fact that $\delta \geq \bar{w}^T y$ for some $y \in \mathcal{A}$.

Suppose that $c^T \bar{w} < 0$ and $\theta_D(\mathcal{F}) > -\infty$. Let $\bar{y}$ a feasible solution of $\theta_D(\mathcal{F})$. It is easy to see:

$$0 \leq \bar{w}^T(c - A^T \bar{y}) = \bar{w}^T c - \bar{y}^T A \bar{w} = c^T \bar{w} < 0,$$

which is a contradiction. We have proved that when $c^T \bar{w} < 0$, then $\theta_D(\mathcal{F}) = -\infty$.

Next we assume that $c^T \bar{w} = 0$. Because $A\bar{w} = 0$, we have $\mathcal{A} \subseteq \{\bar{w}\}^\perp$, and thus $\mathcal{F} \cap \{\bar{w}\}^\perp \cap \mathcal{A} = \mathcal{F} \cap \mathcal{A}$. Finally, the existence of $\bar{f}$ in (13) with $\delta = 0$ ensures $\mathcal{F} \cap \{\bar{w}\}^\perp \subsetneq \mathcal{F}$. □

The following lemma provides a necessary condition to be the minimal cone.

Lemma 2.5. *Let* $\mathcal{F}$ *be a face of* $\mathcal{K}$ *such that* $\mathcal{F} \cap \mathcal{A} = \mathcal{F}_D$. *If relint*$(\mathcal{F}) \cap \mathcal{A} \neq \emptyset$, *then* $\mathcal{F} = \mathcal{K}_{\min}$.

Proof : Note that by assumpiton, $\mathcal{K}_{\min} \neq \emptyset$. Because $\mathcal{F} \cap \mathcal{A} = \mathcal{F}_D$, we have $\mathcal{F} \supseteq \mathcal{K}_{\min}$. Now suppose that $\mathcal{K}_{\min} \subsetneq \mathcal{F}$. This means that $\mathcal{K}_{\min} \cap \text{relint}(\mathcal{F}) = \emptyset$. On the other hand, we have $\text{relint}(\mathcal{F}) \cap \mathcal{A} = \text{relint}(\mathcal{F} \cap \mathcal{A}) = \text{relint}(\mathcal{F}_D)$. Since $\text{relint}(\mathcal{F}_D)$ is nonempty, there exists a feasible solution in $\text{relint}(\mathcal{F})$. This contradicts the fact that $\mathcal{K}_{\min} \supseteq \mathcal{F}_D$. □

In Algorithm 2.2, we have already given a criterion of the termination of FRA. We give other one in the following lemma.

Lemma 2.6. *Assume that* $\mathcal{K}_{\min} \neq \emptyset$*, and let* $\mathcal{F} \supseteq \mathcal{K}_{\min}$ *be a face of* $\mathcal{K}$*. Then* $\mathcal{F} = \mathcal{K}_{\min}$ *if and only if* $\mathcal{F} \subseteq (\ker A \cap H_c^- \cap \mathcal{F}^*)^\perp = (\ker A \cap \ker c^T \cap \mathcal{F}^*)^\perp$.

Proof : Lemma 2.3 means that when $\mathcal{K}_{\min} \neq \emptyset$, $H_c^- = \ker c^T$.

For the if part, suppose that $\mathcal{F} \supsetneq \mathcal{K}_{\min}$. Lemma 2.3 then implies that there exists a $w \in \ker A \cap H_c^- \cap \mathcal{F}^*$, such that $\mathcal{F} \cap \{w\}^\perp \subsetneq \mathcal{F}$. This shows $\mathcal{F} \not\subseteq (\ker A \cap H_c^- \cap \mathcal{F}^*)^\perp$.

For the only-if part, suppose that $\mathcal{F} = \mathcal{K}_{\min}$, and that there exists a snonzero vector $w \in \ker A \cap \ker c^T \cap \mathcal{K}_{\min}^*$ and $s \in \mathcal{K}_{\min}$ such that $s^T w > 0$. Then $\mathcal{K}_{\min} \cap \{w\}^\perp \subsetneq \mathcal{K}_{\min}$, because the former set does not contain s. Furthermore, for any $y \in \mathbb{R}^m$ we have $w^T(c - A^T y) = 0$, and $\mathcal{K}_{\min} \cap \{w\}^\perp \supseteq \mathcal{F}_D$. This contradicts the fact that $\mathcal{K}_{\min}$ is the minimal cone. □

The following theorem ensures that FRA terminates at a finite iteration.

Theorem 2.7. *FRA stops in at most* $\dim(\ker A \cap \ker c^T) + 1$ *iterations by either: (i) detecting a certificate of infeasibility at Step 4, or (ii) finding* $\mathcal{K}_{\min}$ *at Step 2.*

Proof : Note that because $w_{i+1} \in \ker A \cap \ker c^T$ at each iteration unless the infeasibility is detected, we have $\{w_{i+1}\}^\perp \supseteq \mathcal{A}$, and

$$\mathcal{F}_{i+1} \cap \mathcal{A} = \mathcal{F}_i \cap \{w_{i+1}\}^\perp \cap \mathcal{A} = \mathcal{F}_i \cap \mathcal{A} = \cdots = \mathcal{K} \cap \mathcal{A} = \mathcal{F}_D.$$

This means that $\mathcal{F}_i \supseteq \mathcal{K}_{\min}$ for every i.

Assume that, at iteration i, $\mathrm{relint}(\mathcal{F}_i) \cap \mathcal{A} = \emptyset$. Otherwise it follows from Lemma 2.5 that $\mathcal{F}_i = \mathcal{K}_{\min}$. Let w_{i+1} be an indicating vector of Lemma 2.3. We claim that $w_{i+1} \notin \mathrm{span}(w_1, \ldots, w_i)$. If $c^T w_{i+1} < 0$, then the claim is obvious because $c^T w_j = 0$ for $j = 1, \ldots, i$. If $c^T w_{i+1} = 0$, Lemma 2.3 implies that $\mathcal{F}_i \cap \{w_{i+1}\}^\perp \subsetneq \mathcal{F}_i$. Suppose $w_{i+1} \in \mathrm{span}(w_1, \ldots, w_i)$, then

$$\{w_{i+1}\}^\perp \supseteq \bigcap_{j=1}^{i} \{w_j\}^\perp,$$

from which it follows that

$$\mathcal{F}_{i+1} = \mathcal{F}_0 \cap \bigcap_{j=1}^{i+1} \{w_j\}^\perp = \mathcal{F}_0 \cap \left(\bigcap_{j=1}^{i} \{w_j\}^\perp \cap \{w_{i+1}\}^\perp \right) = \mathcal{F}_0 \cap \bigcap_{j=1}^{i} \{w_j\}^\perp = \mathcal{F}_i.$$

This contradicts the fact that $\mathcal{F}_{i+1} \subsetneq \mathcal{F}_i$.

If $\ell^* = \dim(\ker A \cap \ker c^T)$, we have $w_i \in \ker A \cap \ker c^T \cap \mathcal{F}_{i-1}^*$ for all $i = 1, \ldots, \ell^*$. Otherwise FRA must detect the infeasibility of CP (1) and stop. Then at the Step 2 in the case of $i = \ell^* + 1$, $\mathrm{span}(w_1, \ldots, w_\ell^*) = \ker A \cap \ker c^T$, and FRA necessarily stops at Step 2. □

We make two remarks on FRA. First, if we know that $\theta_D(\mathcal{K}) > -\infty$ in advance, then we can replace H_c^- by $\ker c^T$. Second, we have a possibility to choose a nonzero vector w_{i+1} in Step 3 such that $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{w_{i+1}\}^\perp = \mathcal{F}_i$. When $\mathcal{F}_{i+1} = \mathcal{F}_i$, we call such an iteration *void*, and otherwise *valid.* FRA generates a sequence of faces

$$\mathcal{K} = \mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \cdots \supseteq \mathcal{F}_{\ell^*} = \mathcal{K}_{\min},$$

where ℓ^* is the number of the iterations of FRA. On the other hand, let i^* be the number of valid iterations of FRA. Then counting only valid iterations, we obtain a sequence of faces

$$\mathcal{K} = \mathcal{F}_0 \supsetneq \mathcal{F}_1 \supsetneq \cdots \supsetneq \mathcal{F}_{i^*} = \mathcal{K}_{\min}.$$

We obtain the following corollary on the number of valid iterations.

Corollary 2.8. *The number of valid iterations of FRA is bounded by* $\dim(\ker A \cap \ker c^T)$ *and the length of the longest chain of faces in* $\mathcal{K}$*.*

We often encounter the CP (1) where $\mathcal{K}$ is written as a direct product of several convex cones. The following lemma ensures that in that case, all the faces generated by FRA can be formulated as direct products.

Lemma 2.9. *Assume that the convex cone $\mathcal{K}$ in (1) is written as $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_p$, where $\mathcal{K}_1, \ldots, \mathcal{K}_p$ are closed convex cones. If a face $\mathcal{F}$ of $\mathcal{K}$ is written as a direct product $\mathcal{F}_1 \times \cdots \times \mathcal{F}_p$, where each $\mathcal{F}_i$ is a face of $\mathcal{K}_i$ for $i = 1, \ldots, p$, then the face $\mathcal{F}'$ generated by FRA can be formulated as follows:*

$$\mathcal{F}' = (\mathcal{F}_1 \cap \{w_1\}^\perp) \times \cdots \times (\mathcal{F}_p \cap \{w_p\}^\perp),$$

where $w = (w_1, \ldots, w_p)^T \in \ker A \cap \ker c^T \cap \mathcal{F}^*$. Moreover, each set $\mathcal{F}_i \cap \{w_i\}^\perp$ is a face of $\mathcal{K}_i$ for each $i = 1, \ldots, p$.

Proof : Because $\mathcal{F}$ is written as a direct product, so is the dual $\mathcal{F}^*$, and thus $w = (w_1, \ldots, w_p)^T \in \ker A \cap \ker c^T \cap \mathcal{F}^*$ satisfies $w_i \in \mathcal{F}_i^*$ for all $i = 1, \ldots, p$. We obtain $\mathcal{F}' = \mathcal{F} \cap \{w\}^\perp$ by FRA. It is clear that $\mathcal{F}' \supseteq (\mathcal{F}_1 \cap \{w_1\}^\perp) \times \cdots \times (\mathcal{F}_p \cap \{w_p\}^\perp)$. Hence it is sufficient to prove that $\mathcal{F}' \subseteq (\mathcal{F}_1 \cap \{w_1\}^\perp) \times \cdots \times (\mathcal{F}_p \cap \{w_p\}^\perp)$. For $x = (x_1, \ldots, x_p)^T \in \mathcal{F}'$, we have $x_i \in \mathcal{F}_i$ for all $i = 1, \ldots, p$ and $x^T w = \sum_{i=1}^p x_i^T w_i = 0$. It follows from $x_i \in \mathcal{F}_i$ and $w_i \in \mathcal{F}_i^*$ that $x_i^T w_i = 0$ for all $i = 1, \ldots, p$, and thus $x_i \in \{w_i\}^\perp$ for all $i = 1, \ldots, p$. This shows that $\mathcal{F}' = (\mathcal{F}_1 \cap \{w_1\}^\perp) \times \cdots \times (\mathcal{F}_p \cap \{w_p\}^\perp)$.

Because $\mathcal{F}_i$ is a face of $\mathcal{K}_i$, for $x, y \in \mathcal{F}_i \cap \{w_i\}^\perp$, $x + y \in \mathcal{F}_i \cap \{w_i\}^\perp$ implies that $x, y \in \mathcal{F}_i$ and $w_i^T(x + y) = 0$. It follows from $x, y \in \mathcal{F}_i$ and $w_i \in \mathcal{F}_i^*$ that $x, y \in \{w_i\}^\perp$. Therefore $\mathcal{F}_i \cap \{w_i\}^\perp$ is a face of $\mathcal{K}_i$ for all $i = 1, \ldots, p$. □

From Lemma 2.9, we obtain the following fact.

Corollary 2.10. *Let $\mathcal{K}$ be as in Lemma 2.9. Then, all faces $\mathcal{F}_1, \ldots, \mathcal{F}_{\ell^*}$ generated by FRA are*

$$\mathcal{F}_i = \mathcal{F}_{i,1} \times \cdots \times \mathcal{F}_{i,p}$$

for all $i = 1, \ldots, \ell^*$. Moreover, $\mathcal{F}_{i,j}$ is a face of $\mathcal{K}_i$ for all $i = 1, \ldots, \ell^*$ and $j = 1, \ldots, p$.

We give an example to see the behavior of FRA which we give in this paper.

Example 2.11. We denote $\mathcal{K} = \mathrm{cl}(\mathcal{K}_1 \oplus \mathcal{K}_2)$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined as

$$\begin{aligned} \mathcal{K}_1 &= \{x \in \mathbb{R}^3 \mid x_1 \geq \sqrt{x_2^2 + x_3^2}\}, \\ \mathcal{K}_2 &= \{x \in \mathbb{R}^3 \mid x_2 \geq 0, x_1 = x_3 = 0\}. \end{aligned}$$

Then their dual cones are

$$\begin{aligned} \mathcal{K}_1^* &= \mathcal{K}_1 = \{x \in \mathbb{R}^3 \mid x_1 \geq \sqrt{x_2^2 + x_3^2}\}, \\ \mathcal{K}_2^* &= \{x \in \mathbb{R}^3 \mid x_2 \geq 0\}, \end{aligned}$$

and it follows from (10) and (11) that $\mathcal{K}^* = \mathcal{K}_1^* \cap \mathcal{K}_2^*$. For CP (1), we set $A = (1, 0, 1), c = (0, 0, 0)^T$. We apply FRA into the CP (1). To this end, we first need to solve the system $w_1 \in \ker A \cap \ker c^T \cap \mathcal{K}^*$. Because $\ker A \cap \ker c^T = \{(\lambda_1, \lambda_2, -\lambda_1)^T \mid \lambda_1, \lambda_2 \in \mathbb{R}\}$, we have

$$\ker A \cap \ker c^T \cap \mathcal{K}^* = \{(\lambda_1, 0, -\lambda_1)^T \mid \lambda_1 \geq 0\}.$$

We choose $w_1 = (1, 0, -1)^T$ and then

$$\mathcal{F}_1 = \{x \in \mathbb{R}^3 \mid x_1 - x_3 = 0, x_1, x_2 \geq 0\}$$

and

$$\mathcal{F}_1^* = \{\lambda(1, 0, -1)^T \mid \lambda \in \mathbb{R}\} \oplus \{x \in \mathbb{R}^3 \mid x_1, x_2 \geq 0, x_3 = 0\}.$$

$w_2 \in \ker A \cap \ker c^T \cap \mathcal{F}_1^*$ satisfies $(w_2)_2 \geq 0$. Thus, we choose $w_2 = (0, 1, 0)^T$ and then

$$\mathcal{F}_2 = \{x \in \mathbb{R}^3 \mid x_1 - x_3 = 0, x_1 \geq 0, x_2 = 0\}.$$

Because $\dim(\ker A \cap \ker c^T) = 2$, it follows from the second remark of Theorem 2.7 that $\mathcal{F}_2$ is the minimal cone. We can also confirm it by Lemma 2.6. Indeed, the dual $\mathcal{F}_2^*$ is

$$\mathcal{F}_2^* = \{(\lambda_1, \lambda_2, \lambda_3 - \lambda_1)^T \mid \lambda_1, \lambda_2 \in \mathbb{R}, \lambda_3 \geq 0\}.$$

Because $\ker A \cap \ker c^T \subseteq \mathcal{F}_2^*$, $w_3 \in \ker A \cap \ker c^T \cap \mathcal{F}_2^*$ satisfies $w_3 = (\lambda_1, \lambda_2, -\lambda_1)^T$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. For any $x = (x_1, x_2, x_3)^T \in \mathcal{F}_2$, we have $x^T w_3 = (x_1 - x_3)\lambda_1 + x_2\lambda_2 = 0$ for any $w_3 \in \ker A \cap \ker c^T \cap \mathcal{F}_2^*$, which implies $\mathcal{F}_2 \subseteq (\ker A \cap \ker c^T \cap \mathcal{F}_2^*)^\perp$. Therefore from Lemma 2.6, $\mathcal{F}_2$ is the minimal cone $\mathcal{K}_{\min}$ for CP (1) and $\mathcal{K}_{\min} = \mathcal{K} \cap \{w_1\}^\perp \cap \{w_2\}^\perp$.

It should be noted that in general, the minimal cone can be formulated as the intersection of $\mathcal{K}$ and one supporting hyperplane if a convex cone $\mathcal{K}$ is nice. See [6] for the detail. However, because the cone $\mathcal{K}$ in this example is not nice, the minimal cone for CP with the convex cone $\mathcal{K}$ may not be the intersection of the convex cone $\mathcal{K}$ and one supporting hyperplane. Indeed, the minimal cone $\mathcal{K}_{\min}$ in this example is the intersection of the convex cone $\mathcal{K}$ and two supporting hyperplanes. This point is the difference between FRA for nice cone and FRA in this paper.

3. Relationship between FRA and the conic expansion approach

As we have already mentioned in Section 1, the conic expansion approach proposed by Luo *et.al.* [5] and Sturm [11, 12] can also find a new primal-dual CP pair to compute the optimal value of CP (1).

In this section, we will introduce the conic expansion approach and restrict FRA in Section 2. We establish a relationship between the conic expansion approach and the restricted FRA, and we give more elementary proofs of some results on the conic expansion approach in Luo *et.al.* [5] and Sturm [11, 12] by using the relationship.

For simplicity of notation, let $\mathcal{B}$ denote $\ker A \cap \ker c^T$ for CP (1). We define the cone expansion operator $\Gamma_{\mathcal{B}}$ for a closed convex cone $\mathcal{P}$ as follows:

$$\Gamma_{\mathcal{B}}(\mathcal{P}) := \mathrm{cl}(\mathcal{P} \oplus \mathrm{span}(\mathcal{B} \cap \mathcal{P})),$$

where $\oplus$ means the Minkowski sum. We remark that our presentation of the conic expansion looks somewhat simpler than [5] where the closedness of cones are not assumed. Obviously, from the definition of $\Gamma_{\mathcal{B}}$, we have $\Gamma_{\mathcal{B}}(\mathcal{P}) \supseteq \mathcal{P}$ for any closed convex cone $\mathcal{P}$. As $\Gamma_{\mathcal{B}}$ maps a closed convex cone to a closed convex cone, we can consider to apply $\Gamma_{\mathcal{B}}$ repeatedly:

$$\begin{aligned} \Gamma_{\mathcal{B}}^0(\mathcal{P}) &:= \mathcal{P} \\ \Gamma_{\mathcal{B}}^k(\mathcal{P}) &:= \Gamma_{\mathcal{B}}(\Gamma_{\mathcal{B}}^{k-1}(\mathcal{P})) \text{ for } k = 1, 2, \ldots. \\ \Gamma_{\mathcal{B}}^\infty(\mathcal{P}) &:= \lim_{j\to\infty} \Gamma_{\mathcal{B}}^j(\mathcal{P}). \end{aligned}$$

Observe that when $\Gamma_{\mathcal{B}}^{k+1}(\mathcal{P}) = \Gamma_{\mathcal{B}}^k(\mathcal{P})$, then no more strict expansion occurs, i.e, $\Gamma_{\mathcal{B}}^j(\mathcal{P}) = \Gamma_{\mathcal{B}}^k(\mathcal{P})$ for $j \geq k$.

The following lemma gives a necessary and sufficient condition to be $\mathcal{F} = \mathcal{K}_{\min}$.

Lemma 3.1. *Assume that CP (1) is feasible. Let $\mathcal{F}$ be a face of $\mathcal{K}$ including $\mathcal{K}_{\min}$. Then $\Gamma_{\mathcal{B}}(\mathcal{F}^*) = \mathcal{F}^*$ if and only if $\mathcal{F} = \mathcal{K}_{\min}$.*

Proof : Clearly, it follows from the definition of $\Gamma_{\mathcal{B}}$ that $\Gamma_{\mathcal{B}}(\mathcal{F}^*) = \mathcal{F}^*$ if and only if $\mathrm{span}(\mathcal{B} \cap \mathcal{F}^*) \subseteq \mathcal{F}^*$. Considering the duals of both cones, the inclusion is equivalent to $\mathcal{F} \subseteq (\mathcal{B} \cap \mathcal{F}^*)^\perp$. From Lemma 2.6, we obtain the desired result. □

The following theorem shows the relationship between a face generated by FRA and cone by the conic expansion approach.

Theorem 3.2. *We assume that $\mathcal{F}$ is a nonempty face of $\mathcal{K}$. For any $w \in \mathcal{B} \cap \mathcal{F}^*$, the inclusion*

$$(\mathcal{F} \cap \{w\}^\perp)^* \subseteq \Gamma_{\mathcal{B}}(\mathcal{F}^*) \tag{14}$$

holds, with equality if $w \in$ relint$(\mathcal{B} \cap \mathcal{F}^*)$.

Proof : From the definition of $\Gamma_{\mathcal{B}}$ and formulas (10) and (11), we have

$$\Gamma_{\mathcal{B}}(\mathcal{F}^*) = \mathrm{cl}(\mathcal{F}^* \oplus \mathrm{span}(\mathcal{B} \cap \mathcal{F}^*)) = (\mathcal{F} \cap (\mathcal{B} \cap \mathcal{F}^*)^\perp)^*.$$

We obtain $(\mathcal{B} \cap \mathcal{F}^*)^\perp \subseteq \{w\}^\perp$ because $w \in \mathcal{B} \cap \mathcal{F}^*$. This implies (14).

To show the the equality of if-part, we prove $\mathcal{F} \cap \{w\}^\perp \subseteq \mathcal{F} \cap (\mathcal{B} \cap \mathcal{F}^*)^\perp$ if $w \in \mathrm{relint}(\mathcal{B} \cap \mathcal{F}^*)$. From the assumption on w, for any $y \in \mathcal{B} \cap \mathcal{F}^*$, there exist $z \in \mathcal{B} \cap \mathcal{F}^*$ and $0 < \lambda < 1$ such that $w = \lambda y + (1-\lambda) z$. For any $x \in \mathcal{F} \cap \{w\}^\perp$, we have $\lambda\langle x, y\rangle = \langle x, w\rangle - (1-\lambda)\langle x, z\rangle = -(1-\lambda)\langle x, z\rangle$. Because $y, z \in \mathcal{F}^*$, $\langle x, y\rangle = 0$, and this implies $x \in \mathcal{F} \cap (\mathcal{B} \cap \mathcal{F}^*)^\perp$.

□

Using Theorem 3.2, we restrict FRA in Algorithm 2.2 as follows:

Algorithm 3.3. ***FRA-CE*** *(Facial Reduction Algorithm - Conic Expansion)*

Step 1: *Set* $i = 0$ *and* $\mathcal{F}_0 = \mathcal{K}$.
Step 2: *If relint*$(\mathcal{B} \cap \mathcal{F}_i^*) \subseteq$ *span*$(w_1, \dots, w_i)$*, then go to Step 5.*
Step 3: *Find* $w_{i+1} \in$ *relint*$(\mathcal{B} \cap \mathcal{F}_i^*) \setminus$ *span*$(w_1, \dots, w_i)$.
Step 4: *Set* $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{w_{i+1}\}^\perp$ *and* $i = i + 1$*, and go back to Step 2.*
Step 5: *If relint*$(\ker A \cap H_c^- \cap \mathcal{F}_i^*) \subseteq$ *span*$(w_1, \dots, w_i)$*, then stop and return* $\mathcal{F}_i = \mathcal{K}_{\min}$
Step 6: *Otherwise there exists* $w \in (\ker A \cap H_c^- \cap \mathcal{F}_i^*) \setminus$ *span*$(w_1, \dots, w_i)$ *such that* $c^T w < 0$*, and thus CP (1) is infeasible.*

The following theorem ensures that FRA-CE can find the minimal cone $\mathcal{K}_{\min}$ for CP (1) or detects the infeasibility of CP (1).

Theorem 3.4. *The followings hold:*

(i) *All faces* $\mathcal{F}_i$ *generated by FRA-CE satisfy* $\mathcal{F}_i^* = \Gamma_{\mathcal{B}}^i(\mathcal{K}^*)$.
(ii) *If CP (1) is feasible, all faces* $\mathcal{F}_i$ *generated by FRA-CE satisfy*

$$\mathcal{K} = \mathcal{F}_0 \supsetneq \mathcal{F}_1 \supsetneq \cdots \supsetneq \mathcal{F}_{\bar{i}} = \mathcal{F}_{\bar{i}+1} = \cdots = \mathcal{F}_{\bar{\ell}} = \mathcal{K}_{\min}, \tag{15}$$

where $\bar{\ell}$ *is the number of iterations of FRA-CE and is bounded by* $\dim(\ker A \cap \ker c^T)$.
(iii) *If CP (1) is infeasible, FRA-CE detects the infeasibility at the Step 6.*
(iv) $\Gamma_{\mathcal{B}}^k(\mathcal{K}^*) = \Gamma_{\mathcal{B}}^{\bar{\ell}}(\mathcal{K}^*)$ *for all* $k \geq \bar{\ell}$.

Proof : We prove (i). In Algorithm 3.3, we have $\mathcal{F}_{i+1} = \mathcal{F}_i \cap \{w_i\}^\perp$, where $w_i \in \mathrm{relint}(\mathcal{B} \cap \mathcal{F}_i^*)$. Applying Theorem 3.2, we obtain $\mathcal{F}_{i+1}^* = \Gamma_{\mathcal{B}}(\mathcal{F}_i^*)$. Because this holds for all $i = 0, 1, \dots, \bar{\ell} - 1$, we obtain $\mathcal{F}_i^* = \Gamma_{\mathcal{B}}^i(\mathcal{K}^*)$ for all $i = 0, \dots, \bar{\ell}$.

We prove (ii). If CP (1) is feasible, then from the remark of Theorem 2.7, we can replace H_c^- by $\ker c^T$. Because we have $\mathrm{relint}(\mathcal{B} \cap \mathcal{F}_i^*) \subseteq \mathcal{B} \cap \mathcal{F}_i^*$, all faces which are generated by FRA-CE can be also generated by FRA. In addition, because the final face $\mathcal{F}_{\bar{\ell}}$ satisfies $\mathrm{relint}(\mathcal{B} \cap \mathcal{F}_{\bar{\ell}}^*) \subseteq \mathrm{span}(w_1, \dots, w_{\bar{\ell}})$, $\mathcal{F}_{\bar{\ell}}$ also satisfies the condition of termination of FRA in Step 2. Therefore from Theorem 2.7, the final face $\mathcal{F}_{\bar{\ell}}$ is the minimal cone $\mathcal{K}_{\min}$ and it follows that $\bar{\ell}$ is bounded by $\dim(\ker A \cap \ker c^T)$.

We prove (15). We have already proved that FRA-CE finds the minimal cone if CP (1) is feasible. Therefore there exists an $\bar{i} \leq \bar{\ell}$ such that $\mathcal{F}_{\bar{i}} = \mathcal{K}_{\min}$. Let $\bar{i}$ be the minimum number such that $\mathcal{F}_{\bar{i}} = \mathcal{K}_{\min}$. Then for $i < \bar{i}$, because $\mathcal{F}_i \neq \mathcal{K}_{\min}$, it follows from Lemma 3.1 and $\mathcal{F}_{i+1} = (\Gamma_{\mathcal{B}}(\mathcal{F}_i^*)^*)$ that $\mathcal{F}_{i+1} \subsetneq \mathcal{F}_i$. Because $\mathcal{F}_{\bar{i}}$ and the final face $\mathcal{F}_{\bar{\ell}}$ are the minimal cone, it is clear that $\mathcal{F}_i = \mathcal{K}_{\min}$ for all $i = \bar{i}, \dots, \bar{\ell}$.

We prove (iii). Then the final face $\mathcal{F}_{\bar{\ell}}$ generated by FRA-CE satisfies $\mathrm{relint}(\mathcal{B} \cap \mathcal{F}_{\bar{\ell}}^*) \subseteq \mathrm{span}(w_1, \dots, w_{\bar{\ell}})$. Because $\mathcal{A} \cap \mathcal{F}_{\bar{\ell}} \subseteq \mathcal{A} \cap \mathcal{K} = \mathcal{F}_D$, we have $\mathcal{A} \cap \mathcal{F}_{\bar{\ell}} = \emptyset$. We apply Lemma 2.3. Suppose that $w \in \ker A \cap H_c^- \cap \mathcal{F}_{\bar{\ell}}^*$ satisfies (ii) of Lemma 2.3. Then we have $w \in \mathcal{B} \cap \mathcal{F}_{\bar{\ell}}^*$, which implies that $w \in \mathrm{span}(w_1, \dots, w_{\bar{\ell}})$. From the proof of Theorem 2.7, this contradicts $\mathcal{F}_{\bar{\ell}} \cap \{w\}^\perp \subsetneq \mathcal{F}_{\bar{\ell}}$. Therefore, $c^T w < 0$, and thus FRA-CE detects the infeasibility of CP (1) at the Step 6.

We prove (iv). To this end, we prove that $\mathcal{F}_{\bar{\ell}}^* = \Gamma_{\mathcal{B}}(\mathcal{F}_{\bar{\ell}}^*)$. If this holds, we have $\Gamma_{\mathcal{B}}^k(\mathcal{K}^*) = \Gamma_{\mathcal{B}}^{k-\bar{\ell}}(\Gamma_{\mathcal{B}}^{\bar{\ell}}(\mathcal{K}^*)) = \Gamma_{\mathcal{B}}^{k-\bar{\ell}}(\mathcal{F}_{\bar{\ell}}^*) = \mathcal{F}_{\bar{\ell}}^*$ for all $k \geq \bar{\ell}$. This implies that $\Gamma_{\mathcal{B}}^k(\mathcal{K}^*) = \Gamma_{\mathcal{B}}^{\bar{\ell}}(\mathcal{K}^*)$ for all $k \geq \bar{\ell}$.

Clearly, $\mathcal{F}_{\bar{\ell}}^* \subseteq \Gamma_{\mathcal{B}}(\mathcal{F}_{\bar{\ell}}^*)$. Because $\mathcal{F}_{\bar{\ell}}$ is the final face, it satisfies $\mathrm{relint}(\mathcal{B} \cap \mathcal{F}_{\bar{\ell}}^*) \subseteq \mathrm{span}(w_1, \dots, w_{\bar{\ell}})$, and thus $\mathrm{span}(\mathcal{B} \cap \mathcal{F}_{\bar{\ell}}^*) \subseteq \mathrm{span}(w_1, \dots, w_{\bar{\ell}})$. From the definition of the operator $\Gamma_{\mathcal{B}}$, it follows that $\Gamma_{\mathcal{B}}(\mathcal{F}_{\bar{\ell}}^*) \subseteq \mathrm{cl}(\mathcal{F}_{\bar{\ell}}^* + \mathrm{span}(w_1, \dots, w_{\bar{\ell}}))$. The right-hand side is equal to $(\mathcal{F}_{\bar{\ell}} \cap \bigcap_{i=1}^{\bar{\ell}} \{w_i\}^\perp)^*$ because of formula (11). Because $\mathcal{F}_{\bar{\ell}} = \mathcal{K} \cap \bigcap_{i=1}^{\bar{\ell}} \{w_i\}^\perp$, we have $\mathrm{cl}(\mathcal{F}_{\bar{\ell}}^* + \mathrm{span}(w_1, \dots, w_{\bar{\ell}})) = (\mathcal{F}_{\bar{\ell}} \cap \bigcap_{i=1}^{\bar{\ell}} \{w_i\}^\perp)^* = \mathcal{F}_{\bar{\ell}}^*$, and thus $\Gamma_{\mathcal{B}}(\mathcal{F}_{\bar{\ell}}^*) \subseteq \mathcal{F}_{\bar{\ell}}^*$. Therefore we obtain $\Gamma_{\mathcal{B}}(\mathcal{F}_{\bar{\ell}}^*) = \mathcal{F}_{\bar{\ell}}^*$. □

Comparing FRA-CE with FRA, we observe from Theorem 3.4 that if CP (1) is feasible, FRA can generate a finer sequence of faces than FRA-CE. In addition, in the case where CP (1) is infeasible, because FRA check the infeasibility in each iteration, FRA may be able to detect it in fewer iterations than FRA-CE.

We conclude from Theorem 3.4 that the dual of face $\mathcal{F}_k$ generated by FRA-CE is the same as the cone $\Gamma_{\mathcal{B}}^k(\mathcal{K}^*)$ generated by the conic expansion approach for all $k = 0, 1, \dots, \bar{\ell}$. In addition, it follows that $\Gamma_{\mathcal{B}}^k(\mathcal{K}^*) = \Gamma_{\mathcal{B}}^{\bar{\ell}}(\mathcal{K}^*)$ for all $k \geq \bar{\ell}$. Therefore, we can deal with CP $\theta_D((\Gamma_{\mathcal{B}}^k(\mathcal{K}^*))^*)$ for all $k = 0, 1, \dots,$. From (i) and (iv) of Theorem 3.4 and Lemma 2.1, the following corollary follows.

Corollary 3.5. *(Lemma 2.27 and 2.29 in* [12]*) We have* $\Gamma^k_{\mathcal{B}}(\mathcal{K}^*) = \Gamma^{k^*}_{\mathcal{B}}(\mathcal{K}^*)$ *for any* $k \geq k^* := \dim(\mathcal{B})$*. Moreover, the feasible region* $\mathcal{F}_D$ *of CP (1) is equivalent to the feasible regions of CP* $\theta_D((\Gamma^k_{\mathcal{B}}(\mathcal{K}^*))^*)$ *for all* $k = 0, 1, \ldots$*.*

We consider the following CP problem:

$$\theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*) = \sup\left\{ b^T y \;\middle|\; c - A^T y \in (\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^* \right\}.$$

From Corollary 3.5, Luo *et.al.* [5] and Sturm [11, 12] have concluded the following strong duality theorem between CP $\theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*)$ and its dual. We give more elementary proof.

Theorem 3.6. *(Corollary 2.32 in* [12]*) For CP* $\theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*)$*, it holds* $\theta_D = \theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*)$*. Moreover, the followings hold:*

(i) *If* $\theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*)$ *is* $-\infty$*, its dual is either infeasible or unbounded.*
(ii) *If* $\theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*)$ *is finite, its dual is solvable and the duality gap between CP* $\theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*)$ *and its dual is zero.*
(iii) *If* $\theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*)$ *is* $+\infty$*, its dual is infeasible.*

Proof : It follows from Corollary 3.5 that $\theta_D = \theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*)$.

We prove (i). Then we have $\mathcal{F}_{\bar{\ell}} \cap \mathcal{A} = (\Gamma^{\bar{\ell}}_{\mathcal{B}}(\mathcal{K}^*))^* \cap \mathcal{A} = \emptyset$. If its dual is feasible, it has a feasible solution $\bar{x}$ such that $A\bar{x} = b$ and $\bar{x} \in \Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*)$. From Lemma 2.3, there exists $w \in \ker A \cap \Gamma^{\bar{\ell}}_{\mathcal{B}}(\mathcal{K}^*)$ such that $c^T w < 0$. This is found at the Step 6. Because $\Gamma^{\bar{\ell}}_{\mathcal{B}}(\mathcal{K}^*) \subseteq \Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*)$, for any $\alpha \geq 0$, $x(\alpha) = \bar{x} + \alpha w$ is a feasible solution of its dual of CP $\theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*)$, so that its dual is unbounded. Therefore, its dual of CP $\theta_D((\Gamma^\infty_{\mathcal{B}}(\mathcal{K}^*))^*)$ is either infeasible or unbounded.

We prove (ii) and (iii). From the definition of the minimal cone $\mathcal{K}_{\min}$, the set $\mathcal{A} \cap \mathrm{relint}(\mathcal{K}_{\min})$ is nonempty. Then they are well-known that the duality gap is zero and that its dual has an optimal solution if $\theta_D(\mathcal{K}_{\min})$ is less than $+\infty$. They prove (ii) and (iii). □

Although Corollary 3.5 and Theorem 3.6 were proved in [12] from the properties of the operator $\Gamma_{\mathcal{B}}$, we are successful in proving them by the relationship between FRA-CE and the conic expansion approach.

As we have already mentioned, FRA may be able to generate a finer sequence of faces than FRA-CE, *i.e.* the conic expansion approach. We give such an example.

Example 3.7. We consider the following Polynomial Optimization Problem (POP):

$$\left\{ \begin{array}{cl} \inf & x^2 y^2 \\ \text{subject to} & (x, y) \in \mathbb{R}^2. \end{array} \right. \tag{16}$$

We apply Lasserre's SDP relaxation [4] into POP (16). Then we obtain the following SOS problem:

$$\left\{ \begin{array}{cl} \sup & \eta \\ \text{subject to} & x^2y^2 - \eta = u_2(x,y)^T X u_2(x,y) \ \forall (x,y) \in \mathbb{R}^2, X \in \mathbb{S}^6_+, \end{array} \right. \tag{17}$$

where $\mathbb{S}^6$ is the set of 6×6 symmetric matrices, $\mathbb{S}^6_+$ is the set of 6×6 symmetric positive semidefinite matrices and $u_2(x,y) = (1, x, y, x^2, xy, y^2)^T$. For $\alpha \in \mathbb{N}^2_4 := \{\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2 \mid \alpha_1 + \alpha_2 \leq 4\}$, we set $E_\alpha \in \mathbb{S}^6$ and real values b_α as follows:

$$\begin{aligned} (E_\alpha)_{\beta,\gamma} &= \left\{ \begin{array}{ll} 1 & \beta + \gamma = \alpha, \\ 0 & \text{otherwise,} \end{array} \right. \quad \text{for all } \beta, \gamma \in \mathbb{N}^2_2, \\ b_\alpha &= \left\{ \begin{array}{ll} 1 & \alpha = (2,2), \\ 0 & \text{otherwise.} \end{array} \right. \end{aligned}$$

From SOS problem (17), we obtain the following SDP problem:

$$\left\{ \begin{array}{cl} \sup & -E_0 \bullet X \\ \text{subject to} & E_\alpha \bullet X = b_\alpha \ (\alpha \in \mathbb{N}^2_4 \setminus \{0\}), \\ & X \in \mathbb{S}^6_+. \end{array} \right. \tag{18}$$

To apply FRA into SDP (18), we need to convert SDP (18) into the form of CP (1). We define the linear subspace $\mathcal{L} \subseteq \mathbb{S}^6$ associated with SDP (18):

$$\mathcal{L} = \left\{ X \in \mathbb{S}^6 \mid E_\alpha \bullet X = 0 \text{ for all } \alpha \in \mathbb{N}^2_4 \setminus \{0\} \right\}.$$

In addition, let $C \in \mathbb{S}^6$ be a solution of the system $E_\alpha \bullet C = b_\alpha$ for all $\alpha \in \mathbb{N}^2_4 \setminus \{0\}$. For example, the following C satisfies the system:

$$C_{\beta,\gamma} = \begin{cases} 1 & \beta = \gamma = (1,1) \\ 0 & \text{otherwise.} \end{cases}$$

Because this C is positive semidefinite, SDP (18) is feasible, and thus we can skip Step 4 in Algorithm 2.2 and replace H_c^- by $\ker c^T$.

We define the set $\mathcal{S} = \{C + X \in \mathbb{S}^6_+ \mid X \in \mathcal{L}\}$. Then the set $\mathcal{S}$ is equivalent to the feasible region of SDP (18). Let $\{Q_i\}_{i=1}^p \subseteq \mathbb{S}^6$ be a basis of the linear subspace $\mathcal{L}$. We can reformulate the set $\mathcal{S}$ by using the basis $\{Q_i\}_{i=1}^p$:

$$\mathcal{S} = \left\{ C + X \in \mathbb{S}^6 \;\middle|\; X = \sum_{i=1}^p \lambda_i Q_i \text{ for some } \lambda_1, \ldots, \lambda_p \in \mathbb{R} \right\}.$$

Therefore we can rewrite SDP (18), equivalently:

$$\begin{cases} \quad\text{sup} & C \bullet E_0 - \sum_{i=1}^p \lambda_i (E_0 \bullet Q_i) \\ \text{subject to} & C - \sum_{i=1}^p \lambda_i Q_i \in \mathbb{S}^6_+. \end{cases} \tag{19}$$

For SDP (19), the linear subspace corresponding to $\ker A \cap \ker c^T$ in the system (12) is

$$\{X \mid C \bullet X = 0, Q_i \bullet X = 0 \text{ for all } i = 1, \ldots, p\}.$$

Because $\{Q_i\}_{i=1}^p$ is the basis of $\mathcal{L}$, we can denote the linear subspace corresponding to $\ker A \cap \ker c^T$ by E_α:

$$\left\{ X \;\middle|\; X = \sum_{\alpha \in \mathbb{N}^2_4 \setminus \{0\}} y_\alpha E_\alpha \text{ and } y_{(2,2)} = 0. \right\}.$$

Therefore, the element $W \in \mathbb{S}^6$ of the system $\ker A \cap \ker c^T$ can be written as follows:

$$W = \sum_{\alpha \in \mathbb{N}^2_4 \setminus \{0\}} y_\alpha E_\alpha = \left(\begin{array}{cc|cc|cc} 0 & y_{(1,1)} & y_{(1,0)} & y_{(0,1)} & y_{(2,0)} & y_{(0,2)} \\ y_{(1,1)} & 0 & y_{(2,1)} & y_{(1,2)} & y_{(3,1)} & y_{(1,3)} \\ \hline y_{(1,0)} & y_{(2,1)} & y_{(2,0)} & y_{(1,1)} & y_{(3,0)} & y_{(1,2)} \\ y_{(0,1)} & y_{(1,2)} & y_{(1,1)} & y_{(0,2)} & y_{(2,1)} & y_{(0,3)} \\ \hline y_{(2,0)} & y_{(3,1)} & y_{(3,0)} & y_{(2,1)} & y_{(4,0)} & 0 \\ y_{(0,2)} & y_{(1,3)} & y_{(1,2)} & y_{(0,3)} & 0 & y_{(0,4)} \end{array} \right)$$

for some $y_{(1,0)}, y_{(0,1)}, \ldots, y_{(0,4)} \in \mathbb{R}$. The initial face $\mathcal{F}_0$ is $\mathbb{S}^6_+$ and so is the dual $\mathcal{F}_0^*$. Then for any $W^1 \in \ker A \cap \ker c^T \cap \mathcal{F}_0^*$, because the first and second diagonal elements of W are zero, all elements in the first and second rows and columns are zero. Moreover, we obtain $y_{(2,0)} = y_{(0,2)} = 0$, and thus the third and fourth rows and columns are also zero. Therefore, we obtain

$$W^1 = \left(\begin{array}{cc|cc|cc} & & & & & \\ & & & & & \\ \hline & & & & & \\ & & & & & \\ \hline & & & & y_{(4,0)} & \\ & & & & & y_{(0,4)} \end{array} \right), \tag{20}$$

where blanks stand for zero. The matrix W^1 with $y_{(4,0)} > 0$ and $y_{(0,4)} > 0$ is in the relative interior of the set $\ker A \cap \ker c^T \cap \mathcal{F}_0^*$ and then the first face $\mathcal{F}_1$ is

$$\mathcal{F}_1 = \left\{ X \;\middle|\; X = \begin{pmatrix} X_1 & 0 \\ 0^T & O_2 \end{pmatrix} \text{ for some } X_1 \in \mathbb{S}^4_+. \right\},$$

and the dual $\mathcal{F}_1^*$ is

$$\mathcal{F}_1^* = \left\{ W \;\middle|\; W = \begin{pmatrix} W_1 & W' \\ W'^T & W'' \end{pmatrix} \text{ for some } W_1 \in \mathbb{S}^4_+, W' \in \mathbb{R}^{4\times 2} \text{ and } W'' \in \mathbb{S}^2 \right\}.$$

For any $W^2 \in \ker A \cap \ker c^T \cap \mathcal{F}_1^*$, we obtain

$$W^2 = \left(\begin{array}{cc|cc|cc} & & & & y_{(2,0)} & y_{(0,2)} \\ & & & & y_{(3,1)} & y_{(1,3)} \\ \hline & & y_{(2,0)} & & y_{(3,0)} & \\ & & & y_{(0,2)} & & y_{(0,3)} \\ \hline y_{(2,0)} & y_{(3,1)} & y_{(3,0)} & & y_{(4,0)} & \\ y_{(0,2)} & y_{(1,3)} & & y_{(0,3)} & & y_{(0,4)} \end{array}\right). \tag{21}$$

It is clear that W^2 with $y_{(2,0)} > 0$ and $y_{(0,2)} > 0$ is in the relative interior of the set $\ker A \cap \ker c^T \cap \mathcal{F}_1^*$ and $W^2 \notin \mathrm{span}(W^1)$, and then the second face $\mathcal{F}_2$ is

$$\mathcal{F}_2 = \left\{ X \,\middle|\, X = \begin{pmatrix} X_2 & 0 \\ 0^T & O_4 \end{pmatrix} \text{ for some } X_2 \in \mathbb{S}^2_+. \right\},$$

and it is not difficult to verify that the second face $\mathcal{F}_2$ is the minimal cone for SDP (18) by Lemma 2.6. From Theorem 3.2, $\mathcal{F}_1^*$ and $\mathcal{F}_2^*$ are equivalent to the cone $\Gamma_{\mathcal{B}}(\mathcal{K}^*)$ and $\Gamma^2_{\mathcal{B}}(\mathcal{K}^*)$, respectively because we choose W^i from $\mathrm{relint}(\ker A \cap \ker c^T \cap \mathcal{F}_i^*)$.

Although we see that $\mathcal{F}_2$ is the minimal cone for SDP (18) by Lemma 2.6, FRA does not terminate. Indeed, $W \in \ker A \cap \ker c^T \cap \mathcal{F}_2^*$ satisfies

$$W = \left(\begin{array}{cc|cc|cc} & & y_{(1,0)} & y_{(0,1)} & y_{(2,0)} & y_{(0,2)} \\ & & y_{(2,1)} & y_{(1,2)} & y_{(3,1)} & y_{(1,3)} \\ \hline y_{(1,0)} & y_{(2,1)} & y_{(2,0)} & & y_{(3,0)} & y_{(1,2)} \\ y_{(0,1)} & y_{(1,2)} & & y_{(0,2)} & y_{(2,1)} & y_{(0,3)} \\ \hline y_{(2,0)} & y_{(3,1)} & y_{(3,0)} & y_{(2,1)} & y_{(4,0)} & \\ y_{(0,2)} & y_{(1,3)} & y_{(1,2)} & y_{(0,3)} & & y_{(0,4)} \end{array}\right). \tag{22}$$

Clearly, W with $y_{(1,0)} \neq 0$ is not included in $\mathrm{span}(W^1, W^2)$, so that FRA finds W^3 and generate the third face $\mathcal{F}_3$, which is the same as the minimal cone $\mathcal{K}_{\min}$. Although all faces generated after $\mathcal{F}_2$ are the same as the minimal cone $\mathcal{K}_{\min}$, FRA must find W^i until $\ker A \cap H_c^- \cap \mathcal{F}_i^* \subseteq \mathrm{span}(W^1, \ldots, W^i)$. This fact shows that if we add the condition $\mathcal{F}_i \subseteq (\ker A \cap \ker c^T \cap \mathcal{F}_i^*)^\perp$ in the Step 2 of FRA, the algorithm may terminate in fewer iterations than the original FRA and returns the minimal cone.

We next show that FRA can generate a finer sequence of faces for SDP (18) in this example. If we choose W^i from $(\mathcal{B} \cap \mathcal{F}_i^*) \setminus \mathrm{relint}(\mathcal{B} \cap \mathcal{F}_i^*)$, FRA may provide a different sequence of faces from FRA-CE, *i.e.*, the conic expansion approach. For example, if we choose W^1 with $y_{(0,4)} = 0$ at (20), W^1 is not in the relative interior of the set $\ker A \cap \ker c^T \cap \mathcal{F}_0^*$ and the first face $\mathcal{G}_1$ by FRA is

$$\mathcal{G}_1 = \left\{ X \,\middle|\, X = \begin{pmatrix} X_1 & 0 \\ 0^T & 0 \end{pmatrix} \text{ for some } X_1 \in \mathbb{S}^5_+. \right\},$$

and the dual $\mathcal{G}_1^*$ is

$$\mathcal{G}_1^* = \left\{ W \,\middle|\, W = \begin{pmatrix} W_1 & W' \\ W'^T & W'' \end{pmatrix} \text{ for some } W_1 \in \mathbb{S}^5_+, W' \in \mathbb{R}^5 \text{ and } W'' \in \mathbb{R} \right\}.$$

For any $\tilde{W}^2 \in \ker A \cap \ker c^T \cap \mathcal{G}_1^*$, we obtain

$$\tilde{W}^2 = \left(\begin{array}{cc|cc|cc} & & & & & y_{(0,2)} \\ & & & & & y_{(1,3)} \\ \hline & & & & & \\ & & & y_{(0,2)} & & y_{(0,3)} \\ \hline & & & & y_{(4,0)} & \\ y_{(0,2)} & y_{(1,3)} & & y_{(0,3)} & & y_{(0,4)} \end{array}\right).$$

If we choose $\tilde{W}^2$ with $y_{(0,2)} > 0$ and $y_{(4,0)} > 0$, $\tilde{W}^2$ is in the relative interior and we obtain the second face $\mathcal{G}_2$:

$$\mathcal{G}_2 = \left\{ X \,\middle|\, X = \begin{pmatrix} X_2 & 0 \\ 0^T & O_3 \end{pmatrix} \text{ for some } X_2 \in \mathbb{S}^3_+. \right\},$$

Also the dual of $\mathcal{G}_2^*$ is

$$\mathcal{G}_2^* = \left\{ W \,\middle|\, W = \begin{pmatrix} W_2 & W' \\ W'^T & W'' \end{pmatrix} \text{ for some } W_2 \in \mathbb{S}^3_+, W' \in \mathbb{R}^{3\times 3} \text{ and } W'' \in \mathbb{S}^3 \right\}.$$

For any $\tilde{W}^3 \in \ker A \cap \ker c^T \cap \mathcal{G}_2^*$, we obtain

$$\tilde{W}^3 = \left(\begin{array}{cc|cc|cc} & & & y_{(0,1)} & y_{(2,0)} & y_{(0,2)} \\ & & & y_{(1,2)} & y_{(3,1)} & y_{(1,3)} \\ \hline & & y_{(2,0)} & & y_{(3,0)} & y_{(1,2)} \\ y_{(0,1)} & y_{(1,2)} & & y_{(0,2)} & & y_{(0,3)} \\ \hline y_{(2,0)} & y_{(3,1)} & y_{(3,0)} & & y_{(4,0)} & \\ y_{(0,2)} & y_{(1,3)} & y_{(1,2)} & y_{(0,3)} & & y_{(0,4)} \end{array}\right)$$

The matrix $\tilde{W}^3$ with $y_{(2,0)} > 0$ is in the relative interior of the set $\ker A \cap \ker c^T \cap \mathcal{G}_2^*$ and the third face $\mathcal{G}_3$ generated by $\tilde{W}^3$ is

$$\mathcal{G}_3 = \left\{ X \,\middle|\, X = \begin{pmatrix} X_3 & 0 \\ 0^T & O_4 \end{pmatrix} \text{ for some } X_3 \in \mathbb{S}^2_+. \right\},$$

and is equivalent to the minimal cone for SDP (18). As we have already seen, FRA does not terminate because $\ker A \cap \ker c^T \cap \mathcal{G}_3^* \not\subseteq \operatorname{span}(\tilde{W}^1, \tilde{W}^2, \tilde{W}^3)$ and we need to find $\tilde{W}^{i+1} \in (\ker A \cap \ker c^T \cap \mathcal{G}_i^*) \setminus \operatorname{span}(\tilde{W}^1, \dots, \tilde{W}^i)$ until $\ker A \cap \ker c^T \cap \mathcal{G}_i^* \subseteq \operatorname{span}(\tilde{W}^1, \dots, \tilde{W}^i)$.

Figure 3.7 shows the relationships among faces $\mathcal{F}_i$ and $\mathcal{G}_i$ in this example. The dotted arrow means the strict inclusion, *i.e.*, $\mathcal{G}_1 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{G}_2$. From this example, we can observe that FRA can provide a finer sequence of faces than the conic expansion approach.

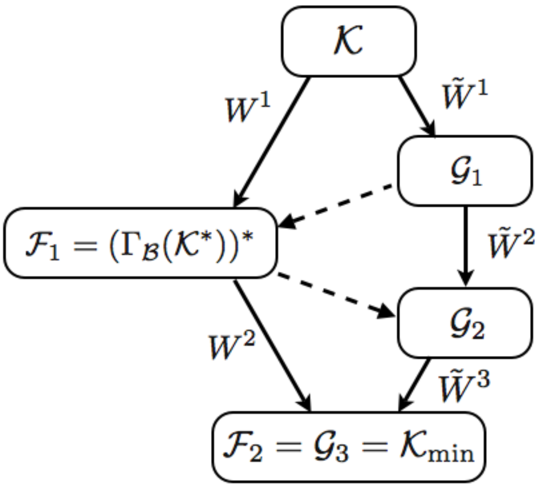

FIGURE 1. Relationships among all faces generated by FRA for Example 3.7

4. APPLICATIONS OF FRA

In this section, we apply FRA and FRA-CE to several important conic programming problems to see how they work.

Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^n$. We denote $I = \{1, 2, \dots, n\}$. We often deal with the case where $\mathcal{K} \subseteq \mathbb{R}^n$ is a direct product of closed convex cones, *i.e.*, $\mathcal{K} = \mathcal{K}_1 \times \cdots \times \mathcal{K}_q$, where $\mathcal{K}_i \subseteq \mathbb{R}^{n_i}$. Here, $n = \sum_{i=1}^q n_i$. For $w \in \mathbb{R}^n$, we write $w = (w_1, \cdots, w_q)$ where $w_i \in \mathbb{R}^{n_i}$. Then we define $J(w, \mathcal{K}) = \{\, i \in I \mid w_i \in \operatorname{relint}(\mathcal{K}_i) \,\}$. We have the following.

Lemma 4.1. *If $\tilde{w} \in$ relint$(\mathcal{L} \cap \mathcal{K})$, then $J(w, \mathcal{K}) \subseteq J(\tilde{w}, \mathcal{K})$ for all $w \in \mathcal{L} \cap \mathcal{K}$.*

Proof : Because $\tilde{w} \in \text{relint}(\mathcal{L} \cap \mathcal{K})$, for any $w \in \mathcal{L} \cap \mathcal{K}$, there exist $z \in \mathcal{L} \cap \mathcal{K}$ and $0 < \lambda < 1$ such that $\tilde{w} = \lambda w + (1-\lambda)z$. Then we have $\tilde{w}_i = \lambda w_i + (1-\lambda)z_i$ for all $i = 1, \ldots, q$. For $i \in J(w, \mathcal{K})$, because $w_i \in \text{relint}(\mathcal{K}_i)$ and $0 < \lambda < 1$, it follows from Theorem 6.1 in [10] that $\tilde{w}_i \in \text{relint}(\mathcal{K}_i)$, and thus $i \in J(\tilde{w}, \mathcal{K})$. ☐

4.1. **FRA for LP.** Let us consider Linear Programming (LP) problems:

$$\begin{cases} \quad \sup & b^T y \\ \text{subject to} & c - A^T y \in \mathbb{R}^n_+, \end{cases} \tag{23}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and $\mathbb{R}^n_+$ denotes the n-dimensional nonnegative orthant. It is well-known that $\mathbb{R}^n_+$ is self-dual, i.e., $(\mathbb{R}^n_+)^* = \mathbb{R}^n_+$. We assume that LP (23) has a feasible solution. Then we can skip the Step 4 in Algorithm 2.2 and replace H_c^- by $\ker c^T$.

We will apply FRA onto LP (23) and show that the first face generated by FRA is the minimal cone for LP if we can compute $w \in \text{relint}(\ker A \cap \ker c^T \cap \mathbb{R}^n_+)$.

For LP, we have $J(w, \mathbb{R}^n_+) = \{i \in I \mid w_i > 0\}$ for any $w = (w_1, \ldots, w_n)^T \in \mathcal{L} \cap \mathbb{R}^n_+$.

The following lemma characterizes the relative interior of the set $\mathcal{L}$ by the maximality of index set $J(w, \mathbb{R}^n_+)$.

Lemma 4.2. *$\tilde{w} \in$ relint$(\mathcal{L} \cap \mathbb{R}^n_+)$ if and only if $J(w, \mathbb{R}^n_+) \subseteq J(\tilde{w}, \mathbb{R}^n_+)$ for all $w \in \mathcal{L} \cap \mathbb{R}^n_+$.*

Proof : The only-if part is obvious from Lemma 4.1 with $K_i = \mathbb{R}_+$.

For if part, let $\delta < \min\{\tilde{w}_j \mid j \in J(\tilde{w}, \mathbb{R}^n_+)\}$. To prove $\tilde{w} \in \text{relint}(\mathcal{L} \cap \mathbb{R}^n_+)$, we will show that $U(\tilde{w}, \delta) \cap \text{span}(\mathcal{L} \cap \mathbb{R}^n_+) \subseteq \mathcal{L} \cap \mathbb{R}^n_+$, where $U(\tilde{w}, \delta)$ is the open ball with center $\tilde{w}$ and radius δ. For $x \in U(\tilde{w}, \delta) \cap \text{span}(\mathcal{L} \cap \mathbb{R}^n_+)$, we have $x \in \text{span}(\mathcal{L} \cap \mathbb{R}^n_+) \subseteq \mathcal{L}$ and then $x = y - z$ for some $y, z \in \mathcal{L} \cap \mathbb{R}^n_+$. In addition, we have $\|\tilde{w} - x\|_2 < \delta$. From the inequality and $J(y, \mathbb{R}^n_+), J(z, \mathbb{R}^n_+) \subseteq J(\tilde{w}, \mathbb{R}^n_+)$, we obtain $y_j = z_j = 0$ for all $j \in I \setminus J(\tilde{w}, \mathbb{R}^n_+)$. This implies that $x_j = 0$ for all $j \in I \setminus J(\tilde{w}, \mathbb{R}^n_+)$. In addition, it follows from $\|\tilde{w} - x\|_2 < \delta$ that $|w_j - x_j| < \delta$ for all $j \in J(\tilde{w}, \mathbb{R}^n_+)$. From these inequalities, we obtain $x_j > 0$ for all $j \in J(\tilde{w}, \mathbb{R}^n_+)$, and thus $x \in \mathbb{R}^n_+$. From $x \in \mathcal{L}$ and $x \in \mathbb{R}^n_+$, it follows that $x \in \mathcal{L} \cap \mathbb{R}^n_+$, which implies that $\tilde{w}$ is in the relative interior of the set $\mathcal{L} \cap \mathbb{R}^n_+$. This completes the proof. ☐

Theorem 4.3. *For LP (23), we choose $\tilde{w} \in$ relint$(\ker A \cap \ker c^T \cap \mathbb{R}^n_+)$. Then $\mathcal{F}_1 = \{x \in \mathbb{R}^n \mid x_j \geq 0 \ (j \in I \setminus J(\tilde{w}, \mathbb{R}^n_+)), x_j = 0 \ (j \in J(\tilde{w}, \mathbb{R}^n_+))\}$. Moreover, $\mathcal{F}_1$ is the minimal cone for LP (23).*

Proof : From Lemma 2.9, the first face $\mathcal{F}_1$ by FRA is

$$\mathcal{F}_1 = (\mathbb{R}_+ \cap \{w_1\}^\perp) \times \cdots \times (\mathbb{R}_+ \cap \{w_n\}^\perp).$$

Let $\mathcal{F}_{1,j} = \mathbb{R}_+ \cap \{w_j\}^\perp$ for all $j = 1, \ldots, n$. Then if $j \in J(\tilde{w}, \mathbb{R}^n_+)$, $\mathcal{F}_{1,j}$ is $\mathbb{R}_+ \cap \{\tilde{w}_j\}^\perp = \{0\}$. Otherwise, $\mathcal{F}_{1,j} = \mathbb{R}_+$. This implies that $\mathcal{F}_1 = \{x \in \mathbb{R}^n \mid x_j \geq 0 \ (j \in I \setminus J(\tilde{w}, \mathbb{R}^n_+)), x_j = 0 \ (j \in J(\tilde{w}, \mathbb{R}^n_+))\}$.

We prove that $\mathcal{F}_1$ is the minimal cone. The dual $\mathcal{F}_1^*$ is

$$\mathcal{F}_1^* = \{w \in \mathbb{R}^n \mid w_j \geq 0 \ (j \in I \setminus J(\tilde{w}, \mathbb{R}^n_+)), w_j \in \mathbb{R} \ (j \in J(\tilde{w}, \mathbb{R}^n_+))\}$$

Let $w' \in \ker A \cap \ker c^T \cap \mathcal{F}_1^*$. If $w'_j > 0$ for some $j \in I \setminus J(\tilde{w}, \mathbb{R}^n_+)$, it contradicts the maximality of $J(\tilde{w}, \mathbb{R}^n_+)$ in Lemma 4.2. Indeed, for sufficiently small $\delta > 0$, $\hat{w} := (1-\delta)\tilde{w} + \delta w' \in \ker A \cap \ker c^T \cap \mathbb{R}^n_+$ and $J(\hat{w}, \mathbb{R}^n_+) \not\subseteq J(\tilde{w}, \mathbb{R}^n_+)$. Therefore, $w'_j = 0$ for all $j \in I \setminus J(\tilde{w}, \mathbb{R}^n_+)$ and we have $\mathcal{F}_1 \subseteq (\mathcal{F}_1^*)^\perp \subseteq (\ker A \cap \ker c^T \cap \mathcal{F}_1^*)^\perp$. From Lemma 2.6, it follows that $\mathcal{F}_1$ is the minimal cone for LP (23). ☐

This theorem shows that the first face by FRA-CE applied to LP is the minimal cone. A similar situation is also observed by Pataki [6]. Notice that, since FRA has more flexibility in choosing w, the first face of FRA is not necessarily the minimal cone.

4.2. **FRA for Second-Order Cone Programming.** In this subsection, we consider FRA for Second-Order Cone Programming (SOCP) problems. In [7], Polik and Terlaky focused on the explicit description of the minimal cone $\mathcal{K}_{\min}$ of SOCP and proposed to solve a conic programming problem whose size is slightly larger than the original SOCP.

We will show that we can reformulate SOCP into a simple SOCP by using solutions $w \in \text{relint}(\ker A \cap \ker c^T \cap \mathcal{F}^*)$ if we can compute them. Specially, we will prove that in the case where SOCP has one second-order cone $\mathcal{K}$, the first face $\mathcal{F}_1 = \mathcal{K} \cap \{w\}^\perp$ is the minimal cone $\mathcal{K}_{\min}$ and SOCP replaced by $\mathcal{K}_{\min}$ is reformulated as a Linear Program (LP) problem if we can choose $w \in \text{relint}(\ker A \cap \ker c^T \cap (\mathcal{K}^* \setminus \{0\}))$.

We first consider FRA for Second-Order Cone Programming (SOCP) problems with a single second-order cone and deal with the following SOCP:

$$\left\{ \begin{array}{ll} \sup & b^T y \\ \text{subject to} & s = c - A^T y \in \mathcal{K}_n, \end{array} \right. \tag{24}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and $\mathcal{K}_n = \{s = (s_0, s_1)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid s_0 \geq \|s_1\|_2 := \sqrt{s_1^T s_1}\}$. We assume that SOCP (24) has a feasible solution. Then we can skip the Step 4 in Algorithm 2.2 and replace H_c^- by $\ker c^T$.

We remark that $\mathcal{K}_n$ is also self-dual with respet to the standard inner product. For $\mathcal{K}_n$, we define:

$$\begin{array}{rcl} \text{int}(\mathcal{K}_n) & = & \{x = (x_0, x_1)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_0 > \|x_1\|_2\}, \\ \text{bd}(\mathcal{K}_n) & = & \{x = (x_0, x_1)^T \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_0 = \|x_1\|_2, x \neq 0\}. \end{array}$$

Any face of $\mathcal{K}_n$ is one of the followings: (i) $\mathcal{K}_n$, (ii) $\text{cone}(u) := \{\lambda u \mid \lambda \geq 0\}$, where $u \in \text{bd}(\mathcal{K}_n^*)$, (iii) $\{0\}$, and (iv) $\emptyset$. The following theorem means that FRA-CE finds the minimal cone of (24) in the first iteration.

Theorem 4.4. *Let* $w = (w_0, w_1)^T \in$ *relint*$(\ker A \cap \ker c^T \cap \mathcal{K}_n^*)$.

(i) *If* $w \in$ *int*$(\mathcal{K}_n^*)$, $\mathcal{F}_1 = \mathcal{K}_{\min} = \{0\}$.
(ii) *If* $w \in$ *bd*$(\mathcal{K}_n^*)$, $\mathcal{F}_1 = \mathcal{K}_{\min} =$ *cone*(u), *where* $u = (w_0, -w_1)^T$.
(iii) *Otherwise* $\mathcal{F}_1 = \mathcal{K}_{\min} = \mathcal{K}_n$.

Proof : We only prove that if $w \in \text{bd}(\mathcal{K}_n^*)$, then $\text{cone}(u)$ is $\mathcal{K}_{\min}$. The other relationships are just straightfoward calculations, and we omit the proof.

Now suppose that $\mathcal{K}_{\min} = \{0\}$, although $\mathcal{F}_1 = \text{cone}(u)$. Then there exists $\tilde{w} \in \ker A \cap \ker c^T \cap \text{cone}(u)^*$ such that $\mathcal{F}_1 \cap \{\tilde{w}\}^\perp = \{0\}$. If $\tilde{w}^T u = 0$, then $\text{cone}(u)$ is not reduced to $\{0\}$, so we assume $\tilde{w}^T u > 0$. Then for sufficiently small $\delta > 0$, $z := \delta \tilde{w} + (1 - \delta) w \in \ker A \cap \ker c^T \cap \text{int}(\mathcal{K}_n^*)$ and $J(z, \mathcal{K}_n^*) = \{1\}$. In contrast, we have $J(w, \mathcal{K}_n^*) = \emptyset$ because $\text{relint}(\mathcal{K}_n^*) = \text{int}(\mathcal{K}_n^*)$. This contradicts Lemma 4.1 with $q = 1$ and $\mathcal{L} = \ker A \cap \ker c^T$. □

Next, we consider SOCP with multiple second order cones:

$$\left\{ \begin{array}{ll} \sup & b^T y \\ \text{subject to} & c_i - A_i^T y \in \mathcal{K}_{n_i} \ (i = 1, \ldots, q), \end{array} \right. \tag{25}$$

where $n = n_1 + \cdots + n_q$, $c_i \in \mathbb{R}^{n_i}$, $A_i \in \mathbb{R}^{m \times n_i}$ for all $i = 1, \ldots, q$, and $b \in \mathbb{R}^m$. We assume that SOCP (25) is feasible. A, c and $\mathcal{K}$ denote the matrix $(A_1, \ldots, A_q)$, the vector $(c_1^T, \ldots, c_q^T)^T$ and the convex cone $\mathcal{K}_{n_1} \times \cdots \times \mathcal{K}_{n_q}$, respectively. We remark that the dual of $\mathcal{K}$ is itself.

Let $w = (w_1, \ldots, w_q)^T \in \ker A \cap \ker c^T \cap (\mathcal{K}_{n_1}^* \times \cdots \times \mathcal{K}_{n_q}^*)$. Because SOCP (25) is feasible, from Lemma 2.9, all faces generated by FRA are also written as a direct product of $\mathcal{K}_{n_i}$, $\text{cone}(u_i)$ and $\{0\}$. This fact shows that some of the cones $\mathcal{K}_{n_i}$ may change from second-order cones into nonnegative cones, and thus the computational cost for solving SOCP (25) may decrease if one can compute $w^i \in \ker A \cap \ker c^T \cap \mathcal{F}_{i-1}^*$ for all $i = 1, \ldots, \ell^*$.

From the remarks of Theorem 2.7, we see that FRA-CE for SOCP (25) requires at most $2q$ iterations. However, we can easily see that the maximum iteration number is bounded by $2q - 1$.

Let Z be the direct product of $q - 2$ sets $\{0\}$, *i.e.*, $Z = \overbrace{\{0\} \times \cdots \times \{0\}}^{q-2}$. Without loss of generality, we consider the following two cases at the $2q - 2$ iteration.

(i) $\mathcal{F}_{2q-2} = Z \times \{0\} \times K_{n_q}$.

(ii) $\mathcal{F}_{2q-2} = Z \times \mathrm{cone}(u_{q-1}) \times \mathrm{cone}(u_q)$.

In the first case, Theorem 4.4 implies that the next face $\mathcal{F}_{2q-1}$ is the minimal cone. In the second case, Theorem 4.3 shows that the next face $\mathcal{F}_{2q-1}$ is the minimal cone. Therefore, we have the following.

Corollary 4.5. *FRA-CE terminates in at most* $2q-1$ *iterations for SOCP (25).*

4.3. **An example of FRA for an SDP problem.** From Theorem 3.6 in Section 3, we have observed that FRA and the conic expansion approach provide a primal-dual pair of CP whose duality gap is zero and whose dual has an optimal solution. This means that in order to compute the optimal value of CP (1), it is effective to apply the interior-point methods for CP replaced by the minimal cone $\mathcal{K}_{\min}$.

In this section, we apply FRA to an ill-conditional SDP problem and show that it produces a simpler problem which can be solved by a usual SDP solver. This indicates the usefulness of FRA to remove numerical difficulty of some SDP problems.

We consider the following POP:

$$\left\{\begin{array}{cl} \inf & x \\ \text{subject to} & x \geq 0, x^2 - 1 \geq 0. \end{array}\right. \tag{26}$$

The optimal value is 1 and the optimal solution is $x = 1$. We apply Lasserre's SDP relaxation to POP (26). Then for $r \geq 1$, we obtain the following SOS problems:

$$\left\{\begin{array}{cl} \sup & p \\ \text{subject to} & x - p = u_r(x)^T X u_r(x) + x u_{r-1}(x)^T Y u_{r-1}(x) \\ & \qquad +(x^2-1) u_{r-1}(x)^T Z u_{r-1}(x) \ (\forall x \in \mathbb{R}), \\ & X \in \mathbb{S}^{r+1}_+, Y, Z \in \mathbb{S}^r_+, \end{array}\right. \tag{27}$$

where $u_k(x) = (1, x, \ldots, x^k)^T$.

For $k = 0, 1, \ldots, 2r$, we set matrices $E_k \in \mathbb{S}^{r+1}$ and $F_k \in \mathbb{S}^r$ to be

$$\begin{array}{rcl} (E_k)_{\alpha,\beta} & = & \left\{\begin{array}{ll} 1 & \alpha+\beta = k, \\ 0 & \text{o.w.} \end{array}\right. \quad \text{for all } 0 \leq \alpha, \beta \leq r, \\ (F_k)_{\alpha,\beta} & = & \left\{\begin{array}{ll} 1 & \alpha+\beta = k, \\ 0 & \text{o.w.} \end{array}\right. \quad \text{for all } 0 \leq \alpha, \beta \leq r-1. \end{array}$$

Note that F_{2r-1} and F_{2r} are $r \times r$ zero matrices.

Using E_k and F_k, we can rewrite (27) as follows:

$$\left\{\begin{array}{cl} \sup & -E_0 \bullet X - F_0 \bullet Z \\ \text{subject to} & E_1 \bullet X + F_0 \bullet Y - F_1 \bullet Z = 1, \\ & E_i \bullet X + F_{i-1} \bullet Y + (F_{i-2} - F_i) \bullet Z = 0, (i = 2, \ldots, 2r-2) \\ & E_{2r-1} \bullet X + F_{2r-2} \bullet Y + F_{2r-3} \bullet Z = 0, \\ & E_{2r} \bullet X + F_{2r-2} \bullet Z = 0, \\ & (X, Y, Z) \in \mathbb{S}^{r+1}_+ \times \mathbb{S}^r_+ \times \mathbb{S}^r_+. \end{array}\right. \tag{28}$$

In [15], it is shown that this problem is numerically very ill-conditioned; the general SDP solvers report wrong optimal value 1, as opposed to the optimal value 0. In fact, [15] shows that it is impossible to calculate the optimal value of (27) if we use a usual floating point precision, and proposed to use the multi-precision SDP solver SDPA-GMP [3].

To apply FRA to SDP (28), we need to compute the set $\ker A \cap H_c^-$ for SDP (28). However, $(X, Y, Z) = (O_{r+1}, F_0, O_r)$ is a feasible solution of SDP (28), and thus we can skip Step 4 in Algorithm 2.2 and replace H_c^- by $\ker c^T$. By applying a similar way to Example 3.7, we can denote the linear subspace corresponding to $\ker A$ by E_k and F_k:

$$\left\{ (W_1, W_2, W_3) \;\middle|\; \begin{array}{c} W_1 = \sum_{i=1}^{2r} y_i E_i, \\ W_2 = \sum_{i=1}^{2r-1} y_i F_{i-1}, \\ W_3 = -y_1 F_1 + \sum_{i=2}^{2r} y_i (F_{i-2} - F_i), \end{array} \quad \text{for some } y_1, \ldots, y_{2r} \in \mathbb{R} \right\}.$$

Also $\ker c^T$ in the system (12) is corresponding to $\{(W_1, W_2, W_3) \mid C_1 \bullet W_1 + C_2 \bullet W_2 + C_3 \bullet W_3 = 0\}$, where (C_1, C_2, C_3) satisfies the linear equalities in SDP (28). For example, (O_{r+1}, F_0, O_r) satisfies the linear equalities, and thus, $(W_1, W_2, W_3) \in \ker A$ satisfies $y_1 = 0$. Therefore, the element $W \in \mathbb{S}^{r+1} \times \mathbb{S}^r \times \mathbb{S}^r$ of the set $\ker A \cap \ker c^T$ can be written as follows:

$$\begin{aligned} W &= (W_1, W_2, W_3), \\ W_1 &= \sum_{i=2}^{2r} y_i E_i = \begin{pmatrix} 0 & 0 & y_2 & \cdots & y_r \\ 0 & y_2 & y_3 & \cdots & y_{r+1} \\ y_2 & y_3 & y_4 & \cdots & y_{r+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_r & y_{r+1} & y_{r+2} & \cdots & y_{2r} \end{pmatrix}, \\ W_2 &= \sum_{i=2}^{2r-1} y_i F_{i-1} = \begin{pmatrix} 0 & y_2 & \cdots & y_r \\ y_2 & y_3 & \cdots & y_{r+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{r-1} & y_r & \cdots & y_{2r-1} \end{pmatrix}, \\ W_3 &= \sum_{i=2}^{2r-2} y_i (F_{i-2} - F_i) + y_{2r-1} F_{2r-3} + y_{2r} F_{2r-2} \\ &= \begin{pmatrix} y_2 & y_3 & \cdots & y_{r+1} - y_{r-1} \\ y_3 & y_4 - y_2 & \cdots & y_{r+2} - y_r \\ \vdots & \vdots & \ddots & \vdots \\ y_{r+1} - y_{r-1} & y_{r+2} - y_r & \cdots & y_{2r} - y_{2r-2} \end{pmatrix} \end{aligned}$$

for some $y_1, \ldots, y_{2r} \in \mathbb{R}$. We observe that the matrix W_1 is the Hankel matrix, *i.e.*, $(W_1)_{i,j} = (W_1)_{i-1,j+1}$ for all $0 \le i, j \le r$. The other matrices are also the Hankel matrices.

The following lemma is useful to analyze elements of the set $\ker A \cap \ker c^T \cap \mathcal{F}^*$ for all faces $\mathcal{F}$ of $\mathcal{K} = \mathbb{S}^{r+1}_+ \times \mathbb{S}^r_+ \times \mathbb{S}^r_+$.

Lemma 4.6. *For $s_0, s_1, \ldots, s_{2r} \in \mathbb{R}$, we consider an $(r+1) \times (r+1)$ Hankel matrix $S = (s_{i+j})_{0 \le i,j \le r}$. Let $q \in \{0, \ldots, r\}$ be fixed. If $s_0 = 0$ and $S_q = (s_{i+j})_{0 \le i,j \le q}$ is positive semidefinite, then $s_p = 0$ for all $p = 1, \ldots, 2q-1$.*

Proof : From $s_0 = 0$ and the positive semidefiniteness of S_q, it follows that $s_p = 0$ for all $p = 1, \ldots, q$. So, it is sufficient to prove that $s_p = 0$ for all $p = q+1, \ldots, 2q-1$. We prove it by induction on p. We assume that $s_p = 0$ for all $p = q+1, \ldots, k$ for some $k \in \{q+1, \ldots, 2q-2\}$ and will show that $s_{k+1} = 0$.

If k is even, then $s_k = (S_q)_{\frac{k}{2},\frac{k}{2}} = 0$ and thus $(S_q)_{\frac{k}{2},p} = 0$ for all $p = 0, \ldots, q$. From the assumption on k, it follows that $\frac{q+3}{2} \le \frac{k}{2} + 1 \le q$. This implies that $0 = (S_q)_{\frac{k}{2},\frac{k}{2}+1} = s_{k+1}$. If k is odd, then $s_{k-1} = (S_q)_{\frac{k-1}{2},\frac{k-1}{2}} = 0$ and thus $(S_q)_{\frac{k-1}{2},p} = 0$ for all $p = 0, \ldots, q$. From the assumption on k, it follows that $\frac{q}{2} + 2 \le \frac{k+3}{2} \le q + \frac{1}{2}$. This implies that $0 = (S_q)_{\frac{k-1}{2},\frac{k+3}{2}} = s_{k+1}$. This completes the proof. □

We are ready to apply FRA to SDP (28). The initial face $\mathcal{F}_0$ of $\mathbb{S}^{r+1}_+ \times \mathbb{S}^r_+ \times \mathbb{S}^r_+$ is $\mathbb{S}^{r+1}_+ \times \mathbb{S}^r_+ \times \mathbb{S}^r_+$ and the dual $\mathcal{F}_0^*$ is also $\mathbb{S}^{r+1}_+ \times \mathbb{S}^r_+ \times \mathbb{S}^r_+$. Then for any $W^1 \in \ker A \cap \ker c^T \cap \mathcal{F}_0^*$, it follows from Lemma 4.6 that $y_i = 0$ for all $i = 2, \ldots, 2r-1$. Hence, for any $W^1 \in \ker A \cap \ker c^T \cap \mathcal{F}_0^*$, we obtain

$$W^1 = (W_1, O_r, W_3), W_1 = \begin{pmatrix} O_r & 0 \\ 0^T & y_{2r} \end{pmatrix}, W_3 = \begin{pmatrix} O_{r-1} & 0 \\ 0^T & y_{2r} \end{pmatrix}.$$

Then the following W^1 satisfies $W^1 \in \operatorname{relint}(\ker A \cap \ker c^T \cap \mathcal{F}_0^*)$:

$$W^1 = (W_1, O_r, W_3), W_1 = \begin{pmatrix} O_r & 0 \\ 0^T & 1 \end{pmatrix}, W_3 = \begin{pmatrix} O_{r-1} & 0 \\ 0^T & 1 \end{pmatrix}.$$

From this W^1, the first face $\mathcal{F}_1$ by FRA is

$$\mathcal{F}_1 = \mathcal{F}_0 \cap \{W^1\}^\perp = \left\{ (X, Y, Z) \;\middle|\; \begin{array}{c} X = \begin{pmatrix} X' & 0 \\ 0^T & 0 \end{pmatrix}, Z = \begin{pmatrix} Z' & 0 \\ 0^T & 0 \end{pmatrix} \\ \text{for some } X', Y \in \mathbb{S}^r_+, Z' \in \mathbb{S}^{r-1}_+ \end{array} \right\}.$$

To find W^2, we need to construct the dual of face $\mathcal{F}_1$. It is

$$\mathcal{F}_1^* = \left\{ (W_1, W_2, W_3) \;\middle|\; \begin{array}{c} W_1 = \begin{pmatrix} W_1' & w_1' \\ {w_1'}^T & w_1'' \end{pmatrix}, W_2 \in \mathbb{S}^r_+, W_3 = \begin{pmatrix} W_3' & w_3' \\ {w_3'}^T & w_3'' \end{pmatrix} \\ \text{for some } W_1' \in \mathbb{S}^r_+, W_3' \in \mathbb{S}^{r-1}_+, w_1' \in \mathbb{R}^r, w_3' \in \mathbb{R}^{r-1}, w_1'', w_3'' \in \mathbb{R} \end{array} \right\}.$$

Then for any $W \in \ker A \cap \ker c^T \cap \mathcal{F}_1^*$, we obtain

$$W = (W_1, W_2, W_3), W_1 = \begin{pmatrix} O_r & 0 & 0 \\ 0^T & 0 & y_{2r-1} \\ 0^T & y_{2r-1} & y_{2r} \end{pmatrix}, W_2 = \begin{pmatrix} O_{r-1} & 0 \\ 0^T & y_{2r-1} \end{pmatrix}, W_3 = \begin{pmatrix} O_r & 0 & 0 \\ 0^T & 0 & y_{2r-1} \\ 0^T & y_{2r-1} & y_{2r} \end{pmatrix}.$$

The following W^2 satisfies $W^2 \in \text{relint}(\ker A \cap \ker c^T \cap \mathcal{F}_1^*)$:

$$W^2 = (O_{r+1}, W_2, O_r), W_2 = \begin{pmatrix} O_r & 0 \\ 0^T & 1 \end{pmatrix}.$$

The second face $\mathcal{F}_2$ generated by W^2 is

$$\mathcal{F}_2 = \left\{ (X, Y, Z) \;\middle|\; \begin{array}{c} X = \begin{pmatrix} X' & 0 \\ 0^T & 0 \end{pmatrix}, Y = \begin{pmatrix} Y' & 0 \\ 0^T & 0 \end{pmatrix}, Z = \begin{pmatrix} Z' & 0 \\ 0^T & 0 \end{pmatrix} \\ \text{for some } X' \in \mathbb{S}^r_+, Y', Z' \in \mathbb{S}^{r-1}_+ \end{array} \right\}.$$

Also the dual of $\mathcal{F}_2$ is

$$\mathcal{F}_2^* = \left\{ (W_1, W_2, W_3) \;\middle|\; \begin{array}{c} W_1 = \begin{pmatrix} W_1' & w_1' \\ {w_1'}^T & w_1'' \end{pmatrix}, W_2 = \begin{pmatrix} W_2' & w_2' \\ {w_2'}^T & w_2'' \end{pmatrix}, W_3 = \begin{pmatrix} W_3' & w_3' \\ {w_3'}^T & w_3'' \end{pmatrix} \\ \text{for some } W_1' \in \mathbb{S}^r_+, W_2', W_3' \in \mathbb{S}^{r-1}_+, w_1' \in \mathbb{R}^r, w_2', w_3' \in \mathbb{R}^{r-1}, w_1'', w_2'', w_3'' \in \mathbb{R} \end{array} \right\}.$$

By repeating FRA for SDP (28), we obtain the following result:

Theorem 4.7. *For SDP (28), the faces generated by FRA are as follows:*

$$\begin{aligned} \mathcal{F}_{2i+1} &= \left\{ (X, Y, Z) \;\middle|\; \begin{array}{c} X = \begin{pmatrix} X' & 0 \\ 0^T & O_{i+1} \end{pmatrix}, Y = \begin{pmatrix} Y' & 0 \\ 0^T & O_i \end{pmatrix}, Z = \begin{pmatrix} Z' & 0 \\ 0^T & O_{i+1} \end{pmatrix} \\ \textit{for some } X', Y' \in \mathbb{S}^{r-i}_+, Z' \in \mathbb{S}^{r-i-1}_+ \end{array} \right\}, \\ \mathcal{F}_{2i+2} &= \left\{ (X, Y, Z) \;\middle|\; \begin{array}{c} X = \begin{pmatrix} X' & 0 \\ 0^T & O_{i+1} \end{pmatrix}, Y = \begin{pmatrix} Y' & 0 \\ 0^T & O_{i+1} \end{pmatrix}, Z = \begin{pmatrix} Z' & 0 \\ 0^T & O_{i+1} \end{pmatrix} \\ \textit{for some } X' \in \mathbb{S}^{r-i}_+, Y', Z' \in \mathbb{S}^{r-i-1}_+ \end{array} \right\} \end{aligned}$$

for all $i = 0, \ldots, r-2$. Thus, we obtain the minimal cone $\mathcal{K}_{\min}$ for (28):

$$\mathcal{K}_{\min} = \mathcal{F}_{2r-1} = \left\{ (X, Y, O_r) \;\middle|\; X = \begin{pmatrix} X_{00} & 0 \\ 0^T & O_r \end{pmatrix}, Y = \begin{pmatrix} Y_{00} & 0 \\ 0^T & O_{r-1} \end{pmatrix} \textit{ for some } X_{00}, Y_{00} \in \mathbb{R}_+ \right\}.$$

From Theorem 4.7, SDP (28) is equivalent to a simpler SDP:

$$\left\{ \begin{array}{ll} \sup & -X_{00} \\ \text{subject to} & X_{00} \geq 0, Y_{00} \geq 0, Y_{00} = 1. \end{array} \right. \tag{29}$$

From SDP (29) and Theorem 3.6, it follows that the optimal value of SDP (28) is 0 and the optimal solution (X, Y, Z) of SDP (28) is (O_{r+1}, F_0, O_r).

Comparing SDP (28) with SDP (29), it is clear that the feasible region of SDP (29) has an interior point and its dual has an optimal solution. In constrast, it is proved in [15] that SDP (28) does not have any interior points and that the dual optimal solution does not exist. Moreover, we observe from SDP (29) that the size of SDP (28) becomes small by applying FRA. These observations show that one should solve SDP (29) rather than SDP (28) to compute the optimal value.

In this example, we can compute $w \in \text{relint}(\ker A \cap \ker c^T \cap \mathcal{F}^*)$, so that we can express the minimal cone explicitly. In general, however, it is necessary to solve the systems $w_{i+1} \in \ker A \cap \ker c^T \cap \mathcal{F}_i^*$ for all $i = 0, 1, \ldots, \ell^*$ and their computations are comparable to solving the original SDP (28), and thus applications of FRA into SDP problems is not always effective.

5. Concluding remarks

We have proposed a facial reduction algorithm for conic programming having general convex cones, and established the relationship between the FRA and the conic expansion approach proposed by Luo *et al.* [5] and Sturm [11, 12]. In particular, FRA-CE is just equivalent with their approach.

In contrast to taking span and the Minkowski sum in the conic expansion approach, our algorithm is more concrete and can be numerically computable.

In general, finding a nonzero solution of (12) is as difficult as solving the original problem itself. However, the examples in this paper show the applicability of our algorithm, and our viewpoint is that our FRA can be used in several contexts. In fact in the forthcoming paper [14], we will show our FRA can be applied to SDP relaxation of polynomial optimization problems to reduce the size of the SDP problms to be solved.

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