Transmission Expansion Planning with Re-design

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Abstract

Expanding an electrical transmission network requires heavy investments that need to be carefully planned, often at a regional or national level. We study relevant theoretical and practical aspects of transmission expansion planning, set as a bilinear programming problem with mixed 0-1 variables. We show that the problem is NP-hard and that, unlike the so-called Network Design Problem, a transmission network may become more efficient after cutting-off some of its circuits. For this reason, we introduce a new model that, rather than just adding capacity to the existing network, also allows for the network to be re-designed when it is expanded. We then turn into different reformulations of the problem, that replace the bilinear constraints by using a "big-M" approach. We show that computing the minimal values for the "big-M" coefficients involves finding the shortest and longest paths between two buses. We assess our theoretical results by making a thorough computational study on real electrical networks. The comparison of various models and reformulations shows that our new model, allowing for re-design, can lead to sensible cost reductions.

Keywords: transmission network, network design, "big-M" formulation.

1 Introduction

Long term transmission expansion planning determines, over an horizon of 10 or more years, optimal investments on new transmission lines that make up an economic and reliable electrical network. In its general form, transmission expansion planning is set as a mixed-integer nonlinear stochastic programming problem that minimizes discounted expected costs of investment, subject to constraints depending on uncertain data, such as future growth of electricity demand and of generation.

Historically, transmission expansion planning stems from centralized systems, with both generation and transmission assets belonging to the government. In this setting, transmission planning should ideally be performed jointly with the generation expansion. However, since the resulting optimization problem would be too complex to handle, electrical transmission and energy generation expansion plans are often determined separately, at least for large power systems. Once both expansion plans are available, they can be used as input for some integrated model of generation and transmission, with simplified features. Alternatively, the output of a simplified integrated model can be used as input of the separate expansion planning problems.

The interest of transmission expansion planning also extends to competitive frameworks. The current deregulation trend often results in a mix of market competition in the generation and distribution sectors, with a centralized regulation for transmission. In this context, the regulating entity is in charge, not only of operating the grid while maximizing energy trade opportunities, but also of defining an expansion plan for the transmission network to remain operational in the future. Whether the power system is centralized or liberalized, transmission expansion planning is a valuable tool for helping the decision-maker in adopting the most appropriate strategies for determining the time, the location, and the type of transmission lines to be built.

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The transmission expansion planning problem is set over an electrical network, designed in the past by taking into account some critical factors, specific to the power system under consideration. The amount of hydropower is crucial in hydro-dominated power systems like Brazil's, because generation sites are usually far away from the consumption centers. Long transmission lines and, hence, important investments, are needed. Also, due to the pluvial regime, the network needs to accommodate various power flows arising in different hydrological conditions. Another important factor is the demand growth rate along the years, especially for countries with significant growth rates, which need large investments and a large portfolio with reinforcement candidates.

The transmission expansion planning optimization problem includes both physical and budget constraints. Operational and investment constraints are often linear, and vary dynamically along the planning horizon. By contrast, expansion transmission constraints are static and nonconvex, generally bilinear. Due to the high complexity and difficulty of the corresponding optimization problem, several simplified models and approximation techniques have been considered; see the review [22]. For example, in [30], the transmission expansion planning problem with security constraints, preventing transmission equipment failure, is set as two-stage stochastic mixed-integer linear program, decomposed by Benders technique and solved by a (multicut) cutting-planes algorithm, [8]. If transmission losses are a concern, they can be treated by a linearization, as in [15, 16].

Due to the restructuration of the electrical sector that affected many countries in the recent years, uncertainty has lately arisen as an important consideration. This impacts the modelling and significantly increases the size and complexity of the optimization problem. Reported results are mostly for small power systems (6 to 30 buses) [35, 12, 25]; see also [10, 11, 17, 43, 9, 23, 24, 34, 15]. When considering larger power systems, the problem size is reduced by some heuristic method, relying on human experts' judgment, as in [37, 29, 28, 10].

In general, the transmission expansion planning problem is solved in two variants, considering or not generation redispatch; see [4, 14, 16]. The case without redispatch requires the planned transmission network to operate correctly for a given set of generation values, computed *apriori* for each generation plant. The variant with redispatch considers generation as a variable in the optimization problem: an economic dispatch and the optimal transmission expansion plan are computed together.

In this work, we propose a transmission expansion planning model that, rather than just adding capacity to the existing network, also allows for the network to be re-designed when it is expanded. Our new modelling introduces more flexibility and is general, in the sense that it can be used for different frameworks, with and without redispatch, and independently of the level of simplification or sophistication of the formulation, including with respect to uncertainty treatement.

The new model with re-design relies on the observation that an existing transmission network, designed in the past, may no longer be optimal in the present and it may become even less well adapted in the future. In the transmission expansion planning problem, electrical power flows in the grid according to the linearized second Kirchoff's law, and has the following peculiar property, unique to electrical networks. Namely, in some configurations, disconnecting an existent transmission line (respectively, adding a new line) does not necessarily decrease (respectively, increase) the network capacity. Our numerical testing shows that allowing for the network to be re-designed while expanding it can result in significant savings.

Our paper is organized as follows. In Section 2, we start with a general transmission expansion planning problem, then present our model with re-design, and comment on alternative models proposed by some authors. As mentioned, the transmission expansion planning problem has bilinear constraints that need to be dealt with. Section 3 contains a mathematical study comparing different disjunctive proposals that can be found in the literature. Some alternative linearization techniques, improving the relaxed transmission expansion planning problem, are also analyzed. In most of the proposals, bilinear constraints are "linearized" by using the "big-M" reformulation from Disjunctive Programming. The problem of choosing suitable values for the corresponding "big-M" coefficients is addressed in Section 4. We first give general minimum values for the models with and without re-design, and then analyze how to exploit the initial network topology to reduce the minimal bounds. Section 5 reports on our numerical testing, including a thorough comparison of the various formulations performances on several grids of real size. The final Section 6 gives the model with re-design when considering $(N-1)$ security constraints, some preliminary numerical experience, and a discussion on how to handle uncertain demand and generation.

List of Symbols

S bus-circuit incidence matrix

$i \in B$ index bus, in the set of buses

$\overline{g}_i$ maximal generation at bus i

d_i load at bus i

$\Omega = \Omega^0 \cup \Omega^1$ set of all circuits

Ω^0 set of existing circuits

Ω^1 set of candidate circuits

$|\Omega^1|$ cardinality of set Ω^1

$i(k)$, $j(k)$ terminal buses of circuit k

γ_k susceptance of circuit k

$\overline{f}_k$ capacity of circuit k

c_k investment cost of circuit k

$k_1 \nparallel k_2$ not parallel circuits

$k_1 \parallel k_2$ parallel circuits

$E = E^0 \cup E^1$ set of all "fat" edges

E^0 set of "fat" edges containing existing circuits

E^1 set of "fat" edges containing candidate circuits

$\overline{x}_{ij}$ maximum number of circuits that can be built between i and j

$\underline{x}_{ij}$ existing number of circuits between i and j

(ij) "fat" edge between i and j

$\ell \in \mathcal{L}_{ij}$ index circuit among all circuits belonging to "fat" edge (ij)

SP_{i-j} shortest path between buses i and j

LP_{i-j} longest path between buses i and j

LP^l_{i-j} longest path between buses i and j not passing through bus l

2 Models for transmission expansion planning

For convenience, we start by formulating a deterministic transmission expansion planning problem without contingencies; in Section 6, we consider how to incorporate uncertainty and $(N-1)$ constraints in the modelling. From the Combinatorial Optimization point of view, the electrical network is an undirected graph (B, Ω) where vertices $i \in B$ are called buses and edges $k \in \Omega$ are called circuits. The set of circuits is partitioned into a subset Ω^0, of existing circuits, and a disjoint subset of candidate circuits, denoted by Ω^1. Circuits are connected to buses in a linear relation given by S, the bus-circuit incidence matrix. For each circuit $k \in \Omega$, indices $i(k)$ and $j(k)$ denote, respectively, the head and the tail of the circuit, while γ_k is the circuit susceptance. The reference bus angle is fixed at $\theta_{ref} = 0$. The grid can have parallel circuits, $k_1, k_2 \in \Omega$, denoted by $k_1 \parallel k_2$, linking the same terminal buses.

2.1 Classical transmission expansion planning problem

The transmission network expansion problem is usually written in the following form:

$$
(\texttt{TEP}) \quad \left\{ \begin{array}{llll}
\min & \sum_{k \in \Omega^1} c_k x_k & & \\
\text{s.t.} & Sf + g = d & & \text{(Load)} \\
 & f_k - \gamma_k(\theta_{i(k)} - \theta_{j(k)}) = 0 & k \in \Omega^0 & \text{(Kirchoff}^0\text{)} \\
 & f_k - \gamma_k x_k(\theta_{i(k)} - \theta_{j(k)}) = 0 & k \in \Omega^1 & \text{(Kirchoff}^1\text{)} \\
 & |f_k| \leq \overline{f}_k & k \in \Omega & \text{(FlowBounds)} \\
 & 0 \leq g_i \leq \overline{g}_i & i \in B & \text{(GenBounds)} \\
 & x_k \in \{0,1\} & k \in \Omega^1 . &
\end{array} \right.
$$

At first glance, problem (TEP) could be considered as a Capacitated Network Design problem, used to model expansion of telecommunication networks [42] and freight transportation networks [13], among others. However, there is one important difference, that has a crucial impact when solving the transmission expansion planning problem. Specifically, most capacitated problems satisfy the following property:

$$
\text{for any given } x \in \{0,1\}^{|\Omega^1|}, \text{ with components } x_k = \left\{ \begin{array}{ll} 1 & \text{for } k \in \Omega' \subset \Omega^1 \\ 0 & \text{for } k \in \Omega^1 \backslash \Omega', \end{array} \right.
$$

$$
\text{if } x \text{ is feasible for (TEP), then any vector } \tilde{x} \in \{0,1\}^{|\Omega^1|} \text{ such that } \tilde{x} \geq x \text{ is also feasible for (TEP).} \tag{1}
$$

Such is not the case for transmission networks. As shown in Figure 1, Property (1) may not hold for (TEP): adding one or more circuits to a functioning network may prevent it from working properly.

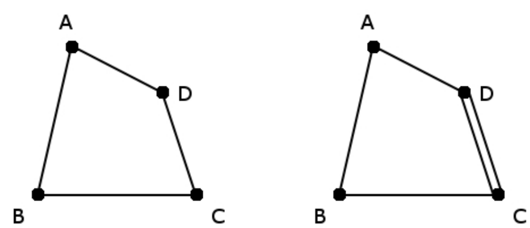

Figure 1: $\overline{g}_A = 100\, MW$, $\overline{g}_B = \overline{g}_C = \overline{g}_D = 0\, MW$, $d_B = d_C = 50\, MW, d_A = d_D = 0\, MW$, $\gamma_{AB} = \gamma_{BC} = 1 \frac{MW}{rad}$ and $\gamma_{CD} = \gamma_{DA} = 2 \frac{MW}{rad}$, and $\overline{f}_{AB} = \overline{f}_{BC} = \overline{f}_{CD} = \overline{f}_{CA} = 50\, MW$. Left network is feasible for $\theta_A = 0\, rad$, $\theta_B = \theta_C = -50\, rad$ and $\theta_D = -25\, rad$, whereas right network is infeasible.

This peculiar feature is in sharp contrast with Capacitated Network Design problems. We shall come back to this issue in Section 2.2. Before, we give a result formalizing the intrinsic difficulty in solving problem (TEP).

Proposition 1 (Complexity of transmission expansion planning)**.** *Problem* (TEP) *is NP-hard.*

Proof. We show how to write a Steiner-tree graph problem in the form (TEP), by suitably choosing the parameters therein. Given an undirected weighted graph defined by vertices in a set V and edges in a set E, a set of terminal vertices $T \subseteq E$, with $|T| \geq 3$, and edge weights $c_k \geq 0$ for all $k \in E$, the *Steiner Problem in Graphs* consists in finding a connected subgraph S (called the *Steiner Tree*) that includes all terminal vertices at minimum edge cost, i.e., $\min \sum_{k \in S} c_k$. This problem is known to be NP-hard, especially for grid graphs, see [21], [2]. Likewise for the single-commodity flow integer formulation of the Steiner problem; see [41]. This formulation expresses the original (undirected weighted graph) problem as a directed weighted graph problem by choosing a "source" terminal vertex t_s offering commodities to the remaining terminal vertices. To see how this last formulation can be cast in the form of problem (TEP), first we let $B = V$, $\Omega = \Omega^1 = E$ and $\Omega^0 = \phi$. Finally, if t denotes the cardinality of the set of terminal vertices $T \subset B$, for an arbitrary source $t_s \in T$, we take

$$
\overline{g}_i = \left\{ \begin{array}{ll} t-1 & i = t_s \\ 0 & i \text{ in } B \backslash \{t_s\} \end{array} \right. , \qquad d_i = \left\{ \begin{array}{ll} 0 & i = \text{ in } B \backslash T \cup \{t_s\} \\ 1 & i \text{ in } T \backslash \{t_s\} \end{array} \right. ,
$$

and, for all $k \in \Omega$, $\bar{f}_k \geq t-1$ and $\gamma_k = 1$. With this data, an optimal solution to (TEP) is nothing but a minimum cost Steiner tree connecting vertices in T. □

2.2 Allowing for the network to be re-designed

In network design problems, new links are added to a network to make it capable of routing given commodities. Typical examples of commodities are passengers using public transportation, merchandise in a vehicle routing problem, data in a telecommunication network, or electricity in a transmission grid.

As mentioned, the peculiar behavior of power flow makes transmission networks very different from the other examples. In particular, for most network design problems, the routing is either decided by some manager, or fixed by a rule aiming at minimizing some utility (congestion, travel time, travel costs). In such circumstances, the fact of adding a new link to a functioning network can never prevent the network from working properly. At worst, the manager can decide not to use that particular link. By constrast, in transmission power systems, the network manager cannot choose which circuits will be used. Only generation dispatch, indirectly affecting the routing, can be chosen (generation levels are control variables, while voltage angles and flows are state variables). The example in Figure 1 shows that, besides being useless, a new link can also make the network inoperational. Similarly, an inoperational network unable to satisfy its load could in some cases start functioning after cutting-off some of its circuits.

The remarks above indicate that, from a modelling point of view, it can be cheaper to allow the network to be re-designed when planning its expansion. The approach is also sensible from a practical point of view. When compared to the high investment required to build new lines, the possibility of cutting some transmission lines, with almost no cost, is worth considering. However, since existing lines can be cut, a model with re-design uses more binary variables and is more difficult from the computational point of view.

The corresponding optimization problem is given by

$$
(\texttt{TEP}_\texttt{R}) \quad \begin{cases} \min & \sum_{k \in \Omega^1} c_k x_k & & \\ \text{s.t.} & Sf + g = d & & \text{(Load)} \\ & f_k - \gamma_k x_k (\theta_{i(k)} - \theta_{j(k)}) = 0 & k \in \Omega & \text{(Kirchoff)} \\ & |f_k| \leq \overline{f}_k & k \in \Omega & \text{(FlowBounds)} \\ & 0 \leq g_i \leq \overline{g}_i & i \in B & \text{(GenBounds)} \\ & x_k \in \{0,1\} & k \in \Omega\,. & \end{cases}
$$

When compared to (TEP), we see that in ($\texttt{TEP}_\texttt{R}$) the bilinear constraints, corresponding to the second Kirchoff's law, are set for all circuits, not only for the new ones. Both problems have the same objective function: only investment cost in building new lines is considered, because the cost of cutting an existing line is negligible. Note, in addition, that the classical model (TEP) can be derived from ($\texttt{TEP}_\texttt{R}$), by adding the constraints $x_k = 1$ for $k \in \Omega^0$ to the re-design problem. This unified approach will be useful in the sequel, when devising solution methods.

In addition to having more binary variables, model ($\texttt{TEP}_\texttt{R}$) is harder to solve than (TEP) because some of the binary variables have null objective cost. As a result, when using an enumeration method, the fathoming of many nodes in the branch-and-bound tree can be significantly delayed. For the same reason, metaheuristics providing very good feasible solutions for (TEP), such as the GRASP described in [6], are no longer applicable to ($\texttt{TEP}_\texttt{R}$), because they are based on selecting circuits by the corresponding investement cost. Finally, as shown in Section 4, the linear relaxation polyhedron for ($\texttt{TEP}_\texttt{R}$) is larger than the one of (TEP). As a result, bounds for ($\texttt{TEP}_\texttt{R}$) may be less tight than for (TEP).

Despite the apparently negative comments above, it is important to keep in mind that, depending on the particular problem, allowing for re-design may have a significant economic impact. Our numerical results on real-life transmission networks show that the model with re-design gives important savings for some configurations.

2.3 Simplified related models

Both (TEP) and ($\texttt{TEP}_\texttt{R}$) can be further complicated by the introduction of $(N-1)$ security constraints. These constraints state that if, for some contingency, any circuit happens to fail (alone), the network must stay functional. We will come back to this issue in Section 6.

In view of the difficulty of the transmission expansion planning problem, even without contingencies, several authors introduced simplified models that we review next. However, in all of the models below, simplification comes at the stake of ending up with a network for which (1) holds. Since this property is not satisfied by a transmission network, for some applications the (simplified) optimal plan computed with such models may need to be modified when the network is actually expanded.

In [5] the model is set to find a minimal cost capacity increase that ensures the network survival to different failures. The network is represented by a graph without parallel edges, and each edge k has an initial capacity, denoted by u_k. Parallel circuits are summed up into a single edge with the corresponding total capacity. In the absence of parallel edges, bilinear constraints can be avoided by replacing, for all $k \in \Omega$, constraints (Kirchoff) and (FlowBounds) by

$$f_k - \gamma_k(\theta_{i(k)} - \theta_{j(k)}) = 0 \qquad \text{and} \qquad |f_k| \leq u_k + \overline{f}_k x_k \,,$$

respectively. Failures are considered in two different variants, depending if they occur simultaneously or in cascade. The first variant is solved by an efficient Benders decomposition scheme. The solution method for the second variant makes use of strong valid inequalities in a cutting planes framework. For both variants, the elimination of parallel circuits allows the authors to solve much bigger instances than the ones handled in our numerical results.

Another simplified model goes back to Garver's transportation model [18], where (Kirchoff^1) is replaced by a flow constraint of the form $|f_k| \leq x_k \overline{f}_k$ for all $k \in \Omega^1$. The resulting mixed-integer linear programming problem is easy to solve by modern solvers, because it is closely related to the so-called single-commodity multi-facility capacitated network design problem. Although unrealistic, the transportation model can provide a better lower bound for (TEP) and ($\texttt{TEP}_\texttt{R}$) than the optimal value of the linear relaxation, see Table 5 in Section 5. Hence, it can be efficiently used in a branch-and-bound process to eliminate portions of the exploration tree.

The third model in our review was proposed in [36] for electricity distribution. Due to the local span of distribution networks, there is one generating unit (only one generation bus) and the network must be a tree (each pair of buses is connected by a single path). In this setting, the model is no longer a simplification, because the actual network satisfies (1).

The tree requirement introduces many combinatorial affine constraints. In counterpart, we show below that a tree network makes the (bilinear) second Kirchoff's law redundant, simplifying substantially the optimization problem (voltage angles disappear from the formulation).

Proposition 2 (Consequence of tree shape)**.** *Suppose the network under consideration is a tree such that*

$$\textit{for any pair of parallel circuits } k_1 \parallel k_2, \textit{ the relation } \overline{f}_{k_1}/\gamma_{k_1} = \overline{f}_{k_2}/\gamma_{k_2} \textit{ holds.} \tag{2}$$

Let Garver's transportation model be given by

$$\left\{\begin{array}{llll} \min & \displaystyle\sum_{k\in\Omega^1} c_k x_k & & \\ s.t. & Sf + g = d & & \\ & |f_k| \leq x_k \overline{f}_k & k \in \Omega^1 & \text{(TranspMod)} \\ & |f_k| \leq \overline{f}_k & k \in \Omega & \\ & 0 \leq g_i \leq \overline{g}_i & i \in B & \\ & x_k \in \{0,1\} & k \in \Omega^1 & \\ & \textit{Tree Network satisfying } (2)\,. & & \text{(Tree)} \end{array}\right.$$

Then any point (x, f, g) is feasible for the transportation problem above if and only if there exists a point (x, f', g, θ') feasible for the transmission expansion planning (TEP) *with the additional constraints (Tree).*

Proof. The necessary condition is straightforward, because the feasible set of the transmission expansion planning (TEP) is contained in the feasible set of the transportation model. To prove the reverse inclusion, given (x, f, g) feasible for the transportation model, we define a point (x', f', g', θ') that is feasible for (TEP), as follows. First, we keep the same design variables, $x'_k = x_k$ for each $k \in \Omega$, and generation variables, $g'_i = g_i$ for each $i \in B$. Then, we consider any circuit $k \in \Omega$ with endpoints i and j. The total flow between i and j is bounded by the total capacity of the circuits connecting i and j, so that their ratio F_{ij} is smaller than one:

$$F_{ij} \equiv \frac{\sum_{h\in\Omega:h\|k} f_h}{\sum_{h\in\Omega^1:h\|k} x_h \overline{f}_h + \sum_{h\in\Omega^0:h\|k} \overline{f}_h} \leq 1.$$

Then, $f'_k = \overline{f}_k F_{ij} \leq \overline{f}_k$ for $k \in \Omega^0$ and $f'_k = x_k \overline{f}_k F_{ij} \leq \overline{f}_k$ for $k \in \Omega^1$ so that f' satisfies the (FlowBounds) constraints. The constraint (Load) for any $b_1 \in B$ is also satisfied, because g_{b_1} is equal to g'_{b_1} and the total flow from b_1 to any $b_2 \in B$ is unchanged: for any $k \in \Omega$ such that $i(k) = b_1$ and $j(k) = b_2$, the total flow between b_1 and b_2 is given by

$$\sum_{h\in\Omega:h||k} f'_h = F_{b_1 b_2} \left(\sum_{h\in\Omega^1:h\|k} x_h \overline{f}_h + \sum_{h\in\Omega^0:h\|k} \overline{f}_h \right) = \sum_{h\in\Omega:h\|k} f_h.$$

The new flow vector f' allows us to set up feasible voltage angles θ' satisfying (Kirchoff^0) and (Kirchoff^1), as follows. First, we choose any bus $ref \in B$ and set $\theta'_{ref} = 0$. Then, we select any built circuit k ($k \in \Omega^1$ and $x_k = 1$, or $k \in \Omega^0$) with $i(k) = ref$ and set $\theta'_{j(k)} = \theta'_{i(k)} - f'_k/\gamma_k = 0 - f'_k/\gamma_k = -F_{i(k)j(k)}\, \overline{f}_k/\gamma_k$. Assumption (2) ensures that choosing $h \parallel k$, instead of k, induces the same angles difference. Next, we select a built circuit $h \nparallel k$ with $i(h) \in \{ref, j(k)\}$ to set up $\theta'_{j(h)}$ in the same way. We repeat this procedure until all voltage angles are set, the tree shape ensuring that each of them shall be set only once. □

3 Linearizing the problem

We now address the problem of defining tight and convex relaxations for the mixed-integer bilinear programming problem ($\mathtt{TEP_R}$). Since (TEP) can be formulated as ($\mathtt{TEP_R}$) plus constraints $x_k = 1$ for $k \in \Omega^0$, the formulations below can be used for both models.

The main difficulty of ($\mathtt{TEP_R}$) arises from its bilinear constraints (Kirchoff), defining the function

$$F(x_k, \theta_{i(k)}, \theta_{j(k)}) := \gamma_k x_k (\theta_{i(k)} - \theta_{j(k)}) \,.$$

This is a bilinear function, neither convex nor concave (its Hessian eigenvalues are constant, equal to 0 and to $\pm\sqrt{2}\gamma_k$). Moreover, there is no quadratic convexification for $F(x_k, \theta_{i(k)}, \theta_{j(k)})$, because the function $F(x_k, \theta_{i(k)}, \theta_{j(k)}) + \lambda(x_k^2 - x_k)$, with Hessian eigenvalues equal to 0 and to $\lambda \pm \sqrt{\lambda^2 + 2\gamma_k^2}$, remains neither convex nor concave, regardless the value of the scalar λ. For this reason, efficient convex mixed-integer nonlinear programming tools, like the method in [32] and its modern implementation FilMint [1], cannot be used in our problem.

Instead, bilinear constraints are "linearized" by using the so-called "big-M"-reformulations for disjunctive programming [33]. Before detailing how to suitably choose such coefficients, we compare two disjunctive approaches that have been used in the literature and give an alternative, third, formulation using "big-M" constraints. To each one of the three formulations corresponds a specific rewriting of bilinear constraints, that yields a different optimization problem, depending if the model of interest is (TEP) or ($\mathtt{TEP_R}$).

3.1 Standard Disjunctive Formulation

Different authors, [31, 40], replace (Kirchoff) by a constraint of the form

$$-M_k(1 - x_k) \leq f_k - \gamma_k(\theta_{i(k)} - \theta_{j(k)}) \leq M_k(1 - x_k) \quad \text{for all } k \in \Omega\,, \tag{3}$$

for some fixed coefficients $M_k > 0$. Flow bounds are written in the form

$$|f_k| \leq x_k \overline{f}_k \qquad \text{for all } k \in \Omega\,. \tag{4}$$

The advantage of this formulation is that its number of variables and constraints grows linearly with the size of the problem. Yet, the formulation is very hard to solve because of the "big-M" coefficients in constraints (3).

3.2 Improved Disjunctive Formulation

A new disjunctive formulation, hopefully tighter than the standard one, and requiring additional continuous variables, was considered in [4]. Each flow is rewritten by using two positive flow variables, as follows:

$$f_k = f_k^+ - f_k^- \qquad \text{for } f_k^+, f_k^- \geq 0 \text{ and } k \in \Omega\,. \tag{5}$$

Likewise for each voltage angles difference:

$$\Delta\theta_k^+ - \Delta\theta_k^- = \theta_{i(k)} - \theta_{j(k)} \qquad \text{for } \Delta\theta_k^+, \Delta\theta_k^- \geq 0 \text{ and } k \in \Omega\,. \tag{6}$$

Using the additional variables in (3) yields the following constraints

$$\begin{array}{l} -M_k(1-x_k) \leq f_k^+ - \gamma_k \Delta\theta_k^+ \leq 0 \\ -M_k(1-x_k) \leq f_k^- - \gamma_k \Delta\theta_k^- \leq 0 \end{array} \quad \text{for all } k \in \Omega\,. \tag{7}$$

With the new variables, flow bounds take the form

$$f_k^+ \leq x_k \overline{f}_k \qquad \text{and} \qquad f_k^- \leq x_k \overline{f}_k \quad \text{for all } k \in \Omega\,. \tag{8}$$

The relation expressing a variable as the difference of its positive and negative parts is a bijection. For this reason, (5) and (8) are equivalent to (4). Since, rather than using the voltage angles, the bijection is used for the voltage angles differences in (6), the feasible set defined by (7) differs from the one defined by (3), as shown next.

3.2.1 Comparing linear relaxations

An important matter when relaxing mixed-integer constraints refers to how close the new feasible set is to the convex hull of the original feasible set, see [26]. A formulation for which the relation is tight is said to be *stronger* than one with a bigger set. To compare the strength of the disjunctive formulations above, we consider their *linear relaxation* polyhedrons, obtained when replacing the $\{0,1\}$ set by the interval $[0,1]$.

Accordingly, we define the polyhedrons

$$\mathcal{P} = \text{conv}\left(\left\{(x,f,g,\theta) \text{ satisfies } \left[\begin{array}{l}(\text{Load}), (\text{Kirchoff}),\\ (\text{FlowBounds}),\\ (\text{GenBounds})\end{array}\right. \quad \text{for some } (f,g,\theta) \text{ and } x \in \{0,1\}^{|\Omega|}\right\}\right),$$

corresponding to the convex hull of feasible vectors for model ($\texttt{TEP}_\texttt{R}$);

$$\mathcal{P}_{3.1} := \left\{(x,f,g,\theta) \text{ satisfies } (\text{Load}), (3), (4), (\text{GenBounds}) \text{ for some } (f,g,\theta) \text{ and } x \in [0,1]^{|\Omega|}\right\},$$

corresponding to the linear relaxation of the standard disjunctive formulation of model ($\texttt{TEP}_\texttt{R}$); and

$$\mathcal{P}_{3.2} := \left\{(x,f,g,\theta) \text{ satisfies } \left[\begin{array}{l}(\text{Load}), (5), (6),\\ (7), (8), (\text{GenBounds})\end{array}\right. \quad \text{for some } (f,g,\theta) \text{ and } x \in [0,1]^{|\Omega|}\right\},$$

corresponding to the linear relaxation of the improved disjunctive formulation of model ($\texttt{TEP}_\texttt{R}$).

We first note that the improved disjunctive formulation is tighter than the standard one. More precisely, in (7), substracting the second equation from the first one, and using (5) and (6), implies satisfaction of (3). Therefore,

$$\mathcal{P}_{3.2} \subseteq \mathcal{P}_{3.1}\,. \tag{9}$$

The following example shows that the inclusion may be strict.

Example 1 (Strict inclusion)**.** Consider a network formed by three buses A, B, and C, with no initial circuits and such that at most one circuit connecting each pair of buses can be built. Suppose, in addition, that the parameters have the values $\overline{g}_A = 100\,MW$, $\overline{g}_B = \overline{g}_C = 0$, $d_A = d_C = 0$, $d_B = 100\,MW$, $\overline{f}_{AB} = \overline{f}_{BC} = \overline{f}_{CA} = 400\,MW$, $\gamma_{AB} = \gamma_{CA} = 1\frac{MW}{rad}$ and $\gamma_{BC} = 0.5\frac{MW}{rad}$ and $c_{AB} = c_{BC} = c_{CA} = 10$. The optimal value to the transmission expansion planning optimization problem ($\mathtt{TEP_R}$) is 10, obtained by constructing only circuit AB: $x_{AB} = 1$, the remaining optimal binary variables being null. The corresponding voltage angles at the optimum are $\theta_A = 0\,rad$ and $\theta_B = -100\,rad$. We show next how to construct a cheaper fractional solution (x, f, g, θ) in $\mathcal{P}_{3.1}$ that does not belong to $\mathcal{P}_{3.2}$.

In Section 4 we give the smallest values for the "big-M" coefficients to ensure a tight relaxation. In particular, by Proposition 3 therein, the minimal value for M_{BC} is $400\,MW$. Consider the fractional vector $x_{AB} = x_{BC} = x_{CA} = 0.25$, with angles $\theta_A = 0$, $\theta_B = \theta_C = -50\,rad$ and flows $f_{AC} = f_{CB} = f_{AB} = 50\,MW$. The corresponding objective function value is 7.5, smaller than the optimal cost of the mixed 0-1 problem.

For the point under consideration, the potential differences $\gamma_{AB}(\theta_A - \theta_B) = \gamma_{AC}(\theta_A - \theta_C) = 50\,MW$ are enough to induce the required flows, whereas $\gamma_{CB}(\theta_C - \theta_B) = 0\,MW$ should not induce any flow. However, since x is fractional, the "big-M" constraints may allow this flow to be routed on the network. Namely, constraint (3) for circuit CB is

$$-300 \leq f_{CB} \leq 300\,,$$

while constraint (7) for circuit CB is

$$f_{CB} = 0. \tag{10}$$

Thus, the flows $f_{AC} = f_{CB} = f_{AB} = 50\,MW$ give a feasible point in $\mathcal{P}_{3.1}$. By contrast, constraints (10) will cut-off the point from $\mathcal{P}_{3.2}$. □

For a linear relaxation to be useful for the optimization problem, its "shadow" projection on the x-variables (see [26]) needs to be tight with respect to the original problem. This means that in the relaxed polyhedrons only the x-components of feasible vectors (x, f, g, θ) matter.

In this sense, although the inclusion (9) ensures a similar relation for the shadow projections, we are in no position to say if the inclusion is strict for the x-variables only. In particular, we now show that for the counter-example above, it is possible to define flows and angles $\tilde{f}, \tilde{\theta}$ satisfying (7) for the fractional values $x_{AB} = x_{BC} = x_{CA} = 0.25$.

Example 2 (No longer strict inclusion)**.** Consider the network in Example 1 and the same fractional vector $x_{AB} = x_{BC} = x_{CA} = 0.25$. Set angles to $\tilde{\theta}_A = \tilde{\theta}_C = 0\,rad$ and $\tilde{\theta}_B = -100\,rad$, and flows to $\tilde{f}_{AB} = 100\,MW$, $\tilde{f}_{AC} = \tilde{f}_{CB} = 0\,MW$. Such flows $\tilde{f}_{AB}$ and $\tilde{f}_{AC}$ are correctly induced by the potential differences, as long as the flow $\tilde{f}_{CB}$ is equal to $50\,MW$. However, recalling that $M_{BC}(1 - x_{BC}) = 300\,MW$, constraint (7) for circuit CB is

$$-250 \leq f_{CB} \leq 50, \tag{11}$$

so that $\tilde{f}_{CB} = 0\,MW$ is feasible for (11) and $(x, \tilde{f}, \tilde{\theta}, g) \in \mathcal{P}_{3.2}$. □

In summary, from relation (9), the linear relaxation of the improved disjunctive formulation is not worse than the one of the standard disjunctive formulation. But it is not known if, when considering only the x-components, the inclusion remains strict (unfortunately, no example is given in [4]). In our computational experience in Section 5, both disjunctive formulations gave identical results, for all the cases in Table 1.

3.3 Breaking Symmetry

In Combinatorial Optimization, it is well known that feasible sets exhibiting symmetry often slow down significantly any branch-and-bound algorithm, due to (useless) exploration of many symmetric nodes. In a transmission network, parallel circuits do induce such a symmetry, making both disjunctive formulations in Sections 3.1 and 3.2 difficult to solve.

Basically, parallel circuits yield feasible points that are indistinguishable by the objective function. Indeed, from a feasible vector involving parallel circuits $k_1 \parallel k_2$, another feasible vector with the same cost can be obtained, simply by swapping indices corresponding to k_1 and k_2.

In order to address this important issue, in what follows we assume the condition below.

Assumption 1. *Any pair of parallel circuits $k_1, k_2 \in \Omega$ has the same capacity, susceptance and cost:*

$$\forall k_1 \parallel k_2 \qquad \overline{f}_{k_1} = \overline{f}_{k_2}\,, \quad \gamma_{k_1} = \gamma_{k_2}\,, \text{ and } c_{k_1} = c_{k_2}\,.$$

All the case studies considered in our numerical experience, and given in Table 1 below, satisfy Assumption 1.

The interest of Assumption 1 is that it allows us to define new circuit sets, by gathering parallel circuits into a single, "fat", edge. We denote such new sets by E^0 and E^1, corresponding to Ω^0 and Ω^1, respectively, with $E = E^0 \cup E^1$ associated to the full set Ω. This re-ordering does not prevent the network from having parallel circuits: to each "fat" edge $(ij) \in E$ we associate an upper bound $\overline{x}_{ij}$ for the number of circuits that can be built. We also denote by $\underline{x}_{ij}$ the initial number of circuits linking i and j. With this notation, instead of using a single index k for a circuit and terminal points $i(k)$ and $j(k)$, each circuit is now determined by a pair (ij, ℓ), referring to the circuit's endpoints $i, j \in B$ and the circuit position $\ell \in \mathcal{L}_{ij} := \{1, \ldots, \underline{x}_{ij} + \overline{x}_{ij}\}$; see Figure 2.

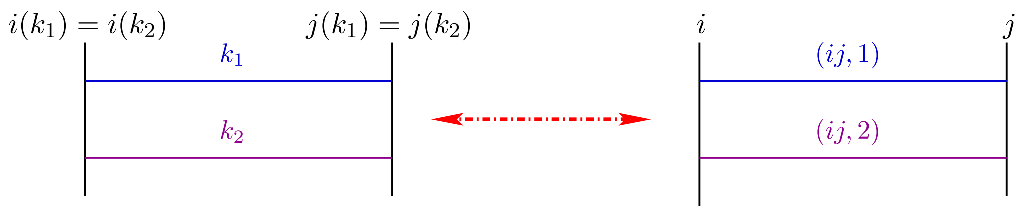

Figure 2: Renaming parallel circuits as part of a single, "fat", edge.

Variables x_k and f_k are renamed accordingly to x_{ij}^{ℓ} and f_{ij}^{ℓ}, and similarly for the investment costs. We show in Section 4 that the actual value used for M_{ij}^{ℓ} is independent of ℓ, so that constraints (3) and (7) are rewritten

$$-M_{ij}(1 - x_{ij}^{\ell}) \le f_{ij}^{\ell} - \gamma_{ij}(\theta_i - \theta_j) \le M_{ij}(1 - x_{ij}^{\ell}) \quad \text{for all } (ij) \in \Omega, \ell \in \mathcal{L}_{ij}\,, \tag{12}$$

and

$$\begin{array}{l} -M_{ij}(1 - x_{ij}^{\ell}) \le f_{ij}^{\ell+} - \gamma_{ij}\Delta\theta_{ij}^{+} \le 0 \\ -M_{ij}(1 - x_{ij}^{\ell}) \le f_{ij}^{\ell-} - \gamma_{ij}\Delta\theta_{ij}^{-} \le 0 \end{array} \quad \text{for all } (ij) \in \Omega, \ell \in \mathcal{L}_{ij}\,, \tag{13}$$

respectively.

Symmetry in the disjunctive formulations can be broken in two different ways:

By ordering parallel candidate circuits: a second circuit can be built only if the first one has been built, and so on:

$$x_{ij}^{\ell+1} \le x_{ij}^{\ell} \qquad (ij) \in E, \ell, \ell+1 \in \mathcal{L}_{ij}\,. \tag{14}$$

These constraints seem to be what in [27] is called "Logical precedence" constraints.

By introducing lexicographical costs: a drawback of the ordering above is the resulting increase in the number of constraints. Instead, parallel circuits can be made distinguishable (and ordered) by assigning to each one of them a different cost, depending on some positive constant ϵ, possibly small:

$$\begin{array}{ll} c_{ij}^{\ell} = (\ell - 1)\epsilon & \forall (ij) \in E\,, 1 \le \ell \le \underline{x}_{ij} \\ c_{ij}^{\ell} = c_{ij} + (\ell - 1)\epsilon & \forall (ij) \in E\,, \underline{x}_{ij} + 1 \le \ell \in \mathcal{L}_{ij}\,. \end{array} \tag{15}$$

In our numerical tests, the improved disjunctive formulation in Section 3.2 did not give competitive results. For this reason, we applied (14) and (15) only to the standard disjunctive formulation from Section 3.1. The lexicographical ordering (15) turned out to be rather poor, at least in our case studies. For instance, the transmission expansion planning for the network "Brazil South", modelled by (TEP) and using the standard disjunctive formulation, took 5 seconds to be solved until optimality. When introducing (15), solution times climbed up to more than 300 seconds.

We mention that CPLEX 11 has an automatic symmetry breaking procedure which can sensibly affect solution times. When this procedure is deactivated, by setting IloCplex.IntParam.Symmetry = 0, the standard disjunctive formulation is much slower than when using the approaches (14) or (15). However, when setting IloCplex.IntParam.Symmetry = -1, the impact of (14) becomes much less expressive.

3.4 Alternative Disjunctive Formulation

We also considered a third disjunctive formulation, grouping together parallel circuits:

$$\left\{\begin{array}{lll}
\min & \sum\limits_{(ij)\in E^1} c_{ij} \sum\limits_{\mathcal{L}_{ij}\ni\ell\geq \underline{x}_{ij}+1} (\ell-\underline{x}_{ij})\, x^\ell_{ij} & \\
\text{s.t.} & Nf+g=d & \\
& -M^\ell_{ij}(1-x^\ell_{ij}) \leq \frac{f_{ij}}{\ell} - \gamma_{ij}(\theta_i-\theta_j) \leq M^\ell_{ij}(1-x^\ell_{ij}) \quad (ij)\in E, \ell\in\mathcal{L}_{ij} & \text{(BigM)}\\
& \sum_{\ell\in\mathcal{L}_{ij}} x^\ell_{ij} \leq 1 \quad (ij)\in E & \text{(SOS1)}\\
& |f_{ij}| \leq \overline{f}_{ij} \sum_{\ell\in\mathcal{L}_{ij}} \ell\, x^\ell_{ij} \quad (ij)\in E & \text{(FlowCap)}\\
& 0\leq g_i \leq \overline{g}_i \quad i\in B & \\
& x^\ell_{ij}\in\{0,1\} \quad ij\in E, \ell\in\mathcal{L}_{ij}\,, &
\end{array}\right.$$

where N is defined from S by selecting only one column for each "fat" edge $(ij) \in E$. Although this formulation significantly reduces the number of flow variables, such potential advantage was not reflected in our computational results; see Section 5. In an effort to improve performance, we also tried CPLEX functionality of using type SOS 1 constraints instead of (SOS1) above. But this option was not effective, probably due to the important increase in the number of nodes to be explored. Setting different values to IloCplex.IntParam.Symmetry did not bring much benefit either.

4 Choosing suitable "big-M" coefficients

The efficient solution of the linearized disjunctive formulations depends strongly on how the coefficients "big-M" are set. Bigger coefficients give less tight polyhedrons, and worse optimal values. It is then worthwhile to compute minimal values, $\underline{M}_{ij}$, such that constraints (12), (13) and (BigM) above are valid for $\mathcal{P}$, for any given value of $\overline{g}$ and d. Recall that an inequality is said to be valid for a polyhedron $\mathcal{Q}$ if $\mathcal{Q}$ is contained in the half space delimited by the inequality.

We first give general minimum values for the models with and without re-design, and then analyze how to exploit the initial network topology (E^0) to reduce the minimal bounds.

In our analysis, paths are always assumed without cycles (they cannot contain twice the same node).

We start with model ($\mathtt{TEP_R}$), allowing re-design, noting that, for any vector $x \in \{0,1\}^{|E|}$, when $\overline{g} = d = 0$, the point $(x, f = 0, \theta = 0, g = 0)$ trivially belongs to $\mathcal{P}$. For this reason, the bound for the "big-M" coefficients should be found for any binary vector x. We now show that the computation of such bound involves solving a longest path problem (see [19]).

Proposition 3. *Suppose Assumption 1 holds and let (ij) be given. Consider constraints* (12) *and* (13) *from Section 3.3. Then the minimal admissible value for M_{ij} such that these constraints are valid for $\mathcal{P}$, for any $\overline{g}, d \geq 0$, is given by $\underline{M}^{(\mathtt{TEP_R})}_{ij} = \gamma_{ij} LP_{i-j}$, the length of the longest path between the buses i and j, computed with costs*

$$\tilde{c}_{b_1b_2} = \frac{\overline{f}_{b_1b_2}}{\gamma_{b_1b_2}}. \tag{16}$$

Proof. For given $(ij) \in E$ and $\ell \in \mathcal{L}_{ij}$, when $x_{ij}^{\ell} = 1$, constraints (12) and (13) imply that $f_{ij}^{\ell} = \gamma_{ij}(\theta_i - \theta_j)$, regardless the value of M_{ij}. Therefore, we only need to consider $x_{ij}^{\ell} = 0$. The flow bounds (4) and (8) force $f_{ij}^{\ell} = 0$. Since the corresponding constraints (12) and (13) state that

$$M_{ij} \geq \gamma_{ij}|\theta_i - \theta_j|\,,$$

we just need to find the largest value of $|\theta_i - \theta_j|$, among all possible network configurations having $x_{ij}^{\ell} = 0$.

To this aim, take $n \neq \ell$ in $\mathcal{L}_{ij}$ and set $x_{ij}^{n} = 1$. The flow on (ij, n) is at most $\overline{f}_{ij}$, so (12) and (13) written for the circuit (ij, n) imply that $\gamma_{ij}|\theta_i - \theta_j| \leq \overline{f}_{ij}$. Thus, M_{ij} must be at least greater than $\overline{f}_{ij}$.

Now, set $x_{ij}^{n} = 0$ for all $n \in \mathcal{L}_{ij}$ such that for some path p between i and j and any link (b_1b_2) in the path it holds that $\sum_n x_{b_1b_2}^{n} \geq 1$. Then, regardless the other values of x, the difference $|\theta_i - \theta_j|$ cannot be greater than:

$$\sum_{(b_1b_2)\in p} |\theta_{b_1} - \theta_{b_2}| \leq \sum_{(b_1b_2)\in p} \frac{\overline{f}_{b_1b_2}}{\gamma_{b_1b_2}}.$$

Since we do not know in advance which path will satisfy the relation $\sum_n x_{b_1b_2}^{n} \geq 1$, we must take the maximum over all paths between i and j. Therefore, $|\theta_i - \theta_j| \leq LP_{i-j}$, with costs given by (16). We are left to show that LP_{i-j} is a minimal value to obtain $\underline{M}_{ij}^{(\mathtt{TEP_R})} = \gamma_{ij}LP_{i-j}$.

To see that LP_{i-j} is a minimal value, it is enough to show that for any path p between i and j,

$$\theta_i - \theta_j = \sum_{(b_1b_2)\in p} \frac{\overline{f}_{b_1b_2}}{\gamma_{b_1b_2}} \tag{17}$$

for at least one generation vector $\overline{g}$ and one demand vector d. Define $\overline{g}$ and d as follows: for each $(b_1b_2) \in p$, $\overline{g}_{b_1} = d_{b_2} = \overline{f}_{b_1b_2}$, and $\overline{g}_r = d_r = 0$ otherwise. Thus, $\theta_{b_1} = \theta_{b_2} + \frac{\overline{f}_{b_1b_2}}{\gamma_{b_1b_2}}$ for each $(b_1b_2) \in p$, yielding (17). □

Note that if the only path from i to j is given by a single candidate circuit from i to j, i.e., by $(ij, 1)$, then there are no constraints on M_{ij} and we can just take $\underline{M}_{ij}^{(\mathtt{TEP_R})} = 0$.

The computation of the minimal value for the third disjunctive formulation can be done in a similar manner.

Corollary 1. *Suppose Assumption 1 holds and let (ij) be given. Consider constraint* (BigM) *from Section 3.4. Then the corresponding minimal admissible value for M_{ij}^{ℓ}, for any $\overline{g}, d \geq 0$, is given by $\underline{M}_{ij}^{(\mathtt{TEP_R}),\ell} = \gamma_{ij}LP_{i-j}$, the length of the longest path between the buses i and j computed with the costs* (16) *for $(b_1b_2) \neq (ij)$, and the following cost for (ij)*

$$\tilde{c}_{ij}^{\ell} = \begin{cases} \frac{\underline{x}_{ij}+\overline{x}_{ij}-\ell}{\ell}\frac{\overline{f}_{ij}}{\gamma_{ij}} & \ell \in \mathcal{L}_{ij}\,, \ell \neq \underline{x}_{ij} + \overline{x}_{ij} \\ \frac{\underline{x}_{ij}+\overline{x}_{ij}-1}{\underline{x}_{ij}+\overline{x}_{ij}}\frac{\overline{f}_{ij}}{\gamma_{ij}} & \ell = \underline{x}_{ij} + \overline{x}_{ij}\,. \end{cases} \tag{18}$$

Proof. When $x_{ij}^{\ell} = 1$, (BigM) forces $f_{ij} = \ell\gamma_{ij}(\theta_i - \theta_j)$, for any value of M_{ij}^{ℓ}. Consider then that $x_{ij}^{\ell} = 0$ for all $n \in \mathcal{L}_{ij}$. By constraint (FlowCap) in Section 3.4, $f_{ij} = 0$ and, like in Proposition 3, constraint (BigM) becomes $M_{ij} \geq \gamma_{ij}|\theta_i - \theta_j|$. As a result, M_{ij}^{ℓ} must be at least greater than the length of the longest path between i and j, in the graph $E\backslash\{(ij)\}$.

Otherwise, let $x_{ij}^{n} = 1$ for some $n \in \mathcal{L}_{ij}$, with $n \neq \ell$. Then, $f_{ij} = n\gamma_{ij}(\theta_i - \theta_j)$, so that (BigM) written for (ij, ℓ) is

$$M_{ij}^{\ell} \geq \left|\gamma_{ij}\left(\frac{n}{\ell} - 1\right)(\theta_i - \theta_j)\right| = \gamma_{ij}\frac{|n-\ell|}{\ell}|\theta_i - \theta_j|\,. \tag{19}$$

This value cannot be greater than $\frac{|n-\ell|}{\ell}\overline{f}_{ij}$. If $\ell < \underline{x}_{ij}+\overline{x}_{ij}$, the right-hand-side of (19) is maximized when $n = \underline{x}_{ij} + \overline{x}_{ij}$. If $\ell = \underline{x}_{ij} + \overline{x}_{ij}$, the right-hand-side of (19) is maximized for $n = 1$. □

Note that, even though the shortest path problem is polynomial and can be solved efficiently by -for instance- Dijkstra's algorithm, the situation is quite different for the longest path. For a graph containing cycles, for example, the problem can be NP-hard. Otherwise, if we could compute in polynomial time the longest path between two adjacent nodes i and j, not passing by trough (ij), with all arc lengths set to 1, we would also be able to find out in polynomial time whether the graph has a Hamiltonian cycle (this is an NP-complete problem, see [2]).

The longest path value LP_{i-j} is often so big that Kirchoff's second law usually fails to hold for the relaxed optimal solution when design variables x are fractional. We now show that the bound can be substantially improved for model ($\texttt{TEP}$), without re-design (and, hence, with less 0-1 variables than ($\texttt{TEP}_\texttt{R}$)).

Because we extend an existing transmission network, we assume the condition below.

Assumption 2. *Let $B^0 \subseteq B$ denote the subset of nodes belonging to edges in E^0. The graph (B^0, E^0) is connected.*

We now consider the convex hull of feasible vectors for model ($\texttt{TEP}$)

$$\tilde{\mathcal{P}} = \text{conv}\left(\left\{(x, f, g, \theta) \text{ satisfies } \left[\begin{array}{l}(\text{Load}), (\text{Kirchoff})^{0,1}, \\ (\text{FlowBounds}), \\ (\text{GenBounds})\end{array}\right. \quad \text{for some } (f, g, \theta) \text{ and } x \in \{0,1\}^{|\Omega^1|}\right\}\right).$$

In the notation gathering parallel circuits, this means that $x^{\ell}_{ij} = 1$ for all $(ij) \in E^0$ and each $\ell \in \mathcal{L}_{ij} = \{1, \ldots, \underline{x}_{ij}\}$.

For costs (16), the improved bound makes use of the shortest path SP_{i-j} between buses i and $j \in E^0$, as well as the longest path LP^k_{i-j} between i and j, not passing through $k \in E^1 \backslash E^0$.

Proposition 4. *Suppose Assumptions 1 and 2 hold, and let (ij) be given. Consider constraints* (12) *and* (13) *from Section 3.3. Then the minimal admissible value for M_{ij} such that these constraints are valid for $\tilde{\mathcal{P}}$, for any $\overline{g}, d \geq 0$, is given by*

$$\underline{M}^{(\texttt{TEP})}_{ij} = \begin{cases} \gamma_{ij} SP_{i-j} & i, j \in B^0 \\ \gamma_{ij} \max_{l \in B^0}(LP^j_{i-l} + SP_{l-j}) & i \notin B^0, j \in B^0 \\ \gamma_{ij} \max\left(LP_{i-j}, \max_{l_1,l_2 \in B^0}(LP^j_{i-l_1} + SP_{l_1-l_2} + LP^i_{l_2-j})\right) & i \notin B^0, j \notin B^0. \end{cases}$$

Proof. When both $i, j \in B^0$, the proof is similar to the one in Proposition 3. First, because $\gamma_{ij}|\theta_i - \theta_j| \leq M_{ij}$, we must compute the maximum feasible value for the differences $|\theta_i - \theta_j|$. Once more, $\sum_{(b_1 b_2) \in p} |\theta_{b_1} - \theta_{b_2}| \leq \sum_{(b_1 b_2) \in p} \frac{\overline{f}_{b_1 b_2}}{\gamma_{b_1 b_2}}$ for any path p in E^0 between i and j. Therefore, we must have $|\theta_i - \theta_j| \leq SP_{i-j}$, since θ cannot induce flows exceeding the capacity of any existing circuit.

When $i \notin B^0, j \in B^0$, any path p from i to j must enter at least once in B^0. Let $l \in B^0$ be the first entry bus and p^1 the sub-path of p from i to l. If $\sum_n x^n_{b_1 b_2} \geq 1$ for each edge $(b_1 b_2) \in p^1$, then

$$|\theta_i - \theta_j| \leq \sum_{(b_1 b_2 \in p^1)} \frac{\overline{f}_{b_1 b_2}}{\gamma_{b_1 b_2}} + |\theta_l - \theta_j| \leq \sum_{(b_1 b_2 \in p^1)} c_{b_1 b_2} + SP_{l-j}.$$

This must be satisfied for any path p from i to j, hence, for any sub-path p_1 from j to $l \in B^0$. Therefore, $|\theta_i - \theta_j| \leq \max_{l \in B^0}(LP^j_{i-l} + SP_{l-j})$, as stated.

When neither i nor j belong to B^0, consider any path p from i to j. If this path does not enter in B^0, then

$$|\theta_i - \theta_j| \leq \sum_{(b_1 b_2) \in p} c_{b_1 b_2}. \tag{20}$$

If this path crosses B^0 at least once, let l_1 be the first entry bus, l_2 be the last exit bus, p^1 the sub-path of p from i to l_1 and p^2 the sub-path of p from l_2 to j. Thus,

$$|\theta_i - \theta_j| \leq \sum_{(b_1 b_2 \in p^1)} c_{b_1 b_2} + |\theta_{l_1} - \theta_{l_2}| + \sum_{(b_1 b_2 \in p^2)} c_{b_1 b_2} \leq \sum_{(b_1 b_2 \in p^1)} c_{b_1 b_2} + SP_{l_1 - l_2} + \sum_{(b_1 b_2 \in p^2)} c_{b_1 b_2}. \tag{21}$$

Finally, taking the maximum of (20) and (21) over all p from i to j and considering a minimality argument, similar to the one in Proposition 3, ends the proof. □

We mention that a result similar to Proposition 4 has already been proved in [7]. However, our result is more compact and general. Moreover, Theorem IV.4 from [7] contains the following (minor) glitch in equation (79) therein. This equation states that if $(i, j) \in E^0$, then $\underline{M}_{ij}^{(\mathtt{TEP})}$ is given by $\overline{f}_{ij}$. Actually, such statement can be made more precise when $SP_{i-j} < \overline{f}_{ij}/\gamma_{ij}$, because in this case $\underline{M}_{ij}^{(\mathtt{TEP})} = \gamma_{ij} SP_{i-j} < \overline{f}_{ij}$.

Once more, the computation of the minimal value for the third disjunctive formulation can be done as for Corollary 1, using the modified costs (16) and (18).

Corollary 2. *Suppose Assumptions 1 and 2 hold and let* (ij) *be given. Consider constraint* (BigM) *from Section 3.4. Then the corresponding minimal admissible value for* M_{ij}^ℓ*, for any* $\overline{g}, d \geq 0$*, is given by*

$$\underline{M}_{ij}^{(\mathtt{TEP}),\ell} = \begin{cases} \gamma_{ij} SP_{i-j} & i\,, j \in B^0 \\ \gamma_{ij} \max_{l \in B^0} (LP_{i-l}^j + SP_{l-j}) & i \notin B^0\,, j \in B^0 \\ \gamma_{ij} \max\left(LP_{i-j},\, \max_{l_1, l_2 \in B^0} (LP_{i-l_1}^j + SP_{l_1-l_2} + LP_{l_2-j}^i)\right) & i \notin B^0\,, j \notin B^0\,, \end{cases}$$

for each $\ell \in \mathcal{L}^{ij}$.

Finally, we show next how the "big-M" constraints can be further strengthened. Consider the "fat" edge $(ij) \in E$. If circuit $(ij, 1)$ is built, $x_{ij}^1 = 1$, the difference $\gamma_{ij}|\theta_i - \theta_j|$ can certainly not exceed $\overline{f}_{ij}$, reducing M_{ij}^ℓ to $\overline{f}_{ij}$ for each $\ell > 1$. Hence, given that $x_{ij}^1 = 1$, we have no longer "big-M" coefficients for the remaining candidate circuits belonging to (ij). Therefore, during the exploration of the branch-and-bound tree, constraints (22) below may yield a linear relaxation that is better than using constraints (12). Without loss of generality, we give the result for the standard disjunctive formulation in Section 3.1.

Proposition 5 (Improved "big-M" constraints)**.** *Suppose Assumption 1 holds and let* (ij) *be given. Consider constraints* (12) *from Section 3.1, and suppose symmetry is broken by using* (14)*. Then for all* $(ij) \in E$ *and* $\ell \in \mathcal{L}_{ij}$*, constraints* (12) *can be replaced by the reinforced constraints*

$$-(M_{ij} - \overline{f}_{ij})(1 - x_{ij}^1) - \overline{f}_{ij}(1 - x_{ij}^\ell) \leq f_{ij}^\ell - \gamma_{ij}(\theta_i - \theta_j) \leq (M_{ij} - \overline{f}_{ij})(1 - x_{ij}^1) + \overline{f}_{ij}(1 - x_{ij}^\ell)\,, \quad (22)$$

which are valid for $\mathcal{P}$*, for any* $\overline{g}, d \geq 0$.

Proof. For $\ell = 1$, (22) is the same as (12). Hence, suppose $\ell > 1$. If $x_{ij}^1 = 0$, then $x_{ij}^\ell = 0$ for each $2 \leq \ell \leq \underline{x}_{ij} + \overline{x}_{ij}$, because of (14), and the left-(respectively, right-) most expression in (22) equals $-M_{ij}$(resp., M_{ij}). If $x_{ij}^1 = x_{ij}^\ell = 1$, then (22) forces $f_{ij}^\ell = \gamma_{ij}(\theta_i - \theta_j)$. Finally, if $x_{ij}^1 = 1$ and $x_{ij}^\ell = 0$, (22) is $\gamma_{ij}|\theta_i - \theta_j| \leq \overline{f}_{ij}$; and constraint (12) for $\ell = 1$ implies that $f_{ij}^1 = \gamma_{ij}(\theta_i - \theta_j)$ so that $\gamma_{ij}|\theta_i - \theta_j| \leq \overline{f}_{ij}$. □

5 Computational Experiments

We make a numerical assesment comparing the different formulations from Section 3 on models (TEP) and ($\mathtt{TEP_R}$), with and without re-design. The main data for our test instances, based on real transmission networks, are reported in Table 1; for full details, we give the corresponding reference in the fourth column of the table. Note that instances "Garver", "IEEE RTS 24-bus", and "Brazil South R" allow redispatch, while the generation variables are fixed for instances "Brazil South" and "Brazil Southeast" (no redispatch is allowed).

The three reformulations from Section 3 gave the same linear relaxation for all cases from Table 1. In order to evaluate the impact of allowing for re-design of the network, we also compared the value of the optimal solutions for some of the models from Section 2. For this comparison, we used the bounds in Section 4 and an alternative bound, simpler to compute, that we detail next.

Remark 1 (Alternative lower bound)**.** *Recall that in Section 4 we gave two types of lower bound for the coefficients* M_{ij} *in each reformulation. The first one (given by Proposition 4 when* $i, j \in B^0$*) is the solution to a shortest path problem, easy to compute, which often has a small value inducing*

name	Topology $\|B\|$	$\|E\|$	Circuits $\|\Omega^0\|$	$\|\Omega^1\|$	Generation/Load $\sum g$ in MW	$\sum d$ in MW	References
Garver	6	15	6	90	1110	760	[3, 18]
IEEE RTS 24-bus	24	34	38	102	10215	8560	[14]
Brazil South	46	79	62	237	6880	6880	[7]
Brazil South R	46	79	62	237	10545	6880	[7]
Brazil Southeast	79	143	156	429	37999	37999	[7]

Table 1: Networks data

a tight linear reformulation of (Kirchoff[1]) *for* (ij). *Such is not the case for the second bound (given by Proposition 4 when* $i \notin B^0$*, and by Proposition 3 for any* $i, j \in B$*), because its computation requires to solve a longest path problem.*

Indeed, being a generalization of the Traveling Salesman Problem*, the longest path problem itself is very difficult to solve, see [19]. Although polynomial algorithms have been proposed for special classes of graphs, see for instance [39], solving the problem for general graphs requires to develop a specialized branch-and-cut algorithm, which is beyond the scope of this work. Moreover, since the value of the second bound,* $\underline{M}_{ij}$*, is already very big, considering an alternative bigger bound neither modifies the quality of the linear relaxation nor decreases the solution times. This remark was confirmed by a set of unreported tests, with increasing values for coefficients* M_{ij}*. Therefore, instead of trying to solve a longest path problem, we use the following alternative upper bound* $\underline{MM}_{ij}$ *for* $\underline{M}_{ij}$*, with value depending on the considered model:*

Model ($\mathtt{TEP_R}$)*with re-design: For each bus* $b \in B$*, let* k_b *be the maximum-cost edge connected to* b*, according to costs* (16)*. Summing up the costs* $\tilde{c}_{k_b}$ *we certainly get an upper bound on the length of any path in the graph* (B, E)*. We can reduce this value, by forbidding to pick up twice the same edge. After ordering the buses* $b_1, \dots, b_{|B|}$*, this is formally written as*

$$\underline{MM}_{ij}^{(\mathtt{TEP_R})} = \gamma_{ij} \sum_{i=1}^{|B|} \max_{(b_i b_j) \in E_{b_i}} \tilde{c}_{b_1 b_2}, \tag{23}$$

where $E_{b_i} = E \backslash \{k_{b_1}, \dots, k_{b_{i-1}}\}$ *for* $i = 2, \dots, |B|$.

Classical transmission expansion planning model ($\mathtt{TEP}$)*: When an initial structure* (B^0, E^0) *is given and* $i \notin B^0$*, the bound* (23) *can be reduced with no additional computational effort to*

$$\underline{MM}_{ij}^{(\mathtt{TEP})} = \gamma_{ij} \left(\max_{b_1, b_2 \in B^0} SP_{b_1 - b_2} + \sum_{i=1}^{|B \backslash B^0|} \max_{(b_i b_j) \in E^1_{b_i}} \tilde{c}_{b_1 b_2} \right),$$

where $E^1_{b_i} = E^1 \backslash \{k_{b_1}, \dots, k_{b_{i-1}}\}$ *for* $i = 2, \dots, |B|$.

All the codes were written in JAVA, using CPLEX concert technology 11 [20], on a computer with an Intel Core 2 Duo processor at 2.40 GHz and 2 GB of RAM memory.

Although we used the MIP black-box solver of CPLEX to handle both ($\mathtt{TEP}$) and ($\mathtt{TEP_R}$), we provided CPLEX with a lower bound for ($\mathtt{TEP}$), and lower and upper bounds for ($\mathtt{TEP_R}$). The lower bound was obtained from the optimal value t of Garvers's transportation model: we added the constraint $\sum_{k \in \Omega^1} c_k x_k \geq t$ to the formulations. Then, solving first ($\mathtt{TEP}$), we could provide CPLEX with a starting feasible solution defining an upper bound. Both bounds reduced significantly the solution times.

Tables 2, 3, and 4 report the results obtained for each formulation from Section 3, for models ($\mathtt{TEP}$), ($\mathtt{TEP_R}$), and Garver's transportation model [18], respectively.

In each table, columns "Standard", "Improved", and "Alternative" stand for formulations from sections 3.1, 3.2, and 3.4, respectively, whereas "Ordering" refers to the "Standard" formulation

name	Standard		Improved		Ordering		Alternative	
	T	nodes	T	nodes	T	nodes	T	nodes
Garver	0.015	5	0.078	8	0.093	11	0.031	45
IEEE	0.23	81	0.92	363	0.56	121	0.67	499
South	5.085	4403	47.75	12149	24.18	5595	48.61	31768
South R	1.51	579	9.079	1823	1.88	395	3.52	1508
Southeast	2438	468428	2888	317314	599	113460	5095	2096088

Table 2: Results for the formulations from Section 3 on (TEP).

with the addition of constraints (14), and "Reinforced" refers to "Ordering", using constraints (22) instead of (12). Solution times are given in seconds, and they exclude the time needed to compute the bounds. We also give the number of nodes explored in the branch-and-bound trees for the different formulations.

name	Standard		Ordering		Reinforced	
	T	nodes	T	nodes	T	nodes
Garver	0.124	0	0.0936	0	0.0468	0
IEEE	50	12756	13	5021	12	3999
South	35091	1368186	19052	952880	20438	1079935
South R	144	14933	31	4408	29	3155

Table 3: Results for the formulations from Section 3 on ($\mathtt{TEP_R}$).

Note that Table 3 does not contain the results for "Brazil Southeast" network, because none of the formulations could solve that instance within 10 hours of computing time. For such a large network, allowing for re-design with the formulations from Section 3 would require to develop a more sophisticated branch-and-cut algorithm.

name	T	nodes
Garver	0.015	0
IEEE	0.031	14
South	2.71	1351
South R	0.078	57
Southeast	0.343	137

Table 4: Garver's transportation model.

Table 5 contains the optimal values found for the models from Sections 2.1, 2.2, and Garver's transportation model. Columns "LPrelax" report the values of the LP relaxations at the root node and the rounded gaps $\frac{Optimal-LPrelax}{Optimal}$. For small instances, reinforced constraints (22) do not improve the solutions times. However, when used in conjunction with a high branching priority for x_{ij}^1, we could obtain a better upper bound for model ($\mathtt{TEP_R}$) of network "Brazil Southeast" (the same branching strategy applied to smaller instances increased the solution times). Apart from this special case, all parameters of CPLEX were left to their default values.

Our results from Table 5 for (TEP) coincide with the best ones reported in the literature. In this sense, any cost below these values can be considered as an improvement. We see on table 5 that the ($\mathtt{TEP_R}$) model for "Brazil South" network induces a cost reduction of **9.67** $= 72.87 - 63.2$ and **8.2** $= 154.4 - 146.2$ for the cases with and without redispatch, respectively. For "Brazil Southeast", the best cost obtained was 405.9 after 10 hours of computing time, with a duality gap of about 29% (the best cost obtained when using (12) instead of (22) was 412). Thus, also for this network, when comparing with the best known values, we have already a cost reduction of **18.9** $= 424.8 - 405.9$. At least on our tested instances, allowing the network to be re-designed can bring important savings in transmission expansion investments.

name	(TEP) Optimal	LPrelax	($\texttt{TEP}_\texttt{R}$) Optimal	LPrelax	Transportation Optimal	LPrelax
Garver	110	99 – 10%	110	99 – 10%	110	99 – 10%
IEEE	152	75 – 50%	152	68.8 – 55%	102	69 – 32 %
South	154.4	82 – 47%	146.2	71.8 – 51%	127	72 – 43%
South R	72.87	41 – 44%	63.2	33 – 48%	53	33 – 38%
Southeast	424.8	173 – 59%	$\leq$ 405.9	120 – N.A.	284	120 – 58%

Table 5: Optimal costs and relaxations.

6 Towards (N-1) reliability

In the previous sections, we investigated some of the difficulties of the transmission expansion problem, leaving aside an important criterion: the designed network must be resistant against the failure of any of its circuits. In principle, $(N-1)$ reliability constraints should ensure that the network remains operational if any of its circuits happened to fail alone. However, in view of (1), it may be too restrictive (thus, too expensive) to require the whole load to be supplied under any circuit failure. This is especially true if the failure of some link would prevent the network from working, whereas not attending to a small portion of the load while the circuit is repaired would keep the network operational.

Therefore, we model the problem as a two-stage stochastic program with continuous recourse, see [8]. First stage variables x_k indicate which circuits are built (or left operational, when $k \in \Omega^0$). At the second stage, for each contingency scenario $h \in \Omega$, we define continuous shortage variables u_i^h, $i \in B$, and associated binary coefficients δ_k^h stating which circuits are operational:

$$\delta_k^h = \begin{cases} 0 & \text{if } h = k \\ 1 & \text{if } h \neq k. \end{cases}$$

In order to take into account the fact that the network must supply the whole load when all of its circuits are operational, we also define the scenario "*all*" (and $\Omega^* = \Omega \cup \{all\}$), corresponding to $u^{all} \equiv 0$ and $\delta^{all} \equiv 1$. Finally, the vector $(f^h, g^h, u^h, \theta^h)$ describes the routing for each scenario $h \in \Omega^*$.

With these additional variables, the $(N-1)$ criterion for ($\texttt{TEP}_\texttt{R}$) has the form:

$$(\texttt{TEP}_\texttt{R}\,\texttt{N}-\texttt{1}) \quad \left\{ \begin{array}{llll} \min & \displaystyle\sum_{k\in\Omega^1} c_k x_k + \sum_{h\in\Omega, b\in B} p_i^h u_i^h & & \\ \text{s.t.} & Sf^h + g^h + u^h = d & h \in \Omega^* & \\ & f_k^h - \gamma_k \delta_k^h x_k(\theta_{i(k)}^h - \theta_{j(k)}^h) = 0 & k \in \Omega,\, h \in \Omega^* & \text{(Kirchoff)} \\ & |f_k^h| \leq \overline{f}_k & k \in \Omega,\, h \in \Omega^* & \text{(FlowBounds)} \\ & 0 \leq g_i^h \leq \overline{g}_i & i \in B,\, h \in \Omega^* & \\ & 0 \leq u_i^h \leq d_i & i \in B,\, h \in \Omega & \\ & 0 \leq u_i^{all} = 0 & i \in B & \\ & x_k \in \{0,1\} & k \in \Omega\,. & \end{array} \right.$$

In the objective function above, each penalty factor p_i^h is an estimation of the practical cost of shortage for bus i, multiplied by the probability of failure h to happen. Then, we can use any of the linearized reformulations from Section 3 to handle the bilinear constraints. For instance, with the standard approach from Section 3.1, we replace constraints (Kirchoff) by

$$-M_k(1-\delta_k^h x_k) \leq f_k^h - \gamma_k(\theta_{i(k)}^h - \theta_{j(k)}^h) \leq M_k(1-\delta_k^h x_k), \qquad \text{for all } k \in \Omega,\, h \in \Omega^*, \tag{24}$$

and constraints (FlowBounds) by

$$|f_k^h| \leq \delta_k^h x_k \overline{f}_k \qquad \text{for all } k \in \Omega,\, h \in \Omega^*, \tag{25}$$

yielding the model introduced in [38] (without re-design).

We mention that in [29] the authors also make a two-stage formulation of the $(N-1)$ criterion. However, their model considers a different bus-circuit incidence matrix S^h for each scenario $h \in \Omega^*$.

Such second-stage matrices are defined by suppressing in S the column related to circuit h (so that $S^{all} = S$). Our simpler recourse formulation ($\mathtt{TEP_R}\,\mathtt{N}-\mathtt{1}$), with a fixed matrix S, should ease the use of Stochastic Programming decomposition algorithms.

Model	time	nodes	optimal	LPrelax
Transportation	0.5	11	116	106.7 – 8%
($\mathtt{TEP}\,\mathtt{N}-\mathtt{1}$)	3.8	31	118.4	106.7 – 9.8%
($\mathtt{TEP_R}\,\mathtt{N}-\mathtt{1}$)	8.3	67	118.4	106.7 – 9.8%

Table 6: $(N-1)$ reliability constraints for Garver's network.

We can see preliminary computational results on Table 6 for Garver's network, using again the MIP solver of CPLEX 11. Model ($\mathtt{TEP}\,\mathtt{N}-\mathtt{1}$) stands for ($\mathtt{TEP_R}\,\mathtt{N}-\mathtt{1}$) with additional constraints $x_k = 1$ for $k \in \Omega^0$, and "Transportation" for ($\mathtt{TEP_R}\,\mathtt{N}-\mathtt{1}$) without (Kirchoff). Since the considered network is small, there is no "slack" for the re-design model to give any improvement: the optimal values of ($\mathtt{TEP}\,\mathtt{N}-\mathtt{1}$) and ($\mathtt{TEP_R}\,\mathtt{N}-\mathtt{1}$) are equal. For this reason, rather than giving insight on the model with re-design, our results in Table 6 should be considered as a validation of our solving methodology for ($\mathtt{TEP_R}\,\mathtt{N}-\mathtt{1}$).

In addition to $(N-1)$ constraints, it is important for the expansion planning problem to consider uncertainty both in the electricity demand and generation. In this case, instead of (or in addition to) considering contigency scenarios $h \in \Omega$, the 2-stage formulation ($\mathtt{TEP_R}\,\mathtt{N}-\mathtt{1}$) makes use of a set $\mathcal{W}$ such that to each scenario $\omega \in \mathcal{W}$ corresponds a demand/generation vector $(d(\omega), g(\omega))$. As in ($\mathtt{TEP_R}\,\mathtt{N}-\mathtt{1}$), the design decisions x must be taken *here-and-now*, while the *wait-and-see* decisions of recourse $(f(\omega), g(\omega), u(\omega), \theta(\omega))$ depend on each scenario $\omega \in \mathcal{W}$. The main difference with the reliability models is that demand/generation scenarios do not need the additional vector δ, because uncertainty is fully characterized by the values of $d(\omega)$ and $g(\omega)$. Finally, one could consider both the $(N-1)$ reliability criterion and different scenarios for demand and generation, see [29].

For networks bigger than "Garver", solving to optimality ($\mathtt{TEP_R}\,\mathtt{N}-\mathtt{1}$) or one of the extensions evocated above needs developing efficient decomposition algorithms, an interesting subject of future research.

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