

On the Power of Robust Solutions in Two-Stage Stochastic and Adaptive Optimization Problems

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We consider a two-stage mixed integer stochastic optimization problem and show that a static robust solution is a good approximation to the fully-adaptable two-stage solution for the stochastic problem under fairly general assumptions on the uncertainty set and the probability distribution. In particular, we show that if the right hand side of the constraints is uncertain and belongs to a *symmetric* uncertainty set (such as hypercube, ellipsoid or norm-ball) and the probability measure is also *symmetric*, then the cost of the optimal fixed solution to the corresponding robust problem is at most twice the optimal expected cost of the two-stage stochastic problem. Furthermore, we show that the bound is tight for symmetric uncertainty sets and can be arbitrarily large if the uncertainty set is not symmetric. We refer to the ratio of the optimal cost of the robust problem and the optimal cost of the two-stage stochastic problem as the *stochasticity gap*. We also extend the bound on the stochasticity gap for another class of uncertainty sets referred to as *positive*.

If both the objective coefficients and right hand side are uncertain, we show that the stochasticity gap can be arbitrarily large even if the uncertainty set and the probability measure are both symmetric. However, we prove that the adaptability gap (ratio of optimal cost of the robust problem and the optimal cost of a two-stage fully-adaptable problem) is at most four even if both the objective coefficients and the right hand side of the constraints are uncertain and belong to a symmetric uncertainty set. The bound holds for the class of *positive* uncertainty sets as well. Moreover, if the uncertainty set is a hypercube (special case of a symmetric set), the adaptability gap is one under an even more general model of uncertainty where the constraint coefficients are also uncertain.

Key words: robust optimization ; stochastic optimization ; adaptive optimization ; stochasticity gap; adaptability gap

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1. Introduction. In most real world problems, several parameters are uncertain at the optimization phase and a solution obtained through a deterministic optimization approach might be sensitive to even slight perturbations in the problem parameters, possibly rendering it highly suboptimal or infeasible. Stochastic optimization that was introduced as early as Dantzig [9] has been extensively studied in the

literature to address uncertainty. A stochastic optimization approach assumes a probability distribution over the uncertain parameters and tries to compute a (two-stage or a multi-stage) solution that optimizes the expected value of the objective function. We refer the reader to several textbooks including Infanger and der Wissenschaftlichen Forschung [10], Kall and Wallace [11], Prékopa [12], Birge and Louveaux [8], and the references therein for a comprehensive view of stochastic optimization. While a stochastic optimization approach addresses the issue of uncertain parameters, it is by and large computationally intractable. Shapiro and Nemirovski [14] give hardness results for two-stage and multistage stochastic optimization problems where they show that multi-stage stochastic optimization is computationally intractable even if approximate solutions are desired. Furthermore, even to solve a two-stage stochastic optimization problem, Shapiro and Nemirovski [14] present an approximate sampling based algorithm where a sufficiently large number of scenarios (depending on the variance of the objective function and the desired accuracy level) are sampled from the assumed distribution and the solution to the resulting sampled problem is argued to provide an approximate solution to the original problem.

More recently, a robust optimization approach has been introduced to address the problem of optimization under uncertainty and has been studied extensively (see Ben-Tal and Nemirovski [3], Bertsimas and Sim [6], Bertsimas and Sim [7]). In a robust optimization approach, the uncertain parameters are assumed to belong to some uncertainty set and the goal is to construct a solution such that the objective value in the worst-case realization of the parameters in the uncertainty set is minimized. A robust optimization approach constructs a single solution that is feasible for all possible realizations of the parameters in the assumed uncertainty set. Therefore, it is a significantly more tractable approach computationally as compared to a stochastic optimization approach. However, it is possible that since a robust optimization approach tries to optimize over the worst-case scenario, it may produce conservative solutions. We point the reader to the survey by Bertsimas et al. [4] and the references therein for an extensive review of the literature in robust optimization.

To address this drawback of robust optimization, approximate adaptive optimization approaches have been considered in the literature where a simpler functional forms (such as an affine policy or linear decision rules) are considered to approximate the optimal decisions. The functional form allows to succinctly represent the solution in each stage for every realization of the uncertain parameters, albeit the loss in optimality. This approach was first considered in Rockafellar and Wets [13] in the context of stochastic optimization and then in robust optimization Ben-Tal et al. [2] and extended to linear systems theory Ben-Tal et al. [1]. In a recent paper, Bertsimas et al. [5] consider a one-dimensional,

box-constrained multi-stage robust optimization problem and show that an affine policy is optimal in this setting. However, in general an affine policy does not necessarily provide a good approximation to the original problem. Moreover, the computation complexity of solving an adaptive optimization problem is significantly higher.

In this paper, we show that under a fairly general model of uncertainty for a two-stage mixed integer optimization problem, a robust optimization approach is a good approximation to solving the corresponding stochastic optimization problem optimally. In other words, the worst-case cost of an optimal solution to the robust two-stage mixed integer optimization problem is not much worse than the expected cost of an optimal solution to the corresponding two-stage stochastic optimization problem when the right hand side of the constraints is uncertain and belongs to a *symmetric* uncertainty set and the probability distribution is also *symmetric* (we also extend our result under milder conditions). Furthermore, a robust optimization problem can be solved efficiently (as compared to stochastic and adaptive) and thus, provides a computationally tractable approach to obtain good approximations to the two-stage stochastic problem. We also show that a robust optimization approach is an arbitrarily bad approximation to the two-stage stochastic optimization problem when both costs and right hand sides are uncertain. However, we show that an optimal solution to the robust problem is a good approximation for a two-stage adaptive optimization problem where the goal is to construct a fully-adaptable solution that minimizes the worst-case cost, even when both costs and right hand sides are uncertain under fairly general assumptions on the uncertainty set.

1.1 Models. We consider the following two-stage stochastic mixed integer optimization problem $\Pi_{Stoch}(b)$:

$$\begin{aligned}
 z_{Stoch}(b) &= \min c^T x + \mathbb{E}_\mu[d^T y(\omega)] \\
 Ax + By(\omega) &\geq b(\omega), \forall \omega \in \Omega \\
 x &\in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1} \\
 y(\omega) &\in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2},
 \end{aligned} \tag{1.1}$$

where $A \in \mathbb{R}^{m \times n_1}$, $B \in \mathbb{R}^{m \times n_2}$, $c \in \mathbb{R}_+^{n_1}$, $d \in \mathbb{R}_+^{n_2}$. Ω denotes the set of scenarios and for any $\omega \in \Omega$, $b(\omega) \in \mathbb{R}_+^m$ denotes the realization of the uncertain values of right hand side of the constraints b and $y(\omega)$ denotes the second-stage decision in scenario ω . Let

$$\mathcal{I}_b(\Omega) = \{b(\omega) \mid \omega \in \Omega\} \subset \mathbb{R}_+^m,$$

be the set of possible values of the uncertain parameters (or the uncertainty set) and μ is a probability measure over the set of scenarios Ω . Also, $\mathbb{E}_\mu[\cdot]$ is the expectation with respect to the probability measure μ .

The corresponding two-stage robust optimization problem, $\Pi_{Rob}(b)$ is as follows:

$$\begin{aligned} z_{Rob}(b) &= \min c^T x + \max_{\omega \in \Omega} (d^T y) \\ Ax + By &\geq b(\omega), \forall \omega \in \Omega \\ x &\in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \\ y &\in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{p_2}. \end{aligned} \tag{1.2}$$

Also, the two-stage adaptive optimization problem, $\Pi_{Adapt}(b)$ is formulated as follows:

$$\begin{aligned} z_{Adapt}(b) &= \min c^T x + \max_{\omega \in \Omega} (d^T y(\omega)) \\ Ax + By(\omega) &\geq b(\omega), \forall \omega \in \Omega \\ x &\in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \\ y(\omega) &\in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{p_2}. \end{aligned} \tag{1.3}$$

Note that we parameterize the problem names with the parameter b that denotes that the right hand side of the constraints is uncertain. We also extend our uncertainty to include cost uncertainty parametrized as (b, d) . The two-stage stochastic optimization problem, $\Pi_{Stoch}(b, d)$ under the new model of uncertainty is as follows:

$$\begin{aligned} z_{Stoch}(b, d) &= \min c^T x + \mathbb{E}_\mu [d(\omega)^T y(\omega)] \\ Ax + By(\omega) &\geq b(\omega), \forall \omega \in \Omega \\ x &\in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \\ y(\omega) &\in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{p_2}, \end{aligned} \tag{1.4}$$

where $A \in \mathbb{R}^{m \times n_1}$, $B \in \mathbb{R}^{m \times n_2}$, $c \in \mathbb{R}_+^{n_1}$. Ω denotes the set of scenarios, where,

$$\mathcal{I}_{(b,d)}(\Omega) = \{(b(\omega), d(\omega)) \mid \omega \in \Omega\} \subset \mathbb{R}_+^{m+n_2},$$

is the uncertainty set and for any $\omega \in \Omega$, $b(\omega) \in \mathbb{R}_+^m$ and $d(\omega) \in \mathbb{R}_+^{n_2}$ are realizations of the uncertain values of right hand side, b , and the second-stage cost vector, d , in scenario ω and μ is a probability measure over the set of scenarios Ω .

The corresponding two-stage robust optimization problem, $\Pi_{Rob}(b, d)$ is as follows:

$$\begin{aligned}
 z_{Rob}(b, d) &= \min c^T x + \max_{\omega \in \Omega} (d(\omega)^T y) \\
 Ax + By &\geq b(\omega), \forall \omega \in \Omega \\
 x &\in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \\
 y &\in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{n_2},
 \end{aligned} \tag{1.5}$$

and the two-stage adaptive optimization problem, $\Pi_{Adapt}(b, d)$ is formulated as follows:

$$\begin{aligned}
 z_{Adapt}(b, d) &= \min c^T x + \max_{\omega \in \Omega} (d(\omega)^T y(\omega)) \\
 Ax + By(\omega) &\geq b(\omega), \forall \omega \in \Omega \\
 x &\in \mathbb{R}_+^{n_1-p_1} \times \mathbb{Z}_+^{p_1} \\
 y(\omega) &\in \mathbb{R}_+^{n_2-p_2} \times \mathbb{Z}_+^{n_2},
 \end{aligned} \tag{1.6}$$

Let us also introduce the following definitions before formally describing our contributions.

DEFINITION 1.1 *A set $H \subset \mathbb{R}^n$ is a hypercube, if there exist $l_i \leq u_i$ for all $i = 1, \dots, n$, such that,*

$$H = \{x \in \mathbb{R}^n \mid l_i \leq x_i \leq u_i, \forall i = 1, \dots, n\}.$$

DEFINITION 1.2 *A set $P \subset \mathbb{R}^n$ is symmetric, if there exists some $u_0 \in P$, such that, for any $z \in \mathbb{R}^n$,*

$$(u_0 + z) \in P \Leftrightarrow (u_0 - z) \in P. \tag{1.7}$$

Note that (1.7) is equivalent to: $x \in P \Leftrightarrow (2u_0 - x) \in P$. A hypercube is a special case of a symmetric set. An ellipsoid, $\mathcal{E}(u, D)$, where $u \in \mathbb{R}^n$ and $D \in \mathbb{R}^{n \times n}$ is a positive semidefinite matrix, and,

$$\mathcal{E}(u, D) = \{u + D^{1/2}v \mid u, v \in \mathbb{R}^n, v^T v \leq 1\},$$

is also an example of a symmetric set that is a commonly used uncertainty set. Another commonly used uncertainty set that is symmetric is a norm-ball $B(x_0, r)$ where $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{R}_+$, and,

$$B(x_0, r) = \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\},$$

where $\|\cdot\|$ denotes some norm (for instance, the euclidean norm). Since most commonly used uncertainty sets are symmetric, our assumption of symmetry on the uncertainty set is not very restrictive. Nevertheless, there are natural uncertainty sets that do not satisfy the assumption of symmetry, such as the following fractional knapsack polytope:

$$P = \left\{ x \in \mathbb{R}_+^n \mid \sum_{j=1}^n x_j \leq k \right\}.$$

It is easy to see that P is not symmetric for $n \geq 2$ and $k = 1$. However, P is a natural uncertainty set that occurs in many settings (for instance, in modeling k -fault tolerance). Therefore, it would be useful to prove a bound for such uncertainty sets as well. We show that our results hold, if we translate P to $P + v$ for some $v \in \mathbb{R}_+^n$ such that v_j is sufficiently large for each $j = 1, \dots, n$. The intuition is that the translation ensures that the possible values of each uncertain parameter are sufficiently far from zero in all scenarios and therefore, the relative change in the parameter value is not large and does not significantly affect the costs. This motivates us to define the following class of convex sets.

DEFINITION 1.3 *A convex set $P \subset \mathbb{R}_+^n$ is positive, if there exists a convex symmetric set $S \subset \mathbb{R}_+^n$ such that $P \subset S$ and the point of symmetry of S is contained in P .*

For instance, consider the following convex set: $P = \{x \in \mathbb{R}_+^n \mid x_1 + \dots + x_n \leq 1\}$. We show that P is not symmetric for $n \geq 2$ (see Lemma 2.4). Also, P is not positive for $n \geq 3$ (for $n = 2$, the point of symmetry of the hypercube containing P is $(1/2, 1/2)$ which belongs to P , and thus, P is positive). However, the set $P + a$ obtained by translating P by $a = (1, 1, \dots, 1)$ is positive (see Figure 1). Consider the following hypersphere:

$$S = \left\{ x \in \mathbb{R}^n \mid \sum_{j=1}^n (x_j - 1)^2 \leq 1 \right\}.$$

It is easy to observe that $S \subset \mathbb{R}_+^n$, and $(P + a) \subset S$. Also, the point of symmetry of S is a which is contained in $P + a$.

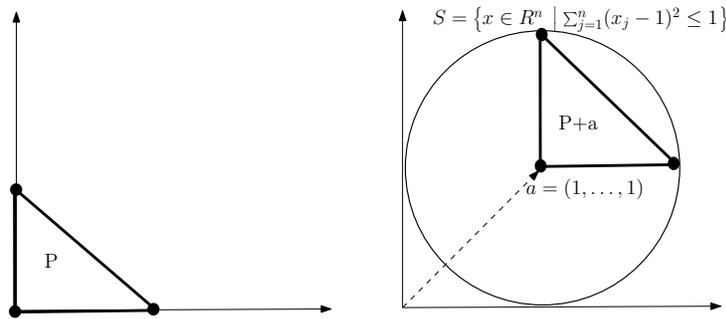


Figure 1: (a) $P = \{x \in \mathbb{R}_+^n \mid x_1 + \dots + x_n \leq 1\}$ is neither symmetric nor positive for $n \geq 3$. (b) $P + a$ (where $a = (1, \dots, 1)$) is positive as the hypersphere $S \subset \mathbb{R}_+^n$ contains $P + a$ and its point of symmetry, a belongs to $P + a$.

Let us also define a symmetric probability measure on a symmetric set.

DEFINITION 1.4 *A probability measure μ on a symmetric set $P \subset \mathbb{R}^n$, where u^0 is the point of symmetry of P , is symmetric, if for any $S \subset P$, $\mu(S) = \mu(\hat{S})$ where $\hat{S} = \{(2u^0 - x) \mid x \in S\}$.*

As an example, the uniform probability measure over any symmetric set $P \subset \mathbb{R}^n$ is symmetric.

1.2 Our Contributions. In this paper, we compare the optimal cost of the robust problem $\Pi_{Rob}(b)$ to the optimal costs of problems $\Pi_{Stoch}(b)$ and $\Pi_{Adapt}(b)$. We refer to the ratio of $z_{Rob}(b)$ and $z_{Stoch}(b)$ (as well as the ratio of $z_{Rob}(b, d)$ and $z_{Stoch}(b, d)$) as the *stochasticity gap* and the ratio of $z_{Rob}(b)$ and $z_{Adapt}(b)$ (as well as the ratio of $z_{Rob}(b, d)$ and $z_{Adapt}(b, d)$) as the *adaptability gap*. Recall that $z_{Rob}(b)$, $z_{Stoch}(b)$ and $z_{Adapt}(b)$ are the optimal costs of $\Pi_{Rob}(b)$, $\Pi_{Stoch}(b)$ and $\Pi_{Adapt}(b)$ respectively and $z_{Rob}(b, d)$, $z_{Stoch}(b, d)$ and $z_{Adapt}(b, d)$ are the optimal costs of $\Pi_{Rob}(b, d)$, $\Pi_{Stoch}(b, d)$ and $\Pi_{Adapt}(b, d)$ respectively.

Stochasticity Gap. We show that the stochasticity gap is at most 2 if the uncertainty set is symmetric (see Definition 1.2) as well as the probability distribution over the uncertainty set is *symmetric* (we further extend to other milder conditions on the probability distribution) and there are no integer decision variables in the second stage, i.e, $p_2 = 0$ in $\Pi_{Stoch}(b)$. This implies that the worst-case cost of an optimal fixed solution $x^* \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}$, $y^* \in \mathbb{R}_+^{n_2}$ for $\Pi_{Rob}(b)$ is at most twice the expected cost of an optimal two-stage solution to $\Pi_{Stoch}(b)$ and thus, the solution $x^*, y(\omega) = y^*$ for all scenarios $\omega \in \Omega$ is a good approximation to $\Pi_{Stoch}(b)$. Moreover, an optimal solution to $\Pi_{Rob}(b)$ can be computed efficiently by solving a single mixed integer optimization problem and does not require any knowledge of the probability distribution μ . This provides a good computationally tractable approximation to the stochastic optimization problem that is intractable in general. Our results hold under the assumptions of symmetry and non-negativity on the uncertainty set. Note that most commonly used uncertainty sets, such as hypercubes (specifying an interval of values for each uncertain parameter), ellipsoids and norm-balls satisfy these assumptions. We also show that the bound on the stochasticity gap holds if the uncertainty set is convex and positive and the probability distribution satisfies a technical condition similar to symmetry. Therefore, we show a surprising approximate equivalence between two-stage robust optimization and two-stage stochastic optimization. We also show that our bound on the stochasticity gap is tight for symmetric uncertainty sets and it can be arbitrarily large if the uncertainty set is not symmetric. Therefore, our results give a nice characterization of when a robust solution is a bounded approximation to the stochastic optimization problem when only the right hand side is uncertain and there are no integer second-stage decision variables. However, for the model with both cost and right hand side uncertainty (problems $\Pi_{Stoch}(b, d)$ and $\Pi_{Rob}(b, d)$), we show that the stochasticity gap (i.e., the ratio of $z_{Rob}(b, d)$ and $z_{Stoch}(b, d)$) can be arbitrarily large even when there are no second-stage integer

decision variables and the uncertainty set as well as the probability distribution are symmetric. In fact, the stochasticity gap is large when only the objective coefficients are uncertain and the right hand side is deterministic.

Adaptability Gap. We show that the adaptability gap ($z_{Rob}(b)/z_{Adapt}(b)$) is at most two if the uncertainty $\mathcal{I}_b(\Omega)$ is symmetric. The bound of two on the adaptability gap holds even if some of the second-stage decision variables are integers in problems $\Pi_{Adapt}(b)$ and correspondingly $\Pi_{Rob}(b)$ unlike the bound on the stochasticity gap which holds only if all the second-stage decision variables are continuous. In fact, the adaptability gap is bounded for problems $\Pi_{Rob}(b, d)$ and $\Pi_{Adapt}(b, d)$ where the formulation also models cost uncertainty along with the right hand side uncertainty unlike the stochasticity gap. Our main results on the adaptability gap are the following:

- (i) If the uncertainty set $\mathcal{I}_{(b,d)}(\Omega)$ is a hypercube (which is a special case of a symmetric uncertainty set), then the adaptability gap is one, i.e., $z_{Rob}(b, d) = z_{Adapt}(b, d)$. This implies, that there is a single solution (x, y) , $x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1}$, $y \in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2}$ which is feasible for all scenarios $\omega \in \Omega$ and the worst-case cost of this solution is exactly equal to the optimal fully-adaptable solution. In fact, we prove this result for an even more general model of uncertainty where we also allow the constraint coefficients to be uncertain. We would like to note that unlike the adaptability gap, the stochasticity gap is two even if the uncertainty set $\mathcal{I}_b(\Omega)$ is a hypercube and the bound is tight in this case as well.
- (ii) For any symmetric uncertainty set $\mathcal{I}_{(b,d)}(\Omega)$, we show that $z_{Rob}(b, d) \leq 4 \cdot z_{Adapt}(b, d)$.
- (iii) We also extend the bound on the adaptability gap for positive uncertainty sets, i.e., $z_{Rob}(b, d) \leq 4 \cdot z_{Adapt}(b, d)$ if $\mathcal{I}_{(b,d)}(\Omega)$ is positive and convex. The bound on the adaptability gap for the case of positive uncertainty sets formalizes the following intuition: the relative change in the optimal cost of a two-stage problem with linear cost function depends on the relative change in the problem parameters and not on the absolute change. If the uncertainty set is positive, all the uncertain parameters are sufficiently far from zero and thus, the relative change in their values can be bounded, a fact that allows us to bound the adaptability gap.
- (iv) For a general convex uncertainty set (neither symmetric nor positive), we show that the adaptability gap can be arbitrarily large. In particular, we construct a convex uncertainty set that is neither symmetric nor positive and $z_{Rob}(b, d) \geq m \cdot z_{Adapt}(b, d)$. This shows that our results give an almost tight characterization of the uncertainty sets where the adaptability gap is bounded.

We would like to note that our bounds on the stochasticity and the adaptability gap hold only for the model with only covering constraints and does not extend to a formulation where some of the constraints are packing. Our results on the stochasticity gap and the adaptability gap for the model where only the right hand side is uncertain are summarized in Table 1.

Uncertainty Set, $\mathcal{I}_b(\Omega)$	Stochasticity Gap $\left(\frac{z_{Rob}(b)}{z_{Stoch}(b)}, p_2 = 0\right)$	Adaptability Gap $\left(\frac{z_{Rob}(b)}{z_{Adapt}(b)}\right)$
Hypercube	2^*	1^*
Symmetric	2^*	2^*
Convex, Positive	2	2
Convex	$\Omega(m)$	$\Omega(m)$

Table 1: Stochasticity Gap and Adaptability Gap for different uncertainty sets for the model with uncertain right hand sides. We assume $\mathcal{I}_b(\Omega)$ lies in the non-negative orthant, objective coefficients $c, d \geq 0$ and $x, y(\omega) \geq 0$. Here * denotes that the bound is tight.

Table 2 summarizes our results when both right hand side and objective coefficients are uncertain.

Uncertainty Set, $\mathcal{I}_{(b,d)}(\Omega)$	Stochasticity Gap $\left(\frac{z_{Rob}(b,d)}{z_{Stoch}(b,d)}\right)$	Adaptability Gap $\left(\frac{z_{Rob}(b,d)}{z_{Adapt}(b,d)}\right)$
Hypercube	$\Omega(n_2)$	1^*
Symmetric	$\Omega(n_2)$	2^*
Convex, Positive	$\Omega(n_2)$	4
Convex	$\Omega(\max\{m, n_2\})$	$\Omega(m)$

Table 2: Stochasticity and Adaptability Gap for the model with both right hand side and costs uncertain. Here * denotes that the bound is tight.

Outline. In Section 2, we present the bound on the stochasticity gap under symmetric uncertainty sets when only the right hand side is uncertain. We present examples that show that the bound is tight for this case and also that the bound can be arbitrarily large for general non-symmetric uncertainty sets in Sections 2.2 and 2.3 and in Section 2.4, we prove the bound on the stochasticity gap for positive sets. In Section 3, we show that the stochasticity gap can be arbitrarily large when the objective coefficients are uncertain even when the uncertainty set and the probability distribution are both symmetric and there are no integer decision variables.

In Section 4, we present our results on the adaptability gap under symmetric uncertainty sets and

extensions to positive uncertainty sets in the model where only the right hand side is uncertain. We also present a tight example that shows that the bound on the adaptability gap is tight for a symmetric uncertainty set when only the right hand side of the constraints is uncertain and an example that shows that the adaptability gap can be arbitrarily large if the uncertainty set is not symmetric. In Section 5, we prove the bound on the adaptability gap for the model where both cost and right hand side are uncertain. The special case of hypercube uncertainty is presented in Section 5.3, where we show that the adaptability gap is one when the uncertainty set is a hypercube even for a more general model where even the constraint coefficients are allowed to be uncertain.

2. Stochasticity Gap under Right Hand Side Uncertainty. In this section, we consider the robust and stochastic problems: $\Pi_{Rob}(b)$ (cf. (1.2)) and $\Pi_{Stoch}(b)$ (cf. (1.1)) where the right hand side of the constraints is uncertain. We show that the worst-case cost of the optimal solution of $\Pi_{Rob}(b)$ is at most two times the expected cost of an optimal solution of $\Pi_{Stoch}(b)$ if the uncertainty set is symmetric. We also show that the bound is tight for symmetric uncertainty sets and the stochasticity gap can be arbitrarily large if the uncertainty set is not symmetric. We further extend the bound on the stochasticity gap for the case of positive uncertainty sets.

2.1 Symmetric Uncertainty Sets. In this section, we prove that under fairly general conditions, the stochasticity gap: $z_{Rob}(b)/z_{Stoch}(b)$ for the two-stage stochastic problem $\Pi_{Stoch}(b)$ and robust problem $\Pi_{Rob}(b)$, is at most two for symmetric uncertainty sets. In particular, we prove the following main theorem.

THEOREM 2.1 *Let $z_{Stoch}(b)$ be the optimal expected cost of $\Pi_{Stoch}(b)$, and $z_{Rob}(b)$ be the optimal worst-case cost of the corresponding problem $\Pi_{Rob}(b)$ where there are no integer decision variables in the second-stage, i.e., $p_2 = 0$ and the uncertainty set, $\mathcal{I}_b(\Omega)$, is symmetric. Let ω^0 denote the scenario such that $b(\omega^0)$ is the point of symmetry of $\mathcal{I}_b(\Omega)$ and the probability measure μ on the set of scenarios Ω satisfies that,*

$$\mathbb{E}_\mu[b(\omega)] \geq b(\omega^0). \quad (2.1)$$

Then,

$$z_{Rob}(b) \leq 2 \cdot z_{Stoch}(b).$$

Recall that $\mathbb{E}_\mu[\cdot]$ denotes the expectation with respect to μ which is a probability measure on the set of scenarios Ω . Since the uncertainty set $\mathcal{I}_b(\Omega)$ is assumed to be symmetric, there exists a point of symmetry,

$b^0 \in \mathcal{I}_b(\Omega)$. The scenario where the realization of the uncertain right hand side is b^0 , is referred to as ω^0 , and $b(\omega^0) = b^0$. We require that the expected value of the uncertain right hand side vector with respect to the probability measure μ is at least $b(\omega^0)$, i.e.,

$$\mathbb{E}_\mu[b_j(\omega)] \geq b_j(\omega^0), \quad j = 1, \dots, m.$$

For instance, consider the following hypercube uncertainty set:

$$\mathcal{I}_b(\Omega) = \{b \in \mathbb{R}_+^m \mid 0 \leq b_j \leq 1, \quad j = 1, \dots, m\}.$$

and each component, b_j , $j = 1, \dots, m$ takes value uniformly at random between 0 and 1 independent of other components. The point of symmetry of the uncertainty set is $b_j^0 = 1/2$ for all $j = 1, \dots, m$ and it is easy to verify that $\mathbb{E}_\mu[b(\omega)] = b^0$. In fact, (2.1) is satisfied for any symmetric probability measure μ (see Definition 1.4) on a symmetric uncertainty set as we show in the following lemma.

LEMMA 2.1 *Let μ be a symmetric probability measure on the symmetric set $S \subset \mathbb{R}^n$ where $u^0 \in \mathbb{R}^n$ is the point of symmetry of S . Let x be a random vector drawn from S with respect to the measure μ . Then,*

$$\mathbb{E}_\mu[x] = u^0.$$

PROOF. We can write the expectation as follows:

$$\begin{aligned} \mathbb{E}_\mu[x] &= \int_{x \in S} x \, d\mu \\ &= \int_{(2u^0 - y) \in S} (2u^0 - y) \, d\mu \end{aligned} \tag{2.2}$$

$$= \int_{y \in S} (2u^0 - y) \, d\mu \tag{2.3}$$

$$\begin{aligned} &= \int_{y \in S} 2u^0 \, d\mu - \int_{y \in S} y \, d\mu \\ &= 2u^0 - \int_{y \in S} y \, d\mu \\ &= 2u^0 - \mathbb{E}_\mu[x], \end{aligned} \tag{2.4}$$

where (2.2) follows from a change of variables, setting $y = 2u^0 - x$. Equation (2.3) follows the symmetry of S and μ . From (2.4), we have that $\mathbb{E}_\mu[x] = u^0$. \square

For a symmetric uncertainty set, the symmetry of the probability measure is natural in most practical settings and thus, (2.1) which is a weaker condition than the symmetry of μ is not a restrictive assumption. We also generalize the bound on the stochasticity gap to the case where the probability measure does not satisfy (2.1) but satisfies a weaker assumption (see Theorem 2.2).

Before proving Theorem 2.1, we show an interesting geometric property for symmetric sets. Consider any symmetric set $S \subset \mathbb{R}^n$. For each $j = 1, \dots, n$, let

$$x_j^h = \max_{x \in S} x_j, \quad (2.5)$$

$$x_j^l = \min_{x \in S} x_j. \quad (2.6)$$

Consider the following hypercube H ,

$$H = \{x \in \mathbb{R}^n \mid x_j^l \leq x_j \leq x_j^h, j = 1, \dots, n\}. \quad (2.7)$$

LEMMA 2.2 For any hypercube $H' \subset \mathbb{R}^n$,

$$S \subset H' \Rightarrow H \subset H'.$$

Therefore, H is the smallest hypercube such that $S \subset H$.

PROOF. Let $H' = \{x \in \mathbb{R}^n \mid p_j \leq x_j \leq q_j\}$ such that $S \subset H'$. Consider $x \in H$. Suppose $x \notin H'$. Therefore, there exists $j \in \{1, 2, \dots, n\}$ such that,

$$x_j > q_j, \text{ or } x_j < p_j.$$

We know that $x_j^l \leq x_j \leq x_j^h$. Suppose $x_j > q_j$ (the other case is similar). Therefore, $q_j < x_j^h$. Consider

$$\beta = \operatorname{argmax}\{x_j \mid x \in S\}.$$

We know that $\beta_j = x_j^h > q_j$. Thus, $\beta \notin H'$; a contradiction. \square

LEMMA 2.3 Let x^0 denote the center of Hypercube H defined in (2.7), i.e., for all $j = 1, \dots, n$,

$$x_j^0 = \frac{x_j^l + x_j^h}{2}.$$

Then x^0 is the point of symmetry of S . Furthermore, $x \leq 2 \cdot x^0$ for all $x \in S$.

PROOF. Suppose the point of symmetry of S is $u^0 \in S$. Therefore, $(u^0 - z) \in S \Rightarrow (u^0 + z) \in S$ for any $z \in \mathbb{R}^n$. We prove that $u^0 = x^0$. For all $j = 1, \dots, n$, let

$$\alpha^j = \operatorname{argmin}\{x_j \mid x \in S\}, \quad (2.8)$$

$$\beta^j = \operatorname{argmax}\{x_j \mid x \in S\}.$$

Note that $\alpha_j^j = x_j^l$ and $\beta_j^j = x_j^h$ (cf. (2.5)-(2.6), see Figure 2). For any $j \in \{1, \dots, m\}$, $\beta^j = u^0 + z^j$ for some $z^j \in \mathbb{R}^n$. Therefore, $u^0 - z^j \in S$ and,

$$x_j^l \leq u_j^0 - z_j^j = 2u_j^0 - (u_j^0 + z_j^j) = 2u_j^0 - \beta_j^j = 2u_j^0 - x_j^h \Rightarrow \frac{x_j^l + x_j^h}{2} \leq u_j^0.$$

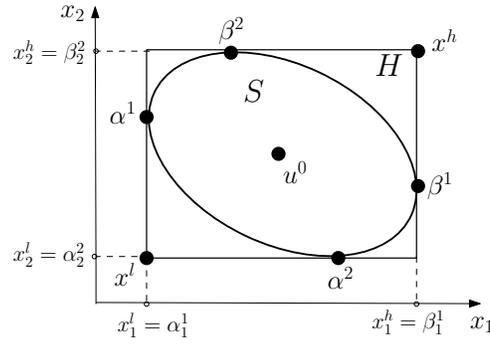


Figure 2: A symmetric set S with point of symmetry u^0 , and the bounding hypercube H . Here x^l and x^h are as defined in (2.5) and (2.6) and $\alpha^1, \alpha^2, \beta^1, \beta^2$ as defined in (2.8).

Also, $\alpha^j = u^0 - y^j$ for some $y^j \in \mathbb{R}^m$ which implies $(u^0 + y^j) \in S$ and,

$$x_j^h \geq u_j^0 + y_j^j = 2u_j^0 - (u_j^0 - y_j^j) = 2u_j^0 - \alpha_j^j = 2u_j^0 - x_j^l \Rightarrow \frac{x_j^l + x_j^h}{2} \geq u_j^0.$$

Therefore,

$$\frac{x_j^l + x_j^h}{2} = u_j^0 = x_j^0, \quad \forall j = 1, \dots, n,$$

which implies that x^0 is the point of symmetry of S .

Consider any $x \in S$. For all $j = 1, \dots, n$,

$$x_j \leq x_j^h \leq (x_j^h + x_j^l) \leq 2 \cdot x_j^0.$$

Therefore, $x \leq 2x^0$ for all $x \in S$. □

PROOF OF THEOREM 2.1. Consider an optimal solution $x^* \in \mathbb{R}^{n_1-p_1} \times \mathbb{Z}_+^{p_1}, y^*(\omega) \in \mathbb{R}_+^{n_2}$ for all $\omega \in \Omega$ for $\Pi_{Stoch}(b)$. Therefore,

$$\begin{aligned} A(2x^*) + B(2y^*(\omega^0)) &= 2(Ax^* + By^*(\omega^0)) \\ &\geq 2b(\omega^0) \end{aligned} \tag{2.9}$$

$$\geq b(\omega), \quad \forall \omega \in \Omega, \tag{2.10}$$

where (2.9) follows from the fact that $(x^*, y^*(\omega^0))$ is a feasible solution for scenario ω^0 . Inequality (2.10) follows as $b(\omega^0)$ is the point of symmetry of $\mathcal{I}_b(\Omega)$ and $b(\omega) \leq 2b(\omega^0)$ for all $b(\omega) \in \mathcal{I}_b(\Omega)$ from Lemma 2.3. Thus, $(2x^*, 2y^*(\omega^0))$ is a feasible solution for $\Pi_{Rob}(b)$, and,

$$z_{Rob}(b) \leq c^T(2x^*) + d^T(2y^*(\omega^0)) = 2 \cdot (c^T x^* + d^T y^*(\omega^0)). \tag{2.11}$$

We know that,

$$Ax^* + By^*(\omega) \geq b(\omega), \quad \forall \omega \in \Omega.$$

If we take the expectation of the above inequality with respect to the probability measure μ , we have,

$$\mathbb{E}_\mu [Ax^* + By^*(\omega)] \geq \mathbb{E}_\mu [b(\omega)] \geq b(\omega^0).$$

Therefore, by the linearity of expectation,

$$\mathbb{E}_\mu [Ax^*] + \mathbb{E}_\mu [By^*(\omega)] = Ax^* + B\mathbb{E}_\mu [y^*(\omega)] \geq b(\omega^0).$$

Therefore, $\mathbb{E}_\mu [y^*(\omega)]$ is a feasible solution for scenario ω^0 which implies that,

$$d^T y^*(\omega^0) \leq d^T \mathbb{E}_\mu [y^*(\omega)], \quad (2.12)$$

as $y^*(\omega^0)$ is an optimal solution for scenario ω^0 . Also,

$$\begin{aligned} z_{Stoch}(b) &= c^T x^* + \mathbb{E}_\mu [d^T y^*(\omega)] \\ &= c^T x^* + d^T \mathbb{E}_\mu [y^*(\omega)] \end{aligned} \quad (2.13)$$

$$\geq c^T x^* + d^T y^*(\omega^0) \quad (2.14)$$

$$\geq \frac{z_{Rob}(b)}{2}, \quad (2.15)$$

where (2.13) follows from the linearity of expectation and (2.14) follows from (2.12). Inequality (2.15) follows from (2.11). \square

Note that while the stochasticity gap is bounded when there are some integer decision variables in the first stage, our bound does not hold in general if there are binary decision variables in the model instead of integer decision variables since we construct a feasible solution to $\Pi_{Rob}(b)$ by scaling the feasible solution for scenario ω^0 by a factor two. We require the symmetry of the uncertainty set in proving that the scaled solution $(2x^*, 2y^*(\omega^0))$ corresponding to the scenario ω^0 , is feasible for $\Pi_{Rob}(b)$ and the condition on the probability measure μ is required to prove that the cost of the fixed solution $(2x^*, 2y^*(\omega^0))$ is not much worse than the optimal expected cost of $\Pi_{Stoch}(b)$. As noted earlier, the assumptions on the uncertainty set and the probability measure are not very restrictive and hold in many natural settings. Furthermore, the bound on the stochasticity gap generalizes even if the Condition 2.1 on the probability measure does not hold, although the bound might be worse. In particular, we prove the following theorem that is a generalization of Theorem 2.1.

THEOREM 2.2 *Let $z_{Stoch}(b)$ be the optimal expected cost of $\Pi_{Stoch}(b)$, and $z_{Rob}(b)$ be the optimal worst-case cost of $\Pi_{Rob}(b)$ where there are no integer decision variables in the second-stage, i.e., $p_2 = 0$ and the uncertainty set, $\mathcal{I}_b(\Omega)$, is symmetric. Let ω^0 denote the scenario such that $b(\omega^0)$ is the point of symmetry of $\mathcal{I}_b(\Omega)$ and suppose for some $\delta > 0$, $\delta \cdot b(\omega^0) \in \mathcal{I}_b(\Omega)$ and the probability measure μ on the set of*

scenarios Ω satisfies that,

$$\mathbb{E}_\mu[b(\omega)] \geq \delta \cdot b(\omega^0). \quad (2.16)$$

Then,

$$z_{Rob}(b) \leq \frac{2}{\delta} \cdot z_{Stoch}(b).$$

PROOF. Let $\tilde{\omega}$ denote the scenario such that $b(\tilde{\omega}) = \delta \cdot b(\omega^0)$. Consider an optimal solution $x^* \in \mathbb{R}^{n_1-p_1} \times \mathbb{Z}_+^{p_1}$, $y^*(\omega) \in \mathbb{R}_+^{n_2}$ for all $\omega \in \Omega$ for $\Pi_{Stoch}(b)$. We show that the solution $(\frac{2x^*}{\delta}, \frac{2y^*(\tilde{\omega})}{\delta})$ is a feasible solution for $\Pi_{Rob}(b)$.

$$\begin{aligned} A \left(\frac{2x^*}{\delta} \right) + B \left(\frac{2y^*(\tilde{\omega})}{\delta} \right) &= \frac{2}{\delta} \cdot (Ax^* + By^*(\tilde{\omega})) \\ &\geq \frac{2}{\delta} \cdot b(\tilde{\omega}) \end{aligned} \quad (2.17)$$

$$= \frac{2}{\delta} \cdot \delta \cdot b(\omega^0) \quad (2.18)$$

$$\begin{aligned} &= 2 \cdot b(\omega^0) \\ &\geq b(\omega), \quad \forall \omega \in \Omega, \end{aligned} \quad (2.19)$$

where (2.17) follows from the feasibility of $(x^*, y^*(\tilde{\omega}))$ for scenario $\tilde{\omega}$ and (2.18) follows as $b(\tilde{\omega}) = \delta \cdot b(\omega^0)$ by definition. Inequality (2.19) follows from the fact that $b(\omega^0)$ is the point of symmetry of $\mathcal{I}_b(\Omega)$ and from Lemma 2.3, $b(\omega) \leq 2b(\omega^0)$ for all $\omega \in \Omega$. Therefore,

$$z_{Rob}(b) \leq c^T \left(\frac{2x^*}{\delta} \right) + d^T \left(\frac{2y^*(\tilde{\omega})}{\delta} \right) = \frac{2}{\delta} (c^T x^* + d^T y^*(\tilde{\omega})). \quad (2.20)$$

We know that,

$$Ax^* + By^*(\omega) \geq b(\omega), \quad \forall \omega \in \Omega.$$

If we take the expectation of the above inequality with respect to the probability measure μ , we have,

$$\begin{aligned} Ax^* + B\mathbb{E}_\mu[y^*(\omega)] &\geq \mathbb{E}_\mu[b(\omega)] \\ &\geq \delta \cdot b(\omega^0) \end{aligned} \quad (2.21)$$

$$= b(\tilde{\omega}), \quad (2.22)$$

where (2.21) follows from (2.16) and (2.22) follows from the definition of scenario $\tilde{\omega}$. Therefore, $\mathbb{E}_\mu[y^*(\omega)]$ is a feasible solution for scenario $\tilde{\omega}$ which implies that,

$$d^T y^*(\tilde{\omega}) \leq d^T \mathbb{E}_\mu[y^*(\omega)], \quad (2.23)$$

as $y^*(\tilde{\omega})$ is an optimal solution for scenario $\tilde{\omega}$. Now,

$$\begin{aligned} z_{Stoch}(b) &= c^T x^* + \mathbb{E}_\mu [d^T y^*(\omega)] \\ &= c^T x^* + d^T \mathbb{E}_\mu [y^*(\omega)] \\ &\geq c^T x^* + d^T y^*(\tilde{\omega}) \end{aligned} \tag{2.24}$$

$$\geq \frac{\delta}{2} \cdot z_{Rob}(b), \tag{2.25}$$

where (2.24) follows from (2.23) and (2.25) follows from (2.20). \square

The bound of two on the stochasticity gap can be further improved if $0 \notin \mathcal{I}_b(\Omega)$. Let $b^l, b^h \in \mathbb{R}_+^m$ be such that for all $j = 1, \dots, m$,

$$b_j^l = \min_{b \in \mathcal{I}_b(\Omega)} b_j, \quad b_j^h = \max_{b \in \mathcal{I}_b(\Omega)} b_j. \tag{2.26}$$

If $b^l \geq \rho \cdot b^h$ for some $\rho \geq 0$, we can improve the bound on the stochasticity gap in Theorem 2.2 by a factor of $(1 + \rho)$ as:

$$b(\omega^0) = \frac{b^l + b^h}{2} \geq \frac{\rho b^h + b^h}{2} = (1 + \rho) \cdot \frac{b^h}{2}.$$

Therefore, if we scale the optimal solution of $\Pi_{Stoch}(b)$ for scenario ω^0 by a factor $2/(1 + \rho)$ (instead of scaling by a factor 2), we obtain a feasible solution for $\Pi_{Rob}(b)$, since:

$$\begin{aligned} \frac{2}{1 + \rho} (Ax^* + By^*(\omega^0)) &\geq \frac{2}{1 + \rho} \cdot b(\omega^0) \\ &= \frac{2}{1 + \rho} \cdot \frac{b^l + b^h}{2} \\ &\geq \frac{2}{1 + \rho} \cdot \frac{b^h \cdot (1 + \rho)}{2} \\ &= b^h \\ &\geq b(\omega), \quad \forall \omega \in \Omega. \end{aligned}$$

Therefor, we have the following theorem.

THEOREM 2.3 *Let $z_{Stoch}(b)$ be the optimal expected cost of $\Pi_{Stoch}(b)$, and $z_{Rob}(b)$ be the optimal worst-case cost of $\Pi_{Rob}(b)$ where there are no integer decision variables in the second-stage, i.e., $p_2 = 0$ and the uncertainty set, $\mathcal{I}_b(\Omega)$, is symmetric. Let ω^0 denote the scenario such that $b(\omega^0)$ is the point of symmetry of $\mathcal{I}_b(\Omega)$ and suppose for some $\delta > 0$, $\delta \cdot b(\omega^0) \in \mathcal{I}_b(\Omega)$ and the probability measure μ on the set of scenarios Ω satisfies Condition (2.16). Let $b^l, b^h \in \mathbb{R}_+^m$ be as defined in (2.26) and suppose $b^l \geq \rho \cdot b^h$ for some $\rho \geq 0$. Then*

$$z_{Rob}(b) \leq \left(\frac{2}{\delta(1 + \rho)} \right) \cdot z_{Stoch}(b).$$

Note that the condition that $b^l \geq \rho b^h$ is trivially satisfied for $\rho = 0$ as $\mathcal{I}_b(\Omega) \subset \mathbb{R}_+^m$ and $b^l \geq 0$. For $\rho = 0$, we get back the bound of Theorem 2.2.

In Theorems 2.1, 2.2, and 2.3, we prove that an optimal solution to the robust problem $\Pi_{Rob}(b)$ is a good approximation for the two-stage stochastic problem $\Pi_{Stoch}(b)$. We next show that an optimal solution to $\Pi_{Rob}(b)$ can be computed efficiently by solving a single mixed integer optimization problem whose size does not depend on the uncertainty set or the number of worst case scenarios. In particular, we prove the following theorem.

THEOREM 2.4 *Let $b^h \in \mathbb{R}_+^m$ be such that, for each $j = 1, \dots, m$,*

$$b_j^h = \max_{b(\omega) \in \mathcal{I}_b(\Omega)} b_j(\omega).$$

Then the optimal solution to $\Pi_{Rob}(b)$ can be obtained by solving the following mixed integer problem, Π :

$$\begin{aligned} z(\Pi) = \min \quad & c^T x + d^T y \\ & Ax + By \geq b^h, \\ & x \in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1} \\ & y \in \mathbb{R}_+^{n_2}. \end{aligned}$$

PROOF. Consider an optimal solution (\hat{x}, \hat{y}) to Π . Clearly, (\hat{x}, \hat{y}) is a feasible solution to $\Pi_{Rob}(b)$ as,

$$A\hat{x} + B\hat{y} \geq b^h \geq b(\omega), \quad \forall \omega \in \Omega.$$

Therefore, $z_{Rob}(b) \leq z(\Pi)$.

Now, consider an optimal solution (x^*, y^*) to $\Pi_{Rob}(b)$. We show that it is a feasible solution to Π . Suppose not. Therefore, there exists $j \in \{1, \dots, m\}$ such that,

$$(Ax^* + By^*)_j < b_j^h.$$

Let,

$$\beta^j = \operatorname{argmax}\{b_j(\omega) \mid b(\omega) \in \mathcal{I}_b(\Omega)\}.$$

Therefore,

$$Ax^* + By^* \geq \beta^j \Rightarrow (Ax^* + By^*)_j \geq \beta_j^j = b_j^h,$$

since β^j is a possible realization of the uncertain right hand side and $\beta_j^j = b_j^h$ (by definition); a contradiction. Therefore, $z(\Pi) \leq z_{Rob}(b)$. \square

Note that the problem Π has only m constraints and $(n_1 + n_2)$ decision variables and thus, the size of Π does not depend on the number of scenarios. Therefore, a good approximate solution to $\Pi_{Stoch}(b)$

can be computed efficiently by solving a single mixed integer optimization problem whose size does not depend on the uncertainty set and even without the knowledge of the probability distribution, as long as it satisfies (2.1) or (2.16) for example.

2.2 A Tight Stochasticity Gap Example for Symmetric Uncertainty Sets.

Here, we present an instance of $\Pi_{Rob}(b)$ and $\Pi_{Stoch}(b)$ where the uncertainty set $\mathcal{I}_b(\Omega)$ is symmetric, such that $z_{Rob}(b) = 2z_{Stoch}(b)$. Consider the following instance where $n_1 = 0, n_2 = n, m = n, A = 0, c = 0, d = (1, \dots, 1), B = I_n$ (here I_n denotes a $n \times n$ identity matrix). Let Ω denote the set of uncertain scenarios and the uncertainty set,

$$\mathcal{I}_b(\Omega) = \{b \in \mathbb{R}_+^n \mid 0 \leq b_j \leq 1, j = 1, \dots, n\}.$$

Also, each $b_j, j = 1, \dots, n$ takes a value uniformly at random between 0 and 1 and independent of other components and the probability measure μ is defined according to this distribution. Therefore,

$$\mathbb{E}_\mu[b(\omega)] = (\mathbb{E}_\mu[b_1(\omega)], \dots, \mathbb{E}_\mu[b_n(\omega)]) = \left(\frac{1}{2}, \dots, \frac{1}{2}\right).$$

Note that $\mathcal{I}_b(\Omega)$ is a hypercube and thus, a symmetric set in the non-negative orthant with $b^0 = (1/2, \dots, 1/2)$ as the point of symmetry. Also, $\mathbb{E}_\mu[b(\omega)] = b^0$. Therefore, the uncertainty set $\mathcal{I}_b(\Omega)$, and the probability measure μ , satisfy the assumptions in Theorem 2.1.

THEOREM 2.5 For $\Pi_{Rob}(b)$ and $\Pi_{Stoch}(b)$ defined above,

$$z_{Rob}(b) = 2 \cdot z_{Stoch}(b).$$

PROOF. Consider any feasible solution (y_1, \dots, y_n) for $\Pi_{Rob}(b)$. For any $(b_1, \dots, b_n) \in \mathcal{I}_b(\Omega)$, we require that $y_j \geq b_j$ for all $j = 1, \dots, n$. We know that $(1, \dots, 1) \in \mathcal{I}_b(\Omega)$. Therefore, $y_j \geq 1$ for all $j = 1, \dots, n$ which implies that $z_{Rob}(b) = n$. Now, consider $\Pi_{Stoch}(b)$ and consider the solution $\hat{y}(\omega) = b(\omega)$. Clearly, the solution $\hat{y}(\omega)$ for all $\omega \in \Omega$ is feasible. Therefore,

$$\begin{aligned} z_{Stoch}(b) &\leq \mathbb{E}_\mu[d^T y(\omega)] \\ &= \sum_{j=1}^n \mathbb{E}_\mu[\hat{y}_j(\omega)] \\ &= \sum_{j=1}^n \mathbb{E}_\mu[b_j(\omega)] \end{aligned} \tag{2.27}$$

$$\begin{aligned} &= \sum_{j=1}^n \frac{1}{2} \\ &= \frac{n}{2}, \end{aligned} \tag{2.28}$$

where (2.27) follows from the fact that $y(\omega) = b(\omega)$ for all $\omega \in \Omega$. Also, from Theorem 2.1, we have that $z_{Rob}(b) \leq 2 \cdot z_{Stoch}(b)$. Therefore, $z_{Rob}(b) = 2 \cdot z_{Adapt}(b)$. \square

2.3 A Large Stochasticity Gap Example for Non-Symmetric Uncertainty

Sets. We show that if the uncertainty set is not symmetric, then the stochasticity gap can be arbitrarily large. Therefore, the assumption of symmetry on the uncertainty set is almost necessary for the stochasticity gap to be bounded.

THEOREM 2.6 *Consider an instance of problem $\Pi_{Rob}(b)$ and $\Pi_{Stoch}(b)$, where, $n_1 = 0$, $n_2 = m = n \geq 3$, $A = 0$, $c = 0$, $d = (0, 0, \dots, 0, 1)$ (an n dimensional vector where only the n^{th} coordinate is one and all others are zero) and $B = I_n$ (I_n is $n \times n$ identity matrix). Let Ω denote the set of scenarios and the uncertainty set,*

$$\mathcal{I}_b(\Omega) = \left\{ b \mid \sum_{j=1}^n b_j \leq 1, b \geq 0 \right\}. \quad (2.29)$$

Let μ be the uniform probability measure on $\mathcal{I}_b(\Omega)$, i.e., for any $\mathcal{S} \subset \Omega$,

$$\mu(\mathcal{S}) = \frac{\text{volume}(\{b(\omega) \mid \omega \in \mathcal{S}\})}{\text{volume}(\mathcal{I}_b(\Omega))}.$$

Then,

$$z_{Rob}(b) \geq (n + 1) \cdot z_{Stoch}(b).$$

We first show that the uncertainty set $\mathcal{I}_b(\Omega)$ is not symmetric.

LEMMA 2.4 *The uncertainty set $\mathcal{I}_b(\Omega)$ defined in (2.29) is not symmetric for $n \geq 2$.*

PROOF. For the sake of contradiction, suppose $\mathcal{I}_b(\Omega)$ is symmetric and let u^0 denote the center of $\mathcal{I}_b(\Omega)$. Since $0 = (u^0 - u^0) \in \mathcal{I}_b(\Omega)$, then $(u^0 + u^0) = 2u^0 \in \mathcal{I}_b(\Omega)$. Therefore,

$$\sum_{j=1}^n 2u_j^0 \leq 1 \Rightarrow u_j^0 \leq \frac{1}{2}, \quad j = 1, \dots, n.$$

Let e_j denote the j^{th} unit vector in \mathbb{R}_+^n where only the j^{th} coordinate is one and all others are zero. Now,

$$e_j = u^0 + (e_j - u^0) \in \mathcal{I}_b(\Omega) \Rightarrow x^j = u^0 - (e_j - u^0) \in \mathcal{I}_b(\Omega) \quad \forall j = 1, \dots, n.$$

If there exists $j \in \{1, \dots, n\}$ such that $u_j^0 < \frac{1}{2}$, then

$$x_j^j = u_j^0 - (1 - u_j^0) = 2u_j^0 - 1 < 0,$$

a contradiction. Therefore, $u_j^0 = \frac{1}{2}$ for all $j = 1, \dots, n$. Now, $2u^0 = (1, \dots, 1)$ which is a contradiction as $2u^0 \in \mathcal{I}_b(\Omega)$ but $(1, \dots, 1) \notin \mathcal{I}_b(\Omega)$. Therefore, $\mathcal{I}_b(\Omega)$ is not symmetric. \square

PROOF OF THEOREM 2.6. Consider the robust problem $\Pi_{Rob}(b)$:

$$\begin{aligned} z_{Rob}(b) &= \min_y \max_{\omega \in \Omega} d^T y \\ I_n y &\geq b(\omega), \forall \omega \in \Omega \\ y &\geq 0. \end{aligned}$$

Since $e_j \in \mathcal{I}_b(\Omega)$ for all $j = 1, \dots, n$, for any feasible solution $y \in \mathbb{R}_+^n$, $I_n y \geq e_j$ for all $j = 1, \dots, n$. Therefore, $y_j \geq 1$ for all $j = 1, \dots, n$ which implies,

$$z_{Rob}(b) \geq 1.$$

Now, consider $\Pi_{Stoch}(b)$:

$$\begin{aligned} z_{Stoch}(b) &= \min_y \mathbb{E}_\mu[d^T y(\omega)] \\ I_n y(\omega) &\geq b(\omega), \forall \omega \in \Omega \\ y(\omega) &\geq 0. \end{aligned}$$

Consider the solution $\hat{y}(\omega) = b(\omega)$ for all $\omega \in \Omega$. Clearly, the solution $\hat{y}(\omega)$ is feasible, as $I_n \hat{y}(\omega) = b(\omega)$ for all $\omega \in \Omega$. Now,

$$\begin{aligned} z_{Stoch}(b) &\leq \mathbb{E}_\mu[d^T \hat{y}(\omega)] \\ &= \mathbb{E}_\mu[\hat{y}_n(\omega)] \\ &= \mathbb{E}_\mu[b_n(\omega)] \end{aligned} \tag{2.30}$$

$$= \frac{\int_{x_1=0}^1 \int_{x_2=0}^{1-x_1} \dots \int_{x_n=0}^{1-(x_1+x_2+\dots+x_{n-1})} x_n dx_n dx_{n-1} \dots dx_1}{\text{volume}(\mathcal{I}_b(\Omega))} \tag{2.31}$$

$$= \frac{\int_{x_1=0}^1 \int_{x_2=0}^{1-x_1} \dots \int_{x_n=0}^{1-(x_1+x_2+\dots+x_{n-1})} x_n dx_n dx_{n-1} \dots dx_1}{\int_{x_1=0}^1 \int_{x_2=0}^{1-x_1} \dots \int_{x_n=0}^{1-(x_1+x_2+\dots+x_{n-1})} dx_n dx_{n-1} \dots dx_1} \tag{2.32}$$

$$\begin{aligned} &= \frac{1/(n+1)!}{1/n!} \\ &= \frac{1}{n+1} \end{aligned}$$

where (2.30) follows as $\hat{y}(\omega) = b(\omega)$ for all $\omega \in \Omega$, and the integrals in the numerator and the denominator of (2.32) follow from standard computation. Therefore, $z_{Rob}(b) \geq (n+1) \cdot z_{Stoch}(b)$. \square

The example in Theorem 2.6 shows that if the uncertainty set is not symmetric, then the optimal cost of $\Pi_{Rob}(b)$ can be arbitrarily large as compared to the optimal expected cost of $\Pi_{Stoch}(b)$. In fact,

this example also shows that the optimal cost of $\Pi_{Adapt}(b)$ can be arbitrarily large as compared to the optimal cost of $\Pi_{Stoch}(b)$. Consider the adaptive problem $\Pi_{Adapt}(b)$ for the instance in Theorem 2.6 and consider the scenario ω' where $b_n(\omega') = 1$ and $b_j(\omega') = 0, j = 1, \dots, n - 1$. Let $y^*(\omega)$ for all $\omega \in \Omega$ be an optimal solution for $\Pi_{Adapt}(b)$. Therefore, for scenario ω' , $I_n y^*(\omega') \geq b(\omega')$. Thus,

$$y_j^*(\omega') = \begin{cases} 1, & \text{if } j = n, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore,

$$z_{Adapt}(b) = \max_{\omega \in \Omega} d^T y^*(\omega) \geq d^T y^*(\omega') = 1,$$

which implies that $z_{Adapt}(b) \geq (n + 1) \cdot z_{Stoch}(b)$. The large gap between the optimal values of $\Pi_{Stoch}(b)$ and $\Pi_{Adapt}(b)$ indicates that allowing a fully-adaptable solution is not the only reason for the large gap between $\Pi_{Stoch}(b)$ and $\Pi_{Rob}(b)$. Instead, the gap is large because the objective function is an expectation in $\Pi_{Stoch}(b)$ while it is the worst case in both $\Pi_{Rob}(b)$ and $\Pi_{Adapt}(b)$.

2.4 Stochasticity Gap for Positive Uncertainty Sets. In this section, we prove a bound on the stochasticity gap when the uncertainty set is not necessarily symmetric. In view of the large stochasticity gap example for a non-symmetric uncertainty set, it is clear that we need additional restrictions on the uncertainty set for the stochasticity gap to be bounded. We prove that the stochasticity gap is at most 2 if the uncertainty set is convex and positive but not necessarily symmetric. Recall that a convex set $P \subset \mathbb{R}_+^n$ is *positive* if there is a convex symmetric $S \subset \mathbb{R}_+^n$ such that $P \subset S$ and the point of symmetry of S belongs to P .

THEOREM 2.7 *Consider the robust and stochastic problems: $\Pi_{Rob}(b)$ and $\Pi_{Stoch}(b)$. Suppose the uncertainty set, $\mathcal{I}_b(\Omega)$, is convex and positive. Let $\mathcal{B} \subset \mathbb{R}_+^n$ be a symmetric uncertainty set containing $\mathcal{I}_b(\Omega)$ such that the point of symmetry, b^0 of \mathcal{B} is contained in $\mathcal{I}_b(\Omega)$. If $\mathbb{E}_\mu[b(\omega)] \geq b^0$, then,*

$$z_{Rob}(b) \leq 2 \cdot z_{Stoch}(b).$$

PROOF. Let ω^0 be the scenario such that $b(\omega^0) = b^0$. Consider an optimal solution $x^*, y^*(\omega), \forall \omega \in \Omega$ to $\Pi_{Stoch}(b)$. Therefore,

$$Ax^* + By^*(\omega) \geq b(\omega), \forall \omega \in \Omega.$$

As in the proof of Theorem 2.1, we take expectation of both sides and we have,

$$\mathbb{E}_\mu[Ax^* + By^*(\omega)] \geq \mathbb{E}_\mu[b(\omega)] \geq b(\omega^0).$$

Therefore, by the linearity of expectation,

$$\mathbb{E}_\mu [Ax^*] + \mathbb{E}_\mu [By^*(\omega)] = Ax^* + B\mathbb{E}_\mu [y^*(\omega)] \geq b(\omega^0).$$

Therefore, $\mathbb{E}_\mu [y^*(\omega)]$ is a feasible solution for scenario ω^0 which implies that,

$$c^T x^* + d^T y^*(\omega^0) \leq c^T x^* + d^T \mathbb{E}_\mu [y^*(\omega)]. \quad (2.33)$$

We next show that the solution $(2x^*, 2y^*(\omega^0))$ is a feasible solution for $\Pi_{Rob}(b)$. We have,

$$\begin{aligned} A(2x^*) + B(2y^*(\omega^0)) &\geq 2b(\omega^0) \\ &\geq b(\omega), \forall \omega \in \Omega, \end{aligned} \quad (2.34)$$

where (2.34) follows from the fact that $b(\omega^0)$ is the center of the symmetric set \mathcal{B} that contains $\mathcal{I}_b(\Omega)$.

Therefore, $b(\omega) \leq 2b(\omega^0)$ for all $\omega \in \Omega$. Hence,

$$\begin{aligned} z_{Rob}(b) &\leq c^T(2x^*) + d^T(2y^*(\omega^0)) \\ &= 2 \cdot (c^T x^* + d^T y^*(\omega^0)) \\ &\leq 2 \cdot (c^T x^* + d^T \mathbb{E}_\mu [y^*(\omega)]) \\ &= 2 \cdot z_{Stoch}(b), \end{aligned} \quad (2.35)$$

where (2.35) follows from (2.33). □

3. Stochasticity Gap under Cost and Right Hand Side Uncertainty. In this section, we show that the stochasticity gap can be arbitrarily large if we consider both cost and right hand side uncertainty even if the uncertainty set and the probability distribution are both symmetric and there are no integer decision variables. In fact, we construct an example with no right hand side uncertainty, no integer decision variables and a single constraint such that the stochasticity gap is arbitrarily large.

THEOREM 3.1 *Consider the following instances of $\Pi_{Stoch}(b, d)$ and $\Pi_{Rob}(b, d)$ where $n_1 = 0, n_2 = n, p_2 = 0, m = 1$ and $c = 0, A = 0$ and $B = [1, 1, \dots, 1] \in \mathbb{R}^{1 \times n}$. Let the uncertainty set $\mathcal{I}_{(b,d)}(\Omega) \subset \mathbb{R}_+^{n+1}$ be given by:*

$$\mathcal{I}_{(b,d)}(\Omega) = \{(b(\omega), d(\omega)) \mid b(\omega) = 1, 0 \leq d_j(\omega) \leq 1, \forall j = 1, \dots, n, \forall \omega \in \Omega\},$$

and each d_j is distributed uniformly at random between 0 and 1 and independent of other coefficients.

Then,

$$z_{Rob}(b, d) \geq (n + 1) \cdot z_{Stoch}(b, d).$$

PROOF. Note that the uncertainty set is a hypercube and thus, symmetric. Let ω^0 denote the scenario corresponding to the point of symmetry of the uncertainty set. Therefore, $d(\omega^0) = (1/2, \dots, 1/2)$

and $b(\omega^0) = 1$. Also, $\mathbb{E}_\mu[d(\omega)] = (1/2, \dots, 1/2) = d(\omega^0)$ and thus, the probability distribution also satisfies (2.1). Consider an optimal solution \hat{y} to $\Pi_{Rob}(b, d)$. Therefore, $(\hat{y}_1 + \dots + \hat{y}_n) \geq 1$, and,

$$z_{Rob}(b, d) = \max_{\omega \in \Omega} (d(\omega))^T \hat{y} \geq (1, \dots, 1)^T \hat{y} \geq 1,$$

since there is a scenario ω such that $d_j(\omega) = 1$ for all $j = 1, \dots, n$. On the other hand, we show that $z_{Stoch}(b, d) \leq 1/(n+1)$. Consider the following solution $\tilde{y}(\omega)$ for all $\omega \in \Omega$ for $\Pi_{Stoch}(b, d)$ where for all scenarios $\omega \in \Omega$ and $j = 1, \dots, n$,

$$\tilde{y}_j(\omega) = \begin{cases} 1, & \text{if } d_j(\omega) = \min(d_1(\omega), \dots, d_n(\omega)), \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to observe that $\tilde{y}(\omega)$ for all $\omega \in \Omega$ is a feasible solution for $\Pi_{Stoch}(b, d)$. Therefore,

$$\begin{aligned} z_{Stoch}(b, d) &\leq \mathbb{E}_\mu[(d(\omega))^T \tilde{y}(\omega)] \\ &= \mathbb{E}_\mu[\min(d_1(\omega), \dots, d_n(\omega))] \end{aligned} \tag{3.1}$$

$$= \frac{1}{n+1}, \tag{3.2}$$

where (3.1) follows from the construction of $\tilde{y}(\omega)$ which implies $(d(\omega))^T \tilde{y}(\omega) = \min(d_1(\omega), \dots, d_n(\omega))$ for all $\omega \in \Omega$. Inequality (3.2) follows from the computation of expected value of the minimum of n independent random variables each uniformly random between 0 and 1. Therefore, $z_{Rob}(b, d) \geq (n+1) \cdot z_{Stoch}(b, d)$. \square

4. Adaptability Gap under Right Hand Side Uncertainty. In this section, we consider the robust and adaptable problems: $\Pi_{Rob}(b)$ (cf. (1.2)) and $\Pi_{Adapt}(b)$ (cf. (1.3)) and show that the worst-case cost of the optimal solution of $\Pi_{Rob}(b)$ is at most two times the worst case cost of an optimal adaptable solution of $\Pi_{Adapt}(b)$ if the uncertainty set is symmetric. Therefore, the adaptability gap is at most two under symmetric right hand side uncertainty. We also show that the bound is tight for symmetric uncertainty sets and the adaptability gap can be arbitrarily large if the uncertainty set is not symmetric as in the case of the stochasticity gap under right hand side uncertainty.

4.1 Adaptability Gap for Symmetric Uncertainty Sets. We show that if the uncertainty set $\mathcal{I}_b(\Omega)$ is symmetric, the adaptability gap is at most two. The bound on the adaptability gap holds even when there are integer decision variables in the second stage unlike the case of the stochasticity gap where integer decision variables are allowed only in the first stage. However, the adaptability gap bound does not hold if there are binary decision variables in the model.

Let us first consider the simpler case where there are no integer decision variables in the second stage. The bound on the adaptability gap in this case follows directly from the bound on the stochasticity gap,

as for any probability measure μ :

$$z_{Stoch}(b) \leq z_{Adapt}(b) \leq z_{Rob}(b).$$

Now, consider a measure μ that satisfies Condition 2.1 in Theorem 2.1, i.e., $\mathbb{E}_\mu[b(\omega)] \geq b(\omega^0)$ where $b(\omega^0)$ is the point of symmetry of $\mathcal{I}_b(\Omega)$. Clearly, $\mathbb{E}_{\mu'}[b(\omega)] = b(\omega^0)$. From Theorem 2.1,

$$z_{Rob}(b) \leq 2 \cdot z_{Stoch}(b),$$

which implies that,

$$z_{Stoch}(b) \leq z_{Adapt}(b) \leq z_{Rob}(b) \leq 2 \cdot z_{Stoch}(b).$$

We prove the bound of 2 on the adaptability gap for the model that allows integer decision variables in the second stage. In particular, we have the following theorem.

THEOREM 4.1 *Let $z_{Rob}(b)$ denote the optimal worst-case cost of $\Pi_{Rob}(b)$ and $z_{Adapt}(b)$ denote the optimal worst-case cost of $\Pi_{Adapt}(b)$ where the uncertainty set $\mathcal{I}_b(\Omega) \subset \mathbb{R}_+^m$ is symmetric. Then,*

$$z_{Rob}(b) \leq 2 \cdot z_{Adapt}(b).$$

PROOF. Let ω^0 denote the scenario such that $b(\omega^0)$ is the point of symmetry of $\mathcal{I}_b(\Omega)$. Let $x^*, y^*(\omega), \forall \omega \in \Omega$ be an optimal solution for $\Pi_{Adapt}(b)$. Then,

$$\begin{aligned} z_{Adapt}(b) &= c^T x^* + \max_{\omega \in \Omega} d^T y^*(\omega) \\ &\geq c^T x^* + d^T y^*(\omega^0). \end{aligned} \tag{4.1}$$

Also,

$$\begin{aligned} A(2x^*) + B(2y^*(\omega^0)) &= 2(Ax^* + By^*(\omega^0)) \\ &\geq 2b(\omega^0) \end{aligned} \tag{4.2}$$

$$\geq b(\omega), \forall \omega \in \Omega, \tag{4.3}$$

where (4.2) follows from the feasibility of the solution $(x^*, y^*(\omega^0))$ for scenario ω^0 . Inequality (4.3) follows from the fact that $b(\omega^0)$ is the point of symmetry of $\mathcal{I}_b(\Omega)$ and from Lemma 2.3, we have that $b(\omega) \leq 2b(\omega^0)$ for all $\omega \in \Omega$. Therefore, $(2x, 2y(\omega^0))$ is a feasible solution for $\Pi_{Rob}(b)$, and,

$$\begin{aligned} z_{Rob}(b) &\leq c^T(2x^*) + \max_{\omega \in \Omega} d^T(2y^*(\omega^0)) \\ &\leq 2 \cdot (c^T x^* + d^T y^*(\omega^0)) \\ &\leq 2 \cdot z_{Adapt}(b), \end{aligned} \tag{4.4}$$

where (4.4) follows from (4.1). □

In fact, we prove a stronger bound on the adaptability gap similar to the bound on the stochasticity gap in Theorem 2.3. We have the following theorem.

THEOREM 4.2 *Let $z_{Rob}(b)$ be the optimal objective value for $\Pi_{Rob}(b)$ and $z_{Adapt}(b)$ be the optimal objective value for $\Pi_{Adapt}(b)$ where the uncertainty set $\mathcal{I}_b(\Omega) \subset \mathbb{R}_+^m$ is symmetric. Let $b^l, b^h \in \mathbb{R}_+^m$ be as defined in (2.26) and $b^l \geq \rho \cdot b^h$ for some $\rho \geq 0$, then*

$$z_{Rob}(b) \leq \left(\frac{2}{1 + \rho} \right) \cdot z_{Adapt}(b).$$

4.2 A Tight Adaptability Gap Example for Symmetric Uncertainty Sets.

We show that the bound of two on the adaptability gap under symmetric right hand side uncertainty like the bound on the stochasticity gap.

Consider the following instance where $n_1 = 0, n_2 = 2, m = 2, A = 0, c = 0, d = (1, 1), B = I_2$ (here I_2 denotes a 2×2 identity matrix). Let Ω denote the set of uncertain scenarios and the uncertainty set,

$$\mathcal{I}_b(\Omega) = \{b \in \mathbb{R}_+^2 \mid b_1 + b_2 = 1\}.$$

Let us first show that $\mathcal{I}_b(\Omega)$ is symmetric.

LEMMA 4.1 *The uncertainty set $\mathcal{I}(\Omega)$ is symmetric with center $u^0 = (\frac{1}{2}, \frac{1}{2})$.*

PROOF. Consider any $z \in \mathbb{R}^2$ such that $(u^0 + z) \in \mathcal{I}(\Omega)$. Therefore,

$$\left(\frac{1}{2} + z_1 \right) + \left(\frac{1}{2} + z_2 \right) = 1 \Rightarrow z_1 + z_2 = 0.$$

Also,

$$0 \leq \left(\frac{1}{2} + z_j \right) \leq 1 \Rightarrow -\frac{1}{2} \leq z_j \leq \frac{1}{2}, \quad j = 1, 2.$$

Therefore,

$$0 \leq \frac{1}{2} - z_j \leq 1, \quad j = 1, 2,$$

and

$$\left(\frac{1}{2} - z_1 \right) + \left(\frac{1}{2} - z_2 \right) = 1 - (z_1 + z_2) = 1 \Rightarrow (u^0 - z) \in \mathcal{I}(\Omega),$$

as $z_1 + z_2 = 0$. □

THEOREM 4.3 *For the robust and adaptable problems $\Pi_{Rob}(b)$ and $\Pi_{Adapt}(b)$ in the above instance,*

$$z_{Rob}(b) = 2 \cdot z_{Adapt}(b).$$

PROOF. Consider any feasible solution (y_1, y_2) for $\Pi_{Rob}(b)$. For any $(b_1, b_2) \in \mathbb{R}_+^2$ such that $b_1 + b_2 = 1$, we require that $y_1 \geq b_1$ and $y_2 \geq b_2$. Therefore, $y_1 = y_2 = 1$ and $z_{Rob}(b) = 2$. Now, consider the adaptable problem $\Pi_{Adapt}(b, d)$ and consider the solution $\hat{y}(\omega) = b(\omega)$. Clearly, the solution $\hat{y}(\omega)$ for all $\omega \in \Omega$ is feasible. Therefore,

$$\begin{aligned} z_{Adapt}(b) &\leq \max_{\omega \in \Omega} d^T y(\omega) \\ &= \max_{\omega \in \Omega} y_1(\omega) + y_2(\omega) \\ &= \max_{\omega \in \Omega} b_1(\omega) + b_2(\omega) \\ &\leq 1. \end{aligned}$$

Also, from Theorem 4.1, we have that $z_{Rob}(b) \leq 2 \cdot z_{Adapt}(b)$. Therefore, $z_{Rob}(b) = 2 \cdot z_{Adapt}(b)$. \square

4.3 A Large Adaptability Gap Example for Non-symmetric Uncertainty

Sets. In this section, we construct an example of a non-symmetric uncertainty set such that the worst case cost of an optimal robust solution is $\Omega(m)$ times the worst case cost of an optimal adaptable solution. Therefore, the adaptability gap is $\Omega(m)$ in this case.

THEOREM 4.4 Consider an instance of problem $\Pi_{Rob}(b)$ and $\Pi_{Adapt}(b)$, where, $n_1 = 0$, $n_2 = n \geq 3$, $p_2 = 0$, $A = 0$, $c = 0$, $d = (1, 1, \dots, 1)$ (an n dimensional vector of all ones) and $B = I_n$ (I_n is $n \times n$ identity matrix). Let Ω denote the set of scenarios and the uncertainty set,

$$\mathcal{I}_b(\Omega) = \left\{ b \mid \sum_{j=1}^m b_j \leq 1, b \geq 0 \right\}. \quad (4.5)$$

Then,

$$z_{Rob}(b) \geq n \cdot z_{Adapt}(b).$$

PROOF. We know that from Lemma 2.4 that the uncertainty set $\mathcal{I}_b(\Omega)$ is not symmetric. Consider the robust problem $\Pi_{Rob}(b)$:

$$\begin{aligned} z_{Rob}(b) &= \min_y \max_{\omega \in \Omega} d^T y \\ I_n y &\geq b(\omega), \forall \omega \in \Omega \\ y &\geq 0. \end{aligned}$$

For any feasible solution $y \in \mathbb{R}_+^m$, $I_n y \geq e_j$ for all $j = 1, \dots, m$ where e_j is the unit vector corresponding to the j^{th} column of I_n ($e_j \in \mathcal{I}(\Omega)$). Therefore, $y_j \geq 1$ for all $j = 1, \dots, m$ which implies,

$$z_{Rob}(b) \geq n.$$

Now, consider $\Pi_{Adapt}(b)$:

$$\begin{aligned} z_{Adapt}(b) &= \min \max_{\omega \in \Omega} d^T y(\omega) \\ I_n y(\omega) &\geq b(\omega), \forall \omega \in \Omega \\ y(\omega) &\geq 0. \end{aligned}$$

Consider any scenario $\omega \in \Omega$ and let $b(\omega)$ be the realizations of b in scenario ω . Then,

$$\sum_{j=1}^n b_j(\omega) \leq 1.$$

Consider the following feasible solution: $\hat{y}(\omega) = b(\omega)$ for all $\omega \in \Omega$. Now,

$$\begin{aligned} z_{Adapt}(b) &\leq \max_{\omega \in \Omega} d^T \hat{y}(\omega) \\ &= \max_{\omega \in \Omega} \sum_{j=1}^n b_j(\omega) \\ &\leq 1. \end{aligned}$$

Therefore, $z_{Rob}(b) \geq n \cdot z_{Adapt}(b)$. □

4.4 Adaptability Gap for Positive Uncertainty Sets. In this section, we extend the

bound on the adaptability gap to the case when the uncertainty set is *positive*.

THEOREM 4.5 *Consider the robust and adaptable problems: $\Pi_{Rob}(b)$ and $\Pi_{Adapt}(b)$. Suppose the uncertainty set, $\mathcal{I}_b(\Omega)$, is convex and positive. Then,*

$$z_{Rob}(b) \leq 2 \cdot z_{Adapt}(b).$$

PROOF. Since $\mathcal{I}_b(\Omega)$ is positive, there exist a convex symmetric set $S \subset \mathbb{R}_+^m$ such that $\mathcal{I}_b(\Omega) \subset S$ and the point of symmetry of S (say u^0) belongs to $\mathcal{I}_b(\Omega)$. Let ω^0 be the scenario corresponding to $u^0 = b(\omega^0)$. Consider an optimal solution $x^*, y^*(\omega), \forall \omega \in \Omega$ to $\Pi_{Adapt}(b)$. We show that the solution $(2x^*, 2y^*(\omega^0))$ is a feasible solution for the robust problem $\Pi_{Rob}(b)$, since,

$$\begin{aligned} A(2x^*) + B(2y^*(\omega^0)) &\geq 2b(\omega^0) \\ &\geq b(\omega), \forall \omega \in \Omega, \end{aligned} \tag{4.6}$$

where (4.6) follows from the fact that $b(\omega^0)$ is the point of symmetry of S and thus, $2b(\omega^0) \geq b$ for all $b \in S \supset \mathcal{I}_b(\Omega)$ from Lemma 2.3. Now,

$$\begin{aligned} z_{Rob}(b) &\leq c^T(2x^*) + d^T(2y^*(\omega^0)) \\ &\leq 2 \cdot (c^T x^* + d^T y^*(\omega^0)) \\ &\leq 2 \cdot z_{Adapt}(b). \end{aligned}$$

□

5. Adaptability Gap under Right Hand Side and Cost Uncertainty. In this section, we bound the adaptability gap for a more general model of uncertainty where both the right hand side of the constraints, b and the objective coefficients, d are uncertain and the second stage decision variables are allowed to be integers. Unlike the stochasticity gap under cost and right hand side uncertainty, the adaptability gap is bounded and is at most four when the uncertainty set is symmetric. The bound can also be extended to the case of positive uncertainty sets. Furthermore, we show that the adaptability gap is one for the special case of hypercube uncertainty sets in an even more general model of uncertainty that allows constraint coefficients to be uncertain. This result is particularly surprising since the bound of two on the stochasticity gap is tight for hypercube right hand side uncertainty (cf. Section 2.2).

5.1 Symmetric Uncertainty Sets. We consider problems: $\Pi_{Adapt}(b, d)$ (cf. (1.6)) and $\Pi_{Rob}(b, d)$ (cf. (1.5)) and show that the adaptability gap is at most four if the uncertainty set is symmetric. In particular, we prove the following theorem.

THEOREM 5.1 *Let $z_{Rob}(b, d)$ denote the optimal worst-case cost of $\Pi_{Rob}(b, d)$ and $z_{Adapt}(b, d)$ denote the optimal worst-case cost of $\Pi_{Adapt}(b, d)$, where the uncertainty set, $\mathcal{I}_{(b,d)}(\Omega) \subset \mathbb{R}_+^{m+n_2}$, is symmetric. Then,*

$$z_{Rob}(b, d) \leq 4 \cdot z_{Adapt}(b, d).$$

PROOF. Let $H = \{(b, d) \in \mathbb{R}_+^{m+n_2} \mid b^l \leq b \leq b^h, d^l \leq d \leq d^h\}$ be the smallest hypercube containing $\mathcal{I}_{(b,d)}(\Omega)$. Let $(b^0, d^0) \in \mathbb{R}_+^{m+n_2}$ be the center of H , i.e.,

$$b^0 = \frac{b^l + b^h}{2}, d^0 = \frac{d^l + d^h}{2}.$$

From Lemma 2.3, we know that $(b^0, d^0) \in \mathcal{I}_{(b,d)}(\Omega)$. Let ω^0 denote the scenario such that $b(\omega^0) = b^0, d(\omega^0) = d^0$. Let $x^*, y^*(\omega), \forall \omega \in \Omega$ be an optimal solution for $\Pi_{Adapt}(b, d)$. Then,

$$\begin{aligned} z_{Adapt}(b, d) &= c^T x^* + \max_{\omega \in \Omega} d(\omega)^T y^*(\omega) \\ &\geq c^T x^* + d(\omega^0)^T y^*(\omega^0) \\ &= c^T x^* + \left(\frac{d^l + d^h}{2} \right)^T y^*(\omega^0) \\ &\geq c^T x^* + \left(\frac{d^h}{2} \right)^T y^*(\omega^0), \end{aligned} \tag{5.1}$$

where (5.1) follows from the fact that $\mathcal{I}(\Omega) \subset \mathbb{R}_+^{m+n_2}$ and thus, $d^l \geq 0$. Similarly, $b^l \geq 0$ and therefore,

$$Ax^* + By^*(\omega^0) \geq b(\omega^0) = \frac{b^l + b^h}{2} \geq \frac{b^h}{2}, \tag{5.2}$$

We claim that $(2x, 2y(\omega^0))$ is a feasible solution for $\Pi_{Rob}(b, d)$ since,

$$A(2x^*) + B(2y^*(\omega^0)) \geq b^h,$$

where the above inequality follows from (5.2). Therefore,

$$\begin{aligned} z_{Rob}(b, d) &\leq c^T(2x^*) + \max_{\omega \in \Omega} d(\omega)^T(2y^*(\omega^0)) \\ &\leq 2(c^T x^* + (d^h)^T y^*(\omega^0)) \end{aligned} \quad (5.3)$$

$$\begin{aligned} &\leq (2 \cdot c^T x^* + (d^h)^T y^*(\omega^0)) + (d^h)^T y^*(\omega^0) \\ &\leq 2 \cdot z_{Adapt}(b, d) + 2 \cdot z_{Adapt}(b, d) \\ &= 4 \cdot z_{Adapt}(b, d), \end{aligned} \quad (5.4)$$

where (5.3) follows from the fact that $d^h \geq d(\omega)$ for all $\omega \in \Omega$ and (5.4) follows from the inequality (5.1) and the fact that $c^T x^* \geq 0$ as $c \in \mathbb{R}_+^{n_1}$ and $x^* \geq 0$. \square

Similar to the bound on the adaptability gap under right hand side uncertainty in Theorem 4.2, we obtain a stronger multiplicative bound on the adaptability gap under cost and right hand side uncertainty. In particular, we have the following theorem.

THEOREM 5.2 *Let $z_{Rob}(b, d)$ be the optimal objective value for $\Pi_{Rob}(b, d)$ and $z_{Adapt}(b, d)$ be the optimal objective value for $\Pi_{Adapt}(b, d)$ where the uncertainty set $\mathcal{I}_{(b,d)}(\Omega)$ is symmetric. Let $(b^l, d^l), (b^h, d^h) \in \mathbb{R}_+^{m+n_2}$ be such that, for all $j = 1, \dots, m$,*

$$b_j^l = \min\{b_j \mid (b, d) \in \mathcal{I}_{(b,d)}(\Omega)\}, \quad b_j^h = \max\{b_j \mid (b, d) \in \mathcal{I}_{(b,d)}(\Omega)\},$$

and for all $j = 1, \dots, n_2$,

$$d_j^l = \min\{d_j \mid (b, d) \in \mathcal{I}_{(b,d)}(\Omega)\}, \quad d_j^h = \max\{d_j \mid (b, d) \in \mathcal{I}_{(b,d)}(\Omega)\},$$

Suppose $b^l \geq \rho \cdot b^h$ and $d^l \geq \rho \cdot d^h$ for some $\rho \geq 0$. Then

$$z_{Rob}(b, d) \leq \left(\frac{4}{1 + \rho} \right) \cdot z_{Adapt}(b, d).$$

PROOF. Let $x^*, y^*(\omega), \forall \omega \in \Omega$ be an optimal solution for $\Pi_{Adapt}(b, d)$. Then, the solution $(\frac{2x^*}{1+\rho}, \frac{2y^*(\omega^0)}{1+\rho})$, where ω^0 is the scenario such that $(b(\omega^0), d(\omega^0))$ is the point of symmetry of $\mathcal{I}_{(b,d)}(\Omega)$, is a feasible solution for $\Pi_{Rob}(b, d)$.

$$\begin{aligned} \frac{2}{1+\rho}(Ax^* + By^*(\omega^0)) &\geq \frac{2}{1+\rho} \cdot \frac{b^l + b^h}{2} \\ &\geq \frac{\delta b^h + b^h}{1+\rho} \\ &= b^h \\ &\geq b(\omega), \quad \forall \omega \in \Omega. \end{aligned}$$

Therefore,

$$z_{Rob}(b, d) \leq \left(\frac{4}{1 + \rho} \right) \cdot z_{Adapt}(b, d).$$

□

5.2 Positive Uncertainty Sets. In this section, we extend the bound on the adaptability gap to the case when the uncertainty set is *positive*.

THEOREM 5.3 *Consider the robust and adaptable problems: $\Pi_{Rob}(b, d)$ and $\Pi_{Adapt}(b, d)$. Suppose the uncertainty set, $\mathcal{I}_{(b,d)}(\Omega)$, is convex and positive. Then,*

$$z_{Rob}(b, d) \leq 4 \cdot z_{Adapt}(b, d).$$

PROOF. Since $\mathcal{I}_{(b,d)}(\Omega)$ is positive, there exist a convex symmetric set $S \subset \mathbb{R}_+^{m+n_2}$ such that $\mathcal{I}_{(b,d)}(\Omega) \subset S$ and the point of symmetry of S (say u^0) belongs to $\mathcal{I}_{(b,d)}(\Omega)$. Let ω^0 be the scenario corresponding to $u^0 = (b(\omega^0), d(\omega^0))$. Consider an optimal solution $x^*, y^*(\omega), \forall \omega \in \Omega$ to $\Pi_{Adapt}(b, d)$. We show that the solution $(2x^*, 2y^*(\omega^0))$ is a feasible solution for the robust problem $\Pi_{Rob}(b, d)$ since,

$$\begin{aligned} A(2x^*) + B(2y^*(\omega^0)) &\geq 2b(\omega^0) \\ &\geq b(\omega), \forall \omega \in \Omega, \end{aligned} \tag{5.5}$$

where inequality (5.5) follows from the fact that $(b(\omega^0), d(\omega^0))$ is the center of the symmetric set S that contains $\mathcal{I}_{(b,d)}(\Omega)$. Therefore, $b(\omega) \leq 2b(\omega^0)$ and $d(\omega) \leq 2d(\omega^0)$ for all $\omega \in \Omega$ from Lemma 2.3. Now,

$$\begin{aligned} z_{Rob}(b, d) &\leq c^T(2x^*) + \max_{\omega \in \Omega} d(\omega)^T(2y^*(\omega^0)) \\ &\leq 2 \cdot c^T x^* + (2 \cdot d(\omega^0))^T(2y^*(\omega^0)) \\ &\leq 4 \cdot z_{Adapt}(b, d). \end{aligned}$$

□

5.3 Special Case of Hypercube Uncertainty Sets. In this section, we consider the case where the uncertainty set is a hypercube and show that the worst case cost of the optimal solution of $\Pi_{Adapt}(b, d)$ is equal to the worst case cost of the optimal solution of $\Pi_{Rob}(b, d)$. In fact, we prove the result for a more general model of uncertainty, one where the coefficients in the constraint matrix are

also uncertain. We consider the following adaptive two-stage mixed integer problem, $\Pi_{Adapt}(A, B, b, d)$:

$$\begin{aligned}
 z_{Adapt}(A, B, b, d) &= \min c^T x + \max_{\omega \in \Omega} d(\omega)^T y(\omega) \\
 A(\omega)x + B(\omega)y(\omega) &\geq b(\omega), \forall \omega \in \Omega \\
 x &\in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1} \\
 y(\omega) &\in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2}, \forall \omega \in \Omega,
 \end{aligned} \tag{5.6}$$

where Ω is the set of uncertain scenarios and $\omega \in \Omega$ refers to a particular scenario. Let $\mathcal{U} = \{(A(\omega), B(\omega), b(\omega), d(\omega)) | \omega \in \Omega\}$ denote the hypercube uncertainty set for scenarios in Ω where $A(\omega) \in \mathbb{R}^{m \times n_1}$, $B(\omega) \in \mathbb{R}^{m \times n_2}$, $b(\omega) \in \mathbb{R}^m$ and $d(\omega) \in \mathbb{R}_+^{n_2}$ are realizations of the uncertain parameters in scenario ω .

The robust counterpart $\Pi_{Rob}(A, B, b, d)$ is formulated as follows:

$$\begin{aligned}
 z_{Rob}(A, B, b, d) &= \min c^T x + \max_{\omega \in \Omega} d(\omega)^T y \\
 A(\omega)x + B(\omega)y &\geq b(\omega), \forall \omega \in \Omega \\
 x &\in \mathbb{R}_+^{n_1 - p_1} \times \mathbb{Z}_+^{p_1} \\
 y &\in \mathbb{R}_+^{n_2 - p_2} \times \mathbb{Z}_+^{p_2}.
 \end{aligned} \tag{5.7}$$

THEOREM 5.4 *Let \mathcal{U} be a hypercube, i.e.,*

$$\mathcal{U} = [l_1, u_1] \times [l_2, u_2] \times \dots \times [l_N, u_N],$$

for some $l_i \leq u_i$ for all $i = 1, \dots, N$ where $N = mn_1 + mn_2 + m + n_2$, then

$$z_{Rob}(A, B, b, d) = z_{Adapt}(A, B, b, d).$$

PROOF. Consider an optimal solution (x^*, y^*) of the problem $\Pi_{Rob}(A, B, b, d)$. Therefore,

$$A(\omega)x^* + B(\omega)y^* \geq b(\omega), \forall \omega \in \Omega.$$

Therefore, the solution $x = x^*, y(\omega) = y^*, \forall \omega$ is a feasible solution for $\Pi_{Adapt}(A, B, b, d)$ which implies,

$$z_{Adapt}(A, B, b, d) \leq c^T x^* + \max_{\omega \in \Omega} d(\omega)^T y^*.$$

Now consider an optimal solution \hat{x} and $\hat{y}(\omega), \forall \omega \in \Omega$ for $\Pi_{Adapt}(A, B, b, d)$. Consider the following realization of the uncertain parameters.

$$\bar{A}_{ij} = \min_{\omega \in \Omega} A_{ij}(\omega), \forall i = 1, \dots, m, j = 1, \dots, n_1 \tag{5.8}$$

$$\bar{B}_{ij} = \min_{\omega \in \Omega} B_{ij}(\omega), \forall i = 1, \dots, m, j = 1, \dots, n_2 \tag{5.9}$$

$$\bar{b}_i = \max_{\omega \in \Omega} b_i(\omega), \forall i = 1, \dots, m \tag{5.10}$$

$$\bar{d}_j = \max_{\omega \in \Omega} d_j(\omega), \forall j = 1, \dots, n_2.$$

Since the uncertainty set is a hypercube, the $\text{vec}(\bar{A}, \bar{B}, \bar{b}, \bar{d})$ is a valid scenario. Let us refer to this scenario as $\bar{\omega}$. Clearly,

$$z_{Adapt}(A, B, b, d) = c^T \hat{x} + \max_{\omega \in \Omega} d(\omega)^T \hat{y}(\omega) \geq c^T \hat{x} + d(\bar{\omega})^T \hat{y}(\bar{\omega}).$$

We claim that $(\hat{x}, \hat{y}(\bar{\omega}))$ is a feasible solution for $\Pi_{Rob}(A, B, b, d)$. Consider any $\omega \in \Omega$. Now,

$$A(\omega)\hat{x} + B(\omega)\hat{y}(\bar{\omega}) \geq \bar{A}\hat{x} + \bar{B}\hat{y}(\bar{\omega}) \quad (5.11)$$

$$\geq b(\bar{\omega}) \quad (5.12)$$

$$\geq b(\omega), \quad (5.13)$$

where (5.11) follows from (5.8) and (5.9) and the fact that $\hat{x}, \hat{y}(\bar{\omega}) \geq 0$. Inequality (5.12) follows from the feasibility of $(\hat{x}, \hat{y}(\bar{\omega}))$ for scenario $\bar{\omega}$ and (5.10). Inequality (5.13) follows from the fact that $b(\bar{\omega}) \geq b(\omega)$ for any $\omega \in \Omega$. Therefore,

$$\begin{aligned} z_{Rob}(A, B, b, d) &\leq c^T \hat{x} + \max_{\omega \in \Omega} d(\omega)^T \hat{y}(\bar{\omega}) \\ &\leq c^T \hat{x} + d(\bar{\omega})^T \hat{y}(\bar{\omega}) \\ &\leq z_{Adapt}(A, B, b, d), \end{aligned} \quad (5.14)$$

where inequality (5.14) follows from the fact that $d(\bar{\omega}) \geq d(\omega)$ for all $\omega \in \Omega$ and $\hat{y}(\bar{\omega}) \geq 0$. \square

Therefore, we obtain the following corollary.

COROLLARY 5.1 *Let $z_{Rob}(b, d)$ be an optimal solution for $\Pi_{Rob}(b, d)$ and $z_{Adapt}(b, d)$ be an optimal solution for $\Pi_{Adapt}(b, d)$, where the uncertainty set, $\mathcal{I}_{(b,d)}(\Omega)$, is a hypercube. Then,*

$$z_{Rob}(b, d) = z_{Adapt}(b, d).$$

6. Conclusions. In this paper, we study the effectiveness of a static-robust solution in approximating two-stage stochastic and adaptive optimization problems and present several surprising positive results under mild restrictions on the model and the uncertainty set. We show that, under fairly general assumptions for the uncertainty set, and the probability measure (namely that both are symmetric), a static-robust solution is a good approximation for the two-stage stochastic optimization problem when the right hand side of the constraints is uncertain and a good approximation for the two-stage adaptive optimization problem when both the right hand side and the costs are uncertain. In other words, both the stochasticity gap (when only the right hand side is uncertain) and the adaptability gap (when both the right hand side and the costs are uncertain) are bounded. We also show that our bounds on the stochasticity and the adaptability gaps are tight for symmetric uncertainty sets.

The assumption of symmetry on the uncertainty set is not very restrictive and is satisfied by most commonly used uncertainty sets such as hypercubes, ellipsoids and norm-balls. Furthermore, we show that if the assumption of symmetry on the uncertainty set is relaxed, both the stochasticity gap can be arbitrarily large which implies that the assumption is “almost” necessary. We also extend the bounds for positive uncertainty sets that are not necessarily symmetric. Therefore, the results in this paper show that the robust optimization approach is more powerful than believed previously and provides a tractable and good approximation for both the two-stage stochastic and the two-stage adaptive optimization problem under fairly general assumptions.

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