A Comparison of Lower Bounds for the Symmetric Circulant Traveling Salesman Problem

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Abstract

When the matrix of distances between cities is symmetric and circulant, the traveling salesman problem (TSP) reduces to the so-called symmetric circulant traveling salesman problem (SCTSP), that has applications in the design of reconfigurable networks, and in minimizing wallpaper waste. The complexity of the SCTSP is open, but conjectured to be NP-hard, and we compare different lower bounds on the optimal value that may be computed in polynomial time. We derive a new linear programming (LP) relaxation of the SCTSP from the semidefinite programming (SDP) relaxation in [E. de Klerk, D.V. Pasechnik, and R. Sotirov. On semidefinite programming relaxation of the traveling salesman problem, *Siam Journal Optimization*, **19**:4, 1559–1573, 2008]. Further, we discuss theoretical and empirical comparisons between this new bound and three well-known bounds from the literature, namely the Held-Karp bound [M. Held and R.M. Karp. The traveling salesman problem and minimum spanning trees. *Operations Research*, **18**:1138–1162, 1970], the 1-tree bound, and the closed-form bound for SCTSP proposed in [Van der Veen, J.A.A. Solvable cases of TSP with various objective functions, PhD thesis, Groningen University, 1992].

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1 Introduction

A (weighted) graph G is called *circulant* if its (weighted) adjacency matrix is circulant. Recall that a circulant matrix has the following form:

$$D = \begin{pmatrix} r_0 & r_1 & r_2 & . & . & r_{n-1} \\ r_{n-1} & r_0 & r_1 & r_2 & . & . \\ . & . & r_0 & r_1 & r_2 & . \\ . & . & . & . & . & . \\ r_2 & . & . & . & r_0 & r_1 \\ r_1 & r_2 & . & . & . & r_0 \end{pmatrix}, \quad (1)$$

that is

$$D_{ij} = r_{j-i \mod n} \quad (i, j = 0, \ldots, n-1).$$

A natural question is whether a given combinatorial optimization problem becomes easier when restricted to circulant graphs.

For example, the maximum clique and minimum graph coloring problems remain NP-hard for circulant graphs, and cannot be approximated to within a constant factor, unless P=NP [3]. It is still an open question if the Hamiltonian directed circuit problem restricted to directed circulant graphs remains NP-hard; see Yang et al [19], Heuberger [10], and Bogdanowicz [1]. On the other hand, the shortest Hamiltonian path problem is polynomial solvable for undirected circulant graphs as shown by Burkard and Sandholzer [2]. Likewise, deciding whether a circulant graph is Hamiltonian may be done in polynomial time [2].

The symmetric circulant traveling salesman problem (SCTSP) is the problem of finding a Hamiltonian circuit of minimum length in a weighted, undirected, circulant graph. As far as we know, the complexity of the SCTSP is still open (see, e.g., [18], [4]). The best known approximation algorithm for SCTSP is a 2-approximation algorithm ([8],[18]). The *bottleneck* TSP problem is known to be polynomially solvable in the circulant case [2].

The study of the circulant TSP is motivated by practical applications, such as reconfigurable network design [15], and minimizing wallpaper waste [6].

In this paper we compare four lower bounds that may be obtained in polynomial time for the SCTSP problem:

1. we introduce a new linear programming (LP) bound derived from a semidefinite programming (SDP) relaxation of TSP due to De Klerk et al. [10].

2. The second lower bound dates back to 1954 and is due to Danzig, Fulkerson and Johnson [5]. Its optimal value coincides with the LP bound of Held and Karp [11] (see, e.g., Theorem 21.34 in [13]), and is commonly known as the Held-Karp bound (HK).

3. The third bound is due to Van der Veen [18] (VdV) and was introduced in 1992 for the SCTSP. It is given as a closed form expression and may be computed in linear time.

4. The fourth bound is the well-known 1-tree (1T) bound for TSP (see, e.g. §7.3 in [4]).

We will show how the bounds 1, 2 and 4 above may be computed more simply for circulant graphs than for general TSP. Subsequently we will perform theoretical and empirical comparisons of the bounds.

Outline

This paper is structured as follows. In Section 2 we review the basic concepts concerning the four lower bounds. In Section 3 we derive the new LP bound from the SDP formulation in [10]. Numerical comparisons between bounds are presented in Section 4 and some theoretical results are proved in Section 5. Conclusions and open problems are listed in Section 6.

Notation and preliminaries

Consider a permutation group on n elements, say $\mathcal{G}$, represented as a multiplicative group of $n \times n$ permutation matrices in the usual way.

Definition 1. *The centralizer ring (or commutant) of the group $\mathcal{G}$ is defined as follows.*

$$\mathcal{A} := \{Y \in \mathbb{R}^{n\times n} : Y = \frac{1}{|\mathcal{G}|}\sum_{P\in\mathcal{G}} P^T X P, \quad X \in \mathbb{R}^{n\times n}\}. \tag{2}$$

An equivalent definition is

$$\mathcal{A} := \{Y \in \mathbb{R}^{n\times n} : PY = YP \quad \forall P \in \mathcal{G}\}.$$

The linear mapping $X \mapsto R(X) := \frac{1}{|\mathcal{G}|}\sum_{P\in\mathcal{G}} P^T X P, \quad X \in \mathbb{R}^{n\times n}$ is called the *group average* or *Reynolds operator*; and $P \in \mathcal{G}$ are the permutation matrices of the permutation matrix representation of $\mathcal{G}$.

We will repeatedly use the following property of the Reynolds operator:

$$\operatorname{trace}(R(X)Y) = \operatorname{trace}(R(Y)X) \quad \forall\, X, Y \in \mathbb{R}^{n\times n}. \tag{3}$$

The centralizer ring of $\mathcal{G}$ has a structure of a matrix *-algebra, i.e. it is a subspace of $\mathbb{R}^{n\times n}$ that is closed under matrix multiplication and taking transposes.

The symmetric circulant matrices may be viewed as the centralizer ring of the dihedral group D_n, and we will repeatedly use this observation in the rest of the paper.

We will denote the standard 0-1 basis of the symmetric circulant matrices by $\{B_0 := I, B_1, \ldots, B_d\}$, where $d := \lfloor n/2 \rfloor$. Thus:

$$(B_k)_{ij} := \begin{cases} 1 & \text{if } i - j = k \mod n \\ 0 & \text{else} \end{cases} \qquad (k = 0, \ldots, d,\ i, j = 1, \ldots, n).$$

The positions of the nonzero entries in B_k are sometimes called the *k-th stripe*, and we will use this terminology.

Since matrix multiplication is also commutative for circulant matrices, the basis can be simultaneously diagonalized by a suitable unitary matrix (called the discrete Fourier transform matrix; see e.g. [9]).

When dealing with circulant matrices, it is usual to introduce some additional notation. If $\{t_0, \ldots, t_m\}$ is a subset of $\{0, 1, \ldots, d\}$ for some $m \leq d$, we define

$$C_n\langle t_1, \ldots, t_m\rangle := \sum_{i=0}^{m} t_i B_{t_i}.$$

Thus we will informally say that the circulant graph $C_n\langle t_1, \ldots, t_m\rangle$ consists of the stripes $t_1, \ldots, t_m$. In other words, we use the same notation for the circulant matrix $C_n\langle t_1, \ldots, t_m\rangle$ and the associated weighted circulant graph.

2 Lower bounds for CSTSP

In this section we discuss four lower bounds for SCTSP.

2.1 SDP/LP bound

Let $K_n(D)$ denote a complete undirected graph on n vertices, with edge lengths (also called *weights* or *costs*) $D_{ij} = D_{ji} > 0$, $\forall i, j = 1, \ldots, n$, where D is called the matrix of distances. The Hamiltonian circuit in $K_n(D)$ of minimum length is often called the *optimal tour*.

It is shown in [12] that the following SDP provides a lower bound on the length of an optimal tour:

$$\min \frac{1}{2} \operatorname{trace}(DX^{(1)})$$

subject to

$$\begin{aligned} X^{(k)} &\geq 0, \quad k = 1, \ldots, d \\ \sum_{k=1}^{d} X^{(k)} &= J - I, \\ I + \sum_{k=1}^{d} \cos(\frac{2ki\pi}{n}) X^{(k)} &\succeq 0, \quad i = 1, \ldots, d \\ X^{(k)} &\in \mathbb{S}^{n\times n}, \quad k = 1, \ldots, d, \end{aligned} \tag{4}$$

where $d = \lfloor \frac{n}{2} \rfloor$ is the diameter of $\mathcal{C}_n$ (i.e. standard circuit on n vertices) and J denotes the all one matrix. Note that this problem involves nonnegative matrix variables $X^{(1)}, \ldots, X^{(d)}$ of order n.

We will see in Section 3 that, if D is circulant, the SDP formulation (4) reduces to an LP problem.

2.2 Held-Karp bound (HK)

One of the best-known linear programming (LP) relaxations of the TSP is the Held-Karp bound, defined as follows.

$$HK := \min \frac{1}{2} \mathrm{trace}(DX)$$

subject to

$$\begin{array}{rcl} Xe & = & 2e, \\ \mathrm{diag}(X) & = & 0, \\ 0 & \leq & X \leq J, \\ \sum_{i \in \mathcal{I}, j \notin \mathcal{I}} X_{ij} & \geq & 2 \ \forall \, \emptyset \neq \mathcal{I} \subset \{1, \ldots, n\}, \end{array} \tag{5}$$

where e denotes the all-ones vector and J the all-ones matrix, as before. The last constraints are called *sub-tour elimination inequalities* and model the fact that a hamiltonian cycle is 2-connected. There are $2^n - 2$ sub-tour elimination inequalities, but even so this problem may be solved in polynomial time using the ellipsoid method; see e.g. Schrijver [16], §58.5.

We will show how to simplify the LP formulation (5) to an equivalent, smaller LP when the distance matrix D is circulant.

The following theorem will allow us to restrict the optimization of (5) to the symmetric circulant matrices.

Theorem 2. *Let $\mathcal{A}$ denote the centralizer ring of a permutation group $\mathcal{G}$ and let $D \in \mathcal{A}$. If we have an optimal solution, X, for problem (5) then there exists an optimal circulant solution, Y, of LP in (5) (i.e. $Y \in \mathcal{A}$).*

Proof. The fact that $D \in \mathcal{A}$ implies that $P^T DP = D$ for all $P \in \mathcal{G}$.

We will show that if X is optimal for (5) then also $Y := R(X)$ is optimal for (5). Recall that $R(X)$ is the image of X under the Reynolds operator.

Since $Pe = e$, $P^T e = e$ and $Xe = 2e$ we have:

$$\begin{array}{rcl} R(X)e & = & \frac{1}{|\mathcal{G}|} \sum_{P \in \mathcal{G}} PXP^T e = \frac{1}{|\mathcal{G}|} \sum_{P \in \mathcal{G}} PXe \\ & = & \frac{1}{|\mathcal{G}|} \sum_{P \in \mathcal{G}} 2Pe = \frac{1}{|\mathcal{G}|} \sum_{P \in \mathcal{G}} 2e = 2e. \end{array}$$

Permuting rows and columns preserves the zero diagonal, therefore diag(X) = 0 implies diag(R(X)) = 0. Moreover, $R(X)$ averages over the permuted entries of X so that $0 \leq R(X) \leq J$ whenever $0 \leq X \leq J$.

To show that $R(X)$ is feasible for (5) we still have to prove that $R(X)$ satisfy the sub-tour elimination constraints.

First notice that if P is a permutation matrix then matrices X and PXP^T are the adjacency matrices of two isomorphic graphs. Thus the minimum cut in the graph having X as adjacency matrix equals the minimum cut in the graph having $R(X)$ as adjacency matrix. Thus we have:

$$\sum_{i\in\mathcal{I},j\notin\mathcal{I}} (PXP^T)_{ij} \quad \geq \quad 2 \ \ \forall\, \emptyset \neq \mathcal{I} \subset \{1,\ldots,n\}.$$

Summing over all $P \in \mathcal{G}$ yields:

$$\sum_{P\in\mathcal{G}} \sum_{i\in\mathcal{I},j\notin\mathcal{I}} (PXP^T)_{ij} \quad \geq \quad 2|\mathcal{G}| \ \ \forall\, \emptyset \neq \mathcal{I} \subset \{1,\ldots,n\}.$$

Thus:

$$\sum_{i\in\mathcal{I},j\notin\mathcal{I}} (R(X))_{ij} \quad \geq \quad 2 \ \ \forall\, \emptyset \neq \mathcal{I} \subset \{1,\ldots,n\},$$

and $R(X)$ is therefore feasible for (5). Moreover, $R(X)$ is optimal since

$$\operatorname{trace}(DR(X)) = \operatorname{trace}(R(D)X) = \operatorname{trace}(DX),$$

by (3), and this concludes the proof of the theorem. □

Recall that, for the SCTSP, the permutation group $\mathcal{G}$ is the dihedral group, and its centralizer ring is the set of symmetric circulant matrices. By Theorem 2, we may restrict the feasible set of (5) to the symmetric circulant matrices whose basis is $\{I = B_0, B_1, \ldots, B_d\}$. Since matrix D has zero on the diagonal we can ignore B_0 and write:

$$X := \sum_{p=1}^{d} x_p B_p \ \text{ and } \ D := \sum_{p=1}^{d} d_p B_p.$$

The objective in (5) reduces to:

$$\min \sum_{p=1}^{d} n d_p x_p,$$

if n is odd. If n is even, the last term becomes $\frac{1}{2} n d_d x_d$ in stead of $n d_d x_d$.

In order to rewrite the sub-tour elimination constraints we will make use of a $\{0,1\}$ matrix denoted by $E_{\mathcal{I}}$. This matrix will have 1 on positions (i,j) and (j,i) if $i \in \mathcal{I}$, $j \notin \mathcal{I}$ and zeros elsewhere. Notice that:

$$\frac{1}{2}\operatorname{trace}(E_{\mathcal{I}}X) = \sum_{i\in\mathcal{I},j\notin\mathcal{I}} X_{ij}.$$

Then the sub-tour the elimination constraints from (5) are equivalent to:

$$\frac{1}{2}\sum_{p=1}^{d} x_p \operatorname{trace}(E_{\mathcal{I}}B_p) \geq 2 \ \ \forall\, \emptyset \neq \mathcal{I} \subset \{1,\ldots,n\}.$$

Notice that $\mathrm{diag}(X) = 0$ is implicit because $x_0 = 0$. Moreover, because $0 \leq X \leq J$ we have $0 \leq x_p \leq 1$, $p = 1, \ldots, d$. We have to split the constraint $Xe = 2e$ into two cases:

- For n odd: $Xe = 2e \Leftrightarrow \sum_{p=1}^{d} x_p B_p e = 2e \Leftrightarrow \sum_{p=1}^{d} x_p = 1$.
- For n even: $Xe = 2e \Leftrightarrow x_d B_d e + \sum_{p=1}^{d-1} x_p B_p e = 2e \Leftrightarrow \frac{1}{2} x_d + \sum_{p=1}^{d-1} x_p = 1$.

We can now write down the simplified equivalent form of (5). For odd n, we have:

$$\min \sum_{p=1}^{d} n d_p x_p,$$

subject to

$$\begin{aligned} \sum_{p=1}^{d} x_p &= 1, \\ x_p &\geq 0, \quad p = 1, \ldots, d \\ \frac{1}{2} \sum_{p=1}^{d} x_p \mathrm{trace}(E_{\mathcal{I}} B_p) &\geq 2 \;\; \forall\, \emptyset \neq \mathcal{I} \subset \{1, \ldots, n\}. \end{aligned} \tag{6}$$

For even n, the last term in the objective function becomes $\frac{1}{2} n d_d x_d$, and the first constraint should be replaced by $\frac{1}{2} x_d + \sum_{p=1}^{d-1} x_p = 1$.

2.3 Van der Veen bound (VdV)

Let $D \in \mathbb{R}^{n \times n}$ be a symmetric circulant matrix and let $r = (r_0, r_1, \ldots, r_{\lfloor \frac{n}{2} \rfloor})$ be the vector that completely determines the entries of D (i.e. the first $d+1$ components on the first row). Recall that $\lfloor \frac{n}{2} \rfloor$=d.

Assume now that $r_0 = 0$ (which is the case for TSP problem) and assume that the r_i's are distinct. Define a permutation Φ such that $\Phi(0) = 0$ and Φ sorts the values of r in ascending order.

Let $\gcd(t_1, \ldots, t_m)$ denote the greatest common divisor of given natural numbers $t_1, \ldots, t_m$. A necessary and sufficient condition for Hamiltonicity of a circulant graph is given by the following theorem.

Theorem 3 (Burkard and Sandholzer [2])**.** *The circulant graph $C_n\langle t_1, \ldots, t_m \rangle$, with vertex set $\{0, 1, \ldots, n-1\}$, consists of $\gcd(n, t_1, \ldots, t_m)$ components ($m \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$). Each component is a graph on $\frac{n}{\gcd(n, t_1, \ldots, t_m)}$ vertices. The vertices in component α ($\alpha = 0, 1, \ldots, \gcd(n, t_1, \ldots, t_m) - 1$) are:*

$$\left\{ (\alpha + k \ \gcd(n, t_1, \ldots, t_m)) \ mod\ n \ \middle|\ k \in 0, 1, \ldots, \frac{n}{\gcd(n, t_1, \ldots, t_m)} - 1 \right\}. \tag{7}$$

Moreover, $C_n\langle t_1, \ldots, t_m \rangle$ is Hamiltonian if and only if $\gcd(n, t_1, \ldots, t_m)$=1.

Let l be the smallest integer such that $\gcd(n, \Phi(1), \ldots, \Phi(l)) = 1$. Then Van der Veen [18] shows that one can construct a Hamiltonian tour only using edges from stripes $\Phi(1), \ldots, \Phi(l)$.

Following [18], we define:

$$\mathcal{GCD}(\Phi(k)) := \gcd(\mathcal{GCD}(\Phi(k-1)), \Phi(k)), \quad k = 1, \ldots, l, \tag{8}$$

and $\mathcal{GCD}(\Phi(0)) := n$.

Further we can assume without loss of generality (see [18]) that:

$$n = \mathcal{GCD}(\Phi(0)) > \mathcal{GCD}(\Phi(1)) > \ldots > \mathcal{GCD}(\Phi(l)) = 1. \tag{9}$$

Then Theorem 7.4.2 from [18] shows that the following value is a lower bound for the SCTSP problem.

$$VdV := \sum_{i=1}^{l} \{(\mathcal{GCD}(\Phi(i-1)) - \mathcal{GCD}(\Phi(i)))r_{\Phi(i)}\} + r_{\Phi(l)}. \tag{10}$$

The term $\sum_{i=1}^{l}\{(\mathcal{GCD}(\Phi(i-1)) - \mathcal{GCD}(\Phi(i)))r_{\Phi(i)}\}$ gives the weight of a shortest Hamiltonian path obtained via the nearest neighbor rule. The last term reflects the fact that each Hamiltonian cycle must include a edge of weight at least $r_{\Phi(l)}$.

2.4 1-tree bound (1T)

Another famous lower bound for TSP is the minimum cost 1-tree bound.

Definition 4. *Let G=(V,E) denote an undirected graph with edge costs c_e, for each $e \in E$ and let $v_1 \in V$. Two edges incident with node v_1 plus a spanning tree of $G \setminus \{v_1\}$ is called a 1-tree in G.*

Definition 5. *Let $G = (V, E)$ denote an undirected graph with edge costs c_e, for each $e \in E$ and let $v_1 \in V$. Let $\delta(v_1)$ denote the set of edges incident to v_1. Now, let $A = min\{c_e + c_f \mid e, f \in \delta(v_1)\}$ and let B be the cost of a minimum spanning tree in $G \setminus \{v_1\}$. Then $A + B$ is a lower bound for the TSP on G, called a 1-tree bound.*

For circulant graphs, one may compute the 1-tree bound in a simpler way than for general graphs, as we will show in Theorem 7. Recall that we can construct a minimum cost spanning tree using the (greedy) Kruskal algorithm. This algorithm starts with an arbitrary edge of lowest cost, and recursively constructs a spanning tree by adding an edge of lowest possible cost to the current forest so that adding this edge does not form a cycle.

As a consequence of Theorem 3, after using all possible edges from the lowest cost stripe, we may assume the Kruskal algorithm has constructed $x := \mathcal{GCD}(\Phi(1))$ components (i.e. disjoint paths). Moreover, by (7) we can describe these disjoint paths as:

$$P_\alpha := \left\{(\alpha + k\ \Phi(1)) \mod n \mid k \in 0, 1, \ldots, \frac{n}{x} - 1\right\} \quad \alpha = 0, \ldots, x-1. \tag{11}$$

An important observation for our purposes is that these paths cover all the vertices; any edge that is subsequently added by the Kruskal algorithm will therefore connect two of these paths.

Now fix v. According to the construction above one has $v \in P_{v \ mod \ x}$. Under the assumption (9), we have $(v+\Phi(i)) \mod n \in P_{(v+\Phi(i)) \mod x}$, for every $i = 2, \ldots, l$. Thus, for each i, the edge $\{v, (v+\Phi(i)) \mod n\}$ connects the paths $P_{v \mod x}$ and $P_{(v+\Phi(i)) \mod x}$.

Lemma 6. *For any $k = 0, \ldots, \frac{n}{\mathcal{GCD}(\Phi(1))}$ and for any $i = 2, \ldots, l$ and $v \in V$, the edge $\{(v+k\Phi(1)) \mod n, (v+k\Phi(1)+\Phi(i)) \mod n\}$ connects the paths $P_{v \mod x}$ and $P_{(v+\Phi(i)) \mod x}$.*

Proof. By (11), $(v+k\Phi(1)) \mod n$ belongs to $P_{v \mod x}$.

For any $i \in \{2, \ldots, l\}$ one has:

$$\begin{aligned}(v+k\Phi(1)+\Phi(i)) \mod n &= ((v+\Phi(i))+k\Phi(1)) \mod n \\ &= ((v+\Phi(i)) \mod n + k\Phi(1)) \mod n.\end{aligned}$$

Since $(v+\Phi(i)) \mod n$ belongs to $P_{(v+\Phi(i)) \mod x}$, using (11) again we have that $(v+k\Phi(1)+\Phi(i)) \mod n \in P_{(v+\Phi(i)) \mod x}$. □

The lemma expresses the fact that one can always connect two distinct paths P_{α_1} and P_{α_2} ($\alpha_1 \neq \alpha_2$) using an edge of cost $r_{\Phi(i)}$, for any $i = 2, \ldots, l$, in more than one way. Now we can prove the following.

Theorem 7. *Let G be a circulant graph on n vertices. Let $\Phi(1)$ denote the stripe of minimum nonzero cost. The value of a minimum cost 1-tree equals the value of a minimum cost spanning tree plus the value of an edge of lowest cost whenever $\Phi(1) \neq \frac{n}{2}$. If n is even and $\Phi(1) = \frac{n}{2}$, then the value of a minimum cost 1-tree equals the value of a minimum cost spanning tree plus the cost of an edge of second lowest cost.*

Proof. We will assume $\gcd(n, \Phi(1)) \neq 1$, since the case $\gcd(n, \Phi(1)) = 1$ is trivial.

Fix $v_1 \in V$, and assume no two stripes have the same cost and $\Phi(1) \neq \frac{n}{2}$. Because of the circulant structure we have two edges of minimum cost with an endpoint at v_1. Start constructing a minimum spanning tree from v_1 using Kruskal's algorithm (denote the first added edge by e_t). Then after adding the edges of minimum cost Kruskal's algorithm has constructed $\gcd(n, \Phi(1))$ disjoint paths covering the vertices of G with edges of lowest cost. After this step any other edge of lowest cost added to the current forest will create a cycle. Call the path with an endpoint at v_1 P_{v_1} and denote the other endpoint of this path by v_2. Connect the paths obtained before using edges of other costs (again using Kruskal's algorithm), but do not allow to connect P_{v_1} via v_1 (this is always possible according to Lemma 6). When the minimum spanning tree is constructed add the edge $e_{12} := v_1 v_2$. Call the resulting structure T.

By construction v_1 has degree 2 in T. The edges that connects v_1 to T are e_{12} end e_t. Notice that both have lowest cost. Therefore $e_t + e_{12}$ is minimum among the sum of the cost of two edges incident to v_1, which shows that T is a 1-tree. Since v_1 was arbitrary chosen we concluded the first part of the proof.

The second part of the proof is similar, and is therefore omitted. □

3 Deriving the new LP bound

In this section we show how to reduce the SDP formulation in (4) to an equivalent LP whenever the distance matrix D is circulant.

The following theorem will allow us to restrict the optimization of (4) to the symmetric circulant matrices, in the case of the SCTSP.

Theorem 8. *Let $\mathcal{A}$ denote the centralizer ring of a permutation group $\mathcal{G}$ and let $D \in \mathcal{A}$. If we have an optimal solution, $X^{(1)}, \ldots, X^{(k)}$, for problem (4) then $\{R(X^{(1)}), \ldots, R(X^{(k)})\} \subset \mathcal{A}$ is also an optimal solution of (4), where R denotes the Reynolds operator of the group $\mathcal{G}$.*

Proof. The fact that $D \in \mathcal{A}$ means that D is invariant under the action of the permutation matrices $P \in \mathcal{G}$, that is $P^T DP = D$ for all $P \in \mathcal{G}$.

We will show that if $X^{(k)}$, $k = 1, \ldots, n$ are feasible for (4) then also $Y^{(k)} := R(X^{(k)})$ are feasible for (4). For simplicity of notation we will show this for a fixed k, but everything holds for any $k = 1, \ldots, d$.

If $X^{(k)} \geq 0$ and symmetric, then by permuting rows and columns and adding elements we obtain again a symmetric, positive matrix, so $R(X^{(k)}) \geq 0$ and $R(X^{(k)}) \in \mathcal{S}^n$.

$R(X^{(k)})$ is a linear mapping therefore $R(\sum_{k=1}^d X^{(k)}) = \sum_{k=1}^d R(X^{(k)})$ and $R(J - I) = R(J) - R(I)$. Notice that $R(J) = J$ and $R(I) = I$. Then we obtain: $\sum_{k=1}^d R(X^{(k)}) = J - I$.

Using $R(I) = I$ and linearity of R, from:

$$I + \sum_{k=1}^{d} \cos(\frac{2ki\pi}{n}) X^{(k)} \succeq 0, \quad i = 1, \ldots, d$$

we obtain:

$$I + \sum_{k=1}^{d} \cos(\frac{2ki\pi}{n}) R(X^{(k)}) \succeq 0, \quad i = 1, \ldots, d.$$

We have seen that $R(X^{(k)})$, $k = 1, \ldots, d$ are feasible. Furthermore,

$$\text{trace}(DR(X^{(1)})) = \text{trace}(R(D)X^{(1)}) = \text{trace}(DX^{(1)}),$$

by (3), and this concludes the proof of the theorem. □

Now let us restrict the feasible set to the circulant matrices. For each $X^{(k)}$, $k = 1, \ldots, d$ we may write

$$X^{(k)} := \sum_{p=1}^{d} x_p^{(k)} B_p, \tag{12}$$

where $\{B_0 = I, B_1, \ldots, B_d\}$ form the standard basis for the symmetric circulant matrices, as before.

The matrix of distances D has zeros on the diagonal, and the variables $x_0^{(k)}$ may therefore be set to zero. Since $B_i's$ are 0-1 matrices $X^{(k)} \geq 0$ is equivalent to $x_p^{(k)} \geq 0$, $k, p = 1, \ldots, d$; and using (12) we obtain the equivalent form of (4):

$$\left.\begin{array}{lrr} \min & \frac{1}{2}\sum_{p=1}^{d} x_p^{(1)} \mathrm{trace}(DB_p) & \\ s.t. & x_p^{(k)} \geq 0, & k, p = 1, \ldots, d \\ & \sum_{k=1}^{d}\sum_{p=1}^{d} x_p^{(k)} B_p = J - I, & \\ & I + \sum_{k=1}^{d}\sum_{p=1}^{d} \cos(\frac{2ki\pi}{n}) x_p^{(k)} B_p \succeq 0, & i = 1, \ldots, d \end{array}\right\} \tag{13}$$

Let Q denote the discrete Fourier transform matrix. Then we may diagonalize the basis matrices via $Q^* B_p Q = \Lambda^{(p)}$, where $\Lambda^{(p)} := \mathrm{diag}(\lambda_j^{(p)})$, $j = 0, \ldots, n-1$ is the diagonal matrix containing the eigenvalues of B_p.

One has:

$$\lambda_j^{(p)} = 2\cos(\frac{2\pi jp}{n}) \ \ p = 1, \ldots, d, \ \ j = 0, \ldots, n-1, \ \ \text{if } n \text{ is odd} \tag{14}$$

and

$$\lambda_j^{(p)} = 2\cos(\frac{2\pi jp}{n}) \ \ p = 1, \ldots, d-1, \ \ j = 0, \ldots, n-1, \ \ \text{if } n \text{ is even} \tag{15}$$

$$\lambda_j^{(d)} = \cos(\frac{2\pi jd}{n}), \ \ j = 0, \ldots, n-1, \ \ \text{if } n \text{ is even.} \tag{16}$$

Because of the simultaneous diagonalization of the B_i's, (13) reduces to an LP problem, as we will now show.

Let us write:

$$D = \sum_{i=1}^{d} d_i B_i. \tag{17}$$

One clearly has:

$$\mathrm{trace}(B_i B_j) = 0 \ \ if \ \ i \neq j.$$

Multiplying (17) by B_p to the right and taking into account that $B_i's$ and D are symmetric, using the previous relation one obtains:

$$\mathrm{trace}(DB_p) = d_p \mathrm{trace}(B_p^2) = cd_p, \tag{18}$$

where $c = 2n$ for $p = 1, \ldots, d$. For n even we have an exception, that is $c = n$ when $p = d$.

We will now transform each linear matrix equality into n linear inequalities. To this end, note that $J - I = \sum_{p=1}^{d} B_p$. Then using the diagonalization, the relation:

$$\sum_{k=1}^{d}\sum_{p=1}^{d} x_p^{(k)} B_p = \sum_{p=1}^{d} B_p$$

reduces to:

$$\sum_{k=1}^{d}\sum_{p=1}^{d} x_p^{(k)} \lambda_j^{(p)} = \sum_{p=1}^{d} \lambda_j^{(p)}, \quad j = 0, \dots, n-1, \tag{19}$$

where the eigenvalues $\lambda_j^{(p)}$ are defined in (14), (15) and (16).

Finally, again using the diagonalization, the d linear matrix inequalities:

$$I + \sum_{k=1}^{d}\sum_{p=1}^{d} \cos(\frac{2ki\pi}{n}) x_p^{(k)} B_p \succeq 0, \quad i = 1, \dots, d$$

reduce to the nd linear inequalities:

$$1 + \sum_{k=1}^{d}\sum_{p=1}^{d} \lambda_j^{(p)} \cos(\frac{2ki\pi}{n}) x_p^{(k)} \geq 0, \quad i = 1, \dots, d, \quad j = 0, \dots, n-1. \tag{20}$$

We can now state the LP reformulation of (13):

$$\left.\begin{array}{lrl} \min & \frac{1}{2}\sum_{p=1}^{d} cd_p x_p^{(1)} & \\ s.t. & x_p^{(k)} \geq 0, & k, p = 1, \dots, d \\ & \sum_{k=1}^{d}\sum_{p=1}^{d} \lambda_j^{(p)} x_p^{(k)} = \sum_{p=1}^{d} \lambda_j^{(p)}, & j = 0, \dots, n-1 \\ & 1 + \sum_{k=1}^{d}\sum_{p=1}^{d} \lambda_j^{(p)} \cos(\frac{2ki\pi}{n}) x_p^{(k)} \geq 0, & i = 1, \dots, d \;\; j = 0, \dots, n-1. \end{array}\right\} \tag{21}$$

4 Numerical comparison between bounds

In this section we present numerical results for the new SDP/LP bound and the other bounds stated in Section 2 (i.e: 1T bound, HK bound and VdV bound); see Table 1. The matrices in Table 1 have dimensions between 6 and 64, and were generated in such a way as to avoid trivial solutions.

The LP problems were solved using the Matlab® toolbox Yalmip [14] together with the optimization solver Sedumi [17]. The optimal values of the SCTSP instances were computed using the Concorde[1] software for TSP. Due to the small sizes of the instances, all the values in the tables could be computed in a few seconds on a standard Pentium IV PC.

A few remarks on Table 1:

- The HK and VdV bounds coincide for all the instances in the table.

[1] The Concorde software is available at `http://www.tsp.gatech.edu/concorde/`

matrix	dimension	SDP/LP	1T	HK	VdV	optimum SCTSP
D1	54	2,114	2,140	2,157	2,157	2,174
D2	18	2.0837	2.1063	2.1392	2.1392	2.1392
D7	28	29.755	33	38	38	38
D8	27	291.738	294	297	297	297
D10	39	547.868	550	552	552	552
D11	57	2,022.715	2,119	2,181	2,181	2,181
Dt1	12	104.84	107	118	118	118
Dt3	22	75,855.77	170,105	340,100	340,100	340,100
D17	36	4,877.80	4,902	4,916	4,916	4,944
D24	26	1,098.86	1,153	1,240	1,240	1,240
D28	30	272.47	296	310	310	310
Dt4	24	123.91	125	126	126	128
Dt6	24	2,448.08	3,095	3,690	3,690	3,690
Dt8	27	7.2462	8	9	9	9
Dt9	25	270,768.63	400,151	500,145	500,145	500,145
Dt10	25	2,862	4,147	5,140	5,140	5,140
Dt11	25	270,765	400,147	500,140	500,140	500,140
Dt12	12	85.85	86	87	87	87
Dt13	12	87.71	88	90	90	90
Dt14	8	57.17	57	58	58	58
Dt15	8	58.34	58	60	60	60
Dt16	6	43.50	43	44	44	44
Dt18	64	25,583	26,901	27,484	27,484	27,484

Table 1: Numerical comparison of the four lower bounds from Section 2 for SCTP instances.

- The HK and VdV bounds give the best bounds in all cases, but do not always equal the optimal value of the SCTSP instance in question.
- The new LP bound is always weaker that the HK and VdV bounds for the test problems, and is even lower that the 1T bound for a few instances. Adding the subtour elimination inequalities to the new LP did not result in better bounds than HK for any of the instances in the table.

The instances from Table 1 are available online at:
`http://lyrawww.uvt.nl/~cdobre/SCTSP_instances.rar`.

5 A theoretical comparison between bounds

Based on the numerical results presented in the previous section, we may conjecture certain relations between the bounds, like VdV = HK $\geq$ SDP/LP.

On the other hand, we have only been able to prove that VdV $\geq$ 1T (cf Theorem 9) and that HK $\geq$ VdV (cf Theorem 11). It is also well-known (see e.g. [4]) that HK $\geq$ 1T. Thus we will obtain the 'sandwich theorem' type result

$$1T \leq VdV \leq HK.$$

Theorem 9. *The VdV bound is at least as good as the one tree ($1T$) bound.*

Proof. Recall that $\Phi(1)$ denotes the stripe of lowest cost. From (10) we have that VdV equals the length of a minimum weight Hamiltonian path plus the weight of an edge of cost $r_{\Phi(l)}$. Moreover, the weight of a minimum Hamiltonian path is always greater or equal than the weight of a minimum weight spanning tree.

The required result now follows from Theorem 7. □

Thus we have VdV $\geq$ 1T. Further, it was shown by de Klerk et al. ([10]) that, for general TSP, HK does not dominate the SDP bound in (4) or vice versa. In the case of the circulant matrices we can state the following theorem, based on the numerical results in Table 1.

Theorem 10. *For SCTSP, the new LP relaxation (21) does not dominate the one tree bound, or, by implication, the Held-Karp bound (5).*

It was not known before whether the SDP bound (4) can be worse than the one tree bound; see [12]. It still remains an open question if the Held-Karp bound dominates the new LP relaxation in the case of SCTSP.

Theorem 11. *For SCTSP, the Held-Karp bound (5) is at least as tight as the Van de Veen bound (10).*

Proof. Let $G = (V, E)$ be a weighted circulant graph with edge weights now denoted by c_e $(e \in E)$, and consider the following equivalent formulation of the Held-Karp bound (5) (details may be found in [4] §7.3):

$$HK := \min \sum_{e \in E} c_e x_e$$

subject to

$$\begin{aligned} \sum_{e \in \delta(S)} x_e &\geq 2, \quad \forall\, S \subset V,\ \ |S| \geq 2 \\ \sum_{e \in \delta(\{v\})} x_e &= 2 \quad \forall\, v \in V \\ 0 &\leq x_e \leq 1 \quad \forall\, e \in E. \end{aligned}$$

We enlarge the feasible set and define a value $p^* \leq HK$ via:

$$p^* := \min \sum_{e \in E} c_e x_e$$

subject to

$$\begin{aligned} \sum_{e \in \delta(S)} x_e &\geq 2, \quad \forall\, S \subset V,\ \ S \neq \emptyset \\ x_e &\geq 0 \quad \forall\, e \in E. \end{aligned}$$

By LP duality theory we have:

$$p^* = \max \sum_{\emptyset \neq S \subset V} 2y_S$$

subject to

$$\begin{aligned} \sum_{S | e \in \delta(S)} y_S &\leq c_e, \quad \forall\, e \in E \qquad (22) \\ y_S &\geq 0 \quad \forall\, e \in E. \end{aligned}$$

We will construct a feasible point of (22) with objective value equal to the value VdV from (10). It then follows that $p^* \geq VdV$, and since $HK \geq p^*$ we will conclude that $HK \geq VdV$ for circulant matrices.

Notice that if $|V| = n$, then the dual formulation in (22) has $2^n - 2$ variables y_S, each corresponding to a nonempty subset of V. Let C_i^k, $k = 0, \dots, l-1$, $i = 1, .., \mathcal{GCD}(\Phi(k))$ denote the connected components of the graph $G_k := \langle \Phi(1), \dots, \Phi(k) \rangle$. In this case C_i^0 represent the vertices of the graph. According to Theorem 3, $C_i^k \neq C_j^l$ if $(i, k) \neq (j, l)$. We will abuse notation by identifying the connected component with its vertices. Define:

$$\begin{aligned} y_{C_i^0} &:= \frac{r_{\Phi(1)}}{2} \;\; for\ i = 1, \dots, n \\ y_{C_i^m} &:= \frac{1}{2}(r_{\Phi(m+1)} - r_{\Phi(m)}), \;\; for\ m = 1, \dots, l-1 \;\; and\ i = 1, \dots, \mathcal{GCD}(\Phi(m)) \\ y_S &:= 0 \text{ otherwise.} \qquad (23) \end{aligned}$$

For a fixed m all the values $y_{C_i^m}$ are equal and nonnegative by definition, since the permutation Φ sorts the value of r in ascending order.

According to Theorem 3 we have for each m exactly $\mathcal{GCD}(\Phi(m))$ nonzero (i.e. strictly positive) $y_{C_i^m}$ variables.

Hence the objective in (22) evaluates to:

$$\begin{aligned} \sum_{\emptyset \neq S \subset V} 2y_S &= \sum_{m=0}^{l-1} 2\mathcal{GCD}(\Phi(m)) y_{C_i^m} \\ &= \mathcal{GCD}(\Phi(0)) r_{\Phi(1)} + \sum_{m=1}^{l-1} \mathcal{GCD}(\Phi(m))(r_{\Phi(m+1)} - r_{\Phi(m)}) \\ &= \sum_{m=1}^{l-1} \{(\mathcal{GCD}(\Phi(m-1)) - \mathcal{GCD}(\Phi(m))) r_{\Phi(m)}\} \\ &\quad + \mathcal{GCD}(\Phi(l-1)) r_{\Phi(l)} \\ &= \sum_{m=1}^{l} \{(\mathcal{GCD}(\Phi(m-1)) - \mathcal{GCD}(\Phi(m))) r_{\Phi(m)}\} + r_{\Phi(l)} =: VdV. \end{aligned}$$

The last equality is due to the fact that $\mathcal{GCD}(\Phi(l)) = 1$.

To show feasibility, first fix an edge $e \in E$ with cost $r_{\Phi(k)}$, with $k \leq l$. Such an edge connects two components of G_m ($m = 0, 1, \ldots, k-1$). Then we have:

$$\begin{aligned} \sum_{S|e\in\delta(S)} y_S &= 2\sum_{m=0}^{k-1} y_{C_i^m} = r_{\Phi(1)} + \sum_{m=1}^{k-1}(r_{\Phi(m+1)} - r_{\Phi(m)}) \\ &= r_{\Phi(1)} + r_{\Phi(k)} - r_{\Phi(1)} = r_{\Phi(k)}. \end{aligned}$$

Now fix an edge $e \in E$ with cost $r_{\Phi(k)}$, with $k > l$. Such an edge connects at most two components of G_m ($m = 0, 1, \ldots, l-1$). Then we have:

$$\sum_{S|e\in\delta(S)} y_S \leq 2\sum_{m=0}^{l-1} y_{C_i^m} = r_{\Phi(l)} < r_{\Phi(k)}.$$

Thus we have constructed a feasible point of (22) with objective value equal to the VdV bound. Therefore HK $\geq$ VdV. □

6 Summary and concluding remarks

The computational complexity of the symmetric circulant traveling salesman problem (SCTSP) remains an open problem.

We have therefore compared four lower bounds for SCTSP that may be computed in polynomial time.

We have been able to show that the Held-Karp bound [11] (see (5)) is as least as tight as a bound by Van der Veen [18] (see (10)) for SCTSP, and that the Van der Veen bound in turn is as least as tight as the minimum weight one tree bound.

Empirically, the Van der Veen bound and Held-Karp bound provided the best lower bounds for all numerical instances that we tested, and actually coincided for all the instances. Since the Van der Veen bound may be computed in linear time, it is clearly the best practical choice of the bounds that we considered.

A new LP bound for SCTSP that we derived from an SDP bound for general TSP by De Klerk et al. [12] proved to be quite weak in practice, but we were unable to prove any theoretical relationships with the other three bounds.

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