

A continuous model for open pit mine planning*

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Abstract

This paper proposes a new mathematical model for the open pit mine planning problem, based on continuous functional analysis. The traditional models for this problem have been constructed by using discrete 0 – 1 decision variables, giving rise to large-scale combinatorial and Mixed Integer Programming (MIP) problems. Instead, we use a continuous approach which allows for a refined imposition of slope constraints associated with geotechnical stability. The model introduced here is posed in a suitable functional space, essentially the real-valued functions that are Lipschitz continuous on a given two dimensional bounded region. We derive existence results and investigate some qualitative properties of the solutions.

Key words: Mine planning, continuous models, functional analysis.

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1 Introduction and motivation

Generally speaking, three different problems are usually considered by mining engineers for the economic valuation, design and planning of open pit mines [9]. The first one is the Final Open Pit (FOP) problem, also called “ultimate pit limit” problem, which aims to find the region of maximal economic value for exploitation under some geotechnical stability constraints and assuming infinite extraction capacity. Another more realistic problem is what we call here the Capacitated Final Open Pit (CFOP) which considers an additional constraint on the total capacity for a one-period exploitation. The third problem is a multi-period version of the latter, which we call here the Capacitated Dynamic Open Pit (CDOP) problem, with the goal of finding an optimal sequence of extracted volumes in a certain finite time horizon for bounded capacities at each period, the optimality criterion being in this case the total discounted profit.

A common practice for the formulation of these problems consists in describing an ore reserve (copper, for example) via the construction of a three-dimensional block model of the mineralization. Each block corresponds to a unitary volume of extraction characterized by some geologic and economic properties which are estimated from sample data. Block models can be represented as directed graphs where nodes are associated with blocks, while arcs correspond to the precedence of these blocks in the ore reserve. The precedence order is induced by some physical and operational constraints as those derived from the geomechanics of slope stability. This discrete approach gives rise naturally to huge combinatorial problems whose mathematical formulations are special large-scale instances of Integer Programming (IP) optimization problems (see, for instance, [3]). This explains why the optimal planning of open pit mines based on block models is usually addressed by approximation methods, heuristics and mixed IP techniques as Linear Programming relaxations of integer variables and branch-and-bound algorithms.

A great number of publications dealing with discrete block modeling for open pit mines have been published since the sixties. A seminal paper by Lerchs and Grossman [11] proposes a practical procedure to obtain the ultimate pit limit, which have been extensively applied in real mines for many years. The capacitated dynamic problem is more difficult to solve and many methods using discrete optimization techniques have been proposed [1, 2, 8]. Some dynamic programming formulations [10, 16] give interesting results, but the applicability of these techniques is still not well established for large-scale problems. Metaheuristic and evolutionary algorithms have also been extensively tested [4, 6].

In this paper we propose an alternative approach to the above mentioned problems based on continuous models for the ore reserve as well as the mining activity. The basic idea is to describe the pit contours through a Lipschitz continuous real-valued function, a *profile* which maps each pair of horizontal coordinates to the corresponding vertical depth. The stability of steep slopes is ensured by a spatially distributed constraint on the local Lipschitz constant of the profile function. The maximal feasible local slope may vary depending on the geotechnical properties of the possibly heterogeneous mineral deposit. The extraction capacity and operational costs

are described by a possibly discontinuous *effort* density, a scalar function defined on the three-dimensional region occupied by the ore reserve. In order to quantify the economic value of an extracted volume described by a given profile function, we consider a *gain* density defined on the deposit, which again may be a discontinuous function.

Our goal here is to develop a complete existence theory and investigate some qualitative properties of the optimal solutions to the proposed continuous versions of the FOP, CFOP and CDOP problems. The numerical resolution of these problems based on strategies from continuous optimization in functional spaces will be a matter for future research.

It is worth mentioning that the first documented continuous model for a parametrized variant of the CFOP problem seems to be the work by Matheron [12], where he explicitly exploited the underlying lattice structure of the set of feasible profiles for his model in order to obtain existence results as well as some interesting characterizations for optimal solutions. More recently, a simple related continuous model was introduced by Morales [13], for underground mines, but no study on existence nor optimality conditions are given there. On the other hand, Guzmán [7] has proposed a continuous model for the FOP problem based on shape and topological optimization using level-set techniques, reporting some computational results for a very simplified instance of the problem, but again no rigorous existence theory is provided.

This paper is organized as follows. In Section 2 we describe the stationary model in terms of continuous profile functions by introducing nonnegativity, boundary and stability conditions, and we prove that such a set of admissible profiles is compact for the uniform-convergence topology. Furthermore we give some structural properties of this set related to the lattice structure induced by pointwise min and max operations. In addition, we introduce some effort and gain functions which are related to the capacity constraints and the profit objective function, respectively. In Section 3 we state the optimization problems for the stationary case, derive some nonconstructive existence results for them and describe some qualitative properties of the corresponding optimal solutions by exploiting the lattice structure of the set of feasible profiles. In Section 4 we introduce a dynamic planning problem with discount rates, investigate some properties of the dynamic feasible set and give an existence result. Finally, in Section 5 we briefly summarize the main contributions of this paper and indicate some lines for future research.

2 The stationary model

2.1 Continuous profile functions

Throughout this paper Ω represents either a bounded connected domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, or a bounded open interval $\Omega = (a, b) \subset \mathbb{R}$ with $\partial\Omega = \{a, b\}$. In any case, the state of excavation at any particular time is defined by a function $p : \overline{\Omega} \rightarrow \mathbb{R}$ called *profile* so that $z = p(x)$ for $x \in \overline{\Omega}$, where the vertical coordinate z indicates the depth of the pit at point x (see Fig. 1). In this paper p , as not stated otherwise, is always an element of the Banach space

of continuous real valued functions $C(\overline{\Omega})$ endowed with the supremum norm given by

$$\|p\|_{\infty} \equiv \sup_{x \in \overline{\Omega}} |p(x)|.$$

The initial state (profile) is defined by a function $p_0 \in C(\overline{\Omega})$ so that all admissible profiles p must satisfy the nonnegativity condition

$$p(x) - p_0(x) \geq 0 \text{ for } x \in \overline{\Omega}. \quad (1)$$

Moreover, we assume that no excavation happens on the boundary of Ω and thus impose the Dirichlet condition

$$p(x) - p_0(x) = 0 \text{ for } x \in \partial\Omega. \quad (2)$$

In other words $p - p_0$ must belong to the nonnegative orthant of the linear space $C_0(\overline{\Omega}) \subset C(\overline{\Omega})$ of continuous functions on Ω that satisfy homogeneous boundary conditions.

The admissible profiles are not only bounded from below by $\underline{z} = \min\{p_0(x) | x \in \overline{\Omega}\}$ but also from above by some $\overline{z} > \underline{z}$ due to physical and operational conditions. Thus for any admissible profile we assume that

$$p(x) \in Z \equiv [\underline{z}, \overline{z}] \text{ for } x \in \overline{\Omega}.$$

Of course, with no loss of generality we may assume $\underline{z} \geq 0$. The general situation is sketched in Fig. 1 for the one dimensional case.

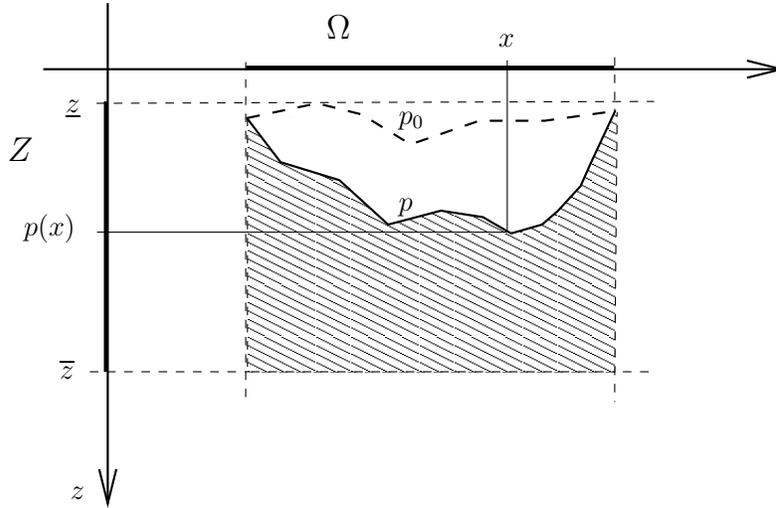


Figure 1: Continuous profile function

2.2 Geotechnical stability condition and compactness

In order to measure the local slope associated with a given profile $p \in C(\overline{\Omega})$, we define

$$L_p(x) \equiv \limsup_{\tilde{x} \rightarrow x \leftarrow \hat{x}} \frac{|p(\hat{x}) - p(\tilde{x})|}{\|\hat{x} - \tilde{x}\|} \quad \text{for } x \in \overline{\Omega}, \quad (3)$$

where $\|\cdot\|$ is the Euclidean norm. One can easily show that for each $p \in C(\overline{\Omega})$ the corresponding function $L_p : \overline{\Omega} \rightarrow [0, \infty]$ is upper semi-continuous. When $L_p(x) < \infty$, from this quantity we get sharp local Lipschitz constants for p around x in the sense that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for any \tilde{x}, \hat{x} with $\|\tilde{x} - x\| < \delta > \|\hat{x} - x\|$ we have

$$|p(\hat{x}) - p(\tilde{x})| \leq (L_p(x) + \varepsilon)\|\hat{x} - \tilde{x}\|,$$

and if L is any local Lipschitz constant for p in a neighborhood of x then $L_p(x) \leq L$.

The key assumption on the admissible profiles p is the pointwise stability condition

$$L_p(x) \leq \omega(x, p(x)) \quad \text{for } x \in \overline{\Omega}, \quad (4)$$

where $\omega : \overline{\Omega} \times Z \rightarrow [0, \infty)$ is an upper bound on the limiting local Lipschitz constant of p , which prescribes the maximal stable local slope and may vary on $\overline{\Omega} \times Z$ depending on the local geotechnical properties of the material. Rather than assuming continuity of the slope function ω we allow for horizontal and vertical jumps, which might be caused by layers of different material. In particular we may have soft and hard material layers next to each other, so that ω may jump upwards as one crosses into the harder material. Wherever ω is discontinuous we may define it as its upper envelope over a neighborhood and thus we can assume with no loss of generality the upper semi-continuity property:

$$\limsup_j \omega(x_j, z_j) \leq \omega(x, z) \quad (5)$$

for all convergent sequences $(x_j, z_j) \rightarrow (x, z) \in \overline{\Omega} \times Z$. This assumption immediately implies that ω attains a maximum

$$\overline{\omega} \equiv \max_{(x,z) \in \overline{\Omega} \times Z} \omega(x, z). \quad (6)$$

Apart from the supremum norm in $C(\overline{\Omega})$ we will utilize the extended-real valued Lipschitz semi-norm

$$\|p\|_{Lip} = \sup_{x \in \overline{\Omega}} L_p(x) \in [0, \infty] \quad \text{for } p \in C(\Omega).$$

Of course, for all $p \in C(\overline{\Omega})$ satisfying our stability condition (4) we have $\|p\|_{Lip} \leq \overline{\omega}$ as defined in (6). The linear space of all $p \in C(\overline{\Omega})$ for which $\|p\|_{Lip}$ is indeed finite is denoted by $Lip(\overline{\Omega})$, which can be endowed with the norm $\|p\|_{1,\infty} \equiv \|p\|_{\infty} + \|p\|_{Lip}$ to obtain the standard Banach space of all Lipschitz functions on $\overline{\Omega}$ (due to the compactness of $\overline{\Omega}$ we do not have to distinguish

between local and global Lipschitz continuity). On the subspace $Lip_0(\overline{\Omega}) \equiv Lip(\overline{\Omega}) \cap C_0(\overline{\Omega})$ the quantity $\|p\|_{Lip}$ defines a proper norm which is equivalent to $\|p\|_{1,\infty}$. In fact, it is not difficult to see that for any $p \in Lip_0(\overline{\Omega})$ we have that $\forall x \in \Omega$, $|p(x)| \leq \|p\|_{Lip} dist_{\Omega}(x, \partial\Omega)$, where $dist_{\Omega}(x, \partial\Omega)$ stands for the distance, relative to Ω , from x to the boundary $\partial\Omega$ of Ω . As Ω is bounded, we conclude that for some constant $C < \infty$ depending only on Ω we have that $\|p\|_{\infty} \leq C\|p\|_{Lip}$. In particular, the embedding $Lip_0(\overline{\Omega}) \rightarrow C_0(\overline{\Omega})$ is continuous. It is well-known that the resulting Banach space $Lip_0(\overline{\Omega})$ is equivalent to the Sobolev space $W_0^{1,\infty}(\Omega)$ (see, for instance, [5]). Furthermore, by virtue of Rademacher's theorem, every $p \in Lip(\overline{\Omega})$ is a.e. differentiable in Ω . It follows that for every p satisfying (4) we have that

$$\|\nabla p(x)\| \leq \omega(x, p(x)) \text{ for a.e. } x \in \Omega.$$

From now on, we assume that p_0 satisfies (4) and we denote by \mathcal{P} the class of all $p \in C(\overline{\Omega})$ satisfying (1), (2) and (4). The main consequence of the u.s.c. assumption (5) on ω is that \mathcal{P} is compact in $C(\overline{\Omega})$ as we will show below. To show why that assumption is necessary for the closeness of \mathcal{P} in $C(\overline{\Omega})$ we consider the following example.

Example 2.1 Take $\Omega = (-1, 1)$ and $\alpha \in [0, 1]$. Set $\omega_{\alpha}(x, z) = 0$ if $x < 0$, α if $x = 0$ and 1 if $x > 0$. This function is discontinuous at $x = 0$ and is not u.s.c. if $\alpha < 1$. For each $\varepsilon > 0$, the profile $p_{\varepsilon}(x) = \max(0, x - \varepsilon)$ satisfies (4) with $\omega = \omega_{\alpha}$ for any $\alpha \in [0, 1]$; nevertheless, its uniform limit $p(x) = \max(x, 0)$ is not admissible for ω_{α} if $\alpha < 1$ but it is so if $\alpha = 1$.

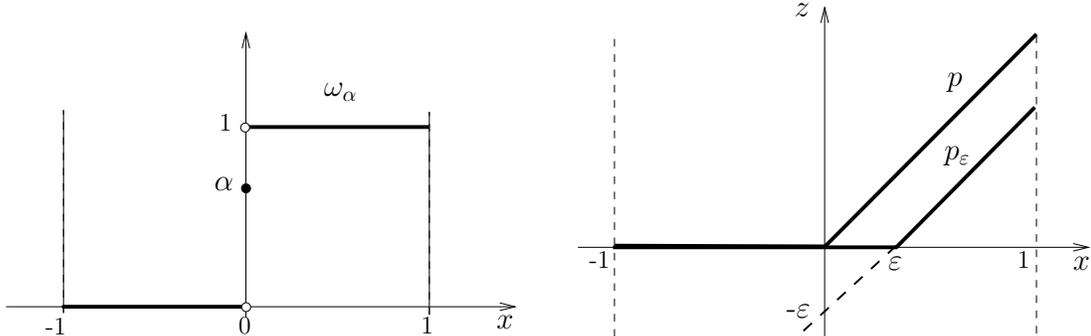


Figure 2: Illustration of Example 2.1

Proposition 2.2 If ω satisfies (5) then \mathcal{P} is compact and has empty interior in $C(\overline{\Omega})$.

Proof. First, we recall that the embedding $Lip_0(\overline{\Omega}) \rightarrow C_0(\overline{\Omega})$ is continuous. As our feasible functions are fixed on $\partial\Omega$ we know that $\mathcal{P} - p_0 \subset C_0(\overline{\Omega})$. The compactness of the closure of \mathcal{P} in $C(\overline{\Omega})$ is a direct consequence of the Arzela-Ascoli theorem: the uniform Lipschitz continuity property ensures the equicontinuity of \mathcal{P} , while all functions in \mathcal{P} have values in the compact interval $Z = [\underline{z}, \overline{z}]$. Now it only remains to show that \mathcal{P} is closed w.r.t. $\|\cdot\|_{\infty}$. Let $p \in p_0 + C_0(\overline{\Omega})$

be a function in the closure of \mathcal{P} w.r.t. $\|\cdot\|_\infty$. By virtue of (5), we have that for given $x \in \Omega$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\tilde{x} \in \Omega$ and $q \in \mathcal{P}$ with $\|\tilde{x} - x\| < \delta$ and $\|q - p\|_\infty < \varepsilon/4$ then

$$\omega(\tilde{x}, q(\tilde{x})) < \omega(x, p(x)) + \varepsilon/4. \quad (7)$$

From this it follows that for all $\tilde{x} \neq \hat{x}$ in the ball $B_\delta(x)$ we have:

$$\frac{|q(\tilde{x}) - q(\hat{x})|}{\|\tilde{x} - \hat{x}\|} \leq \omega(x, p(x)) + \frac{\varepsilon}{2}. \quad (8)$$

In fact, arguing by contradiction, if this inequality were violated by a couple of points $\tilde{x}, \hat{x} = \tilde{x} + \Delta x \in B_\delta(x)$ with $\Delta x \neq 0$, then it would be also violated by at least one of the pairs $(\tilde{x}, \tilde{x} + \Delta x/2)$ or $(\hat{x} - \Delta x/2, \hat{x})$. Recursively, we can continue the bisection process to generate a family of nested segments $\{[\tilde{x}_j, \hat{x}_j]\}_{j \geq 0}$ of length $\|\Delta x\|/2^j$, all of them contained in $B_\delta(x)$, such that the corresponding pairs \tilde{x}_j, \hat{x}_j violates (8) for each j , and moreover one would have that there exists a limit $x_* \in [\tilde{x}_j, \hat{x}_j] \subset B_\delta(x)$ such that $\tilde{x}_j \rightarrow x_*$ and $\hat{x}_j \rightarrow x_*$ as $j \rightarrow \infty$. By construction, using (7) and since $q \in \mathcal{P}$, we get:

$$\omega(x_*, q(x_*)) + \frac{\varepsilon}{4} < \omega(x, p(x)) + \frac{\varepsilon}{2} \leq \frac{|q(\tilde{x}_j) - q(\hat{x}_j)|}{\|\tilde{x}_j - \hat{x}_j\|} \leq \omega(x_*, q(x_*)) + \frac{o(\|\Delta x\|/2^j)}{\|\Delta x\|/2^j}$$

Letting $j \rightarrow \infty$, we obtain a contradiction. Thus (8) holds. Let us return to the function p . Given $\varepsilon \in (0, 1]$, let us consider the corresponding $\delta > 0$ given above. For any $\tilde{x} \neq \hat{x}$ with $\|\tilde{x} - x\| < \delta$ and $\|\hat{x} - x\| < \delta$ there exists a profile $q \in \mathcal{P}$ such that $\|p - q\|_\infty \leq \|\tilde{x} - \hat{x}\|\varepsilon/4$ and therefore

$$\frac{|p(\tilde{x}) - p(\hat{x})|}{\|\tilde{x} - \hat{x}\|} \leq \frac{|q(\tilde{x}) - q(\hat{x})| + 2\|p - q\|_\infty}{\|\tilde{x} - \hat{x}\|} \leq \omega(x, p(x)) + 2\frac{\varepsilon}{2} \leq \omega(x, p(x)) + \varepsilon.$$

Then p satisfies (4) and consequently $p \in \mathcal{P}$. This completes the proof of the compactness of \mathcal{P} .

To see that \mathcal{P} has empty interior w.r.t. the $\|\cdot\|_\infty$ norm we only have to consider a triangle wave function p_t around an admissible profile $p \in \mathcal{P}$. For any $\varepsilon > 0$, scaling the amplitude of p_t we can ensure $\|p_t - p\|_\infty \leq \varepsilon$. But letting the wavelength of p_t goes down, the limiting local slope L_{p_t} increases and thus the stability condition (4) would be violated. Hence in every neighborhood of a feasible profile w.r.t. $\|\cdot\|_\infty$ there are infeasible profiles. \square

Remark: As an immediate consequence of Proposition 2.2 we get that any functional $f : C(\overline{\Omega}) \rightarrow \mathbb{R}$ which is continuous w.r.t. $\|\cdot\|_\infty$ attains on \mathcal{P} a minimum and a maximum. This applies in particular to the distances $f(p) \equiv \|p - \tilde{p}\|_\infty$ for any fixed $\tilde{p} \in L^\infty(\Omega) \supset C(\overline{\Omega}) \supset \mathcal{P}$. Hence we have (non unique) least distance projections from L^∞ to \mathcal{P} .

2.3 Additional conditions on the slope constraint

There are two additional conditions one may want to impose on $\omega(x, z)$ concerning its dependence on the vertical coordinate z . The first one is that the restrictions $\omega_x(z) \equiv \omega(x, z)$ be right

continuous, i.e.

$$\lim_{\tilde{z} \searrow z} \omega_x(\tilde{z}) = \omega_x(z) \text{ for } x \in \overline{\Omega} \quad (9)$$

The physical motivation for this property is the exclusion of some pathological situations as described in the following example.

Example 2.3 Consider two regions with the soft material lying below the hard one as shown in Figure 3.1. If ω is only u.s.c. with respect to all variables, the profile shown in Figure 3.2 would be feasible, but obviously it is not physically stable because the maximal slope of the profile is only supported by the soft material below which only allows a milder slope.

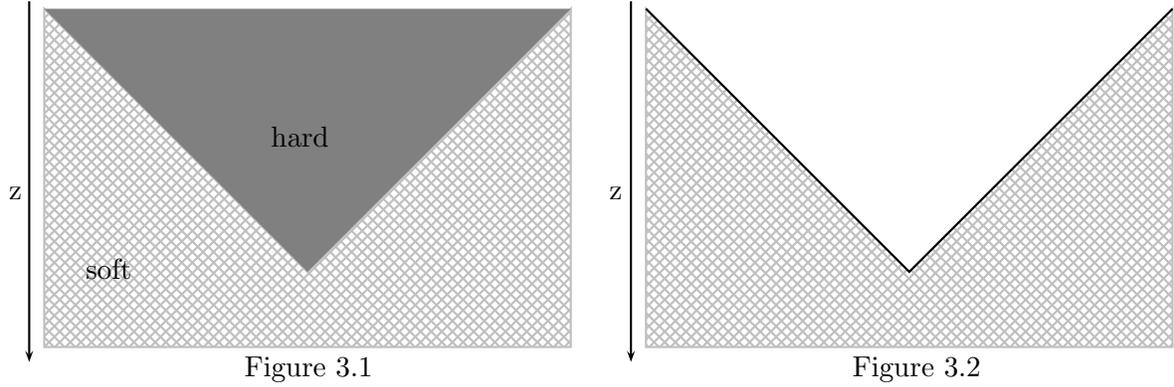


Figure 3: Illustration of Example 2.3

Remark: The assumption of right-continuity w.r.t. z means that the slope constraint can not simply jump up from a soft layer below a hard one, but must build up gradually.

The second additional condition on ω is definitely optional, namely we may require concavity of $\omega(x, z)$ w.r.t. $z \in Z$. This rather strong condition, while clearly not very realistic in the general case, does allow for the possibility of hard material in the middle sandwiched in between soft material on top and below. Of course, one may also consider the case of a concave ω which is monotonically increasing or decreasing w.r.t. z according to the geomechanics of the material.

Lemma 2.4 *If $\omega(x, z)$ is concave w.r.t. z then \mathcal{P} is convex.*

Proof. Take two profiles $p, q \in \mathcal{P}$ and $0 < \alpha < 1$. For any $x \in \overline{\Omega}$, we have that $L_{(1-\alpha)p+\alpha q}(x) \leq (1-\alpha)L_p(x) + \alpha L_q(x) \leq (1-\alpha)\omega(x, p(x)) + \alpha\omega(x, q(x)) \leq \omega(x, (1-\alpha)p(x) + \alpha q(x))$, where the last inequality is a consequence of the concavity property on ω . Thus $(1-\alpha)p + \alpha q \in \mathcal{P}$. \square

We end this section with a sufficient condition for an admissible profile to be in the interior of \mathcal{P} in $Lip(\overline{\Omega})$.

Proposition 2.5 *If ω is continuous on $(\Omega \times Z)$ then any profile $p \in \mathcal{P}$ for which*

$$\varepsilon \equiv \inf_{x \in \Omega} \{\omega(x, p(x)) - L_p(x)\} > 0$$

lies in the interior of \mathcal{P} in $Lip(\overline{\Omega})$.

Proof. Due to the continuity of ω on the compactum Ω there exists a δ such that at all $x \in \Omega$

$$|\omega(x, z) - \omega(x, \tilde{z})| < \varepsilon/2 \text{ if } |z - \tilde{z}| < \delta$$

Pick any q with $\|q - p\|_{Lip} < \varepsilon/8$ and $\|q - p\|_\infty < \delta$. The set of such q is an open neighborhood in \mathcal{P} w.r.t. the Lipschitz norm $\|\cdot\|_{Lip}$. Any $x \in \Omega$ is contained in some ball B_δ such that on that ball $p - q$ has a Lipschitz constant of size $\varepsilon/4$ and p has a Lipschitz constant of size $L_p(x) + \varepsilon/4$. Now we see that for any two points $\tilde{x}, \hat{x} \in B_\delta(x)$

$$\begin{aligned} |q(\tilde{x}) - q(\hat{x})|/\|\tilde{x} - \hat{x}\| &\leq |q(\tilde{x}) - p(\tilde{x}) - (q(\hat{x}) - p(\hat{x})) + p(\tilde{x}) - p(\hat{x})|/\|\tilde{x} - \hat{x}\| \\ &\leq |q(\tilde{x}) - p(\tilde{x}) - (q(\hat{x}) - p(\hat{x}))|/\|\tilde{x} - \hat{x}\| + L_p(x) + \varepsilon/4 \\ &\leq L_p(x) + \varepsilon/2 \leq \omega(x, p(x)) - \varepsilon/2 \leq \omega(x, q(x)) \end{aligned}$$

where the last estimation follows from the condition on the difference of p and q w.r.t. the Supremum norm. Hence q satisfies the slope constraint in that certainly $L_q(x) \leq \omega(x, q(x))$ and thus $q \in \mathcal{P}$. \square

2.4 Some structural properties of the admissible profiles set

The next result establishes some closedness properties of \mathcal{P} under pointwise minima and maxima operations. These properties ensure the connectedness of \mathcal{P} as a subset of $C(\overline{\Omega})$ even when $\omega(x, z)$ is not supposed to be concave w.r.t. z so that \mathcal{P} can be nonconvex.

Proposition 2.6 *Under (5) we have that:*

(i) \mathcal{P} is closed with respect to pointwise minima and maxima in that for any subset $\tilde{\mathcal{P}} \subset \mathcal{P}$ the functions $\underline{p}(x)$ and $\overline{p}(x)$ defined by

$$\underline{p}(x) \equiv \inf\{p(x) | p \in \tilde{\mathcal{P}}\} \quad \text{and} \quad \overline{p}(x) \equiv \sup\{p(x) | p \in \tilde{\mathcal{P}}\} \quad (10)$$

also belong to \mathcal{P} . As a consequence \mathcal{P} contains a unique maximal element $\overline{p}_U \equiv \max_{p \in \mathcal{P}}\{p\}$.

(ii) *If $p, q \in \mathcal{P}$ are such that $p \leq q$ then $q_\tau \equiv \max\{p, \min\{q, \tau\}\} \in \mathcal{P}$ for any "level" $\tau \in Z$. Moreover the path $\tau \rightarrow q_\tau$ is continuous w.r.t. $\|\cdot\|_\infty$.*

(iii) *Any two profiles $p, q \in \mathcal{P}$ are connected via $\min\{p, q\}$ and $\max\{p, q\}$.*

Proof.

(i) First we consider the binary case $\tilde{\mathcal{P}} = \{p, q\} \subset \mathcal{P}$. At points $x \in \Omega$ where $r(x) \equiv \max\{p(x), q(x)\} = p(x) > q(x)$ the same is true for all $\tilde{x} \approx x$ by continuity and we have thus $L_r(x) = L_p(x) \leq \omega(x, p(x)) = \omega(x, r(x))$. At points x where there is a tie $r(x) = p(x) = q(x)$ we find that for any two sequences $y_j \rightarrow x$ and $z_j \rightarrow x$

$$|r(y_j) - r(z_j)| = |\max\{p(y_j), q(y_j)\} - \max\{p(z_j), q(z_j)\}| \leq \max\{|p(y_j) - p(z_j)|, |q(y_j) - q(z_j)|\}.$$

The last inequality follows from the inverse triangle inequality for the supremum norm on \mathbb{R}^2 . After division by $\|y_j - z_j\|$ and taking the limit $y_j \rightarrow x \leftarrow z_j$ we find that

$$L_r(x) \leq \max\{L_p(x), L_q(x)\} \leq \max\{\omega(x, p(x)), \omega(x, q(x))\} = \omega(x, r(x)).$$

Thus $r = \max\{p, q\} \in \mathcal{P}$. The same argument applies to $r = \min\{p, q\} = -\max\{-p, -q\}$ for the slope stability condition. Obviously it follows by induction that maxima and minima of finitely many elements in \mathcal{P} also belong to \mathcal{P} . Now suppose $\tilde{\mathcal{P}}$ contains infinite many elements. First of all we note that from $\bar{p}(x) \equiv \sup\{p(x) | p \in \tilde{\mathcal{P}}\}$ it follows that $\bar{p} \in L^\infty(\bar{\Omega})$. Now we pick a dense subset $\{x_j\}_{j \in \mathbb{N}}$ in Ω . By induction on i we now choose sequences $p_k^{(i)} \in \tilde{\mathcal{P}}$ such that

$$\lim_{k \rightarrow \infty} p_k^{(i)}(x_j) = \bar{p}(x_j) \equiv \sup_{p \in \tilde{\mathcal{P}}} \{p(x_j)\} \quad \text{for } j < i$$

Consider a subsequence $(\tilde{p}_k) \in \tilde{\mathcal{P}}$ such that $\tilde{p}_k(x_i) \rightarrow \bar{p}(x_i)$ and set $p_k^{(i+1)} \equiv \max\{p_k^{(i)}, \tilde{p}_k\}$ so that $p_k^{(i+1)}(x_j) \rightarrow \bar{p}(x_j)$ for $j \leq i$. Now we take the diagonal sequence $p_k^* = p_k^{(k)}$ and get

$$\lim_{k \rightarrow \infty} p_k^*(x_j) = \bar{p}(x_j) \quad \text{for } j \in \mathbb{N}$$

We know for $j \leq k$

$$\bar{p}(x_j) \geq p_k^*(x_j) \geq \underbrace{p_k^{(i+1)}(x_j)}_{\rightarrow \bar{p}(x_j)}$$

and get a subsequence so that for all $j \in \mathbb{N}$, $\bar{p}(x_j) = \lim_{k \rightarrow \infty} p_k^*(x_j)$ by the convergence above. Moreover we can pick a Cauchy subsequence so that without loss of generality $\|\tilde{p} - p_k^*\|_\infty \rightarrow 0$ for some $\tilde{p} \in \mathcal{P}$. Clearly we must have $\tilde{p} = p$ which concludes the proof of (i).

(ii) The assertion is again obvious where all three values $p(x), q(x)$ and τ are distinct. When there is a tie between two we may invoke the same argument as in (i) and then extend it to a three way tie. Since at all $x \in \bar{\Omega}$

$$|q_\tau - q_{\tilde{\tau}}| = |\max\{p, \min\{q, \tau\}\} - \max\{p, \min\{q, \tilde{\tau}\}\}| \leq |\min\{q, \tau\} - \min\{q, \tilde{\tau}\}| \leq |\tau - \tilde{\tau}|$$

we have in fact Lipschitz continuity of q_τ w.r.t. $\tau \in \mathbb{R}$. Consequently, \mathcal{P} is path connected in $C(\bar{\Omega})$ since any two $p, q \in \mathcal{P}$ can be transformed into each other via $\min\{p, q\}$ or $\max\{p, q\}$ by

the path considered above.
(iii) Follows from (i) and (ii). \square

Remark: The path in (ii) is not continuous w.r.t. $\|\cdot\|_{Lip}$. Take for instance $p \equiv 0$ and $q(x) = \frac{1}{4} - (x - \frac{1}{2})^2$ on $\Omega = (0, 1)$ and ω sufficiently large, so that we have $p, q \in \mathcal{P}$ and $q_\tau(x) = \max\{0, \min\{q(x), \tau\}\}$ belongs to \mathcal{P} , but $\|q_\tau - q_0\|_{Lip} = \|q_\tau\|_{Lip} = 1$ for all $\tau > 0$.

The profile modifications used in Proposition 2.6(ii) will be referred to as “level excavations”. They are depicted in Figure 4 and make some practical sense as material is taken away in horizontal layers. While that does not mean optimality when gains are discounted as discussed in section 4 we note that any monotonic chain of feasible profiles $p_0 < p_1 < \dots < p_m \in \mathcal{P}$ can be extended to a feasible path from p_0 to p_m by level excavation between successive profiles $p_j \leq p_{j+1}$.

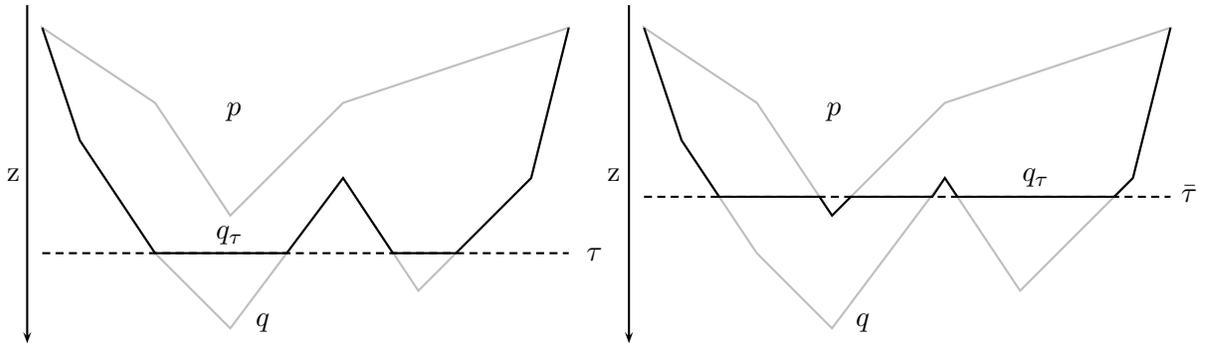


Figure 4: Illustration of level excavations

The next result on level excavations shows that we can regain feasibility from any bounded $q \geq p \in \mathcal{P}$, so there is no danger of getting trapped away from the admissible set.

Proposition 2.7 *Under (5) we have that if $\mathcal{P} \ni p \leq q \in L^\infty(\Omega)$ then*

$$q_\tau \equiv \max\{p, \min\{q, \tau\}\} \in L^\infty(\Omega) \quad \text{for } \tau \in Z$$

and the set $V_\tau \equiv \{x \in \bar{\Omega} \mid L_{q_\tau}(x) > \omega(x, q_\tau(x))\}$ is monotonically growing w.r.t. τ .

Proof. For any particular point $x \in \Omega$ consider the bounds

$$\bar{q} \equiv \limsup_{\tilde{x} \rightarrow x} q(\tilde{x}) \leq \underline{q} \equiv \liminf_{\tilde{x} \rightarrow x} q(\tilde{x}) \geq p(x).$$

If $\bar{q} > \underline{q}$ we must have $L_q(x) = \infty$ and also $L_{q_\tau}(x) = \infty$ as long as $\tau > \underline{q}$ which in turn means $x \in V_\tau$ for $\tau > \underline{q}$. On the other hand it follows for $\tau < \underline{q}$ that $\min\{\tau, q(\tilde{x})\} = \tau$ for all \tilde{x} near x so that clearly $q_\tau(x) = \max\{p(x), \tau\}$ and thus $x \notin V_\tau$. Thus we have monotonicity whether or

not $x \in V_\tau$ for $\tau = \underline{q}$.

Now suppose $\bar{q} = \underline{q}$ which means that $q(x)$ is continuous at x . If $\tau > q(x)$ then for $\tilde{x} \approx x$ we have that $q_\tau(\tilde{x}) = q(\tilde{x})$ and if $\tau < q(x)$ then for $\tilde{x} \approx x$ we have that $q_\tau(\tilde{x}) = \max\{p(\tilde{x}), \tau\}$. Hence we have again $x \notin V_\tau$ if $\tau < q(x)$ and for $\tau > q(x)$ we have $x \in V_\tau \Leftrightarrow L_q(x) > \omega(x, q(x))$. Now the only case left to consider is $L_q(x) \leq \omega(x, q(x))$ where we have to exclude that $x \in V_\tau$ for $\tau = q(x)$. However it follows exactly as in the proof of (i) that for $\tau = q(x)$ $L_r(x) \leq \max\{L_p(x), L_q(x)\} = L_q(x)$ so that we have monotonicity at all cases. \square

2.5 Effort constraints and gain objective function

In addition to ω our model relies on two other given real valued functions, namely

$$e(x, z) \geq e_0 > 0 \quad \text{and} \quad g(x, z) \in \mathbb{R} \quad \text{for} \quad (x, z) \in \Omega \times Z. \quad (11)$$

For any two given profiles $q \geq p$ the integral

$$E([p, q]) \equiv \int_{\Omega} \int_{p(x)}^{q(x)} e(x, z) dz dx$$

represents the “effort” to excavate all the material between profile p and q , which is expected to be bounded by the capacity of the mine operation. On the other hand, the function

$$G([p, q]) \equiv \int_{\Omega} \int_{p(x)}^{q(x)} g(x, z) dz dx$$

represents the total value or “gain” of the material between p and q (without considering a discount rate). Notice that the function $g(x, z)$ may take negative values.

When $p = p_0$ we abbreviate $G(q) \equiv G([p_0, q])$ and $E(q) \equiv E([p_0, q])$. For an ordered triplet $p \leq q \leq r$ with $p, q, r \in \mathcal{P}$ we have additivity in the sense that

$$G([p, r]) = G([p, q]) + G([q, r]) \quad \text{and} \quad E([p, r]) = E([p, q]) + E([q, r]) \quad (12)$$

We only assume the functions e and g to be measurable and uniformly bounded, i.e.

$$e, g \in L^\infty(\Omega \times Z) \quad (13)$$

Hence it is allowed that e and g have jumps due the existence of different types of material in the ore body. Now we can give the basic properties of G and E as follows.

Proposition 2.8 *Under (13) we have that*

- (i) $G(p)$ and $E(p)$ are Lipschitz continuous on $C(\bar{\Omega})$ with constants $\|e\|_\infty |\Omega|$ and $\|g\|_\infty |\Omega|$ respectively, where $|\Omega|$ denotes the area of Ω .

(ii) $G(p)$ and $E(p)$ are Gâteaux differentiable at all $p \in C(\overline{\Omega}) \setminus A$ where A is a meager set in the sense of [14].

(iii) If e (resp. g) is continuous on $\overline{\Omega} \times Z$ then $E(p)$ (resp. $G(p)$) is everywhere Fréchet differentiable. In particular, for any $\Delta p \in C(\overline{\Omega})$ we have that

$$\nabla E(p) \cdot \Delta p = \int_{\Omega} e(x, p(x)) \Delta p(x) dx. \quad (14)$$

(iv) E is convex (resp. G is concave) if e is monotonically increasing (resp. g is monotonically decreasing) w.r.t. z .

All results apply analogously to $G([p, q])$ and $E([p, q])$ when $p \neq p_0$.

Proof. We consider throughout only E without making use of the positivity assumption on e . Thus the results apply analogously to $-g(x, z)$.

(i) Considering two profiles $p, \tilde{p} \in \mathcal{P}$ we get

$$|E(\tilde{p}) - E(p)| \leq \int_{\Omega} |\tilde{p}(x) - p(x)| \sup_{z \in Z} |e(x, z)| dz dx \leq |\Omega| \|e\|_{\infty} \|\tilde{p} - p\|_{\infty}$$

(ii) By Theorem 12 of [14], which is a generalization of Rademacher's theorem, every Lipschitz mapping on an open subset G of a separable Banach space X to a Banach space Y with the Radon-Nikodym property is Gâteaux differentiable at all points of G except those belonging to a meager set A . In our case $X = C(\overline{\Omega})$ is separable and $Y = \mathbb{R}$ has the Radon-Nikodym property, i.e. every Lipschitz map $\mathbb{R} \rightarrow Y$ is differentiable almost everywhere.

(iii) We claim that under continuity of e and g there is an explicit representation for the Fréchet derivative, which in the case of E is given by (14). To establish this we rewrite the difference between E and its proposed linearization as follows:

$$\begin{aligned} & \frac{1}{\|\Delta p\|_{\infty}} \left| \left(E(p + \Delta p) - E(p) - \int_{\Omega} e(x, p(x)) \Delta p(x) dx \right) \right| \\ &= \frac{1}{\|\Delta p\|_{\infty}} \left| \int_{\Omega} \left\{ \left[\int_{p(x)}^{p(x) + \Delta p(x)} e(x, z) dz \right] - e(x, p(x)) \Delta p(x) \right\} dx \right| \\ &= \frac{1}{\|\Delta p\|_{\infty}} \left| \int_{\Omega} e(x, p(x) + \eta \Delta p(x)) \Delta p(x) - e(x, p(x)) \Delta p(x) dx \right| \end{aligned}$$

where $\eta = \eta(x, p(x), \Delta p(x)) \in [0, 1]$ is obtained by the first mean value theorem for integration. Since $e(x, z)$ is uniformly continuous on the compact set $\overline{\Omega} \times Z$ there exist for any $\varepsilon > 0$ a bound $\delta > 0$ such that $\|\Delta p\|_{\infty} < \delta$ implies $|e(x, p + \eta \Delta p) - e(x, p)| \leq \varepsilon$ and thus the last expression on the right hand side is bounded above by $\varepsilon |\Omega|$, which completes the proof.

(iv) The monotonicity assumption implies that the antiderivative $\hat{e}(x, z) \equiv \int_{p_0(x)}^z e(x, \tilde{z}) d\tilde{z}$ is convex w.r.t. z . Hence we find for two profiles $p, \tilde{p} \in \mathcal{P}$

$$\begin{aligned}
E((1 - \alpha)p + \alpha\tilde{p}) &= \int_{\Omega} \int_{p_0(x)}^{(1-\alpha)p(x) + \alpha\tilde{p}(x)} e(x, z) dz dx \\
&= \int_{\Omega} \hat{e}(x, (1 - \alpha)p(x) + \alpha\tilde{p}(x)) dx \\
&\leq \int_{\Omega} (1 - \alpha)\hat{e}(x, p(x)) + \alpha\hat{e}(x, \tilde{p}(x)) dx \\
&= \int_{\Omega} \left((1 - \alpha) \int_{p_0(x)}^{p(x)} e(x, z) dz + \alpha \int_{p_0(x)}^{\tilde{p}(x)} e(x, z) dz \right) dx \\
&= (1 - \alpha)E(p) + \alpha E(\tilde{p})
\end{aligned}$$

For g decreasing we obtain $G((1 - \alpha)p + \alpha\tilde{p}) \geq (1 - \alpha)G(p) + \alpha G(\tilde{p})$ analogously. \square

Remark: The assumption that the effort rate e is monotonically increasing w.r.t. the depth z is natural and realistic; but that the gain rate g be monotonically decreasing w.r.t. z would only apply to very particular deposits. Therefore, in general we cannot expect the gain function G to be concave.

Remark: Without continuity w.r.t z no global Fréchet differentiability is attainable even in terms of $\|\cdot\|_{Lip}$ as we can see from the following example.

Example 2.9 Consider two layers of material to be excavated like in the picture below. Further consider a sequence of profiles passing from the soft material to the hard one horizontally. The effort of this sequence is shown on the right hand side and one can see, that this is not everywhere differentiable. The profile p'' denotes the one which is located directly on the border between the two layers of material. Here it cannot be established any kind of differentiability.

3 Optimal stationary profiles

Using the properties of \mathcal{P} , G and E we have derived in the previous section we can establish existence results for profiles that are optimal in various senses. The continuous formulation we propose for the Final Open Pit problem mentioned in the introduction is the following:

$$(FOP) \quad \max\{G(p) \mid p \in \mathcal{P}\}.$$

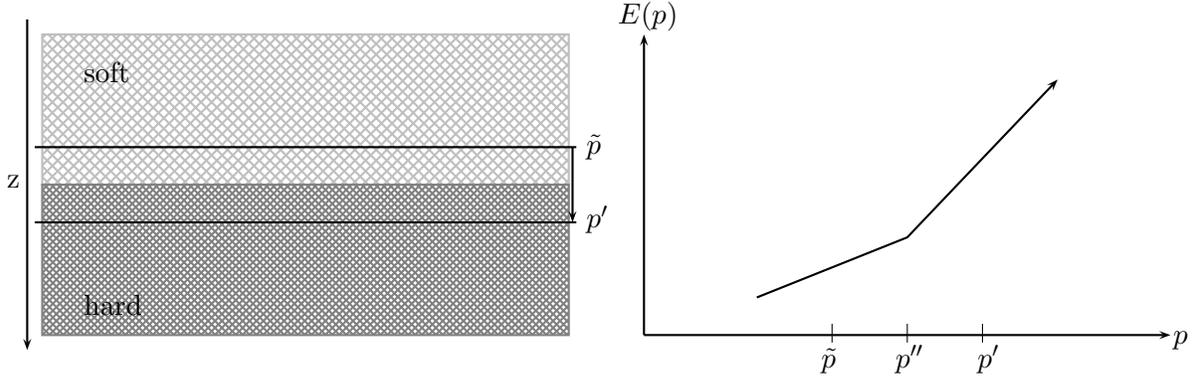


Figure 5: Example of nondifferentiability

Similarly, the continuous Capacitated FOP problem is:

$$(CFOP) \quad \max\{G(p) \mid p \in \mathcal{P}, E(p) \leq \bar{E}\}.$$

The sets of optimal solutions (global maximizers) for these problems are denoted by $S(FOP)$ and $S(CFOP)$, respectively. The following result establishes a property of the gain and effort functions which in particular is useful to investigate the structure of $S(FOP)$.

Lemma 3.1 *For all admissible profiles $p, q \in \mathcal{P}$, whether ordered, optimal or not, we have that $G(p) + G(q) = G(\min\{p, q\}) + G(\max\{p, q\})$ and $E(p) + E(q) = E(\min\{p, q\}) + E(\max\{p, q\})$.*

Proof.

$$\begin{aligned} G([p_0, p]) + G([p_0, q]) &= G([p_0, \min\{p, q\}]) \\ &\quad + G([p_0, \min\{p, q\}]) + \underbrace{G([\min\{p, q\}, p]) + G([\min\{p, q\}, q])}_{=G([\min\{p, q\}, \max\{p, q\}])} \end{aligned}$$

Here everything is done by the decomposition formula for ordered triplet (12) and the fact, that from the minimum of two profiles both excavation and gain are obtained on disjoint areas when we go on excavating to one or the other except the sets where they are the same and thus equivalent to the minimum. \square

Proposition 3.2 *Under the conditions (5) and (13) we have*

(i) $S(FOP)$ is nonempty and contains unique minimal and maximal elements $\underline{p}_g \leq \bar{p}_g$ so that

$$p \in S(FOP) \Rightarrow \underline{p}_g \leq p \leq \bar{p}_g.$$

(ii) For any bound $\bar{E} > 0$ there exists at least one global optimizer of (CFOP).

Proof.

(i) The maximum G_* on \mathcal{P} of G is attained due to the continuity of G and the compactness of \mathcal{P} by Proposition 2.2. On the other hand, Lemma 3.1 implies that the maximum \bar{p}_g and minimum \underline{p}_g over all globally optimal profiles are also optimal. The final assertion follows directly from Proposition 2.6(i).

(ii) The existence follows again from the compactness of \mathcal{P} and the continuity of E and G . \square

Remark: Proposition 2.6(i) yields that \mathcal{P} is a complete lattice as there is a maximal element \bar{p} and a minimal element p_0 , which indeed proves that it is a bounded lattice. By the existence of (10) for each subset as shown in Proposition 2.6(i) we have that all subsets have a joint and a meet. In particular the solution set of FOP is a sublattice of \mathcal{P} .

Remark: $S(\text{FOP})$ need not to be connected nor convex unless g is decreasing w.r.t. z .

Moreover we can use this result to obtain a path of optimal profiles subject to excavation constraints. For each $\lambda \geq 0$ the combined function

$$G_\lambda(p) \equiv G(p) - E(p)/\lambda$$

satisfies all the assumptions we made on $G(p)$ so far. It is in fact concave if this is true for $G(p)$ and $E(p)$ is convex, which follows from $e(x, z)$ and $-g(x, z)$ being monotonically growing. Hence $G_\lambda(p)$ has global minimizers on \mathcal{P} just like G and we obtain a whole path.

The next result follows from the general theory of parametric lattice programming [15] for which we provide a proof for the sake of completeness.

Proposition 3.3 *Under (5) and (13) there exists a path of profiles*

$$p_\lambda \equiv \arg \min\{E(p) \mid p \in \arg \max\{G_\lambda(p) \mid p \in \mathcal{P}\}\} \quad (15)$$

so that

$$0 \leq \lambda < \mu \Rightarrow p_\lambda \leq p_\mu \quad \text{and} \quad p_\lambda \in \arg \max\{G(p) \mid p \in \mathcal{P} \wedge E(p) \leq E(p_\lambda)\}$$

Proof. The existence of the p_λ follows from Proposition 2.8(i), namely G and E are Lipschitz. Let p_λ be defined as the unique minimal element among the global optimizers of G_λ existing by the fact that \mathcal{P} is a complete lattice. To prove the monotonicity consider

$$q = \min\{p_\lambda, p_\mu\}$$

for $\lambda \leq \mu$. By optimality of p_λ we get

$$G(p_\lambda) - E(p_\lambda)/\lambda = G(q) - E(q)/\lambda + G([q, p_\lambda]) - E([q, p_\lambda])/ \lambda = G_\lambda(q) + G_\lambda([q, p_\lambda]) \geq G_\lambda(q)$$

Hence we have

$$0 \leq G_\lambda([q, p_\lambda]) = G([q, p_\lambda]) - E([q, p_\lambda])/\lambda \leq G([q, p_\lambda]) - E([q, p_\lambda])/\mu = G_\mu([q, p_\lambda])$$

and using again the disjointness of $[q, p_\lambda]$ and $[q, p_\mu]$ we find that

$$\begin{aligned} G(\max\{p_\lambda, p_\mu\}) - E(\max\{p_\lambda, p_\mu\})/\mu &= G_\mu(q) + G_\mu([q, p_\lambda]) + G_\mu([q, p_\mu]) \\ &= G_\mu(p_\mu) + G_\mu([q, p_\lambda]) \\ &\geq G_\mu(p_\mu). \end{aligned}$$

By optimality of p_μ for G_μ we obtain $G_\mu([q, p_\lambda]) = 0 \Rightarrow G_\lambda([q, p_\lambda]) = 0$. Hence the first equation reads as $G(p_\lambda) - E(p_\lambda)/\lambda = G(q) - E(q)/\lambda$ and by minimality of p_λ we derive $p_\lambda = q$. Hence $p_\lambda \leq p_\mu$. The last assertion can be checked easily by contradiction. \square

It should be noted that the path p_λ established in Proposition 3.2 is in general not continuous. That can only be expected in nice cases where G is strictly concave and E is strictly convex.

As \mathcal{P} is bounded the same is true for its image $I \equiv (E(p), G(p))_{p \in \mathcal{P}} \subset \mathbb{R}^2$ in the configuration space. The global extrema of E are attained as $E(p_0) = 0$ and $E(p_u) > 0$ where $G(p_u)$ may or may not be also positive. Here p_u denotes the ultimate pit according to Prop. 2.6(i), i.e. the maximal profile bounded only by slope constraints and a given boundary $\partial\Omega$. Every point $(E(p), G(p))$ can be reached from the origin by the level excavation path according to Proposition 2.4(ii) and similar $(E(p_\infty), G(p_\infty))$, where p_∞ denotes an element of the solution set of (FOP), can be reached from it by another level excavation. The slopes of all the paths in the configuration space are bounded by the following result:

Proposition 3.4 *Under our standard assumptions we have for any pair $p \leq q \in \mathcal{P}$ with $p \neq q$*

$$\frac{|G(p) - G(q)|}{E(q) - E(p)} \leq \sigma \equiv \sup\left\{\frac{|g(x, z)|}{e(x, z)} \mid (x, z) \in \bar{\Omega} \times Z\right\} \leq \frac{\|g\|_\infty}{e_0}$$

where e_0 as defined in (11)

Proof. We have that

$$\begin{aligned} |G(q) - G(p)| &= \left| \int_{\Omega} \int_{p(x)}^{q(x)} g(x, z) dz dx \right| \leq \int_{\Omega} \int_{p(x)}^{q(x)} \left| \frac{g(x, z)}{e(x, z)} \right| e(x, z) dz dx \\ &\leq \sup\left\{\frac{|g(x, z)|}{e(x, z)} \mid (x, z) \in \bar{\Omega} \times Z\right\} \int_{\Omega} \int_{p(x)}^{q(x)} e(x, z) dz dx = \sigma(E(q) - E(p)) \end{aligned}$$

\square

like a global maximum but one can do better by simply going to the ultimate gain pit (p_∞) which renders the effort constraint $E(p) \leq \bar{E}_4$ inactive.

4 Dynamic trajectory planning

In the previous section we have established the existence of optimal gain profiles without and with excavation constraints. Rather than solving just this stationary problems one is really interested in an trajectory of profiles that gets to the valuable material as fast as possible.

In other words we are interested in maximizing the present value based with a certain discount function for future earnings.

We consider paths $P : [0, T] \mapsto \mathcal{P}$ that are monotonic, i.e. $s, t \in [0, T]$, with $s \leq t$ imply for $p = P(t)$ and $q = P(s)$ that

$$q(x) \leq p(x) \quad \text{for } x \in \Omega.$$

Naturally, the function $E(P(t))$ must be also monotonically increasing.

We suppose that there exists a function (absolutely continuous)

$$C(t) \equiv \int_0^t c(\tau) d\tau$$

representing the mining capacity in the time interval $[0, t]$, with density $c \in L^\infty(0, T)$ and $c \geq 0$. Finally, we impose the capacity condition on P

$$E([P(s), P(t)]) = E(P(t)) - E(P(s)) \leq C(t) - C(s) = \int_s^t c(\tau) d\tau \quad \text{for } s \leq t, \quad (16)$$

Now we introduce the set of feasible excavation paths:

$$\mathcal{U} = \{P \in C([0, T]; \mathcal{P}) \mid p_0 \leq P(s) \leq P(t), \quad E([P(s), P(t)]) \leq C(t) - C(s) \quad \text{for } s \leq t\}.$$

To prove the existence of maximizers we have to establish that the embedding

$$C([0, T]; \mathcal{P}) \subset C([0, T]; C(\bar{\Omega})) = C([0, T] \times \bar{\Omega})$$

is compact. This will happen again by Arzela Ascoli.

Proposition 4.1 *Under assumption (16), for any $P \in \mathcal{U}$ we have that*

$$\|P(t) - P(s)\|_\infty \leq \left[\frac{\|c\|_\infty}{e_0\pi} + 2\bar{\omega} \right] (t - s)^{1/3}. \quad (17)$$

In particular, the elements of \mathcal{U} are Hölder equicontinuous and hence \mathcal{U} is compact in $C([0, T] \times \bar{\Omega})$ with respect to $\|\cdot\|_\infty$.

Proof. The closeness of \mathcal{U} is direct from its definition together with the continuity of E on $C([0, T]; \mathcal{P})$ endowed with the norm $\|\cdot\|_\infty$. The compactness of \mathcal{U} will follow as a consequence of Arzela-Ascoli theorem. In fact, as for all $t \in [0, T]$ we have $\{P(t)\}_{P \in \mathcal{U}} \subset \mathcal{P}$ with \mathcal{P} being compact by Prop. 2.5, it only remains to prove that \mathcal{U} is equicontinuous. To this end, fix $x_0 \in \Omega$ and take $s, t \in [0, T]$ with $t - s > 0$. Let $P \in \mathcal{U}$ be arbitrary. For any $x \in \Omega$, as $P(s), P(t) \in \mathcal{P}$, by global Lipschitz continuity of each profile we have that

$$P(t)(x) - P(s)(x) \geq -2\bar{\omega}|x - x_0| + P(t)(x_0) - P(s)(x_0).$$

Multiplying by the lower bound $e_0 > 0$ on $e(x, z)$ and integrating with respect to x on the ball $|x - x_0| \leq \delta$ for some $\delta > 0$ sufficiently small which we will choose later on, we get

$$e_0\pi\delta^2 [-2\bar{\omega}\delta + P(t)(x_0) - P(s)(x_0)] \leq \int_{\Omega} \int_{P(s)(x)}^{P(t)(x)} e(x, z) dz dx = E(P(t)) - E(P(s)),$$

where we have supposed that Ω is an open subset of \mathbb{R}^n with $n = 2$ (the case $n = 1$ is similar) and that the ball $B_\delta(x_0)$ is contained in Ω . Therefore

$$e_0\pi\delta^2 [-2\bar{\omega}\delta + P(t)(x_0) - P(s)(x_0)] \leq \int_s^t c(\tau) d\tau \leq \|c\|_\infty (t - s),$$

Hence we deduce that for all $\delta > 0$ small enough, we have

$$0 \leq P(t)(x_0) - P(s)(x_0) \leq \frac{\|c\|_\infty}{e_0\pi} \frac{t - s}{\delta^2} + 2L\delta.$$

If $t - s > 0$ is small then we can take for instance $\delta^3 = t - s$ to obtain

$$0 \leq P(t)(x_0) - P(s)(x_0) \leq \left[\frac{\|c\|_\infty}{e_0\pi} + 2\bar{\omega} \right] (t - s)^{1/3},$$

and since $x_0 \in \Omega$ is arbitrary, it follows that (17) holds. This proves the equicontinuity of \mathcal{U} . \square

Concerning the internal structure of \mathcal{U} , we firstly note that for arbitrary $P, Q \in \mathcal{U}$, neither $\max\{P, Q\}$ nor $\min\{P, Q\}$ need to be elements of \mathcal{U} , as both may violate the capacity constraint (16). For example, consider P and Q as depicted in Fig . 7.

P and Q are excavating material in an unit timeinterval at maximal capacity. The minimum of both would have to excavate twice the allowed value as the domains or the excavation are disjoint. That \mathcal{U} is nevertheless path connected is a consequence of the following Lemma.

Lemma 4.2 *For the set of feasible excavation paths the following holds:*

- (i) Let $P \in \mathcal{U}$ and $p \in \mathcal{P}$ be given. $\Rightarrow \min\{p, P\} \in \mathcal{U} \ni \max\{p, P\}$
- (ii) Any $P(t)$ is connected to the trivial path p_0 via the path $P_r(t) \equiv \min\{p_0 + r, P(t)\}$ for $r \in [0, \bar{z} - \underline{z}]$.

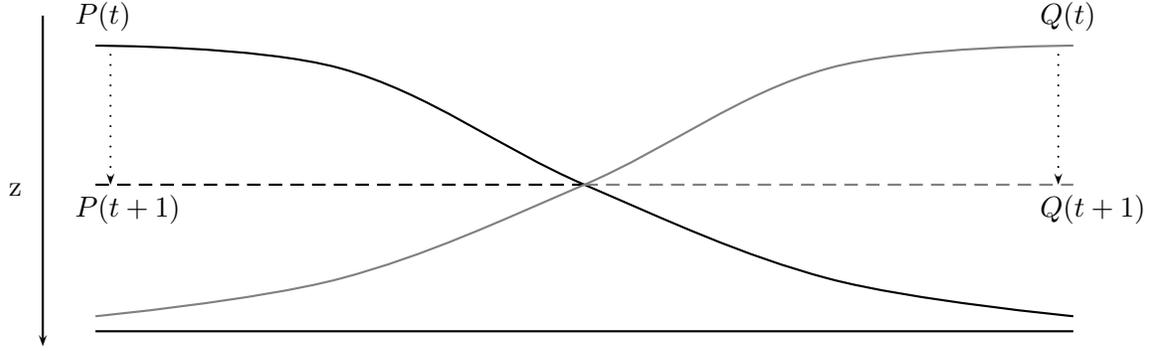


Figure 7: $\min\{P, Q\} \notin \mathcal{U}$

Proof.

(i) We have $p_0 \leq \min\{p, P\}(0)$ and $p_0 \leq \max\{p, P\}(0)$ and by monotonicity for all $s, t \in [0, T]$. Now consider the path $\min\{p, P\}(t)$. We choose $s \leq t \in [0, T]$ arbitrary and define the following subsets of Ω .

$$\begin{aligned}\Omega_1 &\equiv \{x \in \Omega | p(x) < P(s)(x) \leq P(t)(x)\} \\ \Omega_2 &\equiv \{x \in \Omega | P(s)(x) \leq p(x) \leq P(t)(x)\} \\ \Omega_3 &\equiv \{x \in \Omega | P(s)(x) \leq P(t)(x) < p(x)\}\end{aligned}$$

Of course $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$. For (16) we have

$$\begin{aligned}\int_{\Omega} \int_{\min\{p, P\}(s)(x)}^{\min\{p, P\}(t)(x)} e(x, z) dz dx &= \underbrace{\int_{\Omega_1} \int_{p(x)}^{p(x)} e(x, z) dz dx}_{=0} + \underbrace{\int_{\Omega_2} \int_{P(s)(x)}^{p(x)} e(x, z) dz dx}_{\leq \int_{P(s)(x)}^{P(t)(x)} e(x, z) dz dx} + \int_{\Omega_3} \int_{P(s)(x)}^{P(t)(x)} e(x, z) dz dx \\ &\leq \int_{\Omega} \int_{P(s)(x)}^{P(t)(x)} e(x, z) dz dx \leq \int_s^t c(\tau) d\tau\end{aligned}$$

The last estimation holds because P is a feasible path. For max the proof is analogous.

(ii) Follows from (i). \square

Let $\varphi \in C^1(0, T)$ be a monotonically decreasing discount function starting from $\varphi(0) = 1$ and ending at $\varphi(T) < 1$ for some fixed time period $[0, T]$. The standard choice is $\varphi(t) = e^{-\delta t}$ for some discount rate $\delta > 0$.

For paths $P(t, x) \equiv P(t)(x)$ that are smooth in time we may define the present value of the gain

as

$$\hat{G}(P) \equiv \int_0^T \varphi(t) \int_{\Omega} g(x, P(t, x)) dx dP(t) = \int_{\Omega} \int_0^T \varphi(t) g(x, P(t, x)) P'(t) dt dx$$

where the notation dP suggest that $P(t, x)$ must be differentiable w.r.t. t . Integrating by parts we can avoid this requirement and obtain

$$\begin{aligned} \hat{G}(P) &= \varphi(T) \int_{\Omega} \hat{g}(x, P(T, x)) - \hat{g}(x, P(0, x)) dx \\ &\quad + \int_{\Omega} \int_0^T [-\varphi'(t)] [\hat{g}(x, P(t, x)) - \hat{g}(x, P(0, x))] dt dx \end{aligned} \quad (18)$$

where $\hat{g}(x, z)$ is the antiderivative

$$\hat{g}(x, z) \equiv \int_{p_0(x)}^z g(x, \zeta) d\zeta$$

We note that the first term is the total value of the final profile $P(T, x)$ discounted by $\varphi(T)$. The second term, is a correction due to the variation of φ (with $-\varphi'(t) > 0$). This representation of $\hat{G}(P)$ is well defined for every path in $C([0, T] \times \Omega)$.

The optimization problem in the dynamic trajectory planning case, the so called Capacitated Dynamic Open Pit Problem is the following one

$$(CDOP) \quad \max\{\hat{G}(P) \mid P \in \mathcal{U}, P(0) = p_0\}$$

For the objective function \hat{G} we obtain the following results.

Proposition 4.3 *Under the above assumptions for $P, Q \in C([0, T] \times \overline{\Omega})$ we have*

- (i) $|\hat{G}(P) - \hat{G}(Q)| \leq |\Omega| \|g\|_{\infty} \|P - Q\|_{\infty}$
- (ii) *If $g(x, z)$ is decreasing w.r.t. z then \hat{G} is concave on $C([0, T] \times \overline{\Omega})$*
- (iii) *For any $p \in \mathcal{P}$ we have $\hat{G}(P) = \hat{G}(\min\{P, p\}) + \hat{G}(\max\{P, p\})$*

Proof.

(i) Since $\|g\|_{\infty}$ is a Lipschitz constant for \hat{g} , with respect its second argument we can bound the difference in the first term of \hat{G} in (18) by $\varphi(T)|\Omega| \|g\|_{\infty} \|P - Q\|_{\infty}$. Since $-\varphi'(t) \geq 0$ by

assumptions we can similarly bound

$$\begin{aligned}
& \left| \int_{\Omega} \int_0^T -\varphi'(t) [\hat{g}(x, P(t, x)) - \hat{g}(x, Q(t, x))] dt dx \right| \\
& \leq \int_{\Omega} \int_0^T -\varphi'(t) |\hat{g}(x, P(t, x)) - \hat{g}(x, Q(t, x))| dt dx \leq \int_{\Omega} \int_0^T -\varphi'(t) \|g\|_{\infty} \|P - Q\|_{\infty} dt dx \\
& = \int_{\Omega} \|g\|_{\infty} \|P - Q\|_{\infty} (\varphi(0) - \varphi(T)) dx = |\Omega| \|g\|_{\infty} \|P - Q\|_{\infty} (1 - \varphi(T)) dx
\end{aligned}$$

Hence in summing the two terms $\varphi(T)$ cancels out and the Lipschitz constant becomes completely independent of the discount function.

(ii) This is analogous to the proof of Proposition 2.8(ii) so we omit the details.

(iii) Let $\Omega_p(t) \equiv \{x \in \Omega \mid p \leq P(t)\}$ be the subset of the domain where the path is pointwise greater than the given profile. We have

$$\begin{aligned}
\hat{G}(P) &= \varphi(T) \left(\int_{\Omega_p(T)} \hat{g}(x, p(x)) + \int_{p(x)}^{P(T,x)} g(x, \zeta) d\zeta + \int_{\overline{\Omega_p(T)}} \hat{g}(x, P(T, x)) dx \right) \\
&+ \int_0^T [\varphi'(t)] \left(\int_{\Omega_p(t)} \hat{g}(x, p(x)) + \int_{\Omega_p(t)} \int_{p(x)}^{P(t,x)} g(x, \zeta) d\zeta dx + \int_{\overline{\Omega_p(t)}} \hat{g}(x, P(T, x)) dx \right) dt
\end{aligned}$$

This yields to

$$\begin{aligned}
\hat{G}(P) &= \varphi(T) \left(\int_{\Omega_p(T)} \hat{g}(x, p(x)) + \int_{\overline{\Omega_p(T)}} \hat{g}(x, P(T, x)) dx \right) \\
&+ \int_0^T [\varphi'(t)] \left(\int_{\Omega_p(t)} \hat{g}(x, p(x)) dx + \int_{\overline{\Omega_p(t)}} \hat{g}(x, P(T, x)) dx \right) dt \\
&+ \varphi(T) \int_{\Omega_p(T)} \int_{p(x)}^{P(T,x)} g(x, \zeta) d\zeta dx + \int_0^T [\varphi'(t)] \int_{\Omega_p(t)} \int_{p(x)}^{P(t,x)} g(x, \zeta) d\zeta dx dt \\
&= \hat{G}(\min\{P, p\}) + \hat{G}(\max\{P, p\})
\end{aligned}$$

The first two lines of the sum indeed represent the discounted gain of the path P “stopped” at the profile p . \square

By combining Propositions 4.1 and 4.3 we conclude immediately that \hat{G} attains global maximizer on \mathcal{U} . Without any additional assumption and for a limited time horizon $T < \infty$ one can quite easily construct examples where the set of global minimizers is disconnected and may jump around violently when the gain density is slightly perturbed. One only has to think of two areas with high gain in separate places that have nearly identical values and excavation costs.

Due to Lipschitz continuity of \hat{G} on the Banach space $C([0, T] \times \bar{\Omega})$ it has similar differentiability properties as those stated in Proposition 2.8 for G . However as it is typical for optimization it is quite likely that optimizers belong to the meager set of exceptional points where one does not even have Gâteaux differentiability. This can be seen from the following example.

Example 4.4 *Let us consider the triangle an example in which $\omega(x, z) = 2$ and $e(x, z) = 1$ every where such that $e(x, z) \equiv 1 + z$ and $g = \max(0, z - |x|)$. In this case the optimal strategy is to extract the harder material in a horizontal floor in such manner $P(t, x) = \min(1 - |x|, \zeta(t))$ for an increasing value $\zeta(t) = 1 - \sqrt{1 - t}$. Let us fix now $T \leq 1$ and then to “push” the optimal path to the left or right in such manner that $P_\varepsilon(t, x) \equiv P(t, x + \varepsilon)$. Then, the value $\hat{G}(P_\varepsilon)$ is not derivable with respect to the perturbation ε .*

We conclude with a relation with the stationary problem (FOP). Recall that p_∞ is a solution of FOP accordingly with (15).

Proposition 4.5 *Let $\varphi \in C^1([0, T])$ be monotonically decreasing. Then*

- (i) *If $\hat{G}(P) > 0$ for some $P \in \mathcal{U}$ then $G_\infty \equiv G(p_\infty) > 0$.*
- (ii) *If \hat{G}_* is the optimal value of (CDOP) then*

$$\hat{G}_* = \max\{\hat{G}(P) \mid P \in \mathcal{U}_\infty\}$$

where $\mathcal{U}_\infty \equiv \mathcal{U} \cap C([0, T]; \mathcal{P}_\infty)$ for $\mathcal{P}_\infty \equiv \{p \in \mathcal{P} \mid p \leq p_\infty\}$.

Proof.

(i) Suppose $G_\infty = G(p_\infty) \leq 0$ and choose $P \in \mathcal{U}$ arbitrary. Then we have for any t also

$$G(P(t)) = \int_{\Omega} \hat{g}(x, P(t, x)) dx \leq 0.$$

Otherwise we would have $G(P(t)) > G_\infty$ which contradicts the definition of G_∞ . Substituting this relation into the integrated form (18) immediately yields $\hat{G}(P) \leq 0$ which completes the proof of (i).

(ii) Consider a $P \in \mathcal{U}$ with $\Omega_{p_\infty}(t) \neq \emptyset$ from a certain time $t < T$. By Proposition 4.3(iii) we know we can decompose the objective in the following way.

$$\hat{G}(P) = \hat{G}(\min\{P, p_\infty\}) + \varphi(T) \int_{\Omega_p(T)} \int_{p_\infty(x)}^{P(T,x)} g(x, \zeta) d\zeta dx + \int_0^T [\varphi'(t)] \int_{\Omega_p(t)} \int_{p_\infty(x)}^{P(t,x)} g(x, \zeta) d\zeta dx dt$$

The last two terms, representing the gain the path yields between p_∞ and $P(T)$, are an integral over the part, where the path is pointwise larger than p_∞ . Hence, because p_∞ is the solution of (FOP), they can not be positive and for the the stopped path we have $\hat{G}(\min\{P, p_\infty\}) \geq \hat{G}(P)$. Thus any maximizer of \hat{G} is bounded by the solutions of (FOP). \square

Remark: By virtue of Proposition 4.5(ii) for maximizing the discounted gain it suffices to consider feasible profiles such that $p \leq p_\infty$. This is a continuous analogue of a well known property of standard 0–1 discrete formulations for these type of problems, namely that the optimal Final Open Pit is an upper bound on the region to be considered of interest for the nested sequence of profiles, a property which is used to reduce the size of the original block model.

5 Concluding remarks

To the best of our knowledge, the first attempts to give rigorous mathematical formulations of classical open pit mine planning problems based on continuous models date back to the works by Matheron in the 1970s. For instance, in [12] it is developed a general measure-theoretic approach to a parametrized version of the CFOP problem, obtaining the analogous to Propositions 2.6(i), 3.2(i) and 3.3 of this paper. No significant contribution seems to be made since then.

In our approach, the optimization problems are posed in functional spaces and a complete existence theory holds. In fact, under realistic assumptions, the feasible set of profiles in the stationary case as well as the set of paths of them in the dynamical case are compact sets in suitable Banach spaces of real valued functions, which implies the to the existence of solutions as the objective functions are proven to be (Lipschitz) continuous.

In addition, we have provided some interesting structural properties of feasible sets and optimal solutions in the stationary case, and we have developed a qualitative analysis of the behavior in the effort-gain configuration space. We have also obtained some sufficient regularity conditions for differentiability of effort and gain functions, under the assumption that the effort and gain densities are continuous. In general we avoid this assumption as it precludes sharp fronts between layers of different materials, which arise in practice.

We have focused on the existence theory. Another interesting question is how to obtain useful first-order optimality conditions under weak regularity conditions. A possible approach to tackle this problem is to consider duality theory, but up to now we have not made any significant

progress in this direction.

The numerical resolution of the continuous problems is beyond the scope of this paper. Indeed, it is possible to consider various discretization schemes for these problems and apply some numerical methods, but at this stage we only have some very simplified (though encouraging) experiments and more research on numerical methods is needed. In particular, we believe that the established structural properties of the feasible profiles may be exploited from an algorithmic point of view, and we plan to pursue this line of research in the future.

Finally, we would like to make clear that the continuous approach proposed here is not intended to completely substitute the more traditional discrete techniques, but to supplement them with additional tools coming from continuous optimization in functional spaces. On the one hand, as the underlying discrete block models become larger in terms of the number of unitary extraction blocks, the continuous formulations may be viewed as a sort of limiting averaged model, which should provide qualitative and quantitative information about the behavior of optimal solutions from a macroscopic point of view. In particular, continuous optimization is potentially well suited for investigating the sensibility w.r.t. discount rates or extraction effort capacities, which are very interesting issues for future research. On the other hand, the approximate resolution of a macroscopic continuous model might be useful to obtain some insight on how to construct good starting points for discrete problems based on namely microscopic block models.

The connections between the continuous and discrete approaches, in terms of theoretical and algorithmic aspects, and the question on how we might use them from a practical point of view, are clearly key points that will be addressed in our future research.

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