An Interior Proximal Method in Vector Optimization

Kely D. V. Villacorta[*] and *Paulo Roberto Oliveira*[†]

August 13, 2009

Abstract

This paper studies the vector optimization problem of finding weakly efficient points for maps from $\mathbb{R}^n$ to $\mathbb{R}^m$, with respect to the partial order induced by a closed, convex, and pointed cone $C \subset \mathbb{R}^m$, with nonempty interior. We develop for this problem an extension of the proximal point method for scalar-valued convex optimization problem with a modified *convergence sensing conditon* that allows us to construct an interior proximal method for solving POV on nonpolyhedral set.
Keywords: *Interior point methods, vector optimization, $C-$convex, positively lower semicontinouos.*

1 Introduction.

Let C be a closed, convex, and pointed cone in $\mathbb{R}^m$, with $\text{int}(C) \neq \emptyset$, where $\text{int}(C)$ denotes the interior of set C. Then C induces a partial order $\preceq_C$ in $\mathbb{R}^m$, given by $y \preceq_C y'$ if and only if $y' - y \in C$, with its associate relation $\prec_C$, given by $y \prec_C y'$ if and only if $y' - y \in \text{int}(C)$. Our goal is to analize methods to find a weakly efficient minimizer of the following problem

$$(VOP) \qquad C - \min\{F(x) : x \in \Omega\}$$

[*]Department of Systems Engineering and Computer Sciences, COPPE, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil, partially supported by CAPES/CNPq - IEL Nacional - Brasil, `kelydvv@cos.ufrj.br`

[†] Department of Systems Engineering and Computer Sciences, COPPE, Federal University of Rio de Janeiro, Rio de Janeiro, Brazil , `poliveir@cos.ufrj.br`

where Ω is a nonempty convex closed set in $\mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^m \cup \{+\infty_C\}$ is a proper, positively lower semicontinouos (plsc) $C-$convex map.

Recently, Bonnel et al. in (2005) and Ceng and Yao in (2007) considered the extension to vector-valued optimization of several iterative methods for scalar-valued methods. In those extensions, they define the iterates in the vector-valued case by considering the order $\preceq_C$ in Y, where Y is a real Banach space, mimicking, whenever possible, the role of the usual order in $\mathbb{R}$ in the corresponding algorithm for scalar-valued optimization. In the meantime, we admit the possibility that F takes value $+\infty_C$.

The last decade has seen considerable progress in the theory of the proximal point methods for scalar-valued problems, all of them based on generalized distances (Auslender and Teboulle, 2006; Kaplan and Tichatschke, 2004, 2007). Thus, we give next a brief description of this method. Consider the following convex minimization problem:

$$\inf\{f(x) : x \in \Omega\}, \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^m \cup \{+\infty\}$ is a proper, lower semicontinuous convex function. The proximal point method generates a sequence $\{x^k\} \subset \mathbb{R}^n$ corresponding to the recursion

$$g^{k+1} + \beta_k \nabla_1 d(x^{k+1}, x^k) = 0. \tag{2}$$

where $g^{k+1} \in \partial_{\varepsilon_k} f(x^k)$, $\{\beta_k\}$ is a bounded exogenous sequence of positive real numbers (called regularization parameters), $\nabla_1 d(\cdot, y)$ denotes the gradient map of function $d(\cdot, y)$ with respect to the first variable, d is some proximity measure, and x^k the current iterate.

With the choice $d(x, y) = 2^{-1}\|x - y\|^2$ and $\varepsilon_k = 0$, one recovers the proximal algorithm, whose origins can be traced back to the 1960s (see, e.g., Moreau, 1965; Martinet, 1970, 1972; Rockafellar, 1976). In this case, the sequence $\{x^k\}$ produced by the above algorithm does not necessarily belong to $\text{int}(\Omega)$. Thus the proximal term $d(x, y)$ will play the role of a distance-like function, satisfying certain properties, see Section 2, which will force the iterates of the produced sequence to stay in $\text{int}(\Omega)$, and thus automatically eliminate the constraints.

It has been proved in (Auslender and Teboulle, 2006) that the sequence $\{x^k\}$ generated by the proximal point method (2) belongs to $\text{int}(\Omega)$ and converges to the solution of problem (1), pursuant to certain properties.

In the so-called inexact versions of the method, x^k need not be the exact solution of the subproblem in (2), but only an approximate solution of it. Clearly, the inexact version is essential if one wants the convergence results to hold for actual implementations of the method. Inexact version were proposed as early as 1976 (see Rockafellar, 1976), in which the kth subproblem was allowed to be solved within a prescribed tolerance ε_k, and it was demanded that $\sum_{k=0}^{\infty} \varepsilon_k < \infty$. Similar error criteria, always requiring summability of the tolerances, appeared in several papers later on.

Kaplan and Tichatschke in (2007, 2008) use a Bregman function with a modified "convergence sensing condition" that enables them to construct a generalized proximal method for solving (1) on sets that are not necessarily polyhedrals. For this, they admit a successive approximation of the operator ∂f and an inexact calculation of the proximal iterate. This method generates sequences $\{x^k\} \subset \mathbb{R}^n$ and $\{\theta^k\} \subset \mathbb{R}^n$ corresponding to the recursion

$$\theta^{k+1} \in Q^k(x^{k+1}) + \beta_k \nabla_1 D_{\hat{h}}(x^{k+1}, x^k) \tag{3}$$

where $\{\beta_k\}$ is a bounded exogenous sequence of positive real numbers, $D_{\hat{h}}$ is the Bregman distance induced for $\hat{h} = h + \eta$ and $\partial f \subset Q^k \subset \partial_{\varepsilon_k} f$.

It has been proved in (Kaplan and Tichatschke, 2007) that if h, η, θ^k and ε_k satisfy certain properties, the sequence $\{x^k\}$ generated by the proximal point method (3) belongs to $\text{int}(\Omega)$ and converges to some solution of problem (1).

The above discussion refers, of course, to the proximal method for scalar-valued convex optimization. The essence of this paper consists of the extension of both the exact proximal method (2), $\varepsilon_k = 0$, and inexact counterpart (3) to the vector-valued optimization problem introduced at the beginning of this section. Basically, in the exact case the kth subproblem will consist of finding weakly efficient minimizers of

$$F(x) + \beta_k d(x, x^k) e_k \tag{4}$$

restricted to the set $\Omega_k \subset \Omega$ defined as $\Omega_k = \{x \in \Omega : F(x) \preceq_C F(x^k)\}$, where d is a proximal distance on $\text{int}(\Omega)$, e_k is an exogenously selected vector belonging to $\text{int}C$ and such that $\|e_k\| = 1$.

For our inexact version, we consider the positive polar cone $C^* \subset \mathbb{R}^m$, given by $C^* = \{z \in \mathbb{R}^m : \langle y, z\rangle \geq 0 \text{ for all } y \in C\}$, and the indicator function I_{Ω_k}, of set Ω_k, defined as above. We take an exogenous sequence $\{z_k\} \subset C^*$, with $\|z_k\| = 1$ for all $k \in \mathbb{N}$, and define, at iteration k, the function $f_k : \mathbb{R}^n \to \mathbb{R}^m \cup \{+\infty\}$ as

$$f_k(x) = \langle F(x), z_k\rangle + I_{\Omega_k}. \tag{5}$$

Then we take as x^{k+1} any vector $x \in \Omega$ such that there exists $\theta^k \in \mathbb{R}^n$, $\varepsilon_k \in \mathbb{R}_+$ satisfying

$$\theta^{k+1} \in Q^k(x) + \beta_k \langle e_k, z_k\rangle \nabla_1 \tilde{d}(x, x^k), \tag{6}$$

where $\partial f \subset Q^k \subset \partial_{\varepsilon_k} f$ and $\tilde{d}$ is a convenient proximal distance.

We will establish that any sequence generated by either our exact or inexact version converges to a weakly efficient minimizer of F on Ω under the following two assumptions:

(i) F is $C-$convex on Ω, i.e., $F(\lambda x + (1-\lambda)x') \preceq_C \lambda F(x) + (1-\lambda)F(x')$ for all x, $x' \in \Omega$ and all $\lambda \in [0,1]$.

(ii) The set $(F(x^0) - C) \cap F(\Omega)$ is $C-$complete; i.e., for every sequence $\{a_k\} \subset \Omega$, with $a_0 = x^0$, such that $F(a_{n+1}) \preceq_C F(a_k)$ for all $k \in \mathbb{N}$, there exists $a \in \Omega$ such that $F(a) \preceq_C F(a_k)$ for all $k \in \mathbb{N}$.

In the absence of assumption (ii), we establish convergence results, namely, that the generated sequence is a minimizing one for our problem, meaning that $\{F(x_k)\}$ approaches the set of infimal values of F, in a sense explicited in Propositions 4.2 and 4.3 of Section 4.

The paper is organized as follows: Section 2 introduces and recalls some required preliminary material. Section 3 formally states the problem. The exact version of the method is analyzed in Section 4. Section 5 develops the inexact version.

2 Preliminaries

We adopt the following convex analysis notation (Rockafellar, 1970). For a proper convex and lsc function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, its effective domain is set by $\operatorname{dom} f = \{x : f(x) < +\infty\}$, and for all $\epsilon \geq 0$ its $\epsilon-$subdifferential at x is defined by $\partial_\epsilon f(x) = \{g \in \mathbb{R}^n : \forall\, z \in \mathbb{R}^n,\ f(z) + \epsilon \geq f(x) + \langle g, z - x\rangle\}$, which coincides with the usual subdifferential $\partial f = \partial_0 f$ whenever $\epsilon = 0$. We set $\operatorname{dom} \partial f = \{x \in \mathbb{R}^n : \partial f(x) \neq \emptyset\}$. For any closed convex set $S \subset \mathbb{R}^n$, I_S denotes the indicator function of S, $\operatorname{ri}(S)$ its relative interior, $O^+(S)$ its recession cone, and $N_S(x) = \partial I_S(x) = \{\nu \in \mathbb{R}^n : \langle \nu, z - x\rangle \leq 0\ \forall\, z \in S\}$ the normal cone to S at $x \in S$.

Now, we recall some useful properties from convex analisys and on nonnegative sequences.

Lemma 2.1 [Rockafellar, 1970; Corollary 6.5.2] *Let S_1 be a convex set. Let S_2 be a convex set contained in $\overline{S_1}$ but not entirely contained in the relative boundary of S_1. Then $\operatorname{ri}(S_2) \subset \operatorname{ri}(S_1)$.*

Lemma 2.2 [Rockafellar, 1970; Theorem 27.4] *Let f be a proper convex function, and let S be a nonempty convex set. In order that x^* be a point where the infimum of f, as related to S, is attained, it is sufficient that there exists a vector $y^* \in \partial f(x^*)$ such that $-y^*$ is normal to S at x^*. This condition is necessary, as well as sufficent, if $\operatorname{ri}(\operatorname{dom} f)$ intersects $\operatorname{ri}(S)$, or if S is polyhedral and $\operatorname{ri}(\operatorname{dom} f)$ merely intersects S.*

Lemma 2.3 [Polyak, 1987; Lemma 2.2.2] *Let* $\{\xi_k\}$, $\{\upsilon_k\}$ *and* $\{\zeta_k\}$ *nonegative sequences of real numbers satisfying* $\xi_{k+1} \leq (1+\upsilon_k)\xi_k + \zeta_k$ *and such that* $\sum_{k=1}^{\infty} \zeta_k < \infty$, $\sum_{k=1}^{\infty} \upsilon_k < \infty$. *Then, sequence* $\{\xi_k\}$ *converges.*

2.1 Proximal distances

In this part we remember definitions on appropriate proximal distance d and we use a slightly modified of the definition induced proximal distance, H, given in (Auslender and Teboulle, 2006).

Definition 2.1 *A function* $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ *in called a* ***proximal distance*** *with respect to an open nonempty convex set* $S \subset \mathbb{R}^n$ *if for each* $y \in S$ *it satisfies the following properties:*

(P_1) $d(\cdot, y)$ *is proper, lsc, convex, and continuously differentiable on* S*;*

(P_2) $\operatorname{dom} d(\cdot, y) \subset \overline{S}$ *and* $\operatorname{dom} \nabla_1 d(\cdot, y) = S$, *where* $\nabla_1 d(\cdot, y)$ *denotes the gradient map of function* $d(\cdot, y)$ *with respect to the first variable;*

(P_3) $d(\cdot, y)$ *is level bounded on* $\mathbb{R}^n$, *i.e.,* $\lim_{\|x\| \to \infty} d(x, y) = +\infty$*;*

(P_4) $d(y, y) = 0$.

As in (Auslender and Teboulle, 2006), we also denote by $\mathcal{D}(S)$ the family of functions d satisfying Definition 2.1.

The next definition associates to each given $d \in \mathcal{D}(S)$ another function satisfying some convenient properties.

Definition 2.2 *Given* $S \subset \mathbb{R}^n$, *open and convex, and* $d \in \mathcal{D}(S)$, *a function* $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ *is called the* ***induced proximal distance*** *to* d *if* H *is finite valued on* $S \times S$ *and for each* y, $z \in S$ *satisfies*

$$H(y, y) = 0 \tag{7}$$

$$\langle \nabla_1 d(z, y), x - z \rangle \leq H(x, y) - H(x, z) \quad \forall\, x \in S. \tag{8}$$

We also write $(d, H) \in \mathcal{F}(S)$ to specify the triple $[S, d, H]$ that satisfies the premises of Definition 2.2.

Remark 2.1 *Likewise, we will write* $(d, H) \in \mathcal{F}(\overline{S})$ *for the triple* $[\overline{S}, d, H]$ *whenever there exists* H *which is finite valued on* $\overline{S} \times S$*, satisfies* (7)-(8) *for any* $x \in \overline{S}$*, and is such that* $\forall\, x \in \overline{S}$ *one has* $H(x, \cdot)$ *level bounded on* S*. Clearly, one has* $\mathcal{F}(\overline{S}) \subset \mathcal{F}(S)$*.*

Now, we need to make a further assumption on the induced proximal distance H, mimicking the behaviour of norms. For this, we introduce $\mathcal{F}(\overline{S})$ which is a subset of $\mathcal{F}(\overline{S})$.

Definition 2.3 *Given* $S \subset \mathbb{R}^n$*, open and convex, and* $(d, H) \in \mathcal{F}(\overline{S})$*, we denote by* $\mathcal{F}^*(\overline{S})$ *the set of pairs* (d, H) *such that function* H *satisfies the following two additional properties:*

(P_5) *If* $\{y^k\} \subset S$ *is a bounded sequence in* S *and* $\bar{y} \in \overline{S}$ *such that* $\lim_{k \to +\infty} H(\bar{y}, y^k) = 0$*, then* $\lim_{k \to +\infty} y^k = \bar{y}$*;*

(P_6) *If* $\{y^k\} \subset S$ *converges to* y*, then at least one of the relations*

$$\lim_{k \to +\infty} H(y, y^k) = 0, \tag{9}$$

$$\lim_{k \to +\infty} H(\bar{y}, y^k) = +\infty, \quad \forall\, \bar{y} \neq y, \quad \bar{y} \in S \tag{10}$$

holds true.

Remark 2.2 *The last definition is a weaker version convergence sensing condition than that given in* (Auslender and Teboulle, 2006) *and was introduced by Kaplan and Tichatschke in* (2007)*, for the Bregman function. Note also that with this definition we could also work on nonpolyhedral sets.*

In the context of proximal distances, two known choices for d include either Bregman distance (see, e.g., Auslender and Teboulle, 2006; Chen and Teboulle, 1993; Kiwiel, 1997) or $\varphi-$divergences (see, e.g., Auslender and Teboulle, 2006; Teboulle, 1992, 1997). These works have concentrated on the ground set $\overline{S}$ being polyhedral and in particular when $\overline{S}$ is the nonnegative octant in $\mathbb{R}^n$. More recent works have also proposed Bregman-like distances, which introduces a weakened convergence sensing condition (see Kaplan and Tichatschke, 2007, 2008) which enables the work on a nonpolyhedral set.

2.2 Vector optimization problem and related results

We consider maps $G : \mathbb{R}^n \to \mathbb{R}^m \cup \{+\infty_C\}$[1], where $C \subset \mathbb{R}^m$ is a closed, convex, and pointed cone with nonempty interior, which defines a partial order "$\preceq$" in $\mathbb{R}^m$.

We denote by $\operatorname{dom} G = \{x : G(x) \neq +\infty_C\}$ the ***effective domain*** of G. We say that G is ***proper*** if $\operatorname{dom} G \neq \emptyset$.

We extend by continuity every $z \in C^* \setminus \{0\}$ to $\mathbb{R}^m \cup \{-\infty_C, +\infty_C\}$, by putting $\langle \pm\infty_C, z\rangle = \pm\infty$ (see [Bolintinéanu, 2000] for more details).

Lemma 2.4 [Bolintinéanu, 2000; Lemma 2.2] *For all $e \in \operatorname{int}(C)$, we have*

$$\inf\{\langle e, z\rangle : z \in C^*,\ \|z\| = 1\} = \operatorname{dist}(e, \mathbb{R}^m \setminus C).$$

Lemma 2.5 [Bolintinéanu, 2001; Lemma 2.7] *Let $z \in \operatorname{int}(C^*)$. Then*

$$\inf\{\langle z, y\rangle : y \in C,\ \|y\| = 1\} \geq \operatorname{dist}(z, \mathbb{R}^m \setminus C^*).$$

Proposition 2.1 [Luc, 1989] *Let $G : \mathbb{R}^n \to \mathbb{R}^m \cup \{+\infty_C\}$ be a map. G is C-convex if and only if $\langle G(\cdot), z\rangle$ is convex for every $z \in C^*$.*

Definition 2.4 [Luc, 1989] *Let $G : \mathbb{R}^n \to \mathbb{R}^m \cup \{+\infty_C\}$ be a map. We say that G is **positively lower semicontinuous** if $\langle G(\cdot), z\rangle$ is lower semicontinuous for every $z \in C^*$.*

We associate a given set $T \subset \mathbb{R}^m \cup \{-\infty_C, +\infty_C\}$ the following sets:

i) ***the infimal set***:

$$C - \operatorname{INF}(T) = \{y \in \overline{T} : \nexists w \in T \setminus \{y\} \quad \text{such that} \quad w \preceq_C y\}$$

ii) ***the weakly infimal set***:

$$C - \operatorname{INF_w}(T) = \{y \in T : \nexists w \in T \quad \text{such that} \quad w \prec_C y\}$$

Now, given a vector optimization problem of the form

$$(P) \quad C - \min\Big\{G(x) : x \in U\Big\} \tag{11}$$

where $U \subset \mathbb{R}^n$.

[1]The extended space $\mathbb{R}^m \cup \{-\infty_C, +\infty_C\}$ was introduced in (Bolintinéanu, 2000).

Definition 2.5 *A point $x^* \in \mathbb{R}^n$ is called:*

i) ***Efficient*** *(**or Pareto**) if $x^* \in U$ and $G(x^*) \in C - \mathrm{INF}(G(U))$;*

ii) ***Weakly Efficient*** *if $x^* \in U$ and $G(x^*) \in C - \mathrm{INF}_{\mathrm{w}}(G(U))$.*

We denote the sets of efficient (resp., weakly efficient) solutions as $C - \arg\min\{G(x) : x \in U\}$ (resp., $C - \arg\min_{\mathrm{w}}\{G(x) : x \in U\}$).

It is easy to verify that

$$C - \arg\min\{G(x) : x \in U\} \subseteq C - \arg\min_{\mathrm{w}}\{G(x) : x \in U\}.$$

Theorem 2.1 [Luc, 1989] *Assume that $U \subseteq \mathbb{R}^n$ is a convex set and $G : U \to \mathbb{R}^m \cup \{+\infty_C\}$ is a C-convex proper map. Then*

$$C - \arg\min_{\mathrm{w}}\{G(x) : x \in U\} = \bigcup_{z \in C^* \setminus \{0\}} \arg\min\{\langle G(x), z\rangle : x \in U\}.$$

3 The problem

Let $F : \mathbb{R}^n \to \mathbb{R}^m \cup \{+\infty_C\}$ be a proper, C-convex, positively lower semicontinuous map and Ω a convex closed set, with nonempty interior. Consider the vector optimization problem

$$(VOP) \quad C - \min\left\{ F(x) : x \in \Omega \right\}. \tag{12}$$

This problem consists of finding a *feasible* point x^* such that $F(x^*)$ is *weakly efficient* for $F(\Omega)$, i.e., such that

$$\nexists x \in \Omega : \quad F(x) \prec F(x^*).$$

Unless otherwise specified, the following general assumptions on (VOP) will be used.

Assumption A

A1 $\Omega \subseteq \mathrm{dom} F$.

A2 $\exists\, \tilde{z} \in C^*$ such that $-\infty < \langle F(x), \tilde{z}\rangle$, for all $x \in \Omega$.

Remark 3.1 *In practical situations,* Assumption A2 *could be difficult to verify. Thus, it is worth pointing that Graña et al. in* (2008; Corollary 3.1) *establish a result that implies* A2, *i.e.,*

If $F(\Omega)$ *is closed and*

$$C^* \cap \operatorname{int}([O^+(\overline{\operatorname{conv}(F(\Omega))})]^*) \neq \emptyset,$$

then for all $z \in C^* \cap \operatorname{int}([O^+(\overline{\operatorname{conv}(F(\Omega))})]^*)$ *we have that*

$$-\infty < \langle F(x), z\rangle, \quad \forall\, x \in \Omega.$$

Remark 3.2 *Let us consider, for any* $u \in \operatorname{int}(\Omega)$*, the set* $\Omega_u := \{x \in \Omega : F(x) \preceq F(u)\}$*, note that:*

i) $\emptyset \neq \Omega_u \subseteq \Omega$*;*

ii) $u \in \Omega_u \cap \Omega$*;*

iii) F *C-convex, implies convexity of* Ω_u*;*

iv) F *positively lower semicontinuous, implies closedness of* Ω_u*;*

v) $\operatorname{ri}(\Omega_u) \subseteq \operatorname{int}(\Omega)$*, by items* $(i) - (iv)$ *and* Lemma 2.1.

Proposition 3.1 *Let* $d \in \mathcal{D}(\operatorname{int}(\Omega))$ *and for all* $u \in \operatorname{int}(\Omega)$ *consider the vector optimization problem*

$$P(x) \qquad C - \min\{F(x) + d(x,u)e : x \in \Omega_u\} \tag{13}$$

where $e \in \operatorname{int}(C)$*. Then for each* u *fixed, the set* $C - \operatorname{argmin}_{\mathrm{w}}\{F(x) + d(x,u)e : x \in \Omega_u\}$ *is nonempty, and it is a subset of* $\operatorname{int}(\Omega)$*.*

Proof. By assumption A2, there exists $z \in C^*$ and $M \in \mathbb{R}$ such that

$$M < \langle F(x), z\rangle \quad \forall\, x \in \Omega. \tag{14}$$

Now define the following problem

$$\begin{array}{c} \min\ F_z(x) + \langle e, z\rangle d(x,u) \\ \text{subject to} \quad x \in \Omega_u \end{array} \tag{15}$$

where $F_z : \Omega \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$F_z(x) = \langle F(x), z\rangle.$$

Note that (14) is valid in particular for all $x \in \Omega_u \subseteq \Omega$. So

$$M + \langle e, z\rangle d(x,u) \leq F_z(x) + \langle e, z\rangle d(x,u) \quad \forall\, x \in \Omega_u.$$

As the level sets of $d(\cdot, u)$ are bounded on $\mathbb{R}^n$, by (P_3), we have that the level sets of F_z are also bounded.

Note that, from (P_1), $d(\cdot, u)$ is lower semicontinuous, convex, and $\langle e, z\rangle > 0$, because $e \in \operatorname{int}(C)$ and $z \in C^* \setminus \{0\}$, as we then obtain that $F_z + \langle e, z\rangle d(\cdot, u)$ is lower semicontinuous and convex. Therefore, the mimimization in (15) reduces to compact set and the minimum is attained. In view of Theorem 2.1, we have

$$C\text{−}\operatorname{argmin}_{\mathrm{w}}\{F(x)+d(x,u)e : x \in \Omega_u\} = \bigcup_{z \in c^* \setminus \{0\}} \arg\min\{F_z(x)+\langle e, z\rangle d(x,u) : x \in \Omega_u\} \tag{16}$$

thus

$$C - \operatorname{argmin}_{\mathrm{w}}\{F(x) + d(x,u)e : x \in \Omega_u\} \neq \emptyset. \tag{17}$$

Now we prove that

$$C - \operatorname{argmin}_{\mathrm{w}}\{F(x) + d(x,u)e : x \in \Omega_u\} \subset \operatorname{int}(\Omega). \tag{18}$$

We will prove (18) by contradiction. Assume that $\exists\, \hat{z} \in C^* \setminus \{0\}$ such that

$$\emptyset \neq \arg\min\{F_{\hat{z}}(x) + \langle e, \hat{z}\rangle d(x,u) : x \in \Omega_u\} \not\subseteq \operatorname{int}(\Omega). \tag{19}$$

On the other hand, from the first order optimality conditions, for each $x \in \arg\min\{F_{\hat{z}}(x) + \langle e, \hat{z}\rangle d(x,u) : x \in \Omega_u\}$, we have

$$0 \in \partial(F_{\hat{z}} + \langle e, \hat{z}\rangle d(\cdot, u) + I_{\Omega_u})(x).$$

Now, since $\Omega_u \subset \Omega$ and $\operatorname{int}(\Omega) \subset \operatorname{dom} F_{\hat{z}} \cap \operatorname{dom} d(\cdot, u)$, so $\operatorname{ri}(\operatorname{dom} I_{\Omega_u}) \cap \operatorname{ri}(\operatorname{dom} F_{\hat{z}}) \cap (\operatorname{dom} d(\cdot, u)) \neq \emptyset$, we have that the subgradient of the sum is equal to the sum of the subgradients, see, e.g., [Rockafellar, 1970; Theorem 23.8], i.e.

$$\partial(F_{\hat{z}} + \langle e, \hat{z}\rangle d(\cdot, u) + I_{\Omega_u})(x) = \partial F_{\hat{z}}(x) + \langle e, \hat{z}\rangle \nabla_1 d(x,u) + N_{\Omega_u}(x) \tag{20}$$

where $N_{\Omega_u}(x)$ denotes the normal cone of Ω_u in x.

Since $\operatorname{dom} \nabla_1 d(\cdot, u) = \operatorname{int}(\Omega)$, it follows that $x \in \operatorname{int}(\Omega)$ and $N_{\Omega_u}(x) = 0$. Then the zeros of $\partial F_{\hat{z}} + \langle e, \hat{z}\rangle \nabla_1 d(\cdot, u) + N_{\Omega_u}$ belong to $\operatorname{int}(\Omega)$, in contradiction with (19). So (18) must hold. □

4 An exact interior proximal method (EIPM)

The method under consideration generates a sequence $\{x^k\} \subset \text{int}(\Omega)$ corresponding to the recursion: given a current iterate x^k ($x^0 \in \text{int}(\Omega)$ arbitrarily chosen), find $x^{k+1} \in \text{int}(\Omega)$ such that

$$x^{k+1} \in C - \text{argmin}_{\text{w}}\{F(y) + \beta_k d(y, x^k)e_k : y \in \Omega_k\} \tag{21}$$

where d is a proximal distance with respect to $\text{int}(\Omega)$, β_k satisfies $0 < \beta_k < \tilde{\beta}$ for some $\tilde{\beta} > 0$, the vector e_k belongs to $\text{int}(C)$ and $\|e_k\| = 1$, and $\Omega_k := \{x \in \Omega : F(x) \preceq F(x^k)\}$.

Thanks to the Proposition 3.1, the basic algorithm above is well defined.

We now make the following assumption on the map F and the initial iterate x^0.

Assumption B
The set $\left(F(x^0) - C\right) \cap F(\Omega)$ is *C-complete*, meaning that for all sequences $\{a_k\} \subset \Omega$, with $a_0 = x^0$, such that $F(a_{k+1}) \preceq F(a_k)$ for all $k \in \mathbb{N}$, there exists $a \in \Omega$ such that $F(a) \preceq F(a_k)$ for all $k \in \mathbb{N}$.

Define the set of lower bounds of the initial section by E, i.e.

$$E := \{x \in \Omega : F(x) \preceq F(x^k) \quad \forall\, k \in \mathbb{N}\}.$$

Observe that, by assumption B, E is nonempty.

Proposition 4.1 *Let H be an induced proximal distance to d, and let $\{x^k\}$ be a sequence generated by* (EIPM)*. If $(d, H) \in \mathcal{F}(\Omega)$ and assumption B above are satisfied, then we have the following:*

(i) *$H(\bar{x}, x^{k+1}) \le H(\bar{x}, x^k)$ for all k and every $\bar{x} \in E$.*

(ii) *$\{x^k\}$ is bounded with all its limit points being a weakly efficient solution of the VOP.*

Proof. (i) Since $(d, H) \in \mathcal{F}(\Omega)$, by Remark 2.1 and (8) taking $y = x^k$, $z = x^{k+1}$ and $x = \bar{x}$ get

$$\langle \nabla_1 d(x^{k+1}, x^k), \bar{x} - x^{k+1}\rangle \le H(\bar{x}, x^k) - H(\bar{x}, x^{k+1}). \tag{22}$$

As x^{k+1} solves the vector optimization problem in (21), by Theorem 2.1 there exists $z_k \in C^* \setminus \{0\}$ such that x^{k+1} solves the problem

$$\begin{array}{c} \min F_k(x) + \beta_k \langle e_k, z_k\rangle d(x, x^k) \\ \text{subject to } x \in \Omega_k \end{array} \tag{23}$$

where $F_k : \Omega \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$F_k(x) = \langle F(x), z_k \rangle.$$

Since the solution of (23) is not altered through the multiplication of z_k by positive scalar, we can assume, without loss generality, that $\|z_k\| = 1$ for all $k \in \mathbb{N}$. As $\operatorname{ri}(\operatorname{dom}(F_k)) \cap \operatorname{ri}(\operatorname{dom}(d(\cdot, x^k))) \cap \operatorname{ri}(\Omega_k) \neq \emptyset$, according to Lemma 2.2, it follows that x^{k+1} satisfies the first order optimality conditions for problem (23); i.e., there exists $u_k \in \mathbb{R}^n$ such that

$$u_k \in \partial(F_k + \beta_k \langle e_k, z_k \rangle d(\cdot, x^k))(x^{k+1}) \tag{24}$$

and

$$0 \leq \langle x - x^{k+1}, u_k \rangle \tag{25}$$

for all $x \in \Omega_k$. Since $x^{k+1} \in \operatorname{int}(\Omega)$ and from (24) we have

$$u_k = v_k + \beta_k \langle e_k, z_k \rangle \nabla_1 d(x^{k+1}, x^k) \tag{26}$$

for some

$$v_k \in \partial F_k(x^{k+1}). \tag{27}$$

Now take $\bar{x} \in E$. By definition of E, $\bar{x}$ belongs to Ω_k for all $k \in \mathbb{N}$. Combining (25) with $x = \bar{x}$ and (26), we have

$$\begin{aligned} 0 &\leq \langle \bar{v}_k, \bar{x} - x^{k+1} \rangle + \beta_k \langle e_k, z_k \rangle \langle \nabla_1 d(x^{k+1}, x^k), \bar{x} - x^{k+1} \rangle \\ &\leq F_k(\bar{x}) - F_k(x^{k+1}) + \beta_k \langle e_k, z_k \rangle \langle \nabla_1 d(x^{k+1}, x^k), \bar{x} - x^{k+1} \rangle \\ &= \langle F(\bar{x}) - F(x^{k+1}), z_k \rangle + \beta_k \langle e_k, z_k \rangle \langle \nabla_1 d(x^{k+1}, x^k), \bar{x} - x^{k+1} \rangle \\ &\leq \beta_k \langle e_k, z_k \rangle \langle \nabla_1 d(x^{k+1}, x^k), \bar{x} - x^{k+1} \rangle, \end{aligned} \tag{28}$$

using (27) in the second inequality. Now, as $\bar{x}$ belongs to E, we have $F(\bar{x}) - F(x^{k+1}) \preceq 0$. Besides, z_k belongs to $C^* \setminus \{0\}$ and therefore $\langle F(\bar{x}) - F(x^{k+1}), z_k \rangle \leq 0$, giving the fourth inequality.

By (22), (28) and $\beta_k \langle e_k, z_k \rangle > 0$, we have

$$0 \leq \langle \nabla_1 d(x^{k+1}, x^k), \bar{x} - x^{k+1} \rangle \leq H(\bar{x}, x^k) - H(\bar{x}, x^{k+1}) \tag{29}$$

Implying,

$$H(\bar{x}, x^{k+1}) \leq H(\bar{x}, x^k).$$

(ii) $\{H(\bar{x}, x^k)\}$ is nonegative and, from (i) decreasing, for all $\bar{x} \in E$, hence convergent, for all $\bar{x} \in E$.

Since $\{H(\bar{x}, x^k)\}$ is decreasing, we have that $H(\bar{x}, x^k) \leq H(\bar{x}, x^0)$ and therefore $\{x^k\}$ is bounded by Remark 2.1.

Now, we show that the cluster points of $\{x^k\}$ are weakly efficient of VOP.

Since $\{x^k\}$ is bounded, it has cluster points. Let $\hat{x}$ be a cluster point of $\{x^k\}$ and let $\{x^{j_k}\}$ be a sequence convergent to it.

Define $F_z : \mathbb{R}^n \to \mathbb{R}$ as $F_z(x) = \langle F(x), z\rangle$, we claim that

$$F_z(\hat{x}) \leq F_z(x^k) \quad \forall\, z \in C^* \quad \forall\, k \in \mathbb{N}. \tag{30}$$

In fact, as F is positively lower semicontinuous and C- convex, F_z is lower semicontinuous and convex so that $F_z(\hat{x}) \leq \lim\limits_{k\to\infty} F_z(x^{j_k})$. Since $F(x^{k+1}) \preceq F(x^k)$ for all $k \in \mathbb{N}$, we have $F_z(x^{k+1}) \leq F_z(x^k)$ for all $k \in \mathbb{N}$ so that $\lim\limits_{k\to\infty} F_z(x^{j_k}) = \inf\{F_z(x^k)\}$. Thus (30) holds, which implies

$$F(\hat{x}) \preceq F(x^k) \quad \forall\, k \in \mathbb{N}, \tag{31}$$

because $F(x^k) - F(\hat{x}) \in (C^*)^* = C$.

On the other hand, take z_k as chosen in (23). Since $\|z_k\| = 1$ for all k, there exists a cluster point of $\{z_k\}$, say $\bar{z}$, which is a limit of the sequence $\{z_{j_k}\}$. As C^* is closed it follows that $\bar{z}$ belongs to C^*. Thus, we have

$$\begin{array}{rcl}
\langle F(x) - F(\hat{x}), z_{j_k}\rangle & \geq & \langle F(x) - F(x^{j_k+1}), z_{j_k}\rangle \\
& = & F_{j_k}(x) - F_{j_k}(x^{j_k+1}) \\
& \geq & \langle v_{j_k}, x - x^{j_k+1}\rangle \\
& = & \langle u_{j_k}, x - x^{j_k+1}\rangle - \beta_{j_k}\langle e_{j_k}, z_{j_k}\rangle\langle \nabla_1 d(x^{j_k+1}, x^{j_k}), x - x^{j_k+1}\rangle \\
& \geq & -\beta_{j_k}\langle e_{j_k}, z_{j_k}\rangle\langle \nabla_1 d(x^{j_k+1}, x^{j_k}), x - x^{j_k+1}\rangle \\
& \geq & -\beta_{j_k}\langle e_{j_k}, z_{j_k}\rangle(H(x, x^{j_k}) - H(x, x^{j_k+1}))
\end{array} \tag{32}$$

for all $x \in \Omega_{j_k}$, using (31) in the first inequality, (27) in the third, (26) in the fourth equality, (25) in the fifth inequality, and (8) in the sixth.

Note that, since $\{\beta_k\} \subset \mathbb{R}$ is bounded, and $\|z_k\| = \|e_k\| = 1$, we have that $\{\beta_k\langle e_k, z_k\rangle\} \subset \mathbb{R}$ is bounded.

Now, we will prove that $\hat{x}$ is a weakly efficient solution of it VOP by contradiction. Assume there exists $\bar{x} \in \Omega$ such that

$$F(\bar{x}) \prec F(\hat{x}). \tag{33}$$

It follows from (33) that, $\bar{x} \in E$ and $\{H(\bar{x}, x^k)\}$ is a sequence convergent.

Taking (32) with $x = \bar{x}$, we get that the limit of the rightmost expression in (32) as $k \to \infty$ vanishes, and therefore

$$\langle F(\bar{x}) - F(\hat{x}), \bar{z} \rangle \geq 0. \tag{34}$$

Since $\{z_{j_k}\} \subset C^*$, $\|z_{j_k}\| = 1$ and $\bar{z}$ is limit of $\{z_{j_k}\}$, it follows from Lemma 2.4 that $\bar{z} \neq 0$. Then, (34) contradicts the fact that $\bar{z} \in C^* \setminus \{0\}$ and (33). Therefore, $\hat{x}$ must be a weakly efficient solution of VOP. $\square$

Theorem 4.1 *Let* $(d, H) \in \mathcal{F}^*(\Omega)$ *and let* $\{x^k\}$ *be a sequence generated by* (EIPM). *Then, sequence* $\{x^k\}$ *converges to a weakly efficient solution of the VOP.*

Proof. Since that $\{x^k\}$ is bounded, by Proposition 4.1, we consider two subsequences of this with

$$\lim_{k\to\infty} x^{j_k} = \hat{x}, \qquad \lim_{k\to\infty} x^{l_k} = x^*.$$

Note that $\hat{x}$ and $x^* \in E$.

We make use of (P_6), setting $y := x^*$, $y^k := x^{l_k}$ and $\bar{y} := \hat{x}$.

If (9) is valid, then $\lim_{k\to\infty} H(x^*, x^{l_k}) = 0$, and since the whole sequence $\{H(x^*, x^k)\}$ converges, one gets

$$\lim_{k\to\infty} H(x^*, x^k) = 0$$

Now, applying (P_5) with $\bar{y} := x^*$, $y^k := x^{j_k}$, we obtain $x^* = \hat{x}$.

But if (10) holds true and $\hat{x} \neq x^*$, then $\lim_{k\to\infty} H(\hat{x}, x^{l_k}) = +\infty$, which contradicts the convergence of $\{H(\hat{x}, x^k)\}$.

Then $\{x^k\}$ converges to a weakly efficient solution of VOP. $\square$

Remark 4.1 *The smaller set* $\mathcal{F}^*(\Omega)$ *is only necessary to ensure the convergence of the sequence to some weakly efficient solution of* VOP.

Remark 4.2 *Let* $e \in \operatorname{int}(C)$ *be fixed, and let* x^k *be the kth iterate of a sequence generated by EIPM. Then* $x^k \in C - \arg\min_{\mathrm{w}}\{F(x) : x \in \Omega\}$ *if and only if*

$$\inf\{t \in \mathbb{R} : \exists\, x \in \Omega \text{ such that } F(x) \preceq F(x^k) + te\} = 0.$$

The proof of the above remark is obvious when using the definition of a weakly efficient solution and the continuity at 0 of the map $t \mapsto F(x^k) - te - F(x)$ for each fixed $x \in \operatorname{dom}(F)$.

Next we discuss a few issues related to assumption B, i.e., the $C-$completeness hypothesis. First, note that, in scalar minimization problems ($m = 1$, $c = [0, +\infty)$), when the set of minimizers is not empty, every section $(F(x^0) - C) \cap F(\Omega)$ is $C-$complete. We also have the following observation (see Luc, 1989; Lemma 3.5).

Remark 4.3 *Each of the following is a sufficient condition for assumption B:*

1. *The set $(F(x^0) - C) \cap F(\Omega)$ is compact.*
2. *The set $(F(x^0) - C) \cap F(\Omega)$ is weakly compact.*
3. *The set $(F(x^0) - C) \cap F(\Omega)$ has a lower bound (i.e., there exists some $a \in \Omega$ such that, for all $x \in \Omega$ verifying $F(x) \preceq F(x^0)$, we have $F(a) \preceq F(x)$) and the cone C has the Daniell property (i.e., any decreasing net having a lower bound converges to its infimum).*

In practical situations, assumption B could be difficult to verify. Thus, it is interesting to establish some convergence results, albeit weaker than Proposition 4.1, for situations in which this assumption fails to hold. This is the content of the following two propositions.

Proposition 4.2 *Let $F : \mathbb{R}^n \to \mathbb{R}^m \cup \{+\infty_C\}$ be a proper $C-$convex map, positively lower semicontinuous. If $(d, H) \in \mathcal{F}^*$ and sequence $\{x^k\}$, generated by* (EIPM) *(without assuming that B holds) has a cluster point, then it is convergent to a weakly efficient solution of VOP.*

Proof. By assumption, there exists a convergent subsequence $\{x^{j_k}\}$ of $\{x^k\}$ whose limit is some $\hat{x} \in \Omega$. Since sequence $\{F(x^k)\}$ is $C-$nonincreasing, we have that, for all $z \in C^*$, sequence $\{\langle F(x^{j_k}), z\rangle\}$ is nonincreasing and, since the scalar convex function $\langle F(\cdot), z\rangle$ is lower semicontinuous, we obtain that $\langle F(\hat{x}), z\rangle$ is a lower bound for $\{\langle F(x^{j_k}), z\rangle\}$. Since this result holds for all $z \in C^*$, we get that $F(\hat{x}) \preceq F(x^{j_k})$ for all $k \in \mathbb{N}$, implying that $F(\hat{x}) \preceq F(x^k)$. From this point on, the argument continues as in the proof of Proposition 4.1 after (31). □

Proposition 4.3 *Let $F : \mathbb{R}^n \to \mathbb{R}^m \cup \{+\infty_C\}$ be a proper $C-$convex map, positively lower semicontinuous. Assume $(d, H) \in \mathcal{F}^*$ and that $\operatorname{int}(C^*) \neq \emptyset$. Let $\{x^k\}$ be a sequence generated by* (EIMP). *Supose $\{x^k\}$ is unbounded and that the set of finite cluster points of $\{F(x^k)\}$ is nonempty. Then sequence $\{F(x^k)\}$ converges to an element of the weakly infimal set, $C - \mathrm{INF_w}\{F(\Omega)\}$, of VOP.*

Proof. Note first that our assumption implies that $\{x^k\}$ is not convergent. Consider a subsequence $\{F(x^{j_k})\}$ convergent to some $\bar{y} \in \mathbb{R}^m$. We want to prove that the whole sequence $\{F(x^k)\}$ converges to $\bar{y} \in \mathbb{R}^m$. Note that $\bar{y}$ belongs to $\overline{F(\Omega)}$. For every $z \in C^*$, sequence $\{\langle F(x^k), z\rangle\}$ is convergent to $\langle \bar{y}, z\rangle$ as it is nonincreasing and admits a subsequence converging to $\langle \bar{y}, z\rangle$. On the other hand, since $\{F(x^k)\}$ is $C-$nonincreasing, and C is closed, it follows that $\bar{y}$ is a $C-$lower bound for this sequence. Next we take some $z \in \text{int}(C^*)$, and, using Lemma 2.5, we get

$$\langle F(x^k) - \bar{y}, z\rangle \geq \text{dist}(z, \mathbb{R}^m \setminus C^*)\|F(x^k) - \bar{y}\| \tag{35}$$

for all $k \in \mathbb{N}$. Since the left-hand side of (35) goes to zero as $k \to \infty$, and $\text{dist}(z, \mathbb{R}^m \setminus C^*) > 0$, we obtain that $\{F(x^k)\}$ is convergent to $\bar{y}$.

If $\bar{y} \notin \inf_{\text{w}}\{F(\Omega)\}$, then there exists $\bar{x} \in \Omega$ with $F(\bar{x}) \prec \bar{y}$. Using the same argument as in the proof for Proposition 4.2, we get that $\bar{y} \prec F(x^k)$ for all $k \in \mathbb{N}$. Then, we can establish the convergence of $\{x^k\}$ to a weakly efficient solution of VOP with the same argument as in the proof for Proposition 4.1 from (31) on. This contradicts the assumption that $\{x^k\}$ is unbounded, establishing the result. □

5 An inexact interior proximal method (IIPM).

In this part, we consider a generalized proximal method for solving VOP which admits both a sucessive approximation of the subdifferential of the scalar representation, $\partial(\langle F(\cdot), z\rangle + I_{\Omega_k})$, and inexact calculation of the proximal iterates.

The method under consideration generates sequences $\{x^k\} \subset \text{int}(\Omega)$ and $\{w^k\} \subset \mathbb{R}^n$ corresponding to the recursion: given a current iterate x^k ($x^0 \in \text{int}(\Omega)$ is arbitrarily chosen), find $(x^{k+1}, w^{k+1}) \in \text{int}(\Omega) \times \mathbb{R}^n$ such that

$$w^{k+1} \in Q^k(x^{k+1}) + \beta_k\langle e_k, z_k\rangle \nabla_1 \tilde{d}(x^{k+1}, x^k) \tag{36}$$

where $\partial(\langle F(\cdot), z_k\rangle + I_{\Omega_k}) \subset Q^k \subset \partial_{\varepsilon_k}(\langle F(\cdot), z_k\rangle + I_{\Omega_k})$, $\Omega_k = \{x \in \Omega : F(x) \preceq F(x^k)\}$, $\tilde{d}$ is a double regularization, i.e. $\tilde{d} = d_1 + d_2$, with $d_i \in \mathcal{D}(\text{int}(\Omega))$ $i = 1, 2$, $\beta_k \in (0, \tilde{\beta}]$, $\tilde{\beta} < +\infty$, $e_k \in \text{int}(C)$ with $\|e_k\| = 1$, and $z_k \in C^*$, $\|z_k\| = 1$ and $\inf_{x\in\Omega}\langle F(x), z_k\rangle > -\infty$.

We have $\partial f(x) = \partial_0 f(x) \subset \partial f_\varepsilon(x)$ for all convex $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, all $x \in \mathbb{R}^n$, and all $\varepsilon \in \mathbb{R}_+$. Thus, following the proof of Proposition 3.1, as $\tilde{d} \in \mathcal{D}(\text{int}(\Omega))$, we obtain that for $x^k \in \text{int}(\Omega)$ the inclusion

$$0 \in \partial(\langle F(\cdot), z_k\rangle + I_{\Omega_k})(x) + \beta_k\langle e_k, z_k\rangle \nabla_1 \tilde{d}(x, x^k)$$

is solvable. So, for any $w^{k+1} \in \mathbb{R}^n$, the existence of the point x^{k+1} that the pair (x^{k+1}, w^{k+1}) satifies (36) is guaranteed; moreover, $x^{k+1} \in \text{int}(\Omega)$ is secured.

To analyze the convergence of sequence $\{x^k\}$ to a solution of VOP, an additional assumption is required.

Assumption C

C1 $\sum_{k=1}^{\infty} \delta_k < \infty$, where $\delta_k := \max\{\frac{\|w^{k+1}\|}{\beta_k \langle e_k, z_k \rangle}, \frac{\varepsilon_k}{\beta_k \langle e_k, z_k \rangle}\}$;

C2 For each $x \in \Omega$, there exist constants $\alpha(x) > 0$, $c(x)$:

$$H_2(x, \nu) + c(x) \geq \alpha(x)\|x - \nu\|, \quad \forall\, \nu \in \operatorname{int}(\Omega)$$

where H_2 is the induced proximal distance to d_2.

From (Kaplan and Tichatschke, 2004) it follows that condition C2 is valid, in particular, for $H_2 = D_{h_j}$, $j = 1, 2, 3$, the Bregman distance defined by h_j, $j = 1, 2, 3$, where

- $h_1(x) = \sum_{i=1}^{n} |x_i|^{\rho}$ with $\rho > 1$;
- $h_2(x) = \sum_{i=1}^{n} (x_i \ln x_i - x_i)$;
- $h_3(x) = \sum_{i=1}^{n} (x_i \ln(e^{x_i} - 1))$.

Proposition 5.1 *Let H_i be an induced proximal distance to d_i, $i = 1, 2$, and let $\{x^k\}$ be a sequence generated by* (IIPM). *If $(d_i, H_i) \in \mathcal{F}(\Omega)$, $i = 1, 2$ and assumptions B, C above are satisfied, then we have the following:*

(i) *The sequence $\{\tilde{H}(\bar{x}, x^k)\}$ is convergent for every $\bar{x}$ belonging to E, the set of lower bounds of the initial section, where $\tilde{H} = H_1 + H_2$;*

(ii) *$\{x^k\}$ is bounded with all its points being weakly efficient solutions of VOP.*

Proof: (i) From (36) there exists $q^{k+1} \in Q^k(x^{k+1})$ satisfying

$$\langle q^{k+1} + \beta_k \langle e_k, z_k \rangle \nabla_1 \tilde{d}(x^{k+1}, x^k), x - x^{k+1} \rangle \geq -\|w^{k+1}\| \|x - x^{k+1}\|, \quad \forall\, x \in \Omega \quad (37)$$

Since $(d_i, H_i) \in \mathcal{F}(\Omega)$, for $i = 1, 2$, we have, due to Definition 2.2

$$\langle \nabla_1 \tilde{d}(x^{k+1}, x^k), x - x^{k+1} \rangle \leq \tilde{H}(x, x^k) - \tilde{H}(x, x^{k+1}). \quad (38)$$

This yields

$$\tilde{H}(x, x^{k+1}) - \tilde{H}(x, x^k) \leq \frac{1}{\beta_k \langle e_k, z_k \rangle} \langle q^{k+1}, x - x^{k+1} \rangle + \frac{\|w^{k+1}\|}{\beta_k \langle e_k, z_k \rangle} \|x - x^{k+1}\|, \quad \forall x \in \Omega. \tag{39}$$

As $\partial(\langle F(\cdot), z_k \rangle + I_{\Omega_k}) \subset Q^k \subset \partial_{\varepsilon_k}(\langle F(\cdot), z_k \rangle + I_{\Omega_k})$, one gets, for $\bar{x} \in E$, that

$$0 \geq \langle F(\bar{x}), z_k \rangle - \langle F(x^k), z_k \rangle \geq \langle q^{k+1}, \bar{x} - x^{k+1} \rangle - \varepsilon_k.$$

Thus

$$\langle q^{k+1}, \bar{x} - x^{k+1} \rangle \leq \varepsilon_k. \tag{40}$$

and, as consequence of C2, we have

$$\|\bar{x} - x^{k+1}\| \leq \frac{1}{\alpha(\bar{x})} \left[H_2(\bar{x}, x^{k+1}) + c(\bar{x})\right] \leq \frac{1}{\alpha(\bar{x})} \left[\tilde{H}(\bar{x}, x^{k+1}) + c(\bar{x})\right] \tag{41}$$

Now, we take (39) with $x = \bar{x}$, and inserting the above two inequalities lead to

$$\tilde{H}(\bar{x}, x^{k+1}) - \tilde{H}(\bar{x}, x^k) \leq \frac{\delta_k}{\alpha(\bar{x})} \tilde{H}(\bar{x}, x^{k+1}) + \left(1 + \frac{c(\bar{x})}{\alpha(\bar{x})}\right) \delta_k. \tag{42}$$

Assumption C1 provides that $\delta_k/\alpha(\bar{x}) < 1/2$ for $k \geq k_0$ sufficiently large. Therefore,

$$1 \leq \left(1 - \frac{\delta_k}{\alpha(\bar{x})}\right)^{-1} \leq 1 + \frac{2\delta_k}{\alpha(\bar{x})} < 2, \quad \forall k \geq k_0,$$

and (42) results in

$$\tilde{H}(\bar{x}, x^{k+1}) \leq \left(1 + \frac{2\delta_k}{\alpha(\bar{x})}\right) \tilde{H}(\bar{x}, x^k) + 2\left(1 + \frac{c(\bar{x})}{\alpha(\bar{x})}\right) \delta_k. \tag{43}$$

Because $\sum_{k=1}^{\infty} \delta_k < \infty$, Lemma 2.3, applied to (43), guarantees that $\{\tilde{H}(\bar{x}, x^k)\}$ converges for every $\bar{x}$ that belongs to the set of lower bounds of the initial section.

(ii) Since $\{\tilde{H}(\bar{x}, x^k)\}$ converges, we have

$$\tilde{H}(\bar{x}, x^k) \leq \tilde{M} \quad \forall k \in \mathbb{N}$$

where $\tilde{M} := \sup_{k \geq 0} \tilde{H}(\bar{x}, x^k)$.

As $\{x^k\} \subset \operatorname{int}(\Omega)$ and $\tilde{H}(\bar{x}, \cdot)$ is level bounded on $\operatorname{int}(\Omega)$, by Remark 2.1, we have that $\{x^k\}$ is bounded.

Since $\{x^k\}$ is bounded, it has cluster points. We will prove next that all of them are weakly efficient solutions of VOP.

Let $\hat{x}$ be a cluster point of $\{x^k\}$ and let $\{x^{j_k}\}$ be a sequence convergent to it. Following the proof of Propositon 4.1, we obtain that

$$F(\hat{x}) \preceq F(x^k) \qquad k \in \mathbb{N}. \tag{44}$$

On the other hand, since $\|z_k\| = 1$ for all k, there exists a cluster point of $\{z_k\}$, say $\bar{z}$, which is a limit of sequence $\{z_{j_k}\}$. Since C^* is closed, it follows that $\bar{z}$ belongs to C^*. Thus, we have

$$\langle F(x) - F(\hat{x}), z_{j_k}\rangle \geq \langle F(x) - F(x^{j_k+1}), z_{j_k}\rangle \quad \forall\, x \in \Omega. \tag{45}$$

Now, due to the convexity of function $\langle F(\cdot), z_k\rangle + I_{\Omega_k}$ and $q^{k+1} \in Q^k(x^{k+1}) \subset \partial_{\varepsilon_k}(\langle F(\cdot), z_k\rangle + I_{\Omega_k})(x^{k+1})$, relation (37), considering $k = j_k$, it implies that

$$\begin{aligned}\langle F(x) - F(x^{j_k+1}), z_{j_k}\rangle \;&\geq\; -\beta_{j_k}\langle e_{j_k}, z_{j_k}\rangle\langle \nabla_1 \tilde{d}(x^{j_k+1}, x^{j_k}), x - x^{j_k+1}\rangle \\ &\quad -\|w^{j_k+1}\|\|x - x^{j_k+1}\| - \varepsilon_{j_k} \qquad \forall\, x \in \Omega_{j_k}.\end{aligned} \tag{46}$$

From (45), (46), Definition 2.2 and Assumption C we have

$$\begin{aligned}\langle F(x) - F(\hat{x}), z_{j_k}\rangle \;&\geq\; \beta_{j_k}\langle e_{j_k}, z_{j_k}\rangle\big(\tilde{H}(x, x^{j_k+1}) - \tilde{H}(x, x^{j_k})\big) \\ &\quad -\|w^{j_k+1}\|\|x - x^{j_k+1}\| - \varepsilon_{j_k} \qquad (47) \\ &\geq\; \beta_{j_k}\langle e_{j_k}, z_{j_k}\rangle\Big[\tilde{H}(x, x^{j_k+1}) - \tilde{H}(x, x^{j_k}) \\ &\quad -\delta_{j_k+1}(\|x - x^{j_k+1}\| + 1)\Big] \quad \forall\, x \in \Omega_{j_k}. \qquad (48)\end{aligned}$$

Now, assume that $\hat{x}$ is not a weakly efficient solution of VOP, i.e., that there exists $\bar{x} \in \Omega$ such that $F(\bar{x}) \prec F(\hat{x})$.

Since $\bar{x} \in E$, we have that $\{\tilde{H}(\bar{x}, x^k)\}$ is a convergent sequence. Therefore, combining (48) with $x = \bar{x}$, the boundeness of sequences $\{\beta_k\langle e_k, z_k\rangle\}$ and $\{x^k\}$, and $z_{j_k} \to \bar{z}$ and $\delta_k \to 0$, as $k \to \infty$, we obtain that

$$\langle F(\bar{x}) - F(\hat{x}), \bar{z}\rangle \geq 0. \tag{49}$$

By the same argument in Proposition 4.1, $\bar{z} \neq 0$ and it is clear that (49) contradicts the fact that $\bar{z} \in C^*$ and the assumption that $F(\bar{x}) \prec F(\hat{x})$. Thus, such an assumption is false, and $\hat{x}$ must be a weakly efficient solution of VOP. $\square$

Theorem 5.1 *Let $(d_i, H_i) \in \mathcal{F}^*(\Omega)$, $i = 1, 2$, and let $\{x^k\}$ be a sequence generated by* (IIPM). *Then, sequence $\{x^k\}$ converges to a weakly efficient solution of VOP.*

Proof: Because the proof of this theorem is analogous to the proof of Theorem 4.1, we omit it. $\square$

References

[1] AUSLENDER, A. TEBOULLE, M. and BEN-TIBA, S. Interior proximal and multiplier methods based on second order homogeneous kernels. *Mathematics of Operations Research* 1999, 24, 645-668.

[2] AUSLENDER, A. and TEBOULLE, M. Interior gradient and proximal methods for convex and conic optimization. *SIAM Journal on Optimization* 2006, 16, 3, 697-725.

[3] BOLINTINÉANU, S. Approximate efficiency and scalar stationarity in unbounded nonsmooth convex vector optimization problems. *Journal of Optimization Theory and Applications* 2000, 106, 265-296.

[4] BOLINTINÉANU, S. Vector variational principles; $\varepsilon-$efficiency and scalar stationarity. *Journal of Convex Analysis* 2001, 8, 71-85.

[5] BONNEL, H. IUSEM, A. N. and SVAITER, B. F. Proximal methods in vector optimization. *SIAM Journal on Optimization* 2005, 15, 4, 953-970.

[6] CENG, L. and YAO, J. Approximate proximal methods in vector optimization. *European Journal of Operational Research* 2007, 183, 1-19.

[7] CHEN, G. and TEBOULLE, M. Convergence analysis of a proximal-like minimization algorithm using Bregman functions. *SIAM Journal on Optimization* 1993, 3, 3, 538-543.

[8] GRAÑA DRUMMOND, L.M.MACULAN, N. and SVAITER, B. F. On the choice of parameters for the weighting method in vector optimization. *Mathematical Programming, Serie B,* 2008, 111, 201-216.

[9] KAPLAN, A. and TICHATSCHKE, R. On inexact generalized proximal methods with a weakened error tolerance criterion. *Optimization* 2004, 53, 3-17.

[10] KAPLAN, A. and TICHATSCHKE, R. Bregman-like function and proximal methods for variational problems with nonlinear constraints. *Optimization* 2007a, 56, 1-2, 253-265.

[11] KAPLAN, A. and TICHATSCHKE, R. Interior proximal method for variational Inequalities on non-polyhedral sets. *Discussiones Mathematicae: Differential Inclutions, Control and Optimization* 2007b, 27, 71-93.

[12] KAPLAN, A. and TICHATSCHKE, R. Bregman function and auxiliary problem principle. *Optimization Methods & Software* 2008, 23, 1, 95-107.

[13] KIWIEL, K. C. Proximal minimization methods with generalized Bregman functions. *SIAM Journal on Control and Optimization* 2007, 35, 4, 1142-1168.

[14] LUC, D. T. *Theory of Vector Optimization.* Lecture Notes in Economics and Mathematical Systems, 391. Springer-Verlag, Berlin, 1989.

[15] MARTINET, B. Régularisation d'inéquations variationelles par approximations sucessives. *Révue Française d'Informatique et Recherche Opérationelle 1970*, 4, 154-159, 1970.

[16] MARTINET, B. *Algorithmes pous la Résolution de problèmes d'optimisation et minimax* Thèse d'état, Université de Grenoble 1972.

[17] MOREAU, J. . Proximité et dualité dans un espace hilbertien. *Bulletin de la Société Mathématique de France* 1965, 93,273-299.

[18] POLYAK, B. T. *Introduction to Optimization.* Translations Series in Mathematics and Engineering. Optimization Software, Inc., Publications Division, New York, 1987.

[19] ROCKAFELLAR, R. T. *Convex Analysis.* Princeton University Press, 1970.

[20] ROCKAFELLAR, R. T. Monotone operators and the proximal point algorithm. *SIAM Journal on Control and Optimization* 1976, 14, 877-898.

[21] TEBOULLE, M. Entropic proximal mappings with applications to nonlinear programming. *Mathematics of Operations Research* 1992, 17, 3, 670-690.

[22] TEBOULLE, M. Convergence of Proximal-Like Algorithms. *SIAM Journal on Optimization* 1997, 7, 3, 1069-1083.